Generalized quantum field theory: Perturbative computation and perspectives

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We analyze some consequences of two possible interpretations of the action of the ladder operators emerging from generalized Heisenberg algebras in the framework of the second quantized formalism. Within the first interpretation we construct a quantum field theory that creates at any space-time point particles described by a q-deformed Heisenberg algebra and we compute the propagator and a specific first order scattering process. Concerning the second one, we draw attention to the possibility of constructing a theory in which each state of a generalized Heisenberg algebra is interpreted as a particle with different mass.

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I. INTRODUCTION

A class of algebras that is a generalization of the Heisenberg algebra has recently been constructed [1,2]. These algebras are characterized by functionals depending on one of their generators that are called the characteristic functions of the algebra. When this functional is linear with a slope θ the algebra turns into a q-oscillator algebra [3], θ being related to the deformation parameter as $q^2=\tan\theta$. It is worth noticing that when this functional is a general polynomial we obtain the so-called multiparametric deformed Heisenberg algebra.

Concerning their physical interpretation, these generalized algebras describe the Heisenberg-type algebras of one-dimensional quantum systems having arbitrary successive energy levels given by $\epsilon_{n+1}=f(\epsilon_n)$ where $f(x)$ is the characteristic function of the algebra [4]. As examples we mention the nonrelativistic [4] and relativistic [5] square-well potentials in one dimension and the harmonic oscillator on a circle [6]. It is also interesting to mention that the representations of the algebra for a general characteristic function were constructed by studying the stability of the fixed points of this function and of their composing functions [2].

Heisenberg algebra is an essential tool in the second quantization formalism because its generators create and annihilate particle states. In the generalized Heisenberg algebras the generators of the algebra are also ladder operators constructing all different energy levels of a one-dimensional quantum system from a given energy level. In the generalized case the energy difference of any two successive levels is not equal and two possible interpretations of the action of the ladder operators result. In the first case we have an interpretation similar to that in the harmonic oscillator case, but with the difference that the total energy of n particles is not equal to n times the energy of each particle, while in the second interpretation we associate particles of different masses with different energy levels.

It then seems natural to investigate the possibility of extending these interpretations of the ladder operators of generalized Heisenberg algebras in the framework of the second quantized formalism. In this paper we start the analysis of the harmonic-oscillator-like interpretation. We construct a quantum field theory (QFT) having fields that produce at any space-time point particles satisfying a q-deformed Heisenberg algebra and we start the analysis of the perturbation theory for this case. The interest in this QFT comes from the fact that creation and annihilation operators of correlated fermion pairs, in simple many body systems, satisfy a deformed Heisenberg algebra that can be approximated by q oscillators [7].

In Sec. II, we summarize the generalized Heisenberg algebras; in Sec. III we implement a physical realization of a one-parameter deformed Heisenberg algebra. In Sec. IV, a quantum field model following the harmonic-oscillator-like interpretation is presented. The propagator and a specific first order scattering process are computed. In Sec. V, we discuss the possibility of constructing a QFT where each state of its Hilbert space, which is created by the ladder operators of the associated generalized Heisenberg algebra, is interpreted as a particle with different mass. We end this section with some conclusions.

II. GENERALIZED HEISENBERG ALGEBRAS

Let us consider an algebra generated by J_0, A, and $A^\dagger$ and described by the relations [2]

$$J_0 A^\dagger = A^\dagger f(J_0), \tag{1}$$

$$A J_0 = f(J_0) A, \tag{2}$$

$$[A, A^\dagger] = f(J_0) - J_0, \tag{3}$$

where † represents the Hermitian conjugate and, by hypothesis, $J_0^\dagger = J_0$ and $f(J_0)$ is a general analytic function of J_0. It is simple to show that the generators of the algebra trivially satisfy the Jacobi identity [6]. Using the algebraic relations in Eqs. (1)–(3) we see that the operator

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$$C=A^{\dagger}A-J_0=AA^{\dagger}-f(J_0) \tag{4}$$

satisfies

$$[C,J_0]=[C,A]=[C,A^{\dagger}]=0, \tag{5}$$

thus being a Casimir operator of the algebra.

We now analyze the representation theory of the algebra when the function $f(J_0)$ is a general analytic function of J_0. We assume we have an n-dimensional irreducible representation of the algebra given in Eqs. (1)–(3). We also assume that there is a state $|0\rangle$ with the lowest eigenvalue of the Hermitian operator J_0:

$$J_0|0\rangle=\alpha_0|0\rangle. \tag{6}$$

For each value of α_0 we have a different vacuum; thus a better notation could be $|0\rangle_{\alpha_0}$ but, for simplicity, we shall omit the subscript α_0.

