# Thermal and quantum fluctuations around domain walls

Carlos A. A. de Carvalho*

*Instituto de Física, Universidade Federal do Rio de Janeiro, C.P. 68528, Rio de Janeiro, RJ 21945-970, Brazil*
(Received 8 October 2001; published 4 March 2002)

We compute thermal and quantum fluctuations in the background of a domain wall in a scalar field theory at finite temperature using the *exact* scalar propagator in the subspace orthogonal to the wall's translational mode. The propagator makes it possible to calculate terms of any order in the semiclassical expansion of the partition function of the system. The leading term in the expansion corresponds to the fluctuation determinant, which we compute for *arbitrary* temperature in space dimensions 1, 2, and 3. Our results may be applied to the description of thermal scalar propagation in the presence of soliton defects (in polymers, magnetic materials, etc.) and interfaces which are characterized by kinklike profiles. They lead to predictions as to how classical free energy differences, surface tensions, and interface profiles are modified by fluctuations, allowing for comparison with both numerical and experimental data. They can also be used to estimate transition temperatures. Furthermore, the simple analytic form of the propagator may simplify existing calculations, and allow for more direct comparisons with data from scattering experiments.

## I. INTRODUCTION

There are many problems in physics which require quantization around classical backgrounds, such as Schwinger's QED calculation of vacuum polarization around constant electromagnetic fields [1], the semiclassical quantization of field theories [2], and the semiclassical analysis of the *classical* limit [3,4] of quantum mechanics. In fact, the latter two characterize the semiclassical analysis as a two-way road: towards quantization, for the field theorists; towards the classical limit, for learning how classical physics descends from the quantum world.

Taking the field theory lane of that two-way road, we have recently studied the problem of quantizing the simplest of field theories, one-dimensional quantum mechanics, whose (scalar) "field" lives in zero spatial dimension [5–8]. Working in *imaginary time*, i.e., at finite temperature, and with smooth potentials, we were able to show that a full semiclassical series could be constructed from the knowledge of semiclassical propagators in the classical backgrounds of the solutions of the equations of motion, and gave a general prescription for deriving them from those backgrounds. Our construction was similar to the one proposed long ago by DeWitt-Morette [9] in *real time*, i.e., at zero temperature: it generalized to quantum statistical mechanics, where the number of classical solutions is drastically reduced, the results for ordinary quantum mechanics.

This article applies some of our techniques to semiclassical quantization around a very specific background in a very specific scalar field theory, at finite temperature, in one, two, and three spatial dimensions. In that respect, it bears greater similarity to the Schwinger problem than to the semiclassical quantization of field theories. The latter would require finding all classical solutions that satisfy the finite temperature boundary conditions, and a general procedure for constructing semiclassical propagators in their backgrounds. As we

have already stated, in quantum mechanics the general procedure does exist, and it is often possible to find all classical solutions [5–8]. In field theories, however, finding all classical solutions is already a difficult problem, and even if we find them all, no general procedure to obtain semiclassical propagators in their backgrounds is known. The technical reason for that is the presence of spatial dimensions, which transform the fluctuation problem, given by an ordinary differential equation (ODE) in zero spatial dimension, into a partial differential equation (PDE) problem, whose solutions cannot be obtained from the mere knowledge of the background. Even finding the backgrounds involves, in general, a PDE, instead of the ODE of quantum mechanics.

We have thus examined a specific potential of the $\varphi^4$-type, and confined our analysis to a (Euclidean) time-independent solution of the classical equations of motion depending on only one spatial coordinate, a domain wall whose classical profile and dynamics of fluctuations could be *exactly* solved from ODE's. We were then able to construct a semiclassical propagator in the presence of that background, from which, in principle, we can generate any term of the semiclassical series around the domain wall. We note that even in one spatial dimension, where the domain wall has finite Euclidean action, and therefore contributes to the semiclassical approximation, we abdicate the idea of "solving" the theory semiclassically, because there are many other solutions whose semiclassical propagators cannot be obtained from a general recipe. In particular, solutions which only depend on Euclidean time do contribute: they are given by elliptic functions, similar to the ones in quantum mechanics [5–8], but whose fluctuation problems are much harder to solve.

Our motivation for pursuing this problem, besides the technical appeal of obtaining thermal and quantum fluctuations at any temperature from an *exact* expression for the semiclassical propagator, was dictated by its potential physical applications. Indeed, for spatial dimension $d=1$, our formalism can be adapted to describe fluctuations around defects that occur in one-dimensional systems, such as the

*Electronic address: aragao@if.ufrj.br

*solitons* found in polymers (like polyacetylene) [10]; and for $d=2,3$, to describe fluctuations around *interfaces* that separate two distinct phases [11], a description that allows one to calculate how the surface tension, as well as the interface profile are modified by those fluctuations. It can also be used to approximately estimate transition temperatures [12]. Adopting this view, we are able to extract physically relevant results from the study of $d>1$, where the action of the domain wall diverges with the size of the system.

Since the problem is treated at finite temperature $T$, fluctuations incorporate thermal and quantum effects, which justifies the term thermal and quantum fluctuations employed thus far. The background we use, however, does not depend on $T$, as it is a "static" classical solution of the equations of motion, independent of the Euclidean time. That independence guarantees that it remains a *classical* solution at *any* temperature $T$, which is all we require to compute a semiclassical expansion order by order. As we incorporate thermal effects through fluctuations, our classical solution will no longer be a minimum of the (now $T$-dependent) Gibbs free energy (the effective action). Nevertheless, the new minimum may be obtained by extremizing the free energy with an ansatz that adds corrections to the original kink profile. The procedure will be outlined in Sec. V, and will profit from the nice properties of that original profile. Physically, in order for the procedure to be justified, we must be dealing with situations where we start with a rather "massive" background, but whose fluctuations are "light," when both are compared to the temperature $T$. Thus, only the fluctuations would be influenced by the interaction with the heat bath, as in our calculations. This is very much what would be expected in the context of the so-called "adiabatic" approximations, where one can distinguish "heavier" and "lighter" degrees of freedom, as in the physics of polymers. Clearly, we have to assume that boundary conditons at spatial infinity of the domain wall type are externally fixed, which forces us to go to the kink sector.

We should stress that the problem of computing fluctuations around domain wall solutions has played an important role in the physics of interfaces in binary mixtures [11]. In that case, however, temperature was introduced in a phenomenological way, through the "coarse-grained" Ginzburg-Landau free energy, whose mass parameter was assumed temperature dependent. In order to avoid confusion, we will call that temperature $T_{ph}$. Our problem will reduce to that one if we also assume that the mass parameter of our Lagrangian depends on $T_{ph}$, in a manner determined phenomenologically, and, furthermore, that our fluctuations are coupled to a heat bath at *zero* temperature, $T=0$. Then, our methods can be used in the calculations that compute critical exponents to two-loop order [13], which obtain excellent agreement with both numerical and real experiments. Those calculations make use of the semiclassical propagator in the form of a series over eigenmodes. Our closed expression should simplify them, and should allow for an independent check. We, however, treat a different physical situation: one in which a microscopic field theory is forced to have two regions at distinct vacua. For that case, we compute how the interface separating those regions is affected when a heat

bath at $T$ couples to its quantum fluctuations. In the polymer context, at $d=1$, that amounts to deriving how thermal phonons modify a soliton defect. For $d>1$, that can tell us how thermal neutral bosons affect an interface separating regions in different Bose-condensed phases.

We should point out that a number of works have dealt with similar questions: the finite temperature determinant has been computed long ago [12,14], as a product over eigenvalues; Parnachev and Yaffe [15] have carefully studied the same problem at $T=0$, paying special attention to the determinant calculation using a different method. Graham *et al.* [16] have investigated similar problems involving interfaces, resorting to the relation with phase shifts, rather than with the semiclassical propagator. In all cases, it seems possible to establish a precise relation to the approach we adopt. It should be noted that our approach includes a finite temperature $T$, and derives the *exact* semiclassical propagator which, together with a careful treament of collective coordinates, should allow for the computation of correlation functions beyond leading order.

The structure of the article is as follows. In Sec. II, we briefly review how one derives the semiclassical expansion; Sec. II A obtains an *exact* analytic expression for the semiclassical propagator, while Sec. II B uses it to compute fluctuation determinants; in Sec. III, we describe the renormalization procedure, first by regularizing the calculation in Sec. III A, and then by removing its divergences in Sec. III B; in Sec. IV, we discuss the collective coordinates associated with the translational invariance of the solution; in Sec. V, we show how surface tensions and interface profiles are modified by fluctuations, and outline how the first higher order correction can be computed. We also mention how transition temperatures can be estimated from our results. We present our conclusions in Sec. VI.

