Effect of dynamical $SU(2)$ gluons to the gap equation of the Nambu–Jona-Lasinio model in a constant background non-Abelian magnetic field

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In order to estimate the effect of dynamical gluons on chiral condensates, the gap equation of the $SU(2)$ gauged Nambu–Jona-Lasinio model, under a constant background non-Abelian magnetic field, is investigated up to the two-loop order in $2+1$ and $3+1$ dimensions. We set up a general formulation allowing both cases of electric as well as magnetic background fields. We employ the proper time method which gives a gauge-invariant fermionic functional determinant. In $3+1$ dimensions chiral symmetry breaking (χSB) is enhanced by gluons even in a zero background magnetic field and becomes more striking as the background field grows larger. In $2+1$ dimensions gluons also enhance χSB but the dependence on the background field is not simple; dynamical mass is not a monotone function of a background field for a fixed four-Fermi coupling.

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I. INTRODUCTION

The Nambu–Jona-Lasinio (NJL) model [1], the model of four fermion interactions, has been discussed as a possible realistic mechanism of chiral symmetry breaking (χSB) [2,3]. Although the interaction is nonrenormalizable in more than $1+1$ spacetime dimensions, it is regarded as a low-energy effective theory of elementary fermions after integrations of Higgs and/or gauge fields, and has been used as a phenomenological model to describe hadronic spectra, decays, and scattering [4]. (While these properties of hadrons should be derived from quantum chromodynamics, analytical methods on hand are so limited that the dynamics in the low-energy region could not be easily explored.)

In order to grasp the qualitative behavior of a system, it is sometimes useful to examine the response to the external fields or sources. Many such attempts have been made so far, for example, the $O(3)$ gauge-Higgs model in a magnetic-field source [5], fermionic models minimally coupled to strong electromagnetic fields [6], the NJL model minimally coupled to a constant magnetic field and curved spacetime [7] and its extension to the supersymmetric NJL model [8], the instanton motivated four-point interaction model of fermion at finite density [9], and so forth.

Among these, an interesting outcome is found in the study of the NJL model minimally coupled to constant external magnetic fields [10–12], where mass generation occurs even at the weakest attractive interaction between fermions in terms of the so-called "dimensional reduction (DR)" [13]. The case is also investigated by the present authors to find that the origin of DR is the infrared divergencies followed from the fermion loop integral under the influence of the external magnetic fields [12]. The phenomenon is now understood provided universal interactions are short range [14].

Inclusion of dynamical gauge fields is made for QED, in terms of the Schwinger-Dyson equation [15] or renormalization group (RG) [16]. The results show that the dynamical symmetry breaking always occurs with the aid of an external magnetic field.

The motive for this paper is traced to that of Gusynin, Hong, and Shovkovy [17], where $2+1$-dimensional $SU(2)$ gauge theory is investigated in a constant non-Abelian magnetic field by using a constant gauge potential with translational invariance [19], to find that magnetic catalysis of χSB does not occur. The study in $3+1$ dimensions is made in the reference [18], where $SU(N)$ gauged NJL model is handled in the case of weak as well as constant non-Abelian magnetic field; they incorporate the dynamical effect of gluons in RG to reach the conclusion that existence of the external field does not change the condition of χSB. Each result indicates, contrary to our expectation that gluons always trigger χSB (since they do in the low-energy region), that dynamical gluons in the external non-Abelian magnetic field do not play a major role to χSB. This curious situation is our starting point. There needs to be a more detailed analysis.

In this article we study χSB under a constant background non-Abelian magnetic field in a $SU(2)$ gauged NJL model, paying attention to (i) the direct effect of dynamical gluons to the gap equation, not in terms of RG [17,18], and (ii) the effect of the gauge choice to the results, since in the non-Abelian case the situation $B_z^3 = B(\text{constant} \geqslant 0), \text{others} = 0$, the choice [17]

$$A_x^1 = A_y^2 = \sqrt{B}, \quad \text{others} = 0 \tag{1}$$

cannot connect with [18]

$$A_x^3 = -\frac{B}{2}y, \quad A_y^3 = \frac{B}{2}x, \quad \text{others} = 0 \tag{2}$$

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since the remnant gauge transformation, which leaves B_z^3 invariant, that is, any gauge transformations with respect to the third axis, cannot bring the gauge (1) to the gauge (2) or vice versa. To work with the Green's function in momentum space it is convenient to adopt Eq. (1), but this is not a solution of the non-Abelian Maxwell equation in the vacuum:

$$(D_\mu F_{\mu\nu})^a \equiv \partial_\mu F^a_{\mu\nu} + \epsilon^{abc} A^b_\mu F^c_{\mu\nu} = 0. \tag{3}$$

In what follows we rely upon the WKB semiclassical approximation where the classical solution plays a fundamental role. Therefore we should work with the choice of Eq. (2) satisfying Eq. (3). Moreover the choice (2) satisfies a covariantly constant condition

$$(D_\rho F_{\mu\nu})^a \equiv \partial_\rho F^a_{\mu\nu} + \epsilon^{abc} A^b_\rho F^c_{\mu\nu} = 0, \tag{4}$$

enabling us to use the Fock-Schwinger proper time method [20] that was originally developed to handle Abelian background fields and gives us a gauge independent result.

The model is nonrenormalizable so that we must introduce an ultraviolet cutoff Λ which defines the theory with a given bare four-Fermi coupling g. However, it should be gauge invariant when fermions are coupled to gauge fields. In this sense, we utilize a proper time cutoff; in the two-loop order of the effective potential, it respects gauge invariance of the vacuum polarization tensor. There we need an additional regularization, since gluons in the non-Abelian magnetic background suffer from a famous instability [21], due to a tachyonic singularity in the propagator. To avoid it, we introduce a gluon mass M_g whose magnitude is assumed to be larger than that of the background magnetic field. That is, we assume that the magnitude of the background magnetic field is very small, which is different from the assumption of Gusynin *et al* [17].

The article is organized as follows: in Sec. II a general formulation is presented, where we assume generic backgrounds satisfying the covariantly constant condition Eq. (4). In the next sections, Sec. III and Sec. IV, the gap equations in $2+1$ and $3+1$ dimensions are shown. Section V is devoted to discussion. In Appendix A calculations of the kernel are presented and in Appendix B the gluon propagator is represented by means of the proper time method. Then in Appendix C we give an explicit proof that our classical solution satisfies the covariantly constant condition. And finally in Appendix D, in order to ensure the gauge independence of our calculations, that is, their correctness, we study the Ward-Takahashi relation of the vacuum polarization tensor.

II. FORMULATION

In this section, we derive the effective potential of the $SU(2)$ gauged NJL model. The Lagrangian of the system in the Euclidean formulation is given as

$$\mathcal{L} = -\frac{1}{4e^2} F^a_{\mu\nu} F^a_{\mu\nu} - \bar{\psi}\{\gamma_\mu(\partial_\mu - iA_\mu)\}\psi + \frac{g^2}{2}\begin{cases} [(\bar{\psi}\psi)^2 + (\bar{\psi} i\gamma_5\psi)^2], & D=4, \\ [(\bar{\psi}\psi)^2 + (\bar{\psi} i\gamma_4\psi)^2 + (\bar{\psi} i\gamma_5\psi)^2], & D=3, \end{cases} \tag{5}$$

where $A_\mu \equiv A^a_\mu T_a$ with T_a's $(a=1,2,3)$ being the $SU(2)$ generators given by $T_a \equiv \sigma_a/2$, where σ_a's are the Pauli matrices. For the $(2+1)$-dimensional case, a spinorial representation of the Lorentz group is given by two-component spinors, so that corresponding gamma matrices are 2×2. There is no chiral symmetry. In order to be able to discuss chiral symmetry, we introduce an additional flavor such that

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \bar{\psi} \equiv \psi^\dagger \gamma_3 \equiv (\bar{\psi}_1, -\bar{\psi}_2) \equiv (\psi_1^\dagger \sigma_3, -\psi_2^\dagger \sigma_3), \tag{6}$$

with the 4×4 gamma matrices

$$\gamma_\mu = \begin{pmatrix} \sigma_\mu & 0 \\ 0 & -\sigma_\mu \end{pmatrix}; \mu = 1 \sim 3, \quad \gamma_4 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 0 & i\mathbf{1} \\ -i\mathbf{1} & 0 \end{pmatrix}. \tag{7}$$

Chiral symmetry is realized as

$$\psi \to e^{i\alpha\gamma_4}\psi, \quad \psi \to e^{i\beta\gamma_5}\psi, \tag{8}$$

yielding a global $U(2)$ symmetry which is broken by a mass term into $U(1)\times U(1)$.

The partition function of the model is read as

$$\begin{aligned} Z &\equiv \int D[\text{gauge}] D[\psi] D[\bar{\psi}] \exp\left[\int d^D x\, \mathcal{L}\right] \\ &= \int D[\text{gauge}] D[\sigma] D[\boldsymbol{\pi}] D[\psi] D[\bar{\psi}] \\ &\quad \times \exp\left[-\int d^D x \left\{ \frac{1}{4e^2} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2g^2}(\sigma^2 + \boldsymbol{\pi}^2) \right.\right. \\ &\quad \left.\left. + \bar{\psi}\{\gamma_\mu(\partial_\mu - iA_\mu) + (\sigma + i\boldsymbol{\pi}\cdot\boldsymbol{\Gamma})\}\psi \right\}\right], \end{aligned} \tag{9}$$

where the auxiliary fields, σ and $\boldsymbol{\pi}$, have been introduced to erase the four-Fermi interactions,

$$\boldsymbol{\pi}\cdot\boldsymbol{\Gamma} = \begin{cases} \pi\gamma_5 & \text{for } D=4, \\ \pi_1\gamma_4 + \pi_2\gamma_5 & \text{for } D=3, \end{cases} \tag{10}$$

and a measure of gauge fields, $D[\text{gauge}]$, is specified after the following procedures: (i) first, integrate with respect to fermions to give

$$Z=\int D[\text{gauge}]D[\sigma]D[\boldsymbol{\pi}]\exp\Bigg[-\int d^Dx\Bigg\{\frac{1}{4e^2}F^a_{\mu\nu}F^a_{\mu\nu}+\frac{1}{2g^2}(\sigma^2+\boldsymbol{\pi}^2)\Bigg\}+\ln\det[\gamma_\mu(\partial_\mu-iA_\mu)+(\sigma+i\boldsymbol{\pi}\cdot\boldsymbol{\Gamma})]\Bigg]. \tag{11}$$

(Here and hereafter ln, det, and tr designate the functional logarithm, determinant, and trace, respectively.) (ii) Second, set up an ansatz $\sigma(x)=m(:\text{constant}),\boldsymbol{\pi}=0$, under which the equation of gauge fields reads

$$\begin{aligned}\frac{1}{e^2}(\partial_\mu\mathcal{F}^a_{\mu\nu}+\epsilon^{abc}\mathcal{A}^b_\mu\mathcal{F}^c_{\mu\nu})&\equiv\frac{1}{e^2}(\mathcal{D}_\mu\mathcal{F}_{\mu\nu})^a\\&=-\mathrm{tr}\{[\gamma_\mu(\partial_\mu-i\mathcal{A}_\mu)+m]^{-1}(-i\gamma_\nu T_a)\}.\end{aligned} \tag{12}$$

If we take $\mathcal{F}^1_{\mu\nu}=\mathcal{F}^2_{\mu\nu}=0,\mathcal{F}^3_{\mu\nu}=\text{const}$, then

$$\mathcal{A}^i_\mu=0,\quad (i=1,2),\quad \mathcal{A}^3_\mu=-\frac{1}{2}\mathcal{F}^3_{\mu\nu}x_\nu, \tag{13}$$