Let $|m\rangle$ be a normalized eigenstate of J_0,

$$J_0|m\rangle=\alpha_m|m\rangle. \tag{7}$$

Applying Eq. (1) to $|m\rangle$ we have

$$J_0(A^{\dagger}|m\rangle)=A^{\dagger}f(J_0)|m\rangle=f(\alpha_m)(A^{\dagger}|m\rangle). \tag{8}$$

Thus, we see that $A^{\dagger}|m\rangle$ is a J_0 eigenvector with eigenvalue $f(\alpha_m)$. Starting from $|0\rangle$ and applying $A^{\dagger}$ successively to $|0\rangle$, we create different states with the J_0 eigenvalue given by

$$J_0((A^{\dagger})^m|0\rangle)=f^m(\alpha_0)((A^{\dagger})^m|0\rangle), \tag{9}$$

where $f^m(\alpha_0)$ denotes the mth iterate of f. Since the application of $A^{\dagger}$ creates a new vector, the J_0 eigenvalue of which has its iterations of α_0 through f augmented by one unit, it is convenient to define the new vectors $(A^{\dagger})^m|0\rangle$ as proportional to $|m\rangle$ and we then call $A^{\dagger}$ a raising operator. Note that

$$\alpha_m=f^m(\alpha_0)=f(\alpha_{m-1}), \tag{10}$$

where m denotes the number of iterations of α_0 through f.

Following the same procedure for A, applying Eq. (2) to $|m+1\rangle$, we have

$$AJ_0|m+1\rangle=f(J_0)(A|m+1\rangle)=\alpha_{m+1}(A|m+1\rangle), \tag{11}$$

showing that $A|m+1\rangle$ is also a J_0 eigenvector with eigenvalue α_m. Then, $A|m+1\rangle$ is proportional to $|m\rangle$, A being a lowering operator.

Since we consider α_0 the lowest J_0 eigenvalue, we require that

$$A|0\rangle=0. \tag{12}$$

As shown in [1], depending on the function f and its initial value α_0, the J_0 eigenvalue of state $|m+1\rangle$ may be lower than that of state $|m\rangle$. Then, as shown in [2], given an arbitrary analytical function f [and its associated algebra in Eqs. (1)–(3)] in order to satisfy Eq. (12), the allowed values of α_0 are chosen in such a way that the iterations $f^m(\alpha_0)(m\geqslant 1)$ are always bigger than α_0; in other words, Eq. (12) must be checked for every function f, giving consistent vacua for specific values of α_0.

As proven in [2], under the hypothesis stated previously,[1] for a general function f we obtain

$$J_0|m\rangle=f^m(\alpha_0)|m\rangle, \quad m=0,1,2,\ldots, \tag{13}$$

$$A^{\dagger}|m-1\rangle=N_{m-1}|m\rangle, \tag{14}$$

$$A|m\rangle=N_{m-1}|m-1\rangle, \tag{15}$$

where $N_{m-1}^2=f^m(\alpha_0)-\alpha_0$. Note that for each function $f(x)$ the representations are constructed by the analysis of the above equations as was done in [2] for linear and quadratic $f(x)$.

As shown in [2], where the representation theory was constructed in detail for the linear and quadratic functions $f(x)$, the essential tool in order to construct representations of the algebra in Eqs. (1)–(3) for a general analytic function $f(x)$ is the analysis of the stability of the fixed points of $f(x)$ and their composed functions.

We showed in [4] that there is a class of one-dimensional quantum systems described by these generalized Heisenberg algebras. This class is characterized by those quantum systems having energy eigenvalues written as

$$\epsilon_{n+1}=f(\epsilon_n), \tag{16}$$

where ϵ_{n+1} and ϵ_n are successive energy levels and $f(x)$ is a different function for each physical system. This function $f(x)$ is exactly the same function that appears in the construction of the algebra in Eqs. (1)–(3). In the algebraic description of this class of quantum systems, J_0 is the Hamiltonian operator of the system, and $A^{\dagger}$ and A are the creation and annihilation operators. This Hamiltonian and the ladder operators are related by Eq. (4) where C is the Casimir operator of the representation associated with the quantum system under consideration.

III. DEFORMED HEISENBERG ALGEBRA AND ITS PHYSICAL REALIZATION

In this section we are going to discuss the algebra defined by the relations given in Eqs. (1)–(3) for the linear case, i.e., $f(J_0)=rJ_0+s$, $s>0$ [2]. Furthermore, we shall also propose a realization, as in the case of the standard harmonic oscillator, of the ladder operators in terms of the physical operators of the system.

The algebraic relations for the linear case can be rewritten, after the rescaling $J_0\to sJ_0$, $A\to A/\sqrt{s}$, and $A^{\dagger}\to A^{\dagger}/\sqrt{s}$, as

$$[J_0,A^{\dagger}]_r=A^{\dagger}, \tag{17}$$

[1] J_0 is Hermitian and a vacuum state exists.

$$[J_0,A]_{r^{-1}}=-\frac{1}{r}A, \tag{18}$$

$$[A^\dagger,A]=(1-r)J_0-1, \tag{19}$$

where $[a,b]_r\equiv ab-rba$ is the r-deformed commutation of two operators a and b.

It is very simple to realize that for $r=1$ the above algebra is the Heisenberg algebra. In this case the Casimir operator given in Eq. (4) is null. Then, for general r the algebra defined in Eqs. (17)–(19) is a one-parameter deformed Heisenberg algebra and generally speaking the algebra given in Eqs. (1)–(3) is a generalization of the Heisenberg algebra.