## II. THE SEMICLASSICAL EXPANSION

The partition function for a self-interacting scalar field theory in contact with a thermal reservoir at temperature $T$ can be written as ($\beta=1/T$)

$$Z(\beta) = \oint [\mathcal{D}\varphi] e^{-S[\varphi]/\lambda}, \tag{1}$$

$$S[\varphi] = \int_0^\beta d\tau \int d^dx \mathcal{L}[\varphi], \tag{2}$$

$$\mathcal{L}[\varphi] = \frac{1}{2}(\partial_\tau \varphi)^2 + \frac{1}{2}(\boldsymbol{\nabla}\varphi)^2 + \frac{1}{4!}(\varphi^2 - \varphi_v^2)^2. \tag{3}$$

The integral $\oint$ is to be performed over all $\varphi$'s that satisfy the boundary conditions $\varphi(0,\mathbf{x}) = \varphi(\beta,\mathbf{x})$. The fields are rescaled so that the coupling constant $\lambda$ appears as an overall factor.

We shall be interested in domain wall *Ansätze* $\hat{\varphi}(x)$ which only depend on one spatial coordinate $x \equiv x^1$, which will be called longitudinal. As is well known [2], the equation of motion from Eq. (3) is satisfied by a kinklike profile:

$$\hat{\varphi}(x) = \pm \varphi_v \tanh\left[\frac{\varphi_v(x-\bar{x})}{2\sqrt{3}}\right], \tag{4}$$

where $\bar{x}$ denotes the position of the domain wall separating two spatial regions of dimension $d$. The remaining $d-1$ transverse spatial coordinates of $\mathbf{x}$ define a vector $\mathbf{r}$. The $\pm$ refers to kink and antikink, respectively. The classical action of the wall is proportional to the volume of the $(d-1)$-dimensional subspace orthogonal to the $x$-axis, which we will denote by $V_{d-1}$ ($V_0 \equiv 1$):

$$S[\hat{\varphi}] = \frac{2}{3\sqrt{3}} \beta \varphi_v^3 V_{d-1}. \tag{5}$$

We expand the action around the domain wall configuration $\hat{\varphi}$ in order to derive the semiclassical series in that background:

$$\varphi(\tau,\mathbf{x}) = \hat{\varphi}(\tau,\mathbf{x}) + \lambda^{1/2}\eta(\tau,\mathbf{x}), \tag{6}$$

$$\eta(0,\mathbf{x}) = \eta(\beta,\mathbf{x}) = 0. \tag{7}$$

As usual, the fluctuation $\eta$ has to vanish at $\tau=0$ and $\tau=\beta$ because $\hat{\varphi}$ already satisfies the boundary condition. The expression for the partition function becomes

$$Z(\beta) = e^{-S[\hat{\varphi}]/\lambda} \oint [\mathcal{D}\eta] e^{-(S_2+\delta S)}, \tag{8}$$

where both $S_2$ and $\delta S$ depend functionally on $\hat{\varphi}$ and on $\eta$:

$$S_2[\hat{\varphi},\eta] = \int_0^\beta d\tau \int d^d x \left\{ \frac{1}{2}(\partial_\tau\eta)^2 + \frac{1}{2}(\boldsymbol{\nabla}\eta)^2 \right.$$
$$\left. + \frac{1}{12}(3\hat{\varphi}^2 - \varphi_v^2)\,\eta^2 \right\}, \tag{9}$$

$$\delta S[\hat{\varphi},\eta] = \int_0^\beta d\tau \int d^d x \left\{ \frac{\lambda^{1/2}}{3!}\hat{\varphi}\,\eta^3 + \frac{\lambda}{4!}\eta^4 \right\}. \tag{10}$$

If we define $M^2 \equiv \varphi_v^2/3$, and use the explicit form of $\hat{\varphi}$, the quadratic Lagrangian in Eq. (9) can be written

$$\mathcal{L}_2 = \frac{1}{2}\left\{ (\partial_\tau\eta)^2 + (\boldsymbol{\nabla}\eta)^2 + \left[ M^2 - \frac{3}{2}M^2 \operatorname{sech}^2(\xi) \right]\eta^2 \right\}, \tag{11}$$

with $\xi \equiv [\varphi_v(x-\bar{x})/2\sqrt{3}]$ a dimensionless variable.

We may generate a semiclassical series by expanding the $e^{-\delta S}$ term, and using the quadratic propagator in the kink background to Wick-contract the various products of $\eta$ fields that will appear. Note that, when compared with perturbation theory, this expansion will involve different vertices, as well as a different propagator. The latter is already an infinite sum of one-loop perturbative diagrams [6,7]. We now proceed to construct the propagator, a crucial step to obtain the series.

## A. The quadratic propagator

The quadratic propagator of our semiclassical expansion must satisfy [5–7]

$$\left\{ -\partial_\tau^2 - \boldsymbol{\nabla}^2 + \left[ M^2 - \frac{3}{2}M^2 \operatorname{sech}^2(\xi) \right] \right\} G(\tau,\mathbf{x};\tau',\mathbf{x}')$$
$$= \delta(\tau-\tau')\delta^d(\mathbf{x}-\mathbf{x}'), \tag{12}$$

as well as the boundary conditions

$$G(0,\mathbf{x};\tau',\mathbf{x}') = G(\beta,\mathbf{x};\tau',\mathbf{x}') = 0, \tag{13}$$

since the fluctuations vanish at $\tau=0$, $\beta$.

The spectrum of the differential operator in Eq. (12) is that of a Pöschl-Teller potential, which is known exactly [17]. Many applications involving kinklike profiles [11–14,18] have made use of that explicit knowledge of the spectrum. Indeed, in some of them the quadratic propagator was expressed as a sum over the various eigenmodes of the operator, with the respective eigenvalues in the denominator. That, however, leads to very lengthy calculations [11,13], especially if one goes beyond one-loop [13]. Therefore, it is useful to derive a *closed exact* expression for the propagator which is actually the result of that sum. We can do it by realizing that, in transverse momentum space, the problem can be reduced to finding the propagator for an ordinary differential equation.

Fourier transforming the transverse coordinates, as well as the Euclidean time, we obtain the ordinary differential equation

$$\left\{ -\partial_x^2 + \omega_n^2 + \mathbf{k}^2 + \left[ M^2 - \frac{3}{2}M^2 \operatorname{sech}^2(\xi) \right] \right\} \tilde{G}(\kappa_n;x,x')$$
$$= \delta(x-x'), \tag{14}$$

where $\omega_n = 2\pi n/\beta$, and we have defined the dimensionless vector $\boldsymbol{\kappa}_n \equiv (2/M)(\omega_n,\mathbf{k})$. As we will see, $\tilde{G}$ only depends on $\kappa_n = |\boldsymbol{\kappa}_n| = (2/M)\sqrt{\omega_n^2 + \mathbf{k}^2}$. The propagator $G$ is given by

$$G(\tau,\mathbf{x};\tau',\mathbf{x}') = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^{d-1}k}{(2\pi)^{d-1}}$$
$$\times \tilde{G}(\kappa_n;x,x') e^{i\omega_n(\tau-\tau')} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}. \tag{15}$$

In terms of the dimensionless propagator $\tilde{\Gamma} \equiv (M/2)\tilde{G}$, and of the variable $\xi$, defined in Eq. (11), the equation reads

$$\{ -\partial_\xi^2 + 4 + \kappa_n^2 - 6 \operatorname{sech}^2(\xi) \} \tilde{\Gamma}(\kappa_n;\xi,\xi') = \delta(\xi-\xi'). \tag{16}$$

We can solve it by finding two linearly independent solutions, $\phi_1(\xi)$ and $\phi_2(\xi)$, of the homogeneous equation [5–7]

$$\{ -\partial_\xi^2 + 4 + \kappa_n^2 - 6 \operatorname{sech}^2(\xi) \} \phi_i(\xi) = 0. \tag{17}$$

The solutions are obtained in Appendix A:

$$\phi_1(u) = \left(\frac{u}{1-u}\right)^{b_n/2}\left[1 - \frac{6}{b_n+1}u + \frac{12}{(b_n+1)(b_n+2)}u^2\right],$$

$$(18)$$

$$\phi_2(u) = \left(\frac{1-u}{u}\right)^{b_n/2}\left[1 + \frac{6}{b_n-1}u + \frac{12}{(b_n-1)(b_n-2)}u^2\right],$$

$$(19)$$

where $u = (1-\tanh\xi)/2$ and $b_n = \sqrt{4+\kappa_n^2}$. Expressions (18) and (19) are well-defined for $\kappa_n \neq 0$, i.e., $b_n > 2$. We will return to this point later on. The propagator can now be constructed (see Appendix A), yielding

$$\tilde{\Gamma}(\kappa_n; u, u') = \frac{1}{2b_n}\{\phi_1(u)\phi_2(u')\Theta(u'-u)$$

$$+ \phi_1(u')\phi_2(u)\Theta(u-u')\}, \qquad (20)$$

where $\Theta(x)$ is the Heaviside step function. Note that the factors outside the brackets in Eqs. (18) and (19) will cancel in Eq. (20) whenever $u = u'$.