Eq. (12) is fulfilled. (It is easy to prove that the left-hand side vanishes but the proof of the right-hand side is rather lengthy, so it is relegated to Appendix C.) Therefore, our classical solution $\mathcal{A}^a_\mu$ (13) obeys the non-Abelian Maxwell equation (12). We do not take the higher powers of $\delta\sigma$ and $\delta\pi$,

$$\sigma(x)=m+\delta\sigma(x),\qquad \pi(x)=0+\delta\pi(x) \tag{14}$$

but remain with the lowest part. In this section we do not restrict ourselves in the pure magnetic case but in generic cases where both electric and magnetic backgrounds coexist. (iii) Third, we expand the gauge fields around $\mathcal{A}^a_\mu$

$$A^a_\mu=\mathcal{A}^a_\mu+Q^a_\mu, \tag{15}$$

with Q^a_μ being designated as quantum fields. Here, with the aid of the Faddeev-Popov trick, the measure is defined as

$$D[\text{gauge}]\equiv D[Q^a_\mu]\left|\det\frac{\delta G^a}{\delta\theta^b}\right|\exp\left[-\frac{1}{2e^2}\int d^Dx[G^a(x)]^2\right], \tag{16}$$

with

$$G^a(x)\equiv(\mathcal{D}_\mu Q_\mu)^a=\partial_\mu Q^a_\mu+\epsilon^{abc}\mathcal{A}^b_\mu Q^c_\mu. \tag{17}$$

The gauge transformation now reads

$$Q^a_\mu\mapsto Q^a_\mu+(D_\mu\theta)^a, \tag{18}$$

with $D^{ab}_\mu\equiv\delta^{ab}\partial_\mu+\epsilon^{acb}(\mathcal{A}^c_\mu+Q^c_\mu)$, so that the Faddeev-Popov determinant is given by

$$\det\frac{\delta G^a}{\delta\theta^b}=\det(D_\mu\mathcal{D}_\mu)^{ab}. \tag{19}$$

(In what follows, however, it is not necessary to worry about the F-P terms, since they are irrelevant to the gap equation.)

The partition function is given up to $O(Q^2)$, by

$$\begin{aligned}Z[A]=&\exp\left[-\int d^Dx\frac{m^2}{2g^2}+\ln\det[\gamma_\mu(\partial_\mu-i\mathcal{A}^3_\mu T_3)+m]\right]\\&\times\exp\left[-\int d^Dx\frac{1}{4e^2}\mathcal{F}^3_{\mu\nu}\mathcal{F}^3_{\mu\nu}\right]\\&\times\int D[Q^a_\mu]\exp\Bigg[-\frac{1}{2e^2}\int d^Dx\,d^Dy\,Q^a_\mu[\Delta^{-1}_{\mu\nu}+e^2\Pi_{\mu\nu}\\&+e^2(F-P\,\text{term})]^{ab}Q^b_\nu\Bigg],\end{aligned} \tag{20}$$

where Δ^{-1} is the inverse of the gluon propagator

$$(\Delta^{-1})^{ab}_{\mu\nu}\equiv-\delta_{\mu\nu}(\mathcal{D}^2)^{ab}+2\epsilon^{ab3}\mathcal{F}^3_{\mu\nu}, \tag{21}$$

under the gauge (13). In Eq. (21), the symmetric matrix $\epsilon^{ab3}\mathcal{F}^3_{\mu\nu}$ has negative eigenvalues after the diagonalization [21], thus there are tachyonic singularities. This is due to a large magnetic moment of spin-1 particle. (Recall that we are in the Euclidean world so that all gauge fields are considered as magnetic one.) In view of Eq. (21), these tachyonic singularities become harmless if we introduce a gluon mass M_g that should obey

$$(M_g)^2>|\mathcal{F}^3_{\mu\nu}|. \tag{22}$$

The situation would be guaranteed inside a hadronic phase, that is, a confining phase, where it is expected that gluons behave as massive particles [24]. The term in Eq. (20)

$$\begin{aligned}\Pi^{ab}_{\mu\nu}\equiv&-\frac{\delta^2}{\delta Q^a_\mu\delta Q^b_\nu}\ln\det[\gamma_\mu(\partial_\mu-i\mathcal{A}^3_\mu T_3-iQ_\mu)+m]\Bigg|_{Q=0}\\=&-\frac{\delta^2}{\delta Q^a_\mu\delta Q^b_\nu}\mathrm{tr}\ln[\gamma_\mu(\partial_\mu-i\mathcal{A}^3_\mu T_3-iQ_\mu)+m]\Bigg|_{Q=0},\end{aligned} \tag{23}$$

is the vacuum polarization tensor. We omit the isospin index 3 of $\mathcal{A}_\mu^3$ as well as $\mathcal{F}_{\mu\nu}^3$ from now on. (In the above sense our model is not a gauge theory any more even though the proper time regularization enables us to bring a gauge-invariant result for a vacuum polarization tensor; the proof is found in Appendix D.)

Integrating with respect to the quantum field Q_μ, we obtain

$$Z = \exp[-VTv^{(D)}] \times (m - \text{independent terms}), \tag{24}$$

where

$$v^{(D)} \equiv \frac{m^2}{2g^2} + v_1^{(D)} + v_2^{(D)}, \tag{25}$$

with

$$v_1^{(D)} \equiv -\frac{1}{VT}\mathrm{tr}\ln[\gamma_\mu(\partial_\mu - i\mathcal{A}_\mu T_3) + m], \tag{26}$$

$$\begin{aligned} v_2^{(D)} &\equiv \frac{1}{VT}\frac{e^2}{2}\mathrm{tr}(\mathbf{\Pi\Delta}) \\ &= \frac{1}{VT}\frac{e^2}{2}\int d^Dx\, d^Dy\, \Pi_{\mu\nu}^{ab}(x,y)\Delta_{\nu\mu}^{ba}(y,x), \end{aligned} \tag{27}$$

being the one-loop and the two-loop effective potential, respectively. Here V is the $(D-1)$-dimensional volume of the system, and T is the Euclidean time interval. The stationary condition for the effective potential

$$\frac{\partial v^{(D)}}{\partial m} = 0, \tag{28}$$

gives a gap equation,

$$-\frac{(4\pi)^{D/2}}{4g^2\Lambda^{D-2}} = f_1^{(D)}(x) + f_2^{(D)}(x), \quad x \equiv \frac{m^2}{\Lambda^2} \quad (0 \leqslant x \leqslant 1), \tag{29}$$

where Λ is the ultraviolet cutoff, and

$$f_i^{(D)}(x) \equiv \frac{(4\pi)^{D/2}}{2\Lambda^{D-2}}\frac{\partial v_i^{(D)}}{\partial m^2}, \quad i = 1,2. \tag{30}$$

The remaining task is, therefore, the calculation of the effective potential Eqs. (26) and (27). Let us start with the one-loop part. We utilize the proper time method [20]:

$$\begin{aligned} v_1^{(D)} &= -\frac{1}{VT}\mathrm{tr}\ln[\gamma_\mu(\partial_\mu - i\mathcal{A}_\mu T_3) + m] \\ &= -\frac{1}{2VT}\mathrm{tr}\ln\left[-(\partial_\mu - i\mathcal{A}_\mu T_3)^2 - \frac{1}{2}\sigma_{\mu\nu}\mathcal{F}_{\mu\nu}T_3 + m^2\right] \\ &= \frac{1}{2VT}\int_{1/\Lambda^2}^\infty d\tau\tau^{-1}e^{-\tau m^2}\mathrm{tr}\int d^Dx\langle x|e^{-\tau H_0}|x\rangle, \end{aligned} \tag{31}$$

where the ultraviolet cutoff Λ has been introduced, tr is taken only for the spinor and the isospin indices, and

$$H_0 \equiv \Pi_\mu^2 - \frac{1}{2}\sigma_{\mu\nu}\mathcal{F}_{\mu\nu}T_3, \tag{32}$$

with

$$\Pi_\mu \equiv \hat{p}_\mu - \mathcal{A}_\mu(\hat{x})T_3, \tag{33}$$

$[\hat{x}_\mu, \hat{p}_\nu] = i\delta_{\mu\nu}$. Write the kernel as

$$K(x,y;\tau) \equiv \langle x|e^{-\tau H_0}|y\rangle, \tag{34}$$

to find

$$\begin{aligned} K(x,y;\tau) = &\frac{1}{(4\pi\tau)^{D/2}}\left[\det\left(\frac{\sin\tau\mathcal{F}/2}{\tau\mathcal{F}/2}\right)_{\mu\nu}\right]^{-1/2}[K_0(\tau)\mathbf{I} \\ &+ K_3(\tau)T_3]\exp[iT_3\mathcal{C}] \\ &\times\exp\left[-\frac{1}{4}(x-y)_\mu\left(\frac{\mathcal{F}}{2}\cot\frac{\tau\mathcal{F}}{2}\right)_{\mu\nu}(x-y)_\nu\right], \end{aligned} \tag{35}$$

whose derivation is given in Appendix A. In Eq. (35), quantities are defined such that

$$\mathcal{C} \equiv -\frac{1}{2}\mathcal{F}_{\mu\nu}x_\mu y_\nu, \tag{36}$$

$$\begin{aligned} K_0(\tau) \equiv &\cosh\frac{\tau F_+}{2}\cosh\frac{\tau F_-}{2} \\ &- \gamma_5\sinh\frac{\tau F_+}{2}\sinh\frac{\tau F_-}{2}, \end{aligned} \tag{37}$$

$$\begin{aligned} K_3(\tau) \equiv \sigma_{\mu\nu}\Bigg(&N_{\mu\nu}^+\sinh\frac{\tau F_+}{2}\cosh\frac{\tau F_-}{2} \\ &+ N_{\mu\nu}^-\cosh\frac{\tau F_+}{2}\sinh\frac{\tau F_-}{2}\Bigg), \end{aligned} \tag{38}$$

where

$$F_+ \equiv \sqrt{B^2+\boldsymbol{E}^2}, \quad F_- = 0, \qquad B \equiv \mathcal{F}_{12}, \quad \boldsymbol{E} \equiv (\mathcal{F}_{13}, \mathcal{F}_{23}) \tag{39}$$

for $D=3$,

$$F_+ \equiv \frac{\{(|\boldsymbol{B}+\boldsymbol{E}|+|\boldsymbol{B}-\boldsymbol{E}|\}}{2}, \quad \boldsymbol{E} \equiv (\mathcal{F}_{14}, \mathcal{F}_{24}, \mathcal{F}_{34})$$
$$F_- \equiv \frac{\{|\boldsymbol{B}+\boldsymbol{E}|-|\boldsymbol{B}-\boldsymbol{E}|\}}{2}, \quad \boldsymbol{B} \equiv (\mathcal{F}_{23}, \mathcal{F}_{31}, \mathcal{F}_{12}), \tag{40}$$

for $D=4$,

and

$$N^+_{\mu\nu} \equiv \frac{\mathcal{F}_{\mu\nu}}{F_+} \equiv N_{\mu\nu}, \quad N^-_{\mu\nu} = 0, \quad \text{for} \quad D=3, \tag{41}$$

$$N^+_{\mu\nu} \equiv \frac{\mathcal{F}_{\mu\nu}F_+ - \tilde{\mathcal{F}}_{\mu\nu}F_-}{F_+^2 - F_-^2},$$

$$N^-_{\mu\nu} \equiv \frac{\tilde{\mathcal{F}}_{\mu\nu}F_+ - \mathcal{F}_{\mu\nu}F_-}{F_+^2 - F_-^2}, \quad \text{for} \quad D=4, \tag{42}$$

with $\tilde{\mathcal{F}}_{\mu\nu} \equiv \epsilon_{\mu\nu\lambda\rho}\mathcal{F}_{\lambda\rho}/2$ being the dual of $\mathcal{F}_{\mu\nu}$.