It is easy to see for the general linear case that

$$f^m(\alpha_0)=r^m\alpha_0+r^{m-1}+r^{m-2}+\cdots+1$$
$$=r^m\alpha_0+\frac{r^m-1}{r-1}; \tag{20}$$

thus,

$$N_{m-1}^2=f^m(\alpha_0)-\alpha_0=[m]_rN_0^2 \tag{21}$$

where $[m]_r\equiv(r^m-1)/(r-1)$ is the Gauss number of m and $N_0^2=\alpha_0(r-1)+1$.

For infinite-dimensional solutions we must solve the following set of equations:

$$N_m^2>0,\quad \forall m,\quad m=0,1,2,\ldots. \tag{22}$$

Apart from the Heisenberg algebra given by $r=1$, the solutions are [2]

$$\text{type I (unstable fixed point): } r>1 \text{ and } \alpha_0>\frac{1}{1-r} \text{ or}$$
$$\text{type II (stable fixed point): } -1<r<1 \text{ and } \alpha_0<\frac{1}{1-r}, \tag{23}$$

with matrix representations

$$J_0=\begin{pmatrix}\alpha_0 & 0 & 0 & 0 & \ldots\\ 0 & \alpha_1 & 0 & 0 & \ldots\\ 0 & 0 & \alpha_2 & 0 & \ldots\\ 0 & 0 & 0 & \alpha_3 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix},$$

$$A^\dagger=\begin{pmatrix}0 & 0 & 0 & 0 & \ldots\\ N_0 & 0 & 0 & 0 & \ldots\\ 0 & N_1 & 0 & 0 & \ldots\\ 0 & 0 & N_2 & 0 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix},\ A=(A^\dagger)^\dagger. \tag{24}$$

Note that for type I solutions the eigenvalues of J_0, as can easily be computed from Eqs. (13) and (10), go to infinity as we consider eigenvectors $|m\rangle$ with increasing values of m. Instead, for type II solutions the eigenvalues go to the value $1/(1-r)$, the fixed point of f, as the states $|m\rangle$ increase.

It is easy to see that there is a direct relation between the linear Heisenberg algebra given in Eqs. (17)–(19) and the standard q oscillators. In fact, defining [2]

$$J_0=q^{2N}\alpha_0+[N]_{q^2}, \tag{25}$$

$$\frac{A^\dagger}{N_0}=a^\dagger q^{N/2}, \tag{26}$$

$$\frac{A}{N_0}=q^{N/2}a, \tag{27}$$

we see that a, $a^\dagger$, and N satisfy the usual q-oscillator relations [3]

$$aa^\dagger-qa^\dagger a=q^{-N},\quad aa^\dagger-q^{-1}a^\dagger a=q^N,$$
$$[N,a]=-a,\quad [N,a^\dagger]=a^\dagger. \tag{28}$$

Note that Heisenberg algebra is obtained from Eqs. (25)–(27) for $q\to1$ and $\alpha_0=0$.

The next step we have to take is to realize the operators A, $A^\dagger$, and J_0 in terms of physical operators as in the case of the one-dimensional harmonic oscillator, and as was done in [4] and [5] for the square-well potential. To do this, we briefly review the formalism of noncommutative differential and integral calculus on a one-dimensional lattice developed in [8] and [9]. Let us consider a one-dimensional lattice in a momentum space where the momenta are allowed to take only discrete values, say, p_0, p_0+a, p_0+2a, p_0+3a, etc., with $a>0$.

The noncommutative differential calculus is based on the expression [8,9]

$$[p,dp]=dpa, \tag{29}$$

implying that

$$f(p)dg(p)=dg(p)f(p+a) \tag{30}$$

for all functions f and g. We introduce partial derivatives as

$$df(p)=dp(\partial_pf)(p)=(\bar{\partial}_pf)(p)dp, \tag{31}$$

where the left and right discrete derivatives are given by

$$(\partial_pf)(p)=\frac{1}{a}[f(p+a)-f(p)], \tag{32}$$

$$(\bar{\partial}_pf)(p)=\frac{1}{a}[f(p)-f(p-a)], \tag{33}$$

the two possible definitions of derivatives on a lattice. The Leibniz rule for the left discrete derivative can be written as

$$(\partial_p fg)(p)=(\partial_p f)(p)g(p)+f(p+a)(\partial_p g)(p), \quad (34)$$

with a similar formula for the right derivative [8].

Let us now introduce the momentum shift operators

$$T=1+a\partial_p, \quad (35)$$

$$\overline{T}=1-a\overline{\partial_p}, \quad (36)$$

that shift the momentum value by a,

$$(Tf)(p)=f(p+a), \quad (37)$$

$$(\overline{T}f)(p)=f(p-a), \quad (38)$$

and satisfy

$$T\overline{T}=\overline{T}T=\hat{1}, \quad (39)$$

where $\hat{1}$ means the identity on the algebra of functions of p.