The propagator we have just constructed can only be defined for $\kappa_n > 0$, or $b_n > 2$. As is well known [2,17], for $\kappa_0 = 0$ the fluctuation kernel in Eq. (16) admits a zero eigenmode associated to translational symmetry, i.e., a zero eigenvalue solution of Eq. (17) that vanishes at $\xi \to \pm\infty$. Explicitly:

$$\tilde{\eta}_0(u) = \frac{\sqrt{3}}{2}\text{sech}^2[\xi(u)] = 2\sqrt{3}u(1-u). \qquad (21)$$

Consequently, we can only define a propagator in the subspace orthogonal to that spanned by Eq. (21). Thus, we subtract the contribution of that mode to obtain

$$\tilde{\Gamma}'(\kappa_n; u, u')\tilde{\Gamma}(\kappa_n; u, u') - \frac{\tilde{\eta}_0(u)\tilde{\eta}_0(u')}{\kappa_n^2}. \qquad (22)$$

The divergent part of the limit as $\kappa_n \to 0$ of $\tilde{\Gamma}(\kappa_n; u, u')$ is exactly cancelled by the second term on the right-hand side of Eq. (22), so that $\tilde{\Gamma}'(\kappa_n; u, u')$ is well defined and orthogonal to the zero mode subspace in the limit $\kappa_n \to 0$:

$$\lim_{\kappa_n \to 0}\tilde{\Gamma}'(\kappa_n; u, u') = \frac{3}{4}u(1-u)u'(1-u'). \qquad (23)$$

### B. Determinants

The leading term of the semiclassical expansion of the partition function around a domain wall background is obtained by setting $\delta S = 0$ in expression (8). The quadratic path-integral requires collective coordinates for the zero-mode subspace [19,20]. As we will see in Sec. IV, the remaining integration, in the subspace orthogonal to that eigenmode, leads to a fluctuation determinant which will be computed from the quadratic semiclassical propagator for arbitrary temperature $T$ and $d = 1,2,3$.

We again Fourier transform the fluctuation fields in all coordinates but the longitudinal one:

$$\eta(\tau,\mathbf{x}) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\tilde{\eta}(\boldsymbol{\kappa}_n; x)e^{i\omega_n\tau + i\mathbf{k}\cdot\mathbf{r}}. \quad (24)$$

The quadratic part of the action may be written as

$$S_2 = \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\tilde{S}_2(\boldsymbol{\kappa}_n), \qquad (25)$$

$$\tilde{S}_2(\boldsymbol{\kappa}_n) = \frac{1}{2}\int_{-\infty}^{\infty}dx\,\tilde{\eta}(\boldsymbol{\kappa}_n; x)K(\boldsymbol{\kappa}_n)\tilde{\eta}(\boldsymbol{\kappa}_n; x). \qquad (26)$$

We denote by $K(\boldsymbol{\kappa}_n)$ the differential operator in Eq. (14), and *formally* define its determinant as

$$\Delta_{\kappa_n}^{-1/2} = [\det K(\boldsymbol{\kappa}_n)]^{-1/2} = \mathcal{N}\int[\mathcal{D}\tilde{\eta}]e^{-\tilde{S}_2(\boldsymbol{\kappa}_n)}, \qquad (27)$$

where the path integral sums over all fluctuations satisfying $\tilde{\eta}(\boldsymbol{\kappa}_n; \pm\infty) = 0$, and $\mathcal{N}$ is a normalization constant. Denoting by $|\tilde{\eta}_j\rangle$ the eigenmodes (Pöschl-Teller modes, which do not depend on $\boldsymbol{\kappa}_n$), and by $\gamma_{j\kappa_n}^2$ the eigenvalues of $K(\boldsymbol{\kappa}_n)$, respectively, we have

$$K^{-1}(\boldsymbol{\kappa}_n) = \sum_j\frac{|\tilde{\eta}_j\rangle\langle\tilde{\eta}_j|}{\gamma_{j\kappa_n}^2}, \qquad (28)$$

$$\gamma_{j\kappa_n}^2 = \frac{1}{4}M^2\kappa_n^2 + \gamma_j^2, \qquad (29)$$

where $j$ denotes both discrete and continuum Pöschl-Teller eigenstates, and $\gamma_j = \gamma_{j0}$ are their respective eigenvalues. By convention, $j = 0$ denotes the lowest of the eigenstates. As we have already remarked, $\gamma_{00}^2 = 0$, which implies $\Delta_0 = 0$. Therefore, when in the $\kappa_0 = 0$ subspace, we must use a primed determinant $\Delta_0'$ which, by definition, is the product of all but the lowest eigenvalue.

We may now relate ratios of primed determinants (i.e., with the lowest eigenvalue excluded) to the semiclassical propagator of the previous section:

$$\ln\frac{\Delta_{\kappa_n}'}{\Delta_0'} = \int_0^{\kappa_n^2}ds\,\text{Tr}\,\tilde{\Gamma}'(\sqrt{s}), \qquad (30)$$

where the trace is $\int_{-\infty}^{\infty}d\xi\tilde{\Gamma}'(\sqrt{s}; \xi, \xi)$. We are ultimately interested in ratios of determinants in the presence of the domain wall to free ones: for $\kappa_0 = 0$, this is just given by the corresponding ratio for the Pöschl-Teller potential in one-dimensional quantum mechanics, $\ln(\Delta_0'/\Delta_0^F)$, which excludes the zero mode; for $\kappa_n \neq 0$, we may write

$$\ln\frac{\Delta_{\kappa_n}}{\Delta_{\kappa_n}^F} = \ln\kappa_n^2 + \ln\frac{\Delta_{\kappa_n}'}{\Delta_0'} - \ln\frac{\Delta_{\kappa_n}^F}{\Delta_0^F} + \ln\frac{\Delta_0'}{\Delta_0^F}. \qquad (31)$$

The first term in Eq. (31) simply restores the lowest eigenvalue for $\kappa_n \neq 0$, whereas the second and the third can be calculated from the knowledge of propagators: the one in the domain wall background, and the free one. From the calculation outlined in Appendix B, we obtain

$$\ln\frac{\Delta'_{\kappa_n}}{\Delta'_0} - \ln\frac{\Delta^F_{\kappa_n}}{\Delta^F_0} = \ln\left(\frac{b_n-1}{b_n+1}\right) - 2\ln(b_n+2) + \ln 48. \quad (32)$$

We may follow [19] to derive the result for $\kappa_0 = 0$:

$$\ln\frac{\Delta'_0}{\Delta^F_0} = -\ln 48. \quad (33)$$

If we insert Eqs. (32) and (33) into Eq. (31), which is valid for $\kappa_n \neq 0$, we obtain

$$\ln\frac{\Delta_{\kappa_n}}{\Delta^F_{\kappa_n}} = \ln\left(\frac{b_n-1}{b_n+1}\right) + \ln\left(\frac{b_n-2}{b_n+2}\right). \quad (34)$$

The above result includes the $n=0$ subspace of eigenmodes, but requires $\kappa_n \neq 0$, as the $\kappa_0 = 0$ contribution has been explicitly summed to cancel the $\ln 48$ term of Eq. (32).

*Formally*, the logarithm of the complete (primed) determinant is the sum over all possible $\kappa_n$'s ($\kappa_n > 0$) of Eq. (34):

$$\ln\frac{\Delta'}{\Delta^F} = V_{d-1}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\sum_{n=-\infty}^{\infty}\ln\frac{\Delta_{\kappa_n}}{\Delta^F_{\kappa_n}}. \quad (35)$$

Expression (35), as it stands, is meaningless: the sum over $n$ diverges logarithmically, and, for $d>1$, the integral over **k** introduces extra divergences as well. One must undertake a renormalization process in order to arrive at well-defined results. In the next section, we shall regularize, and then renormalize the calculation to obtain a final answer for any $T$ and for $d=1,2,3$, where the theory is renormalizable.