Therefore,

$$v_1^{(D)} = \frac{1}{2VT}\int_{1/\Lambda^2}^{\infty} d\tau \tau^{-1} e^{-\tau m^2} \mathrm{tr} \int d^D x K(x,x;\tau)$$
$$= \frac{4}{(4\pi)^{D/2}} \int_{1/\Lambda^2}^{\infty} d\tau\, \tau^{-D/2-1} e^{-\tau m^2} G_D(\tau F), \tag{43}$$

where

$$G_D(\tau F) \equiv \begin{cases} \dfrac{\tau F_+}{2}\coth\left(\dfrac{\tau F_+}{2}\right) & \text{for} \quad D=3, \\[2ex] \dfrac{\tau^2 F_+ F_-}{4}\coth\left(\dfrac{\tau F_+}{2}\right)\coth\left(\dfrac{\tau F_-}{2}\right) & \text{for} \quad D=4. \end{cases} \tag{44}$$

In order to calculate the two-loop contribution [Eq. (27)] first we express the gluon propagator [Eq. (21)] in terms of the proper time as

$$\Delta^{ab}_{\mu\nu}(x,y) = \int_0^{\infty} d\tau e^{-\tau M_g^2} [(e^{-2i\tau\mathcal{F}})_{\mu\nu} \langle x | e^{-\tau \Pi_\mu^2} | y \rangle]^{ab},$$
$$a,b = 1,2,3, \tag{45}$$

where M_g is the gluon mass [Eq. (22)]. The results whose derivation is relegated to Appendix B are

$$\Delta^{ij}_{\mu\nu}(x,y) = \{(\cos\mathcal{C} + \boldsymbol{\epsilon}\sin\mathcal{C})[\Delta^1_{\mu\nu}(x-y) + \boldsymbol{\epsilon}\Delta^2_{\mu\nu}(x-y)]\}^{ij},$$
$$i,j = 1,2, \tag{46}$$

$$\Delta^{33}_{\mu\nu}(x-y) \equiv \Delta^3_{\mu\nu}(x-y) = \delta_{\mu\nu}\int_0^{\infty} d\tau \frac{e^{-\tau M_g^2}}{(4\pi\tau)^{D/2}} \times \exp\left[-\frac{1}{4\tau}(x-y)^2\right], \tag{47}$$

and others $=0$. (If we work with an Abelian gauge theory, only the $\Delta^3_{\mu\nu}$ term survives. In this sense, we call Eq. (47) the Abelian part.) Here $\mathcal{C}$ has been given in Eq. (36), $\boldsymbol{\epsilon}$ is a 2×2 antisymmetric matrix ($\boldsymbol{\epsilon}^{12} = -\boldsymbol{\epsilon}^{21} = 1$),

$$\Delta^1_{\mu\nu}(x-y) = \int_0^{\infty} d\tau \frac{e^{-\tau M_g^2}}{(4\pi\tau)^{D/2}} (\cos 2\,\tau\mathcal{F})_{\mu\nu} \left[\det\left(\sin\tau\frac{\mathcal{F}}{\tau\mathcal{F}}\right)_{\mu\nu}\right]^{-1/2} \times \exp\left[-\frac{1}{4}(x-y)\cdot(\mathcal{F}\cot\tau\mathcal{F})\cdot(x-y)\right], \tag{48}$$

$$\Delta^2_{\mu\nu}(x-y) = -\int_0^{\infty} d\tau \frac{e^{-\tau M_g^2}}{(4\pi\tau)^{D/2}} (\sin 2\,\tau\mathcal{F})_{\mu\nu} \left[\det\left(\frac{\sin\tau\mathcal{F}}{\tau\mathcal{F}}\right)_{\mu\nu}\right]^{-1/2} \times \exp\left[-\frac{1}{4}(x-y)\cdot(\mathcal{F}\cot\tau\mathcal{F})\cdot(x-y)\right]. \tag{49}$$

Second we need the vacuum polarization tensor [Eq. (23)] for the two-loop effective potential [Eq. (27)], whose proper time expression is found as follows:

$$\mathrm{tr}\ln[\gamma_\mu(\partial_\mu - i\mathcal{A}_\mu T_3 - iQ_\mu) + m]$$
$$= \frac{1}{2}\mathrm{tr}\ln[\gamma_\mu\gamma_\nu(\Pi_\mu - Q_\mu)(\Pi_\nu - Q_\nu) + m^2]$$
$$= -\frac{1}{2}\int_{1/\Lambda^2}^{\infty} du\, u^{-1} e^{-um^2} \mathrm{Tr}(e^{-u(H_0+H_1+H_2)}), \tag{50}$$

where H_0 has been given in Eq. (32), and

$$H_1 \equiv -2Q_\mu\Pi_\mu - [\Pi_\mu, Q_\mu] + i\sigma_{\mu\nu}[\Pi_\nu, Q_\mu], \tag{51}$$

$$H_2 \equiv Q_\mu^2 + \frac{i}{2}\sigma_{\mu\nu}[Q_\mu, Q_\nu], \quad Q_\mu \equiv Q_\mu^a T_a. \tag{52}$$

Expand the final expression of Eq. (50) with respect to the quantum field Q_μ up to the second order to find

$$\mathrm{tr}(e^{-u(H_0+H_1+H_2)}) = \mathrm{tr}(e^{-uH_0}) + I_1 + I_2, \tag{53}$$

with

$$I_1 \equiv -u\,\mathrm{tr}(H_2 e^{-uH_0}) = -u\,\mathrm{tr}\int d^Dx \langle x|H_2 e^{-uH_0}|x\rangle$$
$$= -u\,\mathrm{tr}\int d^Dx H_2(x)K(x,x;u), \tag{54}$$

$$H_2(x) \equiv Q_\mu^2(x) - \frac{1}{2}\sigma_{\mu\nu}\epsilon^{ab3}Q_\mu^a(x)Q_\nu^b(x)T_3, \tag{55}$$

and

$$I_2 \equiv \frac{u}{2}\int_0^u du_1 \mathrm{tr}(H_1 e^{-(u-u_1)H_0}H_1 e^{-u_1H_0})$$
$$= \frac{u^2}{2}\mathrm{tr}\int_0^1 dv\,(H_1 e^{-(1-v)uH_0/2}H_1 e^{-(1+v)uH_0/2})$$
$$= \frac{u^2}{2}\int_0^1 dv\;\mathrm{tr}\int d^Dx\,d^Dy\langle x|H_1 e^{-(1-v)uH_0/2}|y\rangle \times\langle y|H_1 e^{-(1+v)uH_0/2}|x\rangle$$
$$= \frac{u^2}{2}\int_0^1 dv\;\mathrm{tr}\int d^Dx\,d^Dy H_1\left(x,y;\frac{1-v}{2}u\right)\times K\left(x,y;\frac{1-v}{2}u\right)H_1\left(y,x;\frac{1+v}{2}u\right)\times K\left(y,x;\frac{1+v}{2}u\right), \tag{56}$$

$$H_1(x,y;\tau) \equiv -iQ_\mu(x)\left(\frac{\mathcal{F}}{2}\cot\frac{\tau\mathcal{F}}{2}\right)_{\mu\nu}(x-y)_\nu - Q_\mu(x)\mathcal{F}_{\mu\nu}(x-y)_\nu T_3 + i\mathcal{D}_\mu^{ab}Q_\mu^b(x)T_a + \sigma_{\mu\nu}\mathcal{D}_\nu^{ab}Q_\mu^b(x)T_a, \tag{57}$$

where we have utilized the $|x\rangle$ representation such that

$$\langle x|\Pi_\mu e^{-\tau H_0}|y\rangle = \left(-i\partial_\mu^x + \frac{1}{2}\mathcal{F}_{\mu\nu}x_\nu T_3\right)K(x,y;\tau)$$
$$= \left[\frac{i}{2}\left(\frac{\mathcal{F}}{2}\cot\frac{\tau\mathcal{F}}{2}\right)_{\mu\nu}(x-y)_\nu\mathbf{I} + \frac{1}{2}\mathcal{F}_{\mu\nu}(x-y)_\nu T_3\right]K(x,y;\tau), \tag{58}$$

with the aid of Eq. (35), and made a change of variable from u_1 to v, $u_1=(1+v)u/2$, in the first line of Eq. (56). In terms of I_1, I_2, the vacuum polarization tensor [Eq. (23)] reads

$$\Pi_{\mu\nu}^{ab} = \frac{1}{2}\frac{\delta^2}{\delta Q_\mu^a \delta Q_\nu^b}\int_{1/\Lambda^2}^\infty du\,u^{-1}e^{-um^2}[I_1+I_2]. \tag{59}$$

If we write

$$\Pi_{\mu\nu}^{ij}(x,y) = \{(\cos\mathcal{C} + \boldsymbol{\epsilon}\sin\mathcal{C})[\Pi_{\mu\nu}^1(x-y) + \boldsymbol{\epsilon}\Pi_{\mu\nu}^2(x-y)]\}^{ij}, \quad \text{for } i,j=1,2, \tag{60}$$

$$\Pi_{\mu\nu}^{33}(x-y) = \Pi_{\mu\nu}^3(x-y), \quad (\text{Abelian part}), \tag{61}$$

in order to meet the expression of the gluon propagators Eqs. (46) and (47), the two-loop contribution, Eq. (27), is expressed as

$$v_2^{(D)} = \frac{1}{VT}\frac{e^2}{2}\int d^Dx\,d^Dy\,\Pi_{\mu\nu}^{ab}(x,y)\Delta_{\nu\mu}^{ba}(y,x)$$
$$= e^2\int\frac{d^Dp}{(2\pi)^D}\left[\Pi_{\mu\nu}^1(p)\Delta_{\nu\mu}^1(p) - \Pi_{\mu\nu}^2(p)\Delta_{\nu\mu}^2(p) + \frac{1}{2}\Pi_{\mu\nu}^3(p)\Delta_{\nu\mu}^3(p)\right]. \tag{62}$$

Again the third term designates the Abelian contribution.

The explicit forms of $\Pi_{\mu\nu}^1 \sim \Pi_{\mu\nu}^3$ are found, after performing the Fourier transformation, as follows: in $2+1$ dimensions, put

$$\phi^{(3)} \equiv p\cdot\left(\frac{\cos uv\mathcal{F}/2 - \cos u\mathcal{F}/2}{u\mathcal{F}\sin u\mathcal{F}/2}\right)\cdot p, \tag{63}$$

$$\alpha_{\mu\nu}^{\pm} \equiv \left(\frac{\cos u\mathcal{F}/2 \pm \cos uv\mathcal{F}/2}{\sin u\mathcal{F}/2}\right)_{\mu\nu},$$
$$\beta_{\mu\nu} \equiv \left(\frac{\sin uv\mathcal{F}/2}{\sin u\mathcal{F}/2}\right)_{\mu\nu}, \tag{64}$$

and utilize $N_{\mu\nu}$ given in Eq. (41) with obvious abbreviations such that

$$N_{\mu\nu}^2 \equiv N_{\mu\rho}N_{\rho\nu}, \quad N_{\mu\nu}^3 \equiv N_{\mu\rho}N_{\rho\sigma}N_{\sigma\nu},$$
$$(Np)_\mu \equiv N_{\mu\nu}p_\nu, \quad \text{etc.} \tag{65}$$