Introducing the momentum operator P [8],

$$(Pf)(p)=pf(p), \quad (40)$$

we have

$$TP=(P+a)T, \quad (41)$$

$$\overline{T}P=(P-a)\overline{T}. \quad (42)$$

Integrals can also be defined in this formalism. It is shown in Ref. [8] that the property of an indefinite integral

$$\int df=f+\text{periodic function in } a \quad (43)$$

suffices to calculate the indefinite integral of an arbitrary one-form. It can be shown that [8], for an arbitrary function f,

$$\int d\overline{p}f(\overline{p})=\begin{cases} a\sum_{k=1}^{[p/a]} f(p-ka) & \text{if } p\geqslant a, \\ 0 & \text{if } 0\leqslant p<a, \\ -a\sum_{k=0}^{-[p/a]-1} f(p+ka) & \text{if } p<0, \end{cases} \quad (44)$$

where $[p/a]$ is by definition the highest integer $\leqslant p/a$.

All equalities involving indefinite integrals are understood modulo the addition of an arbitrary function periodic in a. The corresponding definite integral is well defined when the length of the interval is a multiple of a. Consider the integral of a function f from p_d to p_u ($p_u=p_d+Ma$, where M is a positive integer) as

$$\int_{p_d}^{p_u} dpf(p)=a\sum_{k=0}^{M} f(p_d+ka). \quad (45)$$

Using Eq. (45), the inner product of two (complex) functions f and g can be defined as

$$\langle f,g\rangle=\int_{p_d}^{p_u} dpf(p)^{*}g(p), \quad (46)$$

where the asterisk indicates complex conjugation of the function f. The norm $\langle f,f\rangle\geqslant 0$ is zero only when f is identically null. The set of equivalence classes[2] of normalizable functions f ($\langle f,f\rangle$ is finite) is a Hilbert space. It can be shown that [8]

$$\langle f,Tg\rangle=\langle \overline{T}f,g\rangle, \quad (47)$$

so that

$$\overline{T}=T^{\dagger}, \quad (48)$$

where $T^{\dagger}$ is the adjoint operator of T. Equations (39) and (48) show that T is a unitary operator. Moreover, it is easy to see that P defined in Eq. (40) is a Hermitian operator and from Eq. (48) one has

$$(i\partial_p)^{\dagger}=i\overline{\partial_p}. \quad (49)$$

Now, we go back to the realization of the deformed Heisenberg algebra Eqs. (17)–(19) in terms of physical operators. We can associate with the one-parameter deformed Heisenberg algebra in Eqs. (17)–(19) the one-dimensional lattice we have just presented.

Observe that we can write J_0 in this case as

$$J_0=q^{2P/a}\alpha_0+[P/a]_{q^2}, \quad (50)$$

where P is given in Eq. (40) and its application to the vector states $|m\rangle$ appearing in Eqs. (13)–(15) gives [4]

$$P|m\rangle=ma|m\rangle, \quad m=0,1,\ldots, \quad (51)$$

where we can write $N=P/a$ with $N|m\rangle=m|m\rangle$. Moreover,

$$\overline{T}|m\rangle=|m+1\rangle, \quad m=0,1,\ldots, \quad (52)$$

where $\overline{T}$ and $T=\overline{T}^{\dagger}$ are defined in Eqs. (35)–(39).

With the definition of J_0 given in Eq. (50) we see that α_n given in Eq. (10) is the J_0 eigenvalue of state $|n\rangle$ as we wanted. Let us now define

$$A^{\dagger}=S(P)\overline{T}, \quad (53)$$

$$A=TS(P), \quad (54)$$

where

$$S(P)^2=J_0-\alpha_0, \quad (55)$$

where α_0, defined in Eq. (6), is the lowest J_0 eigenvalue. Yet, note that Eqs. (41), (42) can also be rewritten as

$$TN=(N+1)T, \quad (56)$$

[2] Two functions are in the same equivalence class if their values coincide on all lattice sites.

$$\overline{T}N=(N-1)\overline{T}. \tag{57}$$

It is easy to realize that A, $A^\dagger$, and J_0 defined in Eqs. (50), (53)–(55) satisfy the one-parameter deformed algebra given in Eqs. (17)–(19). Consider first the relation between J_0 and $A^\dagger$:

$$J_0A^\dagger=\alpha_N S(P)\overline{T}=A^\dagger\alpha_{N+1}, \tag{58}$$

where α_N is $\alpha_m=f^m(\alpha_0)$ in Eq. (20) with the operator N in place of the variable m. In Eq. (58) we have used the realizations in the first equality of the above equation and in the second one we have used Eq. (57). But, from Eq. (10) $\alpha_{N+1}=f(\alpha_N)=f(J_0)$; thus we obtain

$$J_0A^\dagger=A^\dagger f(J_0), \tag{59}$$

that is, Eq. (1) for $f(x)$ linear. Equation (2) is the Hermitian conjugate of Eq. (1), thus its proof using Eqs. (54) and (50) is similar to the previous one. Now, using

$$A^\dagger A=S(P)^2=J_0-\alpha_0, \tag{60}$$

$$AA^\dagger=TS(P)^2\overline{T}=f(J_0)-\alpha_0, \tag{61}$$

for linear $f(x)$, which has the property given in Eq. (10), we get Eq. (3) for $f(x)$ linear, and the proof is complete.