### III. THE RENORMALIZATION PROCEDURE

#### A. Regularization

We shall concentrate our attention on Eq. (35). We begin by examining the behavior of the sum $\mathcal{S}$ over Matsubara frequencies defined as

$$\mathcal{S} \equiv \sum_{n=-N}^{N}\ln\frac{\Delta_{\kappa_n}}{\Delta^F_{\kappa_n}}, \quad (36)$$

where $N$ is a cutoff. If we introduce a dimensionless function $c_k \equiv (\beta/2\pi)\sqrt{M^2+k^2}$, the sum becomes

$$\mathcal{S} = \sum_{n=-N}^{N}\left(\ln\frac{\sqrt{n^2+c_k^2}-\frac{1}{2}c_0}{\sqrt{n^2+c_k^2}+\frac{1}{2}c_0} + \ln\frac{\sqrt{n^2+c_k^2}-c_0}{\sqrt{n^2+c_k^2}+c_0}\right). \quad (37)$$

$\mathcal{S}$ may be rewritten in a convenient way by resorting to expression (C5) of Appendix C, obtained from Plana's formula

[21], which is commonly used in the description of the Casimir effect [22]. For $\kappa_n \neq 0$, it is given as the difference of two terms: $\mathcal{S} = \mathcal{S}_a - \mathcal{S}_b$, where

$$\mathcal{S}_a = \sum_{j=1}^{2}\sum_{\sigma=0}^{1}2e^{i\sigma\pi}\left[\int_0^N dx\,\ln\left(\sqrt{x^2+c_k^2}-\frac{1}{2}je^{i\sigma\pi}c_0\right)\right], \quad (38)$$

$$\mathcal{S}_b = \sum_{j=1}^{2}\int_{c_k}^{\infty}\frac{8\,dy}{e^{2\pi y}-1}\left[\pi-\arctan\left(\frac{2\sqrt{y^2-c_k^2}}{jc_0}\right)\right]. \quad (39)$$

The integral $I$ in the brackets of Eq. (38) can be done exactly [see Eq. (D3) in Appendix D]. For $N$ large, if we neglect terms of order $1/N$, replace $N$ with $\beta\Lambda/2\pi$, and restore dimensional quantities, expression (37) can be split into a zero temperature part $\mathcal{S}_0$,

$$\mathcal{S}_0 = \frac{2\beta k}{\pi}\arcsin\left(\frac{c_0}{c_k}\right) + \frac{2\beta}{\pi}\sqrt{k^2+\frac{3}{4}M^2}\arcsin\left(\frac{c_0}{2c_k}\right)$$
$$-\frac{3}{\pi}\beta M\left[\ln\left(\frac{\beta\Lambda}{\pi c_k}\right)+1\right], \quad (40)$$

and a temperature dependent part $\mathcal{S}_T$,

$$\mathcal{S}_T = \int_{c_k}^{\infty}\frac{8\pi dy}{e^{2\pi y}-1}\left[\arctan\left(\frac{\sqrt{y^2-c_k^2}}{c_0}\right)+\arctan\left(\frac{2\sqrt{y^2-c_k^2}}{c_0}\right)\right]$$
$$+4\ln(1-e^{-2\pi c_k}). \quad (41)$$

In both expressions, the first two terms represent the contribution of the (Pöschl-Teller) bound states in the longitudinal direction, while the last ones correspond to the continuum. For $d=1$, $k=0$, so that the first term of Eq. (40) vanishes.

#### B. Renormalization

The continuum contribution (last term) to the zero temperature part $\mathcal{S}_0$ contains the expected logarithmic divergence. Indeed, at zero temperature we should replace the sum (36) by an integral:

$$\mathcal{S} = \sum_{n=-N}^{N}\ln\frac{\Delta_{\kappa_n}}{\Delta^F_{\kappa_n}}\xrightarrow{T\to 0}\mathcal{I} = \beta\int_{-\Lambda}^{\Lambda}\frac{d\omega}{2\pi}\ln\frac{\Delta_\kappa}{\Delta^F_\kappa}, \quad (42)$$

where $\omega$ is a continuous variable that replaces $\omega_n$, and $\boldsymbol{\kappa} \equiv (2/M)(\omega,\mathbf{k})$. Furthermore, we may write

$$\ln\frac{\Delta_\kappa}{\Delta^F_\kappa} = \mathrm{Tr}\ln[\mathbb{1}-\tilde{\Gamma}_F(\kappa)U], \quad (43)$$

where $\mathbb{1}$ is the identity operator, $\tilde{\Gamma}_F(\kappa;\xi,\xi')$ is given by Eq. (B1), with $b_n$ replaced by $b \equiv \sqrt{4+\kappa^2}$, $U(\xi) \equiv -6\,\mathrm{sech}^2(\xi)$ is the interaction term in Eq. (16), and the trace is defined in Eq. (30). If we expand the logarithm in Eq. (43), we shall have a series of one-loop terms:

$$\ln\frac{\Delta_\kappa}{\Delta_\kappa^F} = -\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}[\widetilde{\Gamma}_F(\kappa)U]^n. \tag{44}$$

Inserting Eq. (44) into Eq. (42), the $n=1$ term $\mathcal{I}_1$ can be readily computed (change the integration variable from $\xi$ to $u$) and identified with the divergent term of Eq. (40):

$$\mathcal{I}_1 = -\beta\int_{-\Lambda}^{\Lambda}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}d\xi\widetilde{\Gamma}_F(\kappa;\xi,\xi)U(\xi)$$

$$= -\frac{3}{\pi}\beta M\ln\left(\frac{\beta\Lambda}{\pi c_k}\right). \tag{45}$$

In order to renormalize the theory, we add a counterterm to the Lagrangian which will absorb that divergent term, and impose renormalization conditions to fix the finite remainder. For simplicity, we will adopt renormalization conditions at zero momentum $\kappa$. In the present case, it suffices to add a mass-type counterterm:

$$\mathcal{L}[\varphi]\rightarrow\mathcal{L}[\varphi]-\frac{1}{2}C_1(\varphi^2-\varphi_v^2), \tag{46}$$

and to require that the two-point one-particle irreducible (1PI) Green function at zero momenta $\Gamma_R^{(2)}$ be given by

$$\widetilde{\Gamma}_R^{(2)}(\mathbf{0},\mathbf{0})=M^2. \tag{47}$$

In order to satisfy Eq. (47), we must subtract the tadpole graph in the kink background, which is also given by Eq. (45) because the free propagator at coincident points can be factored out of the $\xi$ integration. We then obtain the renormalized expression

$$\mathcal{S}_R = \frac{2\beta k}{\pi}\arcsin\left(\frac{c_0}{c_k}\right) + \frac{2\beta}{\pi}\sqrt{k^2+\frac{3}{4}M^2}\arcsin\left(\frac{c_0}{2c_k}\right)$$

$$-\frac{3}{\pi}\beta M + \mathcal{S}_T. \tag{48}$$

For $d=1$, $k=0$, $c_k=c_0$, and there are no additional integrals. In that case, we may verify that the temperature independent part of Eq. (48) coincides with the result quoted in Eq. (3.29) of the first reference in [19]. Indeed, from our calculation we may easily obtain the difference between the energy of the kink and that of the vacuum, i.e., the kink mass:

$$M_{kink} = \frac{1}{\beta}\left(\frac{S[\hat{\varphi}]}{\lambda} + \frac{1}{2}\mathcal{S}_R\right), \tag{49}$$

which at zero temperature yields

$$M_{kink} = \frac{2M^3}{\lambda} + \frac{M}{\sqrt{2}}\left(\frac{1}{2\sqrt{6}} - \frac{3}{\pi\sqrt{2}}\right). \tag{50}$$

The replacements $M=\sqrt{2}m_{raj}$, and $\lambda=6\lambda_{raj}$, needed to account for different definitions of mass and coupling, show that our results coincide with Rajaraman's [19] for $d=1$. We

note that Eqs. (32) and (33) had already made use of that coincidence for the case $d=0$.

For $d=2$, and $d=3$, we still have to integrate over $\mathbf{k}$. This last integral will generate additional divergences in $d=3$ dimensions, which are contained in the $n=2$ term of Eq. (44):

$$\mathcal{I}_2 = V_{d-1}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\beta\int_{-\Lambda}^{\Lambda}\frac{d\omega}{2\pi}\mathcal{J}_2, \tag{51}$$

$$\mathcal{J}_2 \equiv -\frac{1}{2}\int_{-\infty}^{\infty}d\xi_1\int_{-\infty}^{\infty}d\xi_2\widetilde{\Gamma}_F(\kappa;\xi_1,\xi_2)U(\xi_2)$$

$$\times\widetilde{\Gamma}_F(\kappa;\xi_2,\xi_1)U(\xi_1). \tag{52}$$

The calculation of Appendix E leads to

$$\mathcal{J}_2 = -36\left\{\frac{1}{2b^2(b+1)^2} + \frac{1}{6(b+1)^3} + O[1/(b+1)^5]\right\}. \tag{53}$$

Since $b=\sqrt{4+\kappa^2}$, the second term in Eq. (53) dominates in the ultraviolet.