to obtain

$$
\begin{aligned}
\Pi^1_{\mu\nu}(p)=&\frac{F_+}{2(4\pi)^{3/2}}\int_{1/\Lambda^2}^{\infty}du\int_0^1 dv\; u^{1/2}e^{-u(m^2+\phi^{(3)})}\frac{1}{\sinh uF_+/2}\Bigg[-\frac{F_+}{\sinh uF_+/2}\delta_{\mu\nu}+\frac{F_+}{4}\left(\cosh\frac{uF_+}{2}+\cosh\frac{uvF_+}{2}\right)(N\,\alpha^-)_{\mu\nu}\\
&+\frac{F_+}{2}\cosh\frac{uvF_+}{2}(N\,\alpha^+)_{\mu\nu}+\left\{\frac{F_+}{2\sinh uF_+/2}\left(\cosh\frac{uF_+}{2}-\cosh\frac{uvF_+}{2}\right)^2+F_+\sinh\frac{uF_+}{2}\right\}(N^2)_{\mu\nu}-\frac{5F_+}{4}\left(\cosh\frac{uF_+}{2}\right.\\
&\left.-\cosh\frac{uvF_+}{2}\right)(N^3\alpha^-)_{\mu\nu}+F_+\sinh\frac{uvF_+}{2}(N^2\beta)_{\mu\nu}+\cosh\frac{uvF_+}{2}(\beta p)_\mu(\beta p)_\nu-\coth\frac{uF_+}{2}\sinh\frac{uvF_+}{2}\,\delta_{\mu\nu}p\cdot(\beta p)\\
&+\frac{\cosh uF_+/2+\cosh uvF_+/2}{2}\{\delta_{\mu\nu}p^2-p_\mu p_\nu+\delta_{\mu\nu}(\alpha^-p)\cdot(\alpha^-p)-(\alpha^-p)_\mu(\alpha^-p)_\nu\}\\
&+\frac{\cosh uF_+/2-\cosh uvF_+/2}{2}\{\delta_{\mu\nu}(Np)\cdot(Np)-(N^2)_{\mu\nu}p^2+p_\mu(N^2p)_\nu+(N^2p)_\mu p_\nu-3(Np)_\mu(Np)_\nu\\
&+\delta_{\mu\nu}(N\alpha^-p)\cdot(N\alpha^-p)-(N^2)_{\mu\nu}(\alpha^-p)\cdot(\alpha^-p)+(\alpha^-p)_\mu(N^2\alpha^-p)_\nu+(N^2\alpha^-p)_\mu(\alpha^-p)_\nu\\
&-3(N\alpha^-p)_\mu(N\alpha^-p)_\nu\}+\sinh\frac{uvF_+}{2}\{(\beta p)_\mu(N\alpha^-p)_\nu+(N\alpha^-p)_\mu(\beta p)_\nu\}+\sinh\frac{uF_+}{2}\{-2\,\delta_{\mu\nu}p\cdot(N\alpha^-p)\\
&+p_\mu(N\alpha^-p)_\nu+(N\alpha^-p)_\mu p_\nu-(Np)_\mu(\alpha^-p)_\nu-(\alpha^-p)_\mu(Np)_\nu\}\Bigg];
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
\Pi^2_{\mu\nu}(p)=&\frac{F_+}{2(4\pi)^{3/2}}\int_{1/\Lambda^2}^{\infty}du\int_0^1 dv\; u^{1/2}e^{-u(m^2+\phi^{(3)})}\frac{1}{\sinh uF_+/2}\times\Bigg[\frac{F_+}{4}\left(\cosh\frac{uF_+}{2}+3\cosh\frac{uvF_+}{2}\right)N_{\mu\nu}-\frac{3F_+}{4}\left(\cosh\frac{uF_+}{2}\right.\\
&\left.-\cosh\frac{uvF_+}{2}\right)(N^3)_{\mu\nu}-F_+\sinh\frac{uF_+}{2}(N^2\alpha^-)_{\mu\nu}+\frac{\cosh uF_+/2+\cosh uvF_+/2}{2}\{(\alpha^-p)_\mu p_\nu-p_\mu(\alpha^-p)_\nu\}\\
&+\frac{\cosh uF_+/2-\cosh uvF_+/2}{2}\{p_\mu(N^2\alpha^-p)_\nu-(N^2\alpha^-p)_\mu p_\nu+(N^2p)_\mu(\alpha^-p)_\mu-(\alpha^-p)_\mu(N^2p)_\nu\\
&-4N_{\mu\nu}(Np)\cdot(\alpha^-p)-(Np)_\mu(N\alpha^-p)_\nu+(N\alpha^-p)_\mu(Np)_\nu\}+\sinh\frac{uvF_+}{2}\{(\beta p)_\mu(Np)_\nu-(Np)_\mu(\beta p)_\nu\\
&+N_{\mu\nu}p\cdot(\beta p)\}-\sinh\frac{uF_+}{2}\{N_{\mu\nu}p^2+N_{\mu\nu}(\alpha^-p)\cdot(\alpha^-p)+p_\mu(Np)_\nu-(Np)_\mu p_\nu+(\alpha^-p)_\mu(N\alpha^-p)_\nu\\
&-(N\alpha^-p)_\mu(\alpha^-p)_\nu\}\Bigg],
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
\Pi^3_{\mu\nu}(p)=&\frac{F_+}{2(4\pi)^{3/2}}\int_{1/\Lambda^2}^{\infty}du\int_0^1 dv\; u^{1/2}e^{-u(m^2+\phi^{(3)})}\frac{1}{\sinh uF_+/2}\times\Bigg[\frac{\cosh uF_+/2+\cosh uvF_+/2}{2}(\delta_{\mu\nu}p^2-p_\mu p_\nu)\\
&+\sinh\frac{uF_+}{2}\{(Np)_\mu(\alpha^-p)_\nu+(\alpha^-p)_\mu(Np)_\nu\}+\frac{\cosh uF_+/2-\cosh uvF_+/2}{2}\{p^2(N^2)_{\mu\nu}-\delta_{\mu\nu}(Np)\cdot(Np)\\
&-p_\mu(N^2p)_\nu-(N^2p)_\mu p_\nu+3(Np)_\mu(Np)_\nu\}+\cosh\frac{uF_+}{2}\{(\beta p)_\mu(\beta p)_\nu-\beta_{\mu\nu}p\cdot(\beta p)+(\alpha^-p)_\mu(\alpha^-p)_\nu\}\Bigg].
\end{aligned}
\tag{68}
$$

Similarly in $3+1$ dimensions, we introduce

$$
\phi^{(4)}\equiv\frac{\cosh uF_+/2-\cosh uvF_+/2}{uF_+\sinh uF_+/2}(I^+p)\cdot(I^+p)+\frac{\cosh uF_-/2-\cosh uvF_-/2}{uF_-\sinh uF_-/2}(I^-p)\cdot(I^-p),
\tag{69}
$$

where

$$I^{+}_{\mu\nu}\equiv-\frac{F_{+}^{2}(\mathcal{F}^{2})_{\mu\nu}-F_{-}^{2}(\tilde{\mathcal{F}})^{2}_{\mu\nu}}{F_{+}^{4}-F_{-}^{4}},\quad I^{-}_{\mu\nu}\equiv\frac{F_{-}^{2}(\mathcal{F}^{2})_{\mu\nu}-F_{+}^{2}(\tilde{\mathcal{F}})^{2}_{\mu\nu}}{F_{+}^{4}-F_{-}^{4}}[=I^{+}_{\mu\nu}(F_{+}\leftrightarrow F_{-})],\tag{70}$$

which can be regarded as projection operators, obeying

$$I^{+}_{\mu\nu}+I^{-}_{\mu\nu}=\delta_{\mu\nu},\quad (I^{\pm})^{2}_{\mu\nu}=I^{\pm}_{\mu\nu}=-(N^{\pm})^{2}_{\mu\nu},\quad (I^{\pm}N^{\pm})_{\mu\nu}=N^{\pm}_{\mu\nu},\quad (I^{\pm}I^{\mp})_{\mu\nu}=(I^{\pm}N^{\mp})_{\mu\nu}=0,\tag{71}$$

where $N^{\pm}$ have been defined by Eq. (42). Then

$$\begin{aligned}\Pi^{1}_{\mu\nu}(p)=&\frac{F_{+}F_{-}}{4(4\pi)^{2}}\int_{1/\Lambda^{2}}^{\infty}du\int_{0}^{1}dv\;ue^{-u(m^{2}+\phi^{(4)})}\frac{1}{\sinh uF_{+}/2\sinh uF_{-}/2}\Bigg[\left(\frac{F_{+}\cosh uvF_{-}/2}{\sinh uF_{+}/2}-\frac{F_{-}\cosh uF_{+}/2}{\sinh uF_{-}/2}\right)I^{+}_{\mu\nu}\\&+\left(\cosh\frac{uvF_{+}}{2}\cosh\frac{uvF_{-}}{2}-\coth\frac{uF_{+}}{2}\coth\frac{uF_{-}}{2}\sinh\frac{uvF_{+}}{2}\sinh\frac{uvF_{-}}{2}\right)\{(I^{+}p)\cdot(I^{+}p)I^{+}_{\mu\nu}-(I^{+}p)_{\mu}(I^{-}p)_{\nu}\}\\&+2\frac{\cosh uF_{+}/2(\cosh uF_{-}/2-\cosh uvF_{-}/2)}{\sinh^{2}uF_{-}/2}(I^{-}p)\cdot(I^{-}p)I^{+}_{\mu\nu}\\&-2\frac{\cosh uvF_{-}/2(\cosh uF_{+}/2-\cosh uvF_{+}/2)}{\sinh^{2}uF_{+}/2}(I^{+}p)_{\mu}(I^{+}p)_{\nu}+\Bigg(\sinh\frac{uvF_{+}}{2}\sinh\frac{uvF_{-}}{2}\\&-\frac{(1-\cosh uF_{+}/2\cosh uvF_{+}/2)(1-\cosh uF_{-}/2\cosh uvF_{-}/2)}{\sinh uF_{+}/2\sinh uF_{-}/2}\Bigg)(N^{+}p)_{\mu}(N^{-}p)_{\nu}+(+\leftrightarrow-)\Bigg]\end{aligned}\tag{72}$$

$$\begin{aligned}\Pi^{2}_{\mu\nu}(p)=&\frac{F_{+}F_{-}}{4(4\pi)^{2}}\int_{1/\Lambda^{2}}^{\infty}du\int_{0}^{1}dv\;ue^{-u(m^{2}+\phi^{(4)})}\frac{1}{\sinh uF_{+}/2\sinh uF_{-}/2}\Bigg[\frac{F_{-}\sinh uF_{+}/2}{\sinh uF_{-}/2}N^{+}_{\mu\nu}\\&-2\frac{\sinh uF_{-}/2(\cosh uF_{+}/2-\cosh uvF_{+}/2)}{\sinh^{2}uF_{+}/2}(I^{+}p)\cdot(I^{+}p)N^{-}_{\mu\nu}\\&+\coth\frac{uF_{-}}{2}\sinh\frac{uvF_{+}}{2}\sinh\frac{uvF_{-}}{2}\{(I^{+}p)\cdot(I^{+}p)N^{+}_{\mu\nu}+(I^{-}p)_{\mu}(N^{+}p)_{\nu}-(N^{+}p)_{\mu}(I^{-}p)_{\nu}\}\\&+\frac{\cosh uvF_{-}/2(1-\cosh uF_{+}/2\cosh uvF_{+}/2)}{\sinh uF_{+}/2}\times\{(I^{-}p)_{\mu}(N^{+}p)_{\nu}-(N^{+}p)_{\mu}(I^{-}p)_{\nu}-(I^{+}p)_{\mu}(N^{+}p)_{\nu}\\&+(N^{+}p)_{\mu}(I^{+}p)_{\nu}\}+(+\leftrightarrow-)\Bigg],\end{aligned}\tag{73}$$

$$\begin{aligned}\Pi^{3}_{\mu\nu}(p)=&\frac{F_{+}F_{-}}{4(4\pi)^{2}}\int_{1/\Lambda^{2}}^{\infty}du\int_{0}^{1}dv\;ue^{-u(m^{2}+\phi^{(4)})}\frac{1}{\sinh uF_{+}/2\sinh uF_{-}/2}\times\Bigg[\Bigg(\cosh\frac{uvF_{+}}{2}\cosh\frac{uvF_{-}}{2}\\&-\coth\frac{uF_{+}}{2}\coth\frac{uF_{-}}{2}\sinh\frac{uvF_{+}}{2}\sinh\frac{uvF_{-}}{2}\Bigg)\{(I^{-}p)\cdot(I^{-}p)I^{+}_{\mu\nu}-(I^{+}p)_{\mu}(I^{-}p)_{\nu}\}\\&+\frac{2\cosh uF_{-}/2(\cosh uF_{+}/2-\cosh uvF_{+}/2)}{\sinh^{2}uF_{+}/2}\{(I^{+}p)\cdot(I^{+}p)I^{+}_{\mu\nu}-(I^{+}p)_{\mu}(I^{+}p)_{\nu}\}\\&\times\left(\frac{(1-\cosh uF_{+}/2\cosh uvF_{+}/2)(1-\cosh uF_{-}/2\cosh uvF_{-}/2)}{\sinh uF_{+}/2\sinh uF_{-}/2}-\sinh\frac{uvF_{+}}{2}\sinh\frac{uvF_{-}}{2}\right)\\&\times(N^{+}p)_{\mu}(N^{-}p)_{\nu}+(+\leftrightarrow-)\Bigg].\end{aligned}\tag{74}$$