Note that the realization we have found, as shown in Eqs. (53), (54), and (50), is qualitatively different from the realization of the standard harmonic oscillator. This shows that the realization of the ladder operators for the harmonic oscillator means that the deformation parameter $r=1$ is a special case.

IV. DEFORMED QUANTUM FIELD THEORY

We are going to discuss in this section a QFT having as excitations objects described by the one-parameter deformed algebra given in Eqs. (17)–(19). In this QFT the mass spectrum consists of only one particle with mass m. In this case the energy of n particles is not equal to n times the energy of one particle and therefore the energy does not obey the additivity rule. This nonadditivity comes from the fact that q oscillators approximately describe correlated fermion pairs in many body systems [7]. The advantage of this construction is that, being a deformation, we can make contact with the well-known nondeformed model in all steps of the computation by taking the deformation parameter going to 1.

In the momentum space appropriate to the realization of the deformed Heisenberg algebra we discussed, as well as the operator P defined in Eq. (40), one can define two self-adjoint operators as

$$\chi\equiv -i[S(P)(1-a\overline{\partial_p})-(1+a\partial_p)S(P)]=-i(A-A^\dagger), \tag{62}$$

$$Q\equiv S(P)(1-a\overline{\partial_p})+(1+a\partial_p)S(P)=A+A^\dagger, \tag{63}$$

where ∂_p and $\overline{\partial_p}$ are the left and right discrete derivatives defined in Eqs. (32),(33).

It can be checked that the operators P, χ, and Q generate the following algebra on the momentum lattice:

$$[\chi,P]=iaQ, \tag{64}$$

$$[P,Q]=ia\chi, \tag{65}$$

$$[\chi,Q]=2iS(P)[S(P+a)-S(P-a)]. \tag{66}$$

To construct a QFT based on these operators let us now introduce a three-dimensional discrete $\vec{k}$ space,

$$k_i=\frac{2\pi l_i}{L_i}, i=1,2,3, \tag{67}$$

with $l_i=0,\pm1,\pm2,\ldots$ and L_i being the lengths of the three sides of a rectangular box Ω. We introduce for each point of this $\vec{k}$ space the independent r-deformed harmonic oscillator constructed in the last two previous sections so that the deformed operators commute for different three-dimensional lattice points. We also introduce an independent copy of the one-dimensional momentum lattice defined in the previous section for each point of this $\vec{k}$ lattice so that $P_{\vec{k}}^\dagger=P_{\vec{k}}$ and $T_{\vec{k}}$, $\overline{T}_{\vec{k}}$, and $S_{\vec{k}}$ are defined by means of the previous definitions, Eqs. (35), (36), and (55), through the substitution $P\to P_{\vec{k}}$.

It is not difficult to realize that

$$A_{\vec{k}}^\dagger=S_{\vec{k}}\overline{T}_{\vec{k}}, \tag{68}$$

$$A_{\vec{k}}=T_{\vec{k}}S_{\vec{k}}, \tag{69}$$

$$J_0(\vec{k})=q^{2P_{\vec{k}}/a}\alpha_0+[P_{\vec{k}}/a]_{q^2}, \tag{70}$$

satisfy the algebra in Eqs. (17)–(19) for each point of this $\vec{k}$ lattice and the operators $A_{\vec{k}}^\dagger$, $A_{\vec{k}}$, and $J_0(\vec{k})$ commute among them for different points of this $\vec{k}$ lattice.

Now, we define operators χ and Q for each point of the three-dimensional lattice as

$$\chi_{\vec{k}}\equiv -i(T_{-\vec{k}}S_{-\vec{k}}-S_{\vec{k}}\overline{T}_{\vec{k}})=-i(A_{-\vec{k}}-A_{\vec{k}}^\dagger), \tag{71}$$

$$Q_{\vec{k}}\equiv T_{\vec{k}}S_{\vec{k}}+S_{-\vec{k}}\overline{T}_{-\vec{k}}=A_{\vec{k}}+A_{-\vec{k}}^\dagger, \tag{72}$$

such that $\chi_{\vec{k}}^\dagger=\chi_{-\vec{k}}$ and $Q_{\vec{k}}^\dagger=Q_{-\vec{k}}$, exactly as happens in the construction of a spin-0 field for the spin-0 quantum field theory [10]. By means of $\chi_{\vec{k}}$ and $Q_{\vec{k}}$ we define two fields $\phi(\vec{r},t)$ and $\Pi(\vec{r},t)$ as

$$\phi(\vec{r},t)=\sum_{\vec{k}}\frac{1}{\sqrt{2\Omega\omega(\vec{k})}}(A_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{r}}+A_{\vec{k}}e^{i\vec{k}\cdot\vec{r}}), \tag{73}$$