We now analyze expression (51) for each dimension: (i) in $d=1$, the integral over $\omega$ goes like $\int\omega^{-3}d\omega\sim\Lambda^{-2}$, which vanishes as $\Lambda\rightarrow\infty$; (ii) in $d=2$, the integrals over $\omega$ and $k$ lead to $\mathcal{I}_2\sim\Lambda^{-1}$, which also vanishes in that limit; (iii) in $d=3$, however, the integrals over $\omega$ and $k$ lead to $\mathcal{I}_2\sim\ln\Lambda$, which diverges in that limit. Thus, in $d=3$ an additional subtraction is required. The leading ultraviolet behavior of Eq. (51) comes from the second term in Eq. (53):

$$\mathcal{I}_2^{(as)} = V_2\int\frac{d^2k}{(2\pi)^2}\beta\int_{-\Lambda}^{\Lambda}\frac{d\omega}{2\pi}\frac{-6(M/2)^3}{[\sqrt{\omega^2+k^2+M^2}+(M/2)]^3}. \tag{54}$$

We integrate the momenta $\mathbf{k}$ over a sphere of radius $\Lambda$ to obtain [23]

$$\mathcal{I}_2^{(as)} = -\frac{3}{16\pi^2}V_2\beta M^3[\ln(\Lambda^2/M^2)+O(1)]. \tag{55}$$

This is to be compared with the result of integrating the finite terms of Eq. (40) over momenta $\mathbf{k}$ (the divergent term was already cancelled by the first counterterm). Integration by parts leads to elementary integrals; expanding the result for large $\Lambda$, and neglecting terms of order $1/\Lambda^2$, we obtain

$$V_2\int\frac{d^2k}{(2\pi)^2}\mathcal{S}_0 = -\frac{3}{16\pi^2}V_2\beta M^3\ln(\Lambda^2/M^2) - \frac{\sqrt{3}}{48\pi}V_2\beta M^3$$

$$+O(1/\Lambda^2). \tag{56}$$

As expected, the leading ultraviolet behavior is the same, which shows that the $n=2$ term of Eq. (44) is indeed responsible for the new divergence. The renormalization procedure requires that we include an additional counterterm in the

Lagrangian, and use an additional renormalization condition to fix finite parts in $d=3$ (see Appendix F). The Lagrangian becomes

$$\mathcal{L}[\varphi] \rightarrow \mathcal{L}[\varphi] - \frac{1}{2}C_1(\varphi^2 - \varphi_v^2) - \frac{1}{4}C_2(\varphi^2 - \varphi_v^2)^2. \quad (57)$$

$C_1$ and $C_2$ are counterterms that we fix by imposing Eq. (47), as well as

$$\tilde{\Gamma}_R^{(4)}(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) = 1 + \frac{9\lambda}{32\pi^2}. \quad (58)$$

The contribution of the additional counterterm to the effective action in the kink sector is given by

$$\left[ -\frac{1}{2}\int^\Lambda \frac{d^3k}{(2\pi)^3}\int^\Lambda_{-\Lambda}\frac{d\omega}{2\pi}\frac{1}{(\omega^2+k^2+M^2)^2} \right]$$
$$\times \left[ \frac{1}{4}\int^\beta_0 d\tau\int^\infty_{-\infty}d^3x(\hat{\varphi}^2-\varphi_v^2)^2 \right], \quad (59)$$

where the first bracket is identified as the one-loop graph with two external legs at zero momenta (integrated over $w$ in the interval $[-\Lambda,\Lambda]$, and over $\mathbf{k}$ in the two-sphere of radius $\Lambda$, as before), while the second is easily related to the classical kink action. A simple calculation yields

$$-\frac{3}{16\pi^2}V_2\beta M^3\ln(\Lambda^2/M^2) + \frac{3}{16\pi^2}(1-\ln 2)V_2\beta M^3. \quad (60)$$

If we subtract the result of Eq. (60) from Eq. (56), and take the limit $\Lambda\rightarrow\infty$, we finally arrive at

$$\ln\left(\frac{\Delta'}{\Delta^F}\right)_R = -\frac{3}{16\pi^2}\left[1-\ln 2 + \frac{\pi}{3\sqrt{3}}\right]V_2\beta M^3$$
$$+ V_2\int\frac{d^2k}{(2\pi)^2}\mathcal{S}_T. \quad (61)$$

We should point out that expressions for the finite temperature determinant were previously obtained from the explicit knowlegde of the Pöschl-Teller eigenvalues, either by summing over Matsubara frequencies, and then over those eigenvalues [12], or by integrating over transverse momenta in arbitrary dimension first, making use of dimensional regularization, and then summing over Matsubara frequencies [14].

## IV. COLLECTIVE COORDINATES

The domain wall solution breaks translational symmetry along the longitudinal $x$-axis, as it depends on a parameter $\bar{x}$ that characterizes its position. In order to restore that symmetry of the theory, we resort to the method of collective coordinates [19] to sum over all possible values of $\bar{x}$. In $d=1$, there is nothing else to restore. In $d>1$, where the wall

also breaks rotational invariance by singling out a normal direction, consistently with our previous assumptions, the longitudinal direction is externally fixed by the forces that implement the wall boundary conditions at spatial infinity. Therefore, here as well, longitudinal translations are the only degrees of freedom to consider.

The existence of longitudinal translation zero eigenmodes of the fluctuation operator is a clear signal that the symmetry has to be restored. Physically, it means that it costs no energy to go from one solution to a longitudinally translated one. Thus, in the usual manner, in a region of longitudinal length $L$, we introduce in the partition function the identity

$$\int\frac{d\bar{x}}{L}\delta\left(\frac{1}{L\lambda^{1/2}}\int^\beta_0 d\tau\int d^dx\,\eta_0(x-\bar{x})[\varphi(\tau,x,\mathbf{r})\right.$$
$$\left. -\hat{\varphi}(x-\bar{x})]\right)J[\varphi]=1, \quad (62)$$

where the Jacobian is given by

$$J[\varphi] = \lambda^{-1/2}\int^\beta_0 d\tau\int d^dx\,\varphi(\tau,x,\mathbf{r})\frac{\partial}{\partial\bar{x}}\eta_0(x-\bar{x})$$
$$= \lambda^{-1/2}\int^\beta_0 d\tau\int d^dx\,\eta_0(x-\bar{x})\frac{\partial\varphi}{\partial x}. \quad (63)$$

Equation (62) imposes orthogonality to the normalized zero mode $\eta_0$, and integrates over longitudinal positions. The second equality in Eq. (63) comes from an integration by parts. We note that the zero mode is already orthogonal to the domain wall profile since $\eta_0\propto\partial\hat{\varphi}/\partial x$, which implies

$$\int^\infty_{-\infty}dx\,\eta_0(x-\bar{x})\hat{\varphi}(x-\bar{x})\propto[\hat{\varphi}^2]^\infty_{-\infty}=0. \quad (64)$$

Therefore, we may omit the $\hat{\varphi}$ term in Eq. (62). We insert the identity in the expression for the partition function (1), and change the order of integration to obtain

$$Z(\beta) = \int\frac{d\bar{x}}{L}\oint[\mathcal{D}\varphi]e^{-S[\varphi]/\lambda}\delta\left(\frac{1}{L\lambda^{1/2}}\int^\beta_0 d\tau\right.$$
$$\left. \times\int d^dx\,\eta_0(x-\bar{x})\varphi(\tau,x,\mathbf{r})\right)J[\varphi]. \quad (65)$$

We perform the semiclassical expansion around the domain wall according to Eq. (6). Since the action can be written as

$$S[\hat{\varphi}] = \int^\beta_0 d\tau\int d^dx\left(\frac{\partial\hat{\varphi}}{\partial x}\right)^2, \quad (66)$$

the normalized zero eigenmode is simply $\eta_0 = S[\hat{\varphi}]^{-1/2}(\partial\hat{\varphi}/\partial x)$. Using that, and its orthogonality to $\hat{\varphi}$, the expansion yields

$$Z(\beta) = \int \frac{d\bar{x}}{L} e^{-S[\hat{\varphi}]/\lambda} \oint [\mathcal{D}\eta] e^{-S_2} \delta\left(\frac{1}{L} \int_0^\beta d\tau \int d^d x \, \eta_0 \eta\right)$$

$$\times J[\hat{\varphi}, \eta] \sum_{n=0}^\infty \frac{1}{n!} (-\delta S)^n, \qquad (67)$$

with the functional integral being performed over fluctuations that vanish at $\tau = 0, \beta$, and the expanded Jacobian given by

$$J[\hat{\varphi}, \eta] = \left(\frac{S[\hat{\varphi}]}{\lambda}\right)^{1/2} - \int_0^\beta d\tau \int d^d x \frac{\partial \eta_0}{\partial x} \eta$$

$$= \left(\frac{S[\hat{\varphi}]}{\lambda}\right)^{1/2} - S[\hat{\varphi}]^{-1/2} \int_0^\beta d\tau \int d^d x V'[\hat{\varphi}] \eta.$$
$$(68)$$

The leading term in Eq. (67) corresponds to the quadratic approximation $Z_2$ to the partition function in the domain wall background, whose ratio to the free one is given by

$$\frac{Z_2(\beta)}{Z_F} = \int \frac{d\bar{x}}{L} e^{-S[\hat{\varphi}]/\lambda} \left[\frac{\Delta'}{\Delta_F}\right]_R^{-1/2} \left(\frac{S[\hat{\varphi}]}{2\pi\lambda}\right)^{1/2}. \qquad (69)$$

It involves the primed determinant calculated previously, since the delta function restricts us to the subspace orthogonal to the zero eigenmode, has a factor $(S[\hat{\varphi}]/\lambda)^{1/2}$ coming from the Jacobian, and a factor of $(2\pi)^{-1/2}$ from the functional measure which is not cancelled by the corresponding value for the free determinant in the zero mode subspace.