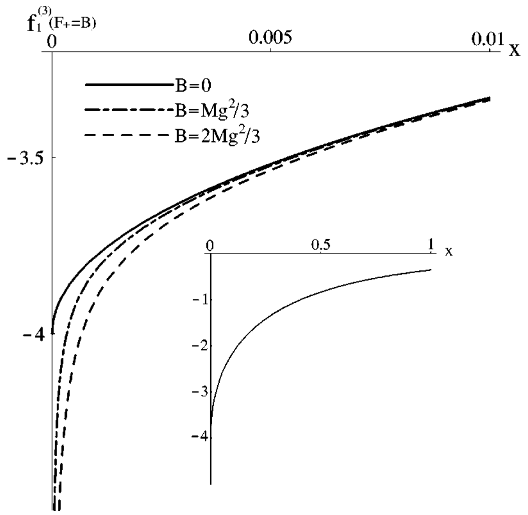

FIG. 1. One-loop contribution to the gap equation in $2+1$ dimensions. The solid line represents $B=0$, the dash-dotted line $M_g^2/3$, and the dotted lines $2M_g^2/3$, respectively. In $x>0.01$, recognized from the small graph, all curves become degenerate. In order to fix the magnitude of the background field, the (dimensionless) gluon mass $y=M_g^2/\Lambda^2$ is set to be 0.01.

(These expressions are so lengthy that we ensure the correctness by checking the gauge invariance of those, that is, the Ward-Takahashi relation in Appendix D.)

Armed with these general results, in the following we consider the magnetic background only and proceed to calculate the gap equation in $2+1$ and then $3+1$ dimensions.

III. THE GAP EQUATION IN 2+1 DIMENSIONS

When the background is purely magnetic, $\boldsymbol{E}=0$, in view of Eq. (39), $F_+\Rightarrow B$. A dimensionless quantity,

$$\mathcal{B}\equiv\frac{B}{\Lambda^2}, \tag{75}$$

is introduced in addition to $x\equiv m^2/\Lambda^2$ in Eq. (29). (Since the coupling constant e has been included to gauge fields the dimension of gauge fields is always one.) The one-loop contribution to the gap equation Eq. (29),

$$-\frac{2(\pi)^{3/2}}{g^2\Lambda}=f_1^{(3)}(x), \tag{76}$$

reads, with the aid of Eqs. (30), (43), and (44) as

$$f_1^{(3)}(x)=\frac{(4\pi)^{3/2}}{2\Lambda}\frac{\partial v_1^{(3)}}{\partial m^2}=-\mathcal{B}\int_1^\infty d\tau\,\tau^{-1/2}e^{-\tau x}\coth\frac{\tau\mathcal{B}}{2}, \tag{77}$$

where τ has been scaled to $\Lambda^2\tau$. We plot $f_1^{(3)}(x)$ in Fig. 1. It is seen that for a fixed four-Fermi coupling g, that is, with respect to a (supposed) horizontal line, mass is a monotone increasing function of magnetic-field strength. It is also noted that the critical coupling g_c, defined by $f_1^{(3)}(x=0)$ in Eq. (76) for a fixed magnetic field, goes to zero when $B\neq 0$. This phenomenon is the so-called "dimensional reduction [13]" and is due to the infrared divergence of the effective potential under the background magnetic field [12].

The two-loop contribution is found as

$$\begin{aligned}f_2^{(3)}(x)=&-\frac{\mathcal{B}^2e^2}{2(4\pi)^{3/2}\Lambda}\int_1^\infty du\int_0^1 dv\int_0^\infty d\tau\,u^{3/2}\\&\times e^{-ux-\tau y}\frac{\sqrt{\mathcal{K}}}{\sinh u\mathcal{B}/2}\\&\times\left[\frac{\mathcal{M}(q_1\cosh 2\tau\mathcal{B}+q_2\sinh 2\tau\mathcal{B}+q_3)}{\cosh\tau\mathcal{B}}+\mathcal{N}q_4\right],\end{aligned} \tag{78}$$

where u and τ have been scaled to $\Lambda^2 u$ and $\Lambda^2\tau$, y, defined as

$$y\equiv\frac{M_g^2}{\Lambda^2}, \tag{79}$$

is a dimensionless gluon mass,

$$\mathcal{K}\equiv\frac{1}{u(1-v^2)/4+\tau},$$

$$\mathcal{M}\equiv\frac{\sinh u\mathcal{B}/2}{\cosh u\mathcal{B}/2-\cosh uv\mathcal{B}/2+\tanh\tau\mathcal{B}\sinh u\mathcal{B}/2},$$

$$\mathcal{N}\equiv\frac{\sinh u\mathcal{B}/2}{\cosh u\mathcal{B}/2-\cosh uv\mathcal{B}/2+\tau\mathcal{B}\sinh u\mathcal{B}/2}, \tag{80}$$

and

$$\begin{aligned}q_1\equiv&\frac{\mathcal{K}}{2}\left(\cosh\frac{u\mathcal{B}}{2}-v\coth\frac{u\mathcal{B}}{2}\sinh\frac{uv\mathcal{B}}{2}\right)\\&+\mathcal{B}\left(\mathcal{M}\cosh\frac{u\mathcal{B}}{2}\cosh\frac{uv\mathcal{B}}{2}\frac{\cosh u\mathcal{B}/2-\cosh uv\mathcal{B}/2}{\sinh^2 u\mathcal{B}/2}\right.\\&\left.+\frac{(1-\cosh u\mathcal{B}/2\cosh uv\mathcal{B}/2)}{\sinh u\mathcal{B}/2}\right),\end{aligned} \tag{81}$$

$$\begin{aligned}q_2\equiv&\frac{\mathcal{K}}{2}\left(\sinh\frac{u\mathcal{B}}{2}-v\sinh\frac{uv\mathcal{B}}{2}\right)\\&+\mathcal{B}\left[\mathcal{M}\frac{\cosh uv\mathcal{B}/2}{\sinh u\mathcal{B}/2}\left(\cosh\frac{u\mathcal{B}}{2}-\cosh\frac{uv\mathcal{B}}{2}\right)-\cosh\frac{uv\mathcal{B}}{2}\right],\end{aligned} \tag{82}$$

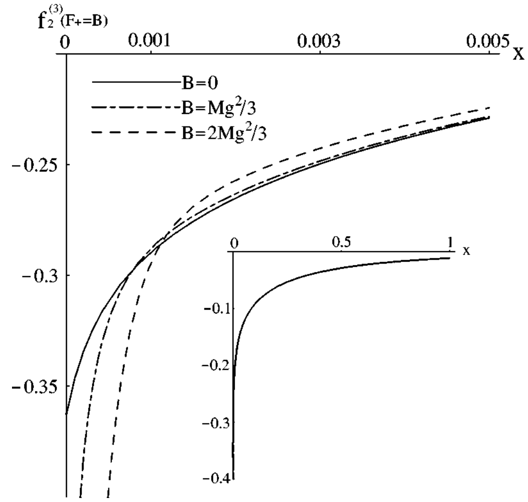

FIG. 2. Two-loop contribution to the gap equation in $2+1$ dimensions. As in Fig. 1, the same line pattern is used. From the smaller graph, we see that all curves become degenerate but remain negative in the whole region $1 \geqslant x \geqslant 0$. The graph is drawn by putting $e^2/(4\pi\Lambda)=0.01$ and $y=0.01$.

$$q_3 \equiv \frac{\mathcal{K}}{4}\left(v^2 \cosh\frac{uv\mathcal{B}}{2} - v \coth\frac{u\mathcal{B}}{2}\sinh\frac{uv\mathcal{B}}{2}\right) + \mathcal{B}\mathcal{M}\frac{\cosh u\mathcal{B}/2 - \cosh uv\mathcal{B}/2}{\sinh^2 u\mathcal{B}/2} + \frac{1}{u}\left(\cosh\frac{uv\mathcal{B}}{2} - \frac{u\mathcal{B}}{2\sinh u\mathcal{B}/2}\right), \tag{83}$$

$$q_4 \equiv \frac{\mathcal{K}}{4}\left(\cosh\frac{uv\mathcal{B}}{2} - v \coth\frac{u\mathcal{B}}{2}\sinh\frac{uv\mathcal{B}}{2}\right) + \mathcal{B}\frac{\mathcal{N}}{2}\left(\frac{\cosh u\mathcal{B}/2 - \cosh uv\mathcal{B}/2}{\sinh^2 u\mathcal{B}/2} + \frac{\cosh uv\mathcal{B}/2 - v \coth u\mathcal{B}/2 \sinh uv\mathcal{B}/2}{2}\right). \tag{84}$$

The q_1, q_2, and q_3 terms come from the first two terms in Eq. (62), that is, from the non-Abelian contribution. Meanwhile q_4 represents the Abelian contribution.

We plot the two-loop part in Fig. 2. (Our choice of the gauge coupling $e^2/(4\pi\Lambda)=0.01$ does guarantee the approximation, since $f_2^{(3)}/f_1^{(3)} \sim 0.1$ by comparing the vertical scale between Figs. 1 and 2.) From the small graph, we can convince that gluons enhance χSB as is expected because all curves remain negative for a whole region, $0 \leqslant x \leqslant 1$. Note that there is a crossover around $x \sim 0.001$; in the region larger than the crossover, x, on some horizontal line (a line with a fixed four-Fermi coupling) is a monotone decreasing function of the magnitude of the background field. In the region smaller than that, x is, however, a increasing function of it, similar to the one-loop case. To see the situation more carefully, we plot the Abelian contribution to the gap equation, that is, the q_4 term in Eq. (78).

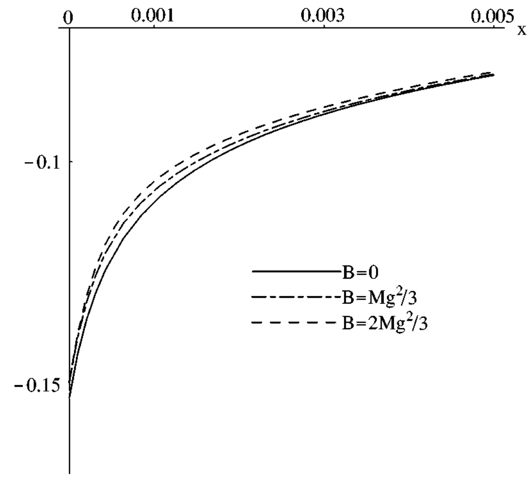

FIG. 3. Abelian part of the two-loop contribution to the gap equation in $2+1$ dimensions. The graph is drawn by putting $e^2/(4\pi\Lambda)=0.01$ and $y=0.01$.