$$\Pi(\vec{r},t)=\sum_{\vec{k}}\frac{i\omega(\vec{k})}{\sqrt{2\Omega\omega(\vec{k})}}(A_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{r}}-A_{\vec{k}}e^{i\vec{k}\cdot\vec{r}}), \tag{74}$$

where $\omega(\vec{k})=\sqrt{\vec{k}^2+m^2}$, m is a real parameter, and Ω is the volume of a rectangular box. Another momentum-type field $\wp(\vec{r},t)$ can be defined as

$$\wp(\vec{r},t)=\sum_{\vec{k}} \sqrt{\frac{\omega(\vec{k})}{2\Omega}}S_{\vec{k}}e^{i\vec{k}\cdot\vec{r}}. \tag{75}$$

By a straightforward calculation, we can show that the Hamiltonian

$$H=\int d^3r[\Pi(\vec{r},t)^2+u|\wp(\vec{r},t)|^2 + \phi(\vec{r},t)(-\vec{\nabla}^2+m^2)\phi(\vec{r},t)], \tag{76}$$

where u is an arbitrary number, can be written as

$$H=\frac{1}{2}\sum_{\vec{k}} \omega(\vec{k})[A_{\vec{k}}^{\dagger}A_{\vec{k}}+A_{\vec{k}}A_{\vec{k}}^{\dagger}+uS_{\vec{k}}(N)^2] =\frac{1}{2}\sum_{\vec{k}} \omega(\vec{k})[S_{\vec{k}}(N+1)^2+(1+u)S_{\vec{k}}(N)^2], \tag{77}$$

where

$$S_{\vec{k}}(N)^2=q^{2N_{\vec{k}}}\alpha_0+[N_{\vec{k}}]_{q^2}-\alpha_0. \tag{78}$$

In order that the energy of the vacuum state becomes zero we replace H in Eq. (77) by

$$H=\frac{1}{2}\sum_{\vec{k}} \omega(\vec{k})[S_{\vec{k}}(N+1)^2+(1+u)S_{\vec{k}}(N)^2 -(q^2-1)\alpha_0-1]. \tag{79}$$

Note that in the limit $q\to 1$ $(\alpha_0\to 0)$, the above Hamiltonian is proportional to the number operator. Furthermore, as can be seen from Eqs. (78) and (79) the energy of this system is nonadditive. The nonadditivity of a free system is not new in field theory; for instance the energy of nontopological solitons has this property (see [10], Chap. 7). In the present case, this nonadditivity of the energy comes from the fact that q oscillators approximately describe correlated fermion pairs in many body systems [7].

The eigenvectors of H form a complete set and span the Hilbert space of this system; they are

$$|0\rangle, \quad A_{\vec{k}}^{\dagger}|0\rangle, \quad A_{\vec{k}}^{\dagger}A_{\vec{k}'}^{\dagger}|0\rangle \quad \text{for} \quad \vec{k}\neq\vec{k}', \quad (A_{\vec{k}}^{\dagger})^2|0\rangle, \ldots, \tag{80}$$

where the state $|0\rangle$ satisfies as usual $A_{\vec{k}}|0\rangle=0$ [see Eq. (12)] for all $\vec{k}$ and $A_{\vec{k}},A_{\vec{k}}^{\dagger}$ for each $\vec{k}$ satisfy the q-deformed Heisenberg algebra Eqs. (17)–(19).

The time evolution of the fields can be studied by means of Heisenberg's equation for $A_{\vec{k}}^{\dagger}$, $A_{\vec{k}}$, and $S_{\vec{k}}$. Define

$$E(N_{\vec{k}})\equiv J_0(\vec{k})=q^{2N_{\vec{k}}}\alpha_0+[N_{\vec{k}}]_{q^2} \tag{81}$$

and

$$h(N_{\vec{k}})\equiv\frac{1}{2}(1+u+r)(E(N_{\vec{k}}+1)-E(N_{\vec{k}})]. \tag{82}$$

Thus, using Eqs. (77) or (79) and (17)–(19) we obtain

$$[H,A_{\vec{k}}^{\dagger}]=\omega(\vec{k})A_{\vec{k}}^{\dagger}h(N_{\vec{k}}). \tag{83}$$

We can solve Heisenberg's equation for the q-deformed case, obtaining

$$A_{\vec{k}}^{\dagger}(t)=A_{\vec{k}}^{\dagger}(0)e^{i\omega(\vec{k})h(N_{\vec{k}})t}. \tag{84}$$