## V. PHYSICAL APPLICATIONS AND HIGHER ORDERS

The result obtained in the preceding section has an immediate physical application in the calculation of the interface tension in $d=2$ and $d=3$. If our system is confined to a longitudinal length $L$, and a transverse cross section $V_{d-1}$, the free energy difference per unit cross section between the situation with and without the domain wall defines the interface (surface) tension [11]

$$\sigma \equiv -\lim_{V_{d-1} \to \infty} \frac{1}{\beta V_{d-1}} \lim_{L \to \infty} \ln\left(\frac{Z}{Z_F}\right), \qquad (70)$$

where $Z$ denotes the partition function in the presence of the wall, and $Z_F$ the free partition function, without the domain wall. The lowest order result (69), with the integral over $\bar{x}$ taken from $-L/2$ to $L/2$ gives

$$\sigma = \lim_{V_{d-1} \to \infty} \frac{1}{\beta V_{d-1}} \left\{\frac{S[\hat{\varphi}]}{\lambda} + \frac{1}{2} \ln\left(\frac{\Delta'}{\Delta_F}\right)_R\right\}, \qquad (71)$$

where we have left out terms vanishing as $\ln V_{d-1}/V_{d-1}$ or faster. Another way of deriving the interface tension is through its relationship with the energy splitting between ground and first excited state of the field theory, when its vacuum degeneracy is broken by the finite volume [24]. The

tunneling between the previously degenerate states can be computed using domain walls [25,26]. This calculation was performed [27] using a dilute gas of kinks and antikinks [28], and yields

$$\Delta E = 2 e^{-S[\hat{\varphi}]/\lambda} \left[\frac{\Delta'}{\Delta_F}\right]_R^{-1/2} \left(\frac{S[\hat{\varphi}]}{2\pi\lambda}\right)^{1/2}, \qquad (72)$$

an expression that comes from the exponentiation of Eq. (69), which results from summing over kinks and antikinks. Our results incorporate finite temperature effects into the expressions obtained in Ref. [27]. Again, we stress that our starting point is a field theory that has no phenomenological temperature dependence in its Lagrangian, unlike treatments that start from a phenomenological Ginzburg-Landau coarse grained free energy [11,13]. Clearly, for $d=1$, a case that can be applied to the study of soliton backgrounds in one-dimensional polymers, we would be comparing the free energy difference with and without the soliton background ($V_0 = 1$), and finding how it is affected by thermal phonons.

The result obtained for the surface tension also provides us with an indirect determination of the temperature of the second order transition that will restore the $\varphi \to -\varphi$ symmetry. Strictly speaking, it determines the so-called "percolation temperature," the temperature at which the surface tension vanishes, which coincides with the critical temperature in $d=3$, although it is smaller in $d=2$ [29]. In fact, the statistical mechanics literature just compares free energies with and without domain wall boundary conditions, and establishes that the signal for the symmetry restoring transition is given when these two are equal. That means that the system will prefer to form walls separating regions at different vacua, whose condensation leads to a vanishing order parameter. Our calculation is just the semiclassical version of that. That estimate has already been made long ago [12], using a different method, as already explained.

Another physical information we can extract from our calculations is how the soliton ($d=1$), or interface ($d=2,3$) profiles are modified by thermal and quantum fluctuations. In lowest order, we may use the one-loop expression for the effective action,

$$\mathcal{A}[\langle \varphi \rangle] = \frac{S[\langle \varphi \rangle]}{\lambda} + \frac{1}{2} \ln(\Delta[\langle \varphi \rangle]), \qquad (73)$$

where $\langle \varphi(\mathbf{x}) \rangle$ is the expectation value of the field, and $\mathcal{A}$ is the Legendre transform of the free energy. We should look for extrema of Eq. (73), so we functionally differentiate with respect to $\langle \varphi(\mathbf{x}) \rangle$. The resulting equation will just be the classical equation of motion plus a one-loop correction which involves the propagator in the $\langle \varphi(\mathbf{x}) \rangle$ background. Neglecting the one-loop part, we know that the domain wall profile is a solution. Thus, the ansatz

$$\langle \varphi(\mathbf{x}) \rangle = \hat{\varphi}(x) + \lambda^{1/2} \delta\langle \varphi(\mathbf{x}) \rangle, \qquad (74)$$

leads to an equation for the correction term that involves the propagator in the domain wall background, for which we have an explicit expression. As a result, we will see how the

thermal and quantum fluctuations change the profile, and should be able to compare that prediction with scattering data from radiation (of appropriate wavelength) scattered off the domain wall.

We close this section with comments about higher order corrections. In our construction, these are generated by expanding the $e^{-\delta S}$ term in Eq. (8), and using the quadratic propagator in the domain wall background to Wick-contract the various products of $\eta$ fields that will appear. There is also an additional vertex that comes from the second term of the Jacobian in Eq. (68), which is linear in $\eta$, and proportional to $\lambda^{1/2}$. The first correction to the lowest order result for the partition functions involves graphs which were computed in the second article of Ref. [13], and represent a two-loop correction. Those same graphs can now be calculated with our propagator either at $T=0$ or at finite temparature ($\tau$ integrals going from 0 to $\beta$). In order to check and extend those results, one has to use the same renormalization scheme, conveniently chosen to facilitate comparisons with Monte Carlo data [13].

Numerical calculations will be required in applying even our lowest order results, as well as to allow for comparisons with experimental or Monte Carlo data. The two-loop calculation just mentioned will also require a numerical effort. We plan to undertake such efforts in the near future.

## VI. CONCLUSIONS

We have shown how one can systematically compute quantum fluctuations around a domain wall background which is temperature independent, but whose fluctuations interact with a heat bath at temperature $T$. The fluctuation determinant, the leading term in our semiclassical expansion, already makes the effective action $T$ dependent, and one can obtain an extremum for it that corresponds to a dressed $T$-dependent version of the original domain wall. As higher orders are included, that dependence will obviously change. The determinant also leads to the calculation of the surface tension, which again will depend on $T$ through the fluctuations. In both cases, one can follow the computations reviewed in Ref. [11], omitting their $T_{ph}$ dependence in the paramenters of the Lagrangian, and replacing the determinants with our $T$-dependent ones. Also, the temperature $T$ at which the surface tension vanishes can be used to estimate the phase transition to the unbroken phase [12].

As we have already mentioned, results for $d=1$ can be applied to describe how thermal phonons affect soliton backgrounds in polymer physics (examples involving magnetic materials are also potential applications), while those for $d=2,3$ are applicable to Bose systems which are forced to condense in different phases in two different regions separated by a domain wall. The simple analytic form of the propagator in the domain wall background is quite useful in calculations, and should lead to the determination of physical quantities measured in scattering experiments.

In principle, the methods we use could be applied to other theories, as long as the background solutions are sufficiently symmetric so as to depend on only one spatial variable (a radial one, for instance). The basic ingredient required is the knowledge, exact or approximate, of two linearly independent solutions of the fluctuation ODE at zero eigenvalue, from which we construct the semiclassical propagator. Theories with solitonic (kinks, vortices, monopoles, etc.) or instantonic backgrounds often have such a property.

We hope, in the very near future, to present detailed numerical calculations of the various quantities mentioned in Sec. V. From them, we expect to establish how realistic our description is, by confronting it with available experimental data.

*Note added in proof.* After completing this work, we learned that Baacke and co-workers have published a series of articles on related problems. Two recent ones are [31] and [32]. Also, a recent paper by Graham [33] deals with a modified version of this problem, using phase shifts to compute determinants.