From Fig. 3, x, on some horizontal line, is a monotone decreasing function of the background field everywhere for a fixed g. Therefore, the increasing tendency of Fig. 2 in $x < 0.001$ comes from the non-Abelian parts in Eq. (78).

IV. THE GAP EQUATION IN 3+1 DIMENSIONS

In $3+1$ dimensions, when $\boldsymbol{E}=0$

$$F_+ \Rightarrow B, \quad F_- \Rightarrow 0. \tag{85}$$

Again employing the dimensionless quantity $\mathcal{B}=B/\Lambda^2$, we have the gap equation of one-loop contribution,

$$-\frac{4(\pi)^2}{g^2\Lambda^2} = f_1^{(4)}(x), \tag{86}$$

with

$$f_1^{(4)}(x) = \frac{(4\pi)^2}{2\Lambda^2}\frac{\partial v_1^{(4)}}{\partial m^2} = -\mathcal{B}\int_1^\infty d\tau\, \tau^{-1} e^{-\tau x}\coth\frac{\tau\mathcal{B}}{2}, \tag{87}$$

which is plotted in Fig. 4. All curves become degenerate again where x is large. For a fixed four-Fermi coupling g, mass is a monotone increasing function of magnetic-field strength. Moreover the critical coupling goes to zero even under infinitesimal magnetic fields [13,12].

The two-loop contribution is given by

$$f_2^{(4)}(x) = -\frac{\mathcal{B}^2 e^2}{2(4\pi)^2}\int_1^\infty du \int_0^1 dv \int_0^\infty d\tau\, u\, e^{-ux-\tau y}\,\frac{\mathcal{K}}{\sinh u\mathcal{B}/2} \times \left[\frac{\mathcal{M}(p_1\cosh 2\tau\mathcal{B} + p_2 \sinh 2\tau\mathcal{B} + p_3)}{\cosh \tau\mathcal{B}} + \mathcal{N}p_4\right], \tag{88}$$

where $\mathcal{K}$, $\mathcal{M}$, and $\mathcal{N}$ are the same as Eq. (80) and

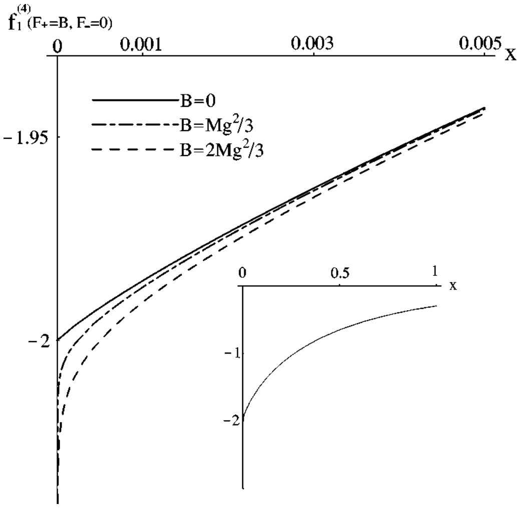

FIG. 4. One-loop contribution to the gap equation in $3+1$ dimensions. The solid line, dash-dotted line, and dotted line designate $B=0, M_g^2/3$, and $2M_g^2/3$, respectively. The smaller graph shows the whole structure, $1 \geqslant x \geqslant 0$. We again set $M_g^2/\Lambda^2=0.01$.

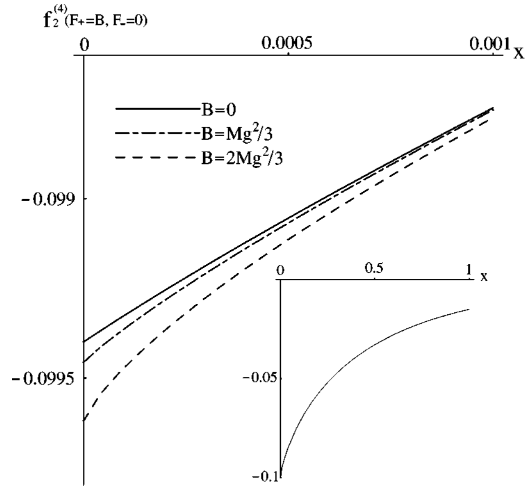

FIG. 5. Two-loop contribution to the gap equation in $3+1$ dimensions. In the smaller graph the whole structure, $0 \leqslant x \leqslant 1$, is shown. The same line pattern is used for different curves by putting $e^2/4\pi=0.01$ and $M_g^2/\Lambda^2=0.01$.

$$p_1 \equiv \mathcal{K}(1-v^2)\cosh\frac{u\mathcal{B}}{2}+\mathcal{B}\mathcal{M}\left(\cosh\frac{uv\mathcal{B}}{2}-v\coth\frac{u\mathcal{B}}{2}\right.$$
$$\left.\times\sinh\frac{uv\mathcal{B}}{2}-\frac{\cosh u\mathcal{B}/2-\cosh uv\mathcal{B}/2}{\sinh^2 u\mathcal{B}/2}\right)-\frac{2}{u}\left(\cosh\frac{u\mathcal{B}}{2}\right.$$
$$\left.-\frac{u\mathcal{B}}{2\sinh u\mathcal{B}/2}\right), \tag{89}$$

$$p_2 \equiv \mathcal{K}(1-v^2)\sinh\frac{u\mathcal{B}}{2}+\mathcal{B}\mathcal{M}\left(\frac{\cosh u\mathcal{B}/2\cosh uv\mathcal{B}/2-1}{\sinh u\mathcal{B}/2}\right.$$
$$\left.-v\sinh\frac{uv\mathcal{B}}{2}\right)-\frac{2\sinh u\mathcal{B}/2}{u}, \tag{90}$$

$$p_3 \equiv \mathcal{K}\left(\cosh\frac{uv\mathcal{B}}{2}-v\coth\frac{u\mathcal{B}}{2}\sinh\frac{uv\mathcal{B}}{2}-\frac{1-v^2}{2}\cosh\frac{uv\mathcal{B}}{2}\right)$$
$$+2\mathcal{B}\mathcal{M}\frac{\cosh u\mathcal{B}/2-\cosh uv\mathcal{B}/2}{\sinh^2 u\mathcal{B}/2}+\frac{2}{u}\left(\cosh\frac{uv\mathcal{B}}{2}\right.$$
$$\left.-\frac{u\mathcal{B}}{2\sinh u\mathcal{B}/2}\right), \tag{91}$$

$$p_4 \equiv \frac{\mathcal{K}}{2}\left(\cosh\frac{uv\mathcal{B}}{2}-v\coth\frac{u\mathcal{B}}{2}\sinh\frac{uv\mathcal{B}}{2}+\frac{1-v^2}{2}\cosh\frac{u\mathcal{B}}{2}\right)$$
$$+\mathcal{B}\frac{\mathcal{N}}{2}\left(\cosh\frac{uv\mathcal{B}}{2}-v\coth\frac{u\mathcal{B}}{2}\sinh\frac{uv\mathcal{B}}{2}\right.$$
$$\left.+\frac{\cosh u\mathcal{B}/2-\cosh uv\mathcal{B}/2}{\sinh^2 u\mathcal{B}/2}\right). \tag{92}$$

The graph is shown in Fig. 5. (The choice of the gauge coupling $e^2/4\pi=0.01$ again guarantees our approximation, since $f_2^{(4)}/f_1^{(4)} \sim 0.05$. It should be noted that our result of $B=0$ is consistent with that of Kondo *et al.* [22].) All curves, shown in the smaller graph, remain negative and become degenerate for $x \geqslant 0.001$.

Therefore, gluons enhance χSB everywhere even in no background $B=0$, which fits our expectation. Contrary to the case in $2+1$ dimensions, mass is a monotone increasing function of magnetic-field strength everywhere for a fixed four-Fermi coupling g.

V. DISCUSSION

In this paper we discuss the effect of dynamical $SU(2)$ gluons to the gap equation of the NJL model under the influence of the constant background non-Abelian magnetic field. The two-loop calculations make expressions considerably complicated but correctness of the results is guaranteed by checking the Ward-Takahashi relation in Appendix D. In $3+1$ dimensions, as is seen from Figs. 4 and 5, gluons play the same role as fermions in the one-loop, that is, they enhance χSB. Moreover, the dependence of gluons on the background field is also the same as a fermion in the one loop; dynamical mass grows larger as the background magnetic field becomes stronger. The result is consistent with our expectation but different from [18] where RG with the one-loop calculation was employed. In $2+1$ dimensions, the situation is unchanged that gluons enhance χSB even under the influence of the background field, contrary to the work of [17]. The dependence of gluons on the background magnetic field, however, is not so simple as in $3+1$ dimensions; as is seen from Fig. 2 when dynamical mass is tiny the background field increases it, but in a well-broken region, that is, in a region where dynamical mass is large, the background field resists a mass to grow. The difference between $2+1$ and $3+1$ dimensions is due to that of the u dependence in Eqs. (78) and (88); by making the scale transformation to u and τ such that

$$u \mapsto \frac{u}{\mathcal{B}}, \quad \tau \mapsto \frac{\tau}{\mathcal{B}}, \tag{93}$$

quantities, $\mathcal{K},\mathcal{M},\mathcal{N}$ [Eq. (80)], scale as

$$\mathcal{K} \mapsto \mathcal{B}K, \quad \mathcal{M} \mapsto \mathcal{M}, \quad \mathcal{N} \mapsto \mathcal{N}, \tag{94}$$

so that q_i and p_i, $(i=1\text{–}4)$, Eq. (81)–Eq. (84) and Eq. (89)–Eq. (92) transform

$$q_i \mapsto \mathcal{B}q_i, \quad p_i \mapsto \mathcal{B}p_i. \tag{95}$$

Therefore,

$$\begin{aligned} f_1^{(3)} &\mapsto \sqrt{\mathcal{B}} f_1^{(3)}, \\ f_1^{(4)} &\mapsto \mathcal{B} f_1^{(4)}, \\ f_2^{(4)} &\mapsto \mathcal{B} f_2^{(4)}, \end{aligned} \tag{96}$$

which shows a monotone character with respect to $\mathcal{B}$, convincing us of the results of Figs. 1 , 4, and 5, qualitatively (because there is still $\mathcal{B}$ dependence in $f_1^{(3)}, f_1^{(4)}$, and $f_2^{(4)}$). But

$$f_2^{(3)} \mapsto f_2^{(3)}, \tag{97}$$

which implies that dependence on $\mathcal{B}$ is due only to the detailed structure of integrand of expression (78), that is, we cannot extract a simple monotone behavior in this case.

The second point we wish to discuss is on the instability of the gluon functional determinant; we have avoided this by introducing gluon mass M_g, which is always assumed bigger than the magnitude of the background magnetic field B, $M_g^2 > B$. Physically, it is interpreted that the energy of non-Abelian particles could become lower and lower as the background magnetic field grows larger and larger. There is no lower limit in the system. The situation is exactly the same in the constant electric-field case, where the vacuum becomes unstable due to successive pair productions. We have treated this pathological instability by considering only an external electric field whose magnitude E is less than that of the dynamical mass squared; $m^2 > E$ [12]. The point is that the setup itself—"field theories under constant background field"—is pathological. The system is not closed; energy is continuously supplied from the outer environment. However, even in these pathological environments, we could still think about those background effects if we suppose that gluons are massive in a confining phase and that the magnitude of background fields is smaller than the gluon mass squared.

The final point to discuss is beyond the tree approximation of the auxiliary fields σ and π; in most cases of the NJL study, fermions are assumed to have N components with N being supposed infinite finally. However, in the actual situation, N is finite so that $O(1/N)$ corrections should be taken into account. A study in a simpler model [23] says that the approximation becomes more and more accurate if we incorporate higher-order terms. Thus going beyond the one loop of the auxiliary fields is captivating and the work in this direction is in progress.