Note that for $q\to 1$ and $u=0$ we have $h(N_{\vec{k}})\to 1$ and Eq. (84) gives the correct result for this undeformed case. It is not difficult to realize, using Eqs. (17)–(19), that for this linear case we have $[f(J_0)-J_0]A=q^{-2}A[f(J_0)-J_0]$. Taking the Hermitian conjugate of Eq. (84) and using the result just mentioned we have

$$A_{\vec{k}}(t)=A_{\vec{k}}(0)e^{-iq^{-2}\omega(\vec{k})h(N_{\vec{k}})t}, \tag{85}$$

also giving the correct undeformed limit. Furthermore, we easily see that the operators $P_{\vec{k}}$ and $S_{\vec{k}}$ are time independent. We emphasize that the extra term $h(N_{\vec{k}})$ in the exponentials depends on the number operator, this being the main difference from the undeformed case. The Fourier transformation Eq. (73) can then be written as

$$\phi(\vec{r},t)=\alpha(\vec{r},t)+\alpha(\vec{r},t)^{\dagger}, \tag{86}$$

where

$$\alpha(\vec{r},t)=\sum_{\vec{k}} \frac{1}{\sqrt{2\Omega\,\omega(\vec{k})}}A_{\vec{k}}e^{i\vec{k}\cdot\vec{r}-iq^{-2}\omega(\vec{k})h(N_{\vec{k}})t}, \tag{87}$$

$A_{\vec{k}}$ in Eq. (87) is time independent, and $\alpha(\vec{r},t)^{\dagger}$ is the Hermitian conjugate of $\alpha(\vec{r},t)$.

The Feynman propagator $D_F^N(x_1,x_2)$ defined, as usual, as the Dyson-Wick contraction between[3] $\phi(x_1)$ and $\phi(x_2)$, can be computed using Eqs. (17)–(19) and (86),(87):

$$D_F^N(x_1,x_2)=\sum_{\vec{k}} \frac{e^{i\vec{k}\cdot\Delta\vec{r}_{12}}}{2\Omega\,\omega(\vec{k})}[S_{\vec{k}}(N+1)^2e^{\mp i\omega(\vec{k})h(N_{\vec{k}})\Delta t_{12}} -S_{\vec{k}}(N)^2e^{\mp i\omega(\vec{k})h(N_{\vec{k}}-1)\Delta t_{12}}], \tag{88}$$

where $\Delta t_{12}=t_1-t_2$, $\Delta\vec{r}_{12}=\vec{r}_1-\vec{r}_2$, the minus sign in the exponent holds when $t_1>t_2$, and the positive sign when $t_2>t_1$. Note that when $q\to 1$, $h(N_{\vec{k}})\to 1$, and $S_{\vec{k}}(N+1)^2-S_{\vec{k}}(N)^2\to 1$, the standard result for the propagator is recovered. It is also simple to obtain the following integral representation for the Feynman propagator:

[3] $x_i\equiv(\vec{r}_i,t_i)$.

$$D_F^N(x)=\frac{-i}{(2\pi)^4}\int\frac{S_{\vec{k}}(N+1)^2 e^{i\vec{k}\cdot\vec{r}-ik_0 h(N_{\vec{k}})t}d^4k}{k^2+m^2}-(N\to N-1), \tag{89}$$

where in the second part of the right hand side of the above equation we have the first part with $N\to N-1$. Again, note that the $q\to 1$ limit of the integral representation of the above Feynman propagator gives the usual propagator. We point out that this propagator is not a simple c number since it depends on the number operator N.

We shall now discuss the first order scattering process $1+2\to 1'+2'$ where the initial state is

$$|1,2\rangle \equiv A^\dagger_{p_1}A^\dagger_{p_2}|0\rangle \tag{90}$$

and the final state is

$$|1',2'\rangle \equiv A^\dagger_{p_1'}A^\dagger_{p_2'}|0\rangle, \tag{91}$$

where A_{p_i} and $A^\dagger_{p_i}$ satisfy the algebraic relations in Eqs. (17)–(19). These particles are supposed to be described by the Hamiltonian given in Eq. (76) with an interaction given by $\lambda\int:\phi(\vec{r},t)^4:d^3r$. To the lowest order of λ, we have

$$\langle 1',2'|S|1,2\rangle=-i\lambda\int d^4x\langle 1',2'|:\phi^4(x):|1,2\rangle. \tag{92}$$

We use Eq. (86) in Eq. (92) and put it in normal order. Since the exponentials now have number operators we must take the exponentials it outside the matrix elements, obtaining

$$\begin{aligned}\langle 1',2'|S|1,2\rangle\\ =\frac{-6i\lambda}{4\Omega^2}\int d^4x\sum_{\vec{k}_1\cdots\vec{k}_4}\frac{1}{\sqrt{\omega_{\vec{k}_1}\cdots\omega_{\vec{k}_4}}}\\ \times\langle 0|A_{\vec{p}_1'}A_{\vec{p}_2'}A^\dagger_{\vec{k}_1}A^\dagger_{\vec{k}_2}A_{\vec{k}_3}A_{\vec{k}_4}A^\dagger_{\vec{p}_1}A^\dagger_{\vec{p}_2}|0\rangle\\ \times e^{-i(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)\cdot\vec{r}+iW(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)t},\end{aligned} \tag{93}$$

where

$$\begin{aligned}W(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)&=\omega(\vec{k}_1)h_1(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\\ &+\omega(\vec{k}_2)h_2(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\\ &-\omega(\vec{k}_3)h_3(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\\ &-\omega(\vec{k}_3)h_3(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\end{aligned} \tag{94}$$