## APPENDIX A

In this appendix, we compute the solutions to Eq. (17), and use them to construct the propagator (20). The standard substitution [17] $\phi(\xi)=e^{-a\xi}\cosh^{-b_n}(\xi)F(\xi)$, with $a=0$ and $b_n=\sqrt{4+\kappa_n^2}$, and the change of variable $u=(1-\tanh\xi)/2$ lead to a hypergeometric equation for $F$:

$$u(1-u)\frac{d^2F}{du^2}+[(b_n+1)-2(b_n+1)u]$$

$$\times\frac{dF}{du}+[6-b_n(b_n+1)]F=0. \tag{A1}$$

The general solution is the linear combination

$$F(u)=c_1\,_2F_1(b_n-2,b_n+3;1+b_n;u)$$

$$+c_2u^{-b_n}\,_2F_1(3,-2;1-b_n;u), \tag{A2}$$

where $_2F_1(A,B;C;u)$ is the hypergeometric function. Using the identity [23]

$$_2F_1(A,B;C;u)=(1-u)^{C-A-B}\,_2F_1(C-A,C-B;C;u), \tag{A3}$$

we obtain

$$F(u)=c_1(1-u)^{-b_n}\,_2F_1(3,-2;1+b_n;u)$$

$$+c_2u^{-b_n}\,_2F_1(3,-2;1-b_n;u). \tag{A4}$$

Both series terminate, and we are led to the solutions in Eqs. (18) and (19). The propagator can now be constructed from the functions

$$\Omega(\xi,\xi') \equiv \phi_1(\xi)\phi_2(\xi') - \phi_1(\xi')\phi_2(\xi), \qquad (A5)$$

$$W(\xi) \equiv \phi_1(\xi)\phi_2'(\xi) - \phi_1'(\xi)\phi_2(\xi). \qquad (A6)$$

It is given by [5–7]

$$\tilde{\Gamma}(\kappa_n;\xi,\xi') = \frac{\Omega(\infty,\xi_>)\Omega(\xi_<,-\infty)}{W(\xi)\Omega(-\infty,\infty)}, \qquad (A7)$$

where $\xi_{<(>)} \equiv \min(\max)(\xi,\xi')$. In terms of the variable $u$, noting that $\xi > \xi' \Leftrightarrow u < u'$, $\xi \to \infty \Rightarrow u \to 0$, and $\xi \to -\infty \Rightarrow u \to 1$, we obtain

$$\tilde{\Gamma}(\kappa_n;u,u') = \frac{\phi_1(u_<)\phi_2(u_>)}{W(u)}. \qquad (A8)$$

The propagator does vanish when $u,u' = 0,1$. Also, $W(u)$ is a constant, easy to compute at $u = 0$; we find $W(u) = 2b_n$. We can then write expression (20).

### APPENDIX B

In this appendix, we evaluate Eq. (32) from the propagators. The free one, which can also be obtained from the methods described previously, is given by

$$\tilde{\Gamma}_F(\kappa_n;\xi,\xi') = \frac{1}{2b_n}e^{-b_n|\xi-\xi'|}. \qquad (B1)$$

Changing variables to $b = \sqrt{4+s}$ and $u = (1-\tanh\xi)/2$, the second term in Eq. (31) becomes

$$\ln\frac{\Delta_{\kappa_n}'}{\Delta_0'} = \int_2^{b_n}db\,b\int_0^1\frac{du}{u(1-u)}[\tilde{\Gamma}'(\sqrt{b^2-4};u,u)$$

$$-\tilde{\Gamma}_F(\sqrt{b^2-4};u,u)]. \qquad (B2)$$

The expression inside the brackets is easily computed from the propagators, yielding

$$\frac{6u(1-u)}{b(b^2-1)} - \frac{12(b^2+2b+3)u^2(1-u)^2}{b(b+2)(b^2-1)}. \qquad (B3)$$

Doing the integrals in Eq. (B2), we arrive at Eq. (32).

### APPENDIX C

In this appendix, we shall make use of Plana's formula [21,30] to compute the sum (37), which can be decomposed into sums of the form

$$\mathcal{S}(r,e^{i\sigma\pi}s) \equiv \sum_{n=-N}^N \ln(\sqrt{n^2+r^2}+e^{i\sigma\pi}s), \qquad (C1)$$

with $r > s > 0$, and $\sigma = 0,1$.

Plana's formula can be derived from Cauchy's theorem applied to two contours in the complex plane [21]: one that runs counterclockwise in the upper half-plane, from $(x,y) = (n_1,n_1+i\infty)$ to the real axis along a perpendicular, then along the real axis, and finally perpendicularly away from it to $(n_2,n_2+i\infty)$, avoiding the integers on the real segment $(n_1,n_2)$ (semicircling them), as well as the corners $(n_1,0)$ and $(n_2,0)$ (with $\pi/2$ arcs); the other is its mirror reflection on the real axis. For a function $\phi(z)$ analytic and bounded for $n_1 \leq \text{Re}(z) \leq n_2$, we have

$$\frac{\phi(n_1)}{2} + \sum_{n=n_1}^{n_2}\phi(n) + \frac{\phi(n_1)}{2}$$

$$= \int_{n_1}^{n_2}dx\,\phi(x) + \frac{1}{i}\int_0^\infty dy\,\frac{\Phi(n_2,y)-\Phi(n_1,y)}{e^{2\pi y}-1}, \qquad (C2)$$

where $\Phi(n,y) \equiv \phi(n+iy) - \phi(n-iy)$. Defining $\phi(z) = \ln(\sqrt{z^2+r^2}+e^{i\sigma\pi}s)$, we may rewrite Eq. (C1) as

$$\mathcal{S}(r,e^{i\sigma\pi}s) = 2\sum_{n=0}^N \ln(\sqrt{n^2+r^2}+e^{i\sigma\pi}s) - \ln(r+e^{i\sigma\pi}s), \qquad (C3)$$

and use Eq. (C2) with $n_1 = 0^+$ (in order to avoid the square-root cut at $\text{Re}(z) = 0$) and $n_2 = N$. The contribution to the second integral of Eq. (C2) involving $\Phi(N,y)$ can be expanded for large $N$, and shown to behave as $1/N$. The one involving $\Phi(0^+,y)$ has to be split into two pieces: $|y| > r$ and $|y| < r$. The latter piece yields zero, whereas the former uses

$$\phi(0^+ \pm iy) = \frac{1}{2}\ln[y^2-(r^2-s^2)]$$

$$\pm i\left[\sigma\pi + e^{i\sigma\pi}\arctan\left(\frac{\sqrt{y^2-r^2}}{s}\right)\right]. \quad (C4)$$

Neglecting terms that vanish as $N \to \infty$, we finally arrive at

$$\mathcal{S}(r,e^{i\sigma\pi}s) = 2\int_0^N dx\,\ln(\sqrt{x^2+r^2}+e^{i\sigma\pi}s)$$

$$+ \ln(\sqrt{N^2+r^2}+e^{i\sigma\pi}s) - 4\int_r^\infty \frac{dy}{e^{2\pi y}-1}$$

$$\times\left[\sigma\pi + e^{i\sigma\pi}\arctan\left(\frac{\sqrt{y^2-r^2}}{s}\right)\right], \qquad (C5)$$

where $r > s > 0$, and $\sigma = 0,1$. It is easy to verify that Eq. (C5) correctly reproduces the results for $\mathcal{S}(r,0)$ and for the sum $\Sigma_{\sigma=0}^1\mathcal{S}(r,e^{i\sigma\pi}s)$, which can be obtained straightforwardly.

### APPENDIX D

In this appendix, we calculate the integral $I$ that appears inside the brackets of Eq. (38):

$$I(r, e^{i\sigma\pi}s) = \int_0^N dx \ln(\sqrt{x^2 + r^2} + e^{i\sigma\pi}s). \quad \text{(D1)}$$

Changing variable to $q(x) = \sqrt{x^2 + r^2} + e^{i\sigma\pi}s$, we integrate by parts to obtain

$$I = [\sqrt{(q - e^{i\sigma\pi}s)^2 - r^2} \ln q]_{q(0)}^{q(N)}$$
$$- \int_{q(0)}^{q(N)} \frac{dq}{q} \sqrt{(q - e^{i\sigma\pi}s)^2 - r^2}. \quad \text{(D2)}$$

The last integral can be found in [23]. Finally

$$I = N \ln(\sqrt{N^2 + r^2} + e^{i\sigma\pi}s) - N + e^{i\sigma\pi}s\left[\ln\left(\frac{\sqrt{N^2 + r^2} + N}{r}\right)\right]$$
$$- \sqrt{r^2 - s^2}\left\{\arcsin\left[\frac{r^2 + e^{i\sigma\pi}s\sqrt{N^2 + r^2}}{r(\sqrt{N^2 + r^2} + e^{i\sigma\pi}s)}\right] - \frac{\pi}{2}\right\}. \quad \text{(D3)}$$

### APPENDIX E

We now compute the integral $\mathcal{J}_2$, defined in Eq. (52). Again resorting to the change of variables $\xi \to u$,

$$\mathcal{J}_2 = \frac{-36}{b^2} \int_0^1 du_1 \left(\frac{u_1}{1 - u_1}\right)^{-b} \int_0^{u_1} du_2 \left(\frac{u_2}{1 - u_2}\right)^b, \quad \text{(E1)}$$

which is equal to [23]

$$\mathcal{J}_2 = \frac{-36}{b^2(b+1)} \int_0^1 du_1 u_1 (1 - u_1)^b {}_2F_1(b, b+1; b+2; u_1). \quad \text{(E2)}$$