APPENDIX A: CALCULATION OF KERNEL BY THE PROPER TIME

In this appendix we derive the expression of the kernel (34)

$$K(x,y,\tau) = \langle x|e^{-\tau H_0}|y\rangle = e^{\tau\sigma_{\mu\nu}\mathcal{F}_{\mu\nu}T_3/2}\langle x|e^{-\tau\Pi_\mu^2}|y\rangle. \tag{A1}$$

Because of the covariantly constant condition (4), the matrix element $\langle x|e^{-\tau\Pi_\mu^2}|y\rangle$ can be calculated exactly the same way as the Abelian case [20]

$$\begin{aligned} \langle x|e^{-\tau\Pi_\mu^2}|y\rangle = \frac{1}{(4\pi\tau)^{D/2}} \exp[iT_3\mathcal{C}]\left[\det\left(\frac{\sin\tau\mathcal{F}/2}{\tau\mathcal{F}/2}\right)_{\mu\nu}\right]^{-1/2} \\ \times \exp\left[-\frac{1}{4}(x-y)_\mu\left(\frac{\mathcal{F}}{2}\cot\frac{\tau\mathcal{F}}{2}\right)_{\mu\nu}(x-y)_\nu\right], \end{aligned} \tag{A2}$$

with

$$\mathcal{C} \equiv -\frac{1}{2}\mathcal{F}_{\mu\nu}x_\mu y_\nu. \tag{A3}$$

The remaining task is therefore the calculation of $\exp[(\tau/2)\sigma_{\mu\nu}\mathcal{F}_{\mu\nu}T_3]$; in $2+1$ dimensions, the gamma matrices are given as

$$\begin{aligned} \gamma_\mu = \begin{pmatrix} \sigma_\mu & 0 \\ 0 & -\sigma_\mu \end{pmatrix}, \quad \frac{\sigma_{\mu\nu}}{2} \equiv \frac{1}{4i}[\gamma_\mu,\gamma_\nu] = \frac{\epsilon_{\mu\nu\rho}}{2}\begin{pmatrix} \sigma_\rho & 0 \\ 0 & \sigma_\rho \end{pmatrix} \\ \equiv \epsilon_{\mu\nu\rho}J_\rho, \quad \mu,\nu,\rho = 1,2,3, \end{aligned} \tag{A4}$$

where J_μ's satisfy

$$[J_\mu, J_\nu] = i\epsilon_{\mu\nu\rho}J_\rho, \quad \{J_\mu, J_\nu\} = \frac{\delta_{\mu\nu}}{2}\mathbf{I}. \tag{A5}$$

In terms of J_μ's

$$\frac{1}{2}\sigma_{\mu\nu}\mathcal{F}_{\mu\nu} = E_2J_1 - E_1J_2 + BJ_3, \quad B \equiv \mathcal{F}_{12}; \boldsymbol{E} \equiv (\mathcal{F}_{13}, \mathcal{F}_{23}). \tag{A6}$$

From Eqs. (A5) and (A6), we obtain

$$\left(\frac{1}{2}\sigma_{\mu\nu}\mathcal{F}_{\mu\nu}T_3\right)^2 = \left(\frac{\sqrt{B^2+\boldsymbol{E}^2}}{2}\right)^2\mathbf{I} \equiv \left(\frac{F_+}{2}\right)^2\mathbf{I}. \tag{A7}$$

Meanwhile,

$$\begin{aligned} \frac{1}{2}\sigma_{\mu\nu}\mathcal{F}_{\mu\nu}T_3 = \sigma_{\mu\nu}\frac{\mathcal{F}_{\mu\nu}}{F_+}\frac{F_+}{2}T_3 \equiv \sigma_{\mu\nu}N_{\mu\nu}T_3\frac{F_+}{2}, \\ N_{\mu\nu} \equiv \frac{\mathcal{F}_{\mu\nu}}{F_+}. \end{aligned} \tag{A8}$$

Therefore in $2+1$ dimensions

$$\exp\left[\frac{\tau}{2}\sigma_{\mu\nu}\mathcal{F}_{\mu\nu}T_3\right]=\cosh\frac{\tau F_+}{2}\mathbf{I}+\sigma_{\mu\nu}N_{\mu\nu}\sinh\frac{\tau F_+}{2}T_3. \tag{A9}$$

In $3+1$ dimensions, first write

$$\begin{cases} F_+\equiv\dfrac{\{|\boldsymbol{B}+\boldsymbol{E}|+|\boldsymbol{B}-\boldsymbol{E}|\}}{2}, & \boldsymbol{E}\equiv(\mathcal{F}_{14},\mathcal{F}_{24},\mathcal{F}_{34}),\\ F_-\equiv\dfrac{\{|\boldsymbol{B}+\boldsymbol{E}|-|\boldsymbol{B}-\boldsymbol{E}|\}}{2}, & \boldsymbol{B}\equiv(\mathcal{F}_{23},\mathcal{F}_{31},\mathcal{F}_{12})\end{cases} \tag{A10}$$

and introduce the antisymmetric tensors, $N^{\pm}_{\mu\nu}=-N^{\pm}_{\nu\mu}$, such that

$$N^+_{\mu\nu}\equiv\frac{\mathcal{F}_{\mu\nu}F_+-\tilde{\mathcal{F}}_{\mu\nu}F_-}{F_+^2-F_-^2},\quad N^-_{\mu\nu}\equiv\frac{\tilde{\mathcal{F}}_{\mu\nu}F_+-\mathcal{F}_{\mu\nu}F_-}{F_+^2-F_-^2},$$

$$\tilde{\mathcal{F}}_{\mu\nu}\equiv\frac{\epsilon_{\mu\nu\lambda\rho}}{2}\mathcal{F}_{\lambda\rho}, \tag{A11}$$

which satisfy

$$(N^{\pm}N^{\mp})_{\mu\nu}\equiv N^{\pm}_{\mu\lambda}N^{\mp}_{\lambda\nu}=0,\quad \frac{\epsilon_{\mu\nu\lambda\rho}}{2}N^{\pm}_{\lambda\rho}=N^{\mp}_{\mu\nu}, \tag{A12}$$

$$(N^{\pm}_{\mu\nu})^2=2, \tag{A13}$$

where the second relation can be verified by using Eqs. (A10) and

$$(\mathcal{F}_{\mu\nu})^2=(\tilde{\mathcal{F}}_{\mu\nu})^2=2(F_+^2+F_-^2),\quad \mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}_{\mu\nu}=4F_+F_-. \tag{A14}$$

With the aid of $N^{\pm}_{\mu\nu}$, $\mathcal{F}_{\mu\nu}$ is expressed as

$$\mathcal{F}_{\mu\nu}=F_+N^+_{\mu\nu}+F_-N^-_{\mu\nu}, \tag{A15}$$

giving

$$\frac{1}{2}\sigma_{\mu\nu}\mathcal{F}_{\mu\nu}=\frac{1}{2}(F_+\sigma_{\mu\nu}N^+_{\mu\nu}+F_-\sigma_{\mu\nu}N^-_{\mu\nu}),$$

$$\sigma_{\mu\nu}\equiv\frac{[\gamma_\mu,\gamma_\nu]}{2i}. \tag{A16}$$

By noting

$$[\sigma_{\mu\nu},\sigma_{\lambda\rho}]=2i(\delta_{\mu\lambda}\sigma_{\nu\rho}-\delta_{\mu\rho}\sigma_{\nu\lambda}-\delta_{\nu\lambda}\sigma_{\mu\rho}+\delta_{\nu\rho}\sigma_{\mu\lambda}), \tag{A17}$$

and Eq. (A12), we find

$$[\sigma_{\mu\nu}N^+_{\mu\nu},\sigma_{\lambda\rho}N^-_{\lambda\rho}]=0. \tag{A18}$$

Therefore,

$$\exp\left[\frac{1}{2}\sigma_{\mu\nu}\mathcal{F}_{\mu\nu}T_3\right]=\exp\left[\frac{F_+}{2}\sigma_{\mu\nu}N^+_{\mu\nu}T_3\right]\exp\left[\frac{F_-}{2}\sigma_{\mu\nu}N^-_{\mu\nu}T_3\right]. \tag{A19}$$

Also by noting

$$\{\sigma_{\mu\nu},\sigma_{\lambda\rho}\}=2(\delta_{\mu\lambda}\delta_{\nu\rho}-\delta_{\mu\rho}\delta_{\nu\lambda}-\epsilon_{\mu\nu\lambda\rho}\gamma_5), \tag{A20}$$

and Eq. (A13),

$$(\sigma_{\mu\nu}N^{\pm}_{\mu\nu})^2=4. \tag{A21}$$

Hence

$$\left(\frac{F_\pm}{2}\sigma_{\mu\nu}N^{\pm}_{\mu\nu}T_3\right)^2=\left(\frac{F_\pm}{2}\right)^2\mathbf{I}, \tag{A22}$$

yielding

$$\exp\left[\frac{F_\pm}{2}\sigma_{\mu\nu}N^{\pm}_{\mu\nu}T_3\right]=\cosh\frac{F_\pm}{2}\mathbf{I}+\sigma_{\mu\nu}N^{\pm}_{\mu\nu}\sinh\frac{F_\pm}{2}T_3. \tag{A23}$$

Finally utilizing

$$\sigma_{\mu\nu}N^+_{\mu\nu}\sigma_{\rho\lambda}N^-_{\rho\lambda}=-4\gamma_5, \tag{A24}$$

we obtain

$$\begin{aligned}\exp\left[\frac{\tau}{2}\sigma_{\mu\nu}\mathcal{F}_{\mu\nu}T_3\right]=&\left(\cosh\frac{\tau F_+}{2}\cosh\frac{\tau F_-}{2}\right.\\&\left.-\gamma_5\sinh\frac{\tau F_+}{2}\sinh\frac{\tau F_-}{2}\right)\mathbf{I}\\&+\sigma_{\mu\nu}\left(N^+_{\mu\nu}\sinh\frac{\tau F_+}{2}\cosh\frac{\tau F_-}{2}\right.\\&\left.+N^-_{\mu\nu}\cosh\frac{\tau F_+}{2}\sinh\frac{\tau F_-}{2}\right)T_3\end{aligned} \tag{A25}$$

$$=K_0(\tau)\mathbf{I}+K_3(\tau)T_3, \tag{A26}$$

$$K_0(\tau)\equiv\cosh\frac{\tau F_+}{2}\cosh\frac{\tau F_-}{2}-\gamma_5\sinh\frac{\tau F_+}{2}\sinh\frac{\tau F_-}{2}, \tag{A27}$$

$$K_3(\tau)\equiv\sigma_{\mu\nu}\left(N^+_{\mu\nu}\sinh\frac{\tau F_+}{2}\cosh\frac{\tau F_-}{2}+N^-_{\mu\nu}\cosh\frac{\tau F_+}{2}\sinh\frac{\tau F_-}{2}\right). \tag{A28}$$

APPENDIX B: GLUON PROPAGATOR IN TERMS OF THE PROPER TIME

In this appendix we show the proper time representation of the gluon propagator (21)

$$(\Delta^{-1})^{ab}_{\mu\nu} = -\delta_{\mu\nu}(\mathcal{D}^2)^{ab} + 2i\mathcal{F}_{\mu\nu}[\mathrm{ad}(T_3)]^{ab}, \tag{B1}$$

where we have introduced a slightly different notation from Eq. (21)

$$\mathcal{D}^{ab}_{\mu} = \delta^{ab}\partial_\mu - i\mathcal{A}_\mu[\mathrm{ad}(T_3)]^{ab}, \quad [\mathrm{ad}(T_3)] \equiv \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{B2}$$