and

$$\begin{aligned}h_1(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)&=h(\delta^3_{\vec{k}_1,\vec{k}_2}-\delta^3_{\vec{k}_1,\vec{k}_3}-\delta^3_{\vec{k}_1,\vec{k}_4}+\delta^3_{\vec{k}_1,\vec{p}_1}+\delta^3_{\vec{k}_1,\vec{p}_2}),\\ h_2(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)&=h(-\delta^3_{\vec{k}_2,\vec{k}_3}-\delta^3_{\vec{k}_2,\vec{k}_4}+\delta^3_{\vec{k}_2,\vec{p}_1}+\delta^3_{\vec{k}_2,\vec{p}_2}),\\ h_3(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)&=h(-1-\delta^3_{\vec{k}_3,\vec{k}_4}+\delta^3_{\vec{k}_3,\vec{p}_1}+\delta^3_{\vec{k}_3,\vec{p}_2}),\\ h_4(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)&=h(-1+\delta^3_{\vec{k}_4,\vec{p}_1}+\delta^3_{\vec{k}_4,\vec{p}_2}).\end{aligned} \tag{95}$$

The matrix elements in Eq. (93) can be handled using the algebraic relations in Eqs. (17)–(19). At this point the integral in Eq. (93) can be computed, giving

$$\begin{aligned}&\langle 1',2'|S|1,2\rangle\\ &=\frac{-24(2\pi)^4 ih(0)^2\lambda}{\Omega^2(1+u+q^2)^4\sqrt{\omega_{\vec{p}_1}\omega_{\vec{p}_2}\omega_{\vec{p}'_1}\omega_{\vec{p}'_2}}}\\ &\times[h(0)^2\delta^4(P_{1,a}+P_{2,a}-P'_{1,a}-P'_{2,a})\\ &+h(0)h(\delta^3_{\vec{p}_1',\vec{p}_2'})\delta^4(P_{1,b}+P_{2,b}-P'_{1,b}-P'_{2,b})\\ &+h(0)h(\delta^3_{\vec{p}_1,\vec{p}_2})\delta^4(P_{1,c}+P_{2,c}-P'_{1,c}-P'_{2,c})\\ &+h(\delta^3_{\vec{p}_1,\vec{p}_2})h(\delta^3_{\vec{p}'_1,\vec{p}'_2})\delta^4(P_{1,d}+P_{2,d}-P'_{1,d}-P'_{2,d})],\end{aligned} \tag{96}$$

where

$$\begin{aligned}&P_{1,a}=(\vec{p}_1,\omega_{\vec{p}_1}h(0)),\quad P_{2,a}=(\vec{p}_2,\omega_{\vec{p}_2}h(\delta^3_{\vec{p}_1,\vec{p}_2})),\\ &P'_{1,a}=(\vec{p}_1',\omega_{\vec{p}_1'}h(\delta^3_{\vec{p}_1',\vec{p}_2'})),\quad P'_{2,a}=(\vec{p}_2',\omega_{\vec{p}_2'}h(0));\\ &P_{1,b}=(\vec{p}_1,\omega_{\vec{p}_1}h(0)),\quad P_{2,b}=(\vec{p}_2,\omega_{\vec{p}_2}h(\delta^3_{\vec{p}_1,\vec{p}_2})),\\ &P'_{1,b}=(\vec{p}_1',\omega_{\vec{p}_1'}h(0)),\quad P'_{2,b}=(\vec{p}_2',\omega_{\vec{p}_2'}h(\delta^3_{\vec{p}_1',\vec{p}_2'}));\\ &P_{1,c}=(\vec{p}_1,\omega_{\vec{p}_1}h(\delta^3_{\vec{p}_1,\vec{p}_2})),\quad P_{2,c}=(\vec{p}_2,\omega_{\vec{p}_2}h(0)),\\ &P'_{1,c}=(\vec{p}_1',\omega_{\vec{p}_1'}h(\delta^3_{\vec{p}_1',\vec{p}_2'})),\quad P'_{2,c}=(\vec{p}_2',\omega_{\vec{p}_2'}h(0));\\ &P_{1,d}=(\vec{p}_1,\omega_{\vec{p}_1}h(\delta^3_{\vec{p}_1,\vec{p}_2})),\quad P_{2,d}=(\vec{p}_2,\omega_{\vec{p}_2}h(0)),\\ &P'_{1,d}=(\vec{p}_1',\omega_{\vec{p}_1'}h(0)),\quad P'_{2,d}=(\vec{p}_2',\omega_{\vec{p}_2'}h(\delta^3_{\vec{p}_1',\vec{p}_2'})).\end{aligned} \tag{97}$$

Note that when $q\to 1$ we have $h\to 1$, $P_{i,a}=P_{i,b}=P_{i,c}=P_{i,d}$, $P'_{i,a}=P'_{i,b}=P'_{i,c}=P'_{i,d}$ ($i=1,2$), $u=0$, and Eq. (96) becomes