This last integral can also be done [23], and we finally obtain

$$\mathcal{J}_2 = \frac{-36}{b^2(b+1)^2(b+2)} {}_3F_2(b, b+1, 2; b+2, b+3; 1). \quad \text{(E3)}$$

After some rearrangement, we derive

$$\mathcal{J}_2 = -36\left[\zeta(2, b+1) + \frac{1}{2b^2} - \frac{1}{b}\right], \quad \text{(E4)}$$

involving Riemann's zeta function $\zeta(z, q) \equiv \Sigma_0^\infty (n + q)^{-z}$. Plana's formula of Appendix C may be used in the zeta function, yielding

$$\zeta(2, b+1) = \frac{1}{2(b+1)^2} + \frac{1}{(b+1)}$$
$$+ \int_0^\infty \frac{dy}{e^{2\pi y} - 1} \frac{4(b+1)y}{[y^2 + (b+1)^2]^2}, \quad \text{(E5)}$$

so that the result for $\mathcal{J}_2$ is

$$\mathcal{J}_2 = -36\left\{\frac{1}{2b^2(b+1)^2} + \int_0^\infty \frac{dy}{e^{2\pi y} - 1} \frac{4(b+1)y}{[y^2 + (b+1)^2]^2}\right\}. \quad \text{(E6)}$$

If we expand the last fraction in Eq. (E6) in a power series in $y$, use the definition of the Bernouilli numbers,

$$\frac{B_{2n}}{4n} \equiv (-1)^{n-1} \int_0^\infty \frac{dy}{e^{2\pi y} - 1} y^{2n-1}, \quad \text{(E7)}$$

and the fact that $B_2 = 1/6$, we finally obtain Eq. (53).

### APPENDIX F

We compute the effective potential at zero temperature for $d = 3$, in order to illustrate the workings of the renormalization procedure. The calculation can be viewed as a special case of the one performed in the text, with the kink being traded for a constant background $\varphi_c$. In that case, determinants are simple to compute, and we formally obtain

$$V_{eff}(\varphi_c) = \frac{1}{4!}(\varphi_c^2 - \varphi_v^2)^2 + \frac{\lambda}{2} \int \frac{d^4k}{(2\pi)^4} \ln\left[1 + \frac{\varphi_c^2 - \varphi_v^2}{2(k^2 + M^2)}\right]. \quad \text{(F1)}$$

The subtraction of counterterms leads to

$$V_{eff}(\varphi_c) \to V_{eff}(\varphi_c) - \frac{1}{2} C_1(\varphi_c^2 - \varphi_v^2) - \frac{1}{4} C_2(\varphi_c^2 - \varphi_v^2)^2, \quad \text{(F2)}$$

which will yield the renormalized expression if we use a cutoff $\Lambda$ to regularize Eq. (F2), and fix renormalization conditions. Since we may write the renormalized effective potential $V_R$ as

$$V_R(\varphi_c) = \sum_{n=1}^\infty \frac{1}{n!} \tilde{\Gamma}_R^{(n)}(\mathbf{0}, \dots, \mathbf{0})(\varphi_c - \varphi_v)^n, \quad \text{(F3)}$$

it is easy to show that the renormalization conditions (47) and (58) determine $C_1$ and $C_2$:

$$C_1 = \lambda \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + M^2)}, \quad \text{(F4)}$$

$$C_2 = -\frac{\lambda}{2} \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + M^2)^2}. \quad \text{(F5)}$$

Clearly, both expressions correspond to one-loop graphs at zero external momenta. The renormalization procedure adopted in the text also uses zero-momentum subtractions in expansions in $(\varphi^2 - \varphi_v^2)/2$, but in the kink sector. In that case, the effective action in the kink sector replaces the effective potential, while the expansion (F3) becomes a functional one

$$\mathcal{A}_R[\hat{\varphi}] = \sum_{n=1}^{\infty} \frac{1}{n!} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \int \frac{d^4 p_n}{(2\pi)^4} \widetilde{\Gamma}_R^{(n)}(\mathbf{p_1}, \ldots, \mathbf{p_n})$$

$$\times [\hat{\varphi}(\mathbf{p_1}) - \varphi_v] \cdots [\hat{\varphi}(\mathbf{p_n}) - \varphi_v], \qquad \text{(F6)}$$

with $\hat{\varphi}(\mathbf{p})$ standing for the Fourier transform of the kink background. To set the record straight, we note that the renormalization condition for $\widetilde{\Gamma}_R^{(4)}$ which appears in [12] is mistaken. It should be corrected to Eq. (58) multiplied by $\lambda$.

---

[1] J. Schwinger, Phys. Rev. **82**, 664 (1951), reprinted in *Selected Papers in Quantum Electrodynamics*, edited by J. Schwinger (Dover, New York, 1958).

[2] R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **10**, 4114 (1974); **10**, 4130 (1974); **10**, 4138 (1974); **11**, 3424 (1975).

[3] M. Gutzwiller, in *Chaos and Quantum Physics*, Les Houches Session LII, edited by M. J. Giannoni, A. Voros, and J. Zinn-Justin (Elsevier, Amsterdam, 1992).

[4] M. Berry, in *Chaos and Quantum Physics*, Les Houches Session LII, edited by M. J. Giannoni, A. Voros, and J. Zinn-Justin (Elsevier, Amsterdam, 1992).

[5] C. A. A. de Carvalho, R. M. Cavalcanti, E. S. Fraga, and S. E. Jorás, Ann. Phys. (N.Y.) **273**, 146 (1999).

[6] C. A. A. de Carvalho and R. M. Cavalcanti, in *Trends in Theoretical Physics II*, edited by H. Falomir, R. E. Gamboa Saraví, and F. A. Schaposnik, AIP Conf. Proc. No. 484 (AIP, Woodbury, NY, 1999).

[7] C. A. A. de Carvalho, R. M. Cavalcanti, E. S. Fraga, and S. E. Jorás, Phys. Rev. E **61**, 6392 (2000).

[8] C. A. A. de Carvalho, R. M. Cavalcanti, E. S. Fraga, and S. E. Jorás, quant-ph/0105076.

[9] C. DeWitt-Morette, Commun. Math. Phys. **28**, 47 (1972); **37**, 63 (1974); Ann. Phys. (N.Y.) **97**, 367 (1976).

[10] Yu Lu, in *Solitons and Polarons in Conducting Polymers* (World Scientific, Singapore, 1988); C. A. A. de Carvalho, Nucl. Phys. **B324**, 729 (1989).

[11] D. Jasnow, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1986), Vol. 10, p. 270.

[12] C. A. A. de Carvalho, G. C. Marques, A. J. da Silva, and I. Ventura, Nucl. Phys. B: Field Theory Stat. Syst. **B265** [FS15], 45 (1986).

[13] G. Münster, Nucl. Phys. **B340**, 559 (1990); Phys. Lett. A **238**, 265 (1998).

[14] C. G. Bollini and J. J. Giambiagi, Rev. Bras. Fis., volume dedicated to M. Schönberg (1984), p. 364.

[15] A. Parnachev and L. G. Yaffe, Phys. Rev. D **62**, 105034 (2000).

[16] N. Graham, R. L. Jaffe, M. Quandt, and H. Weigel, Phys. Rev. Lett. **87**, 131601 (2001).

[17] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).

[18] E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, Phys. Rev. D **15**, 1544 (1977); B. B. Pant and D. Jasnow, Phys. Rev. B **29**, 2728 (1984); E. Brézin and S. Feng, *ibid.* **29**, 472 (1984).

[19] R. Rajaraman, Phys. Rep., Phys. Lett. **21C**, 227 (1975); *Solitons and Instantons: An Introduction to Solitons and Instantons in Quantum Field Theory* (North-Holland, Amsterdam, 1989), and references therein.

[20] B. Felgaser, *Geometry, Particles, and Fields* (Springer-Verlag, New York, 1998).

[21] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, England, 1996).

[22] V. M. Mostepanenko and N. N. Trunov, *The Casimir Effect and its Applications* (Clarendon, Oxford, 1997).

[23] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1965).

[24] C. Domb, Adv. Phys. **9**, 149 (1960).

[25] M. E. Fisher, J. Phys. Soc. Jpn. Suppl. **26**, 87 (1969).

[26] V. Privman and M. E. Fisher, J. Stat. Phys. **33**, 385 (1983).

[27] G. Münster, Nucl. Phys. **B324**, 630 (1989).

[28] S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, England, 1985).

[29] J. L. Lebowitz and C. E. Pfister, Phys. Rev. Lett. **46**, 1031 (1981).

[30] H. Bateman and A. Erdélyi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953).

[31] A. Surig, Phys. Rev. D **57**, 5049 (1998).

[32] J. Baacke and H. Spencer, Phys. Rev. D **60**, 054017 (1999).

[33] N. Graham, hep-th/0112148.