The proper time representation is obtained as usual.

$$\Delta^{ab}_{\mu\nu}(x,y) = \int_0^\infty d\tau e^{-\tau M_g^2}[\langle x|e^{-\tau\Pi^2}|y\rangle]^{ac}(e^{-2i\tau\mathcal{F}[\mathrm{ad}(T_3)]})^{cb}_{\mu\nu}, \tag{B3}$$

where

$$\Pi^2 \equiv (\Pi^{ab}_\mu)^2, \quad \Pi^{ab}_\mu \equiv \delta^{ab}\hat{p}_\mu - \mathcal{A}_\mu(\hat{x})[\mathrm{ad}(T_3)]^{ab}, \tag{B4}$$

and we have introduced the gluon mass M_g to avoid the tachyonic singularity. For $a=b=3$, it reads

$$\Delta^{33}_{\mu\nu} \equiv \Delta^3_{\mu\nu} = (-\delta_{\mu\nu}\partial^2)^{-1}, \tag{B5}$$

which is the free propagator. Therefore we obtain Eq. (47):

$$\Delta^3_{\mu\nu}(x-y) = \delta_{\mu\nu}\int_0^\infty d\tau \frac{e^{-\tau M_g^2}}{(4\pi\tau)^{D/2}} \exp\left[-\frac{1}{4\tau}(x-y)^2\right]. \tag{B6}$$

In $a,b=1,2$ use $i,j=1,2$ and utilize the result in Appendix A [Eq. (A2)] and $[\mathrm{ad}(T_3)]^{ij} = -i\epsilon^{ij}$ to find

$$\begin{aligned}[\langle x|e^{-\tau\Pi^2}|y\rangle]^{ij} &= \frac{1}{(4\pi\tau)^{D/2}}(\delta^{ij}\cos\mathcal{C} + \epsilon^{ij}\sin\mathcal{C})\left[\det\left(\frac{\sin\tau\mathcal{F}}{\tau\mathcal{F}}\right)_{\mu\nu}\right]^{-1/2} \\ &\quad\times\exp\left[-\frac{1}{4}(x-y)_\mu(\mathcal{F}\cot\tau\mathcal{F})_{\mu\nu}(x-y)_\nu\right],\end{aligned} \tag{B7}$$

where $\mathcal{C}$ is given in Eq. (A3). Finally by noting that

$$(e^{-2i\tau\mathcal{F}[\mathrm{ad}(T_3)]})^{ij}_{\mu\nu} = \delta^{ij}(\cos 2\tau\mathcal{F})_{\mu\nu} - \epsilon^{ij}(\sin 2\tau\mathcal{F})_{\mu\nu}, \tag{B8}$$

the relations (46)–(49) are obtained.

APPENDIX C: PROOF THAT OUR CLASSICAL SOLUTION SATISFIES THE COVARIANTLY CONSTANT CONDITION

In this appendix we show that the right-hand side of Eq. (12) vanishes:

$$\begin{aligned}&\mathrm{tr}[(\gamma_\mu(\partial_\mu - i\mathcal{A}_\mu T_3) + m)^{-1}(-i\gamma_\nu T_a)] \\ &\quad = \mathrm{tr}[S_A(x,x)(-i\gamma_\nu T_a)] = 0,\end{aligned} \tag{C1}$$

where $S_A(x,y)$ is the fermion propagator under the background fields;

$$[\gamma_\mu(\partial_\mu - i\mathcal{A}_\mu T_3) + m]S_A(x,y) = \delta^D(x-y). \tag{C2}$$

$S_A(x,y)$ can be expressed, by using the proper time method, as

$$\begin{aligned}S_A(x,y) &= \langle x|(i\gamma_\mu\Pi_\mu + m)^{-1}|y\rangle \\ &= \langle x|(-i\gamma_\mu\Pi_\mu + m) \\ &\quad\times\left(\Pi_\rho^2 - \frac{1}{2}\sigma_{\rho\lambda}\mathcal{F}_{\rho\lambda}T_3 + m^2\right)^{-1}|y\rangle \\ &= \int_0^\infty d\tau e^{-\tau m^2}\langle x|(-i\gamma_\mu\Pi_\mu + m)e^{-\tau H_0}|y\rangle \\ &= \int_0^\infty d\tau e^{-\tau m^2}\left[\frac{1}{2}\gamma_\mu\left(\frac{\mathcal{F}}{2}\cot\frac{\tau\mathcal{F}}{2}\right)_{\mu\nu}(x-y)_\nu\mathbf{I}\right. \\ &\quad\left. - \frac{i}{2}\gamma_\mu\mathcal{F}_{\mu\nu}(x-y)_\nu T_3 + m\right]K(x,y;\tau),\end{aligned} \tag{C3}$$

where $K(x,y;\tau)$ is the kernel of Eq. (A1). Therefore

$$\begin{aligned}&\mathrm{tr}[S_A(x,x)(-i\gamma_\nu T_a)] \\ &\quad = -2im\int_0^\infty d\tau e^{-\tau m^2}\frac{1}{(4\pi\tau)^{D/2}} \\ &\quad\quad\times\left[\det\left(\frac{\sin\tau\mathcal{F}/2}{\tau\mathcal{F}/2}\right)_{\mu\nu}\right]^{-1/2}\mathrm{tr}[e^{\tau/2\sigma_{\mu\nu}\mathcal{F}_{\mu\nu}T_3}\gamma_\nu T_a] = 0,\end{aligned} \tag{C4}$$

since the trace for the gamma matrices vanishes because the total number of those is odd.

APPENDIX D: THE WARD-TAKAHASHI RELATION OF VACUUM POLARIZATION

In this appendix it is shown that the vacuum polarization function satisfies the Ward-Takahashi relation

$$(\mathcal{D}^x_\mu)^{ab}\Pi^{bc}_{\mu\nu}(x,y)=0, \tag{D1}$$

where

$$\mathcal{D}^{ab}_\mu=\delta^{ab}\partial_\mu-\epsilon^{ab3}\mathcal{A}_\mu,$$

with $\mathcal{A}_\mu$ being a background field

$$\mathcal{A}_\mu=-\frac{1}{2}\mathcal{F}_{\mu\nu}x_\nu. \tag{D2}$$

The Ward-Takahashi relation (D1) is separated into

$$\partial_\mu\Pi^3_{\mu\nu}(x,y)=0, \tag{D3}$$

and

$$[\delta^{ij}\partial^x_\mu-\epsilon^{ij3}\mathcal{A}_\mu(x)]\Pi^{jk}_{\mu\nu}(x,y)=0, \quad \text{for } i,j,k=1,2. \tag{D4}$$

The first relation (D3) can easily be checked by noting Eq. (68) and Eq. (74) so that the second relation (D4) must be examined. In view of the fact that $\Pi^{ij}_{\mu\nu}(x,y)$ can be written as

$$\Pi^{ij}_{\mu\nu}(x,y)=\{(\cos\mathcal{C}+\boldsymbol{\epsilon}\sin\mathcal{C})[\Pi^1_{\mu\nu}(x-y)+\boldsymbol{\epsilon}\Pi^2_{\mu\nu}(x-y)]\}^{ij}, \tag{D5}$$

the matrix relation reduces to

$$\partial^x_\mu\Pi^1_{\mu\nu}(x-y)+\mathcal{A}_\mu(x-y)\Pi^2_{\mu\nu}(x-y)=0, \tag{D6}$$

$$\partial^x_\mu\Pi^2_{\mu\nu}(x-y)-\mathcal{A}_\mu(x-y)\Pi^1_{\mu\nu}(x-y)=0. \tag{D7}$$

Utilizing the series expansion with respect to the background gauge field, we show, up to $O(F)$, that these relations indeed hold: first note that $\Pi^1_{\mu\nu}$ and $\Pi^2_{\mu\nu}$ are polynomials of even and odd powers of F, respectively. Thus in $O(1)$ Eq. (D6) reads

$$\partial_\mu\Pi^1_{\mu\nu}|_{F=0}=0, \tag{D8}$$

which is fulfilled, since from Eq. (66) and Eq. (72), $\Pi^1_{\mu\nu}$ has been given by

$$\Pi^1_{\mu\nu}(p)|_{F=0}=\frac{1}{(4\pi)^{D/2}}\int_{1/\Lambda^2}^\infty du\int_0^1 dv\,\frac{1-v^2}{u^{D/2-1}}\times\exp\left[-u\left(m^2+\frac{1-v^2}{4}p^2\right)\right]\{p^2\delta_{\mu\nu}-p_\mu p_\nu\}. \tag{D9}$$

Next in $O(F)$,

$$\partial_\mu\Pi^2_{\mu\nu}|_{O(F)}-\mathcal{A}_\mu(x-y)\Pi^1_{\mu\nu}(x,y)|_{F=0}=0, \tag{D10}$$

which becomes in the momentum space

$$p_\mu\Pi^2_{\mu\nu}(p)\Big|_{O(F)}+\frac{1}{2}\mathcal{F}_{\mu\rho}\frac{\partial}{\partial p_\rho}\Pi^1_{\mu\nu}(p)\Big|_{F=0}=0, \tag{D11}$$

where $\Pi^2_{\mu\nu}(p)|_{O(F)}$ can be found from expression (67) in $2+1$ dimensions

$$\begin{aligned}\Pi^2_{\mu\nu}(p)|_{O(F)}&=\frac{1}{(4\pi)^{3/2}}\int_{1/\Lambda^2}^\infty du\int_0^1 dv\,u^{-1/2}\\&\quad\times\exp\left[-u\left(m^2+\frac{1-v^2}{4}p^2\right)\right]\\&\quad\times\left[\mathcal{F}_{\mu\nu}-\frac{u(1-v^2)}{4}[p_\mu(\mathcal{F}p)_\nu-(\mathcal{F}p)_\mu p_\nu\right.\\&\quad\left.+2p^2\mathcal{F}_{\mu\nu}]\right],\end{aligned} \tag{D12}$$

and from Eq. (73) in $3+1$ dimensions

$$\begin{aligned}\Pi^2_{\mu\nu}(p)|_{O(F)}&=\frac{1}{(4\pi)^2}\int_{1/\Lambda^2}^\infty du\int_0^1 dv\,\exp\left[-u\left(m^2+\frac{1-v^2}{4}p^2\right)\right]\\&\times\left[F_+\left\{\frac{1}{u}N^+_{\mu\nu}-\frac{(1-v^2)}{2}(I^-p)\cdot(I^-p)N^+_{\mu\nu}\right.\right.\\&-\frac{1+v^2}{4}[(I^-p)_\mu(N^+p)_\nu-(N^+p)_\mu(I^-p)_\nu\\&-(I^+p)_\mu(N^+p)_\nu+(N^+p)_\mu(I^+p)_\nu]\\&+\frac{v^2}{2}[(I^+p)\cdot(I^+p)N^+_{\mu\nu}+(I^-p)_\mu(N^+p)_\nu\\&\left.\left.-(N^+p)_\mu(I^-p)_\nu]\right\}+(+\leftrightarrow-)\right],\end{aligned} \tag{D13}$$

respectively. With the use of Eq. (71), the left-hand side of Eq. (D11) is shown to vanish,

$$\begin{aligned}\text{LHS}&=\frac{1}{2(4\pi)^{D/2}}\int_{1/\Lambda^2}^\infty du\int_0^1 dv\,\frac{e^{-um^2}}{u^{D/2-1}}(\mathcal{F}p)_\nu\\&\quad\times\frac{d}{dv}\left[v(1-v^2)\exp\left(-\frac{u(1-v^2)}{4}p^2\right)\right]\\&=0.\end{aligned} \tag{D14}$$

Therefore we can convince ourselves that the Ward-Takahashi relation is satisfied in each order of the background field [24].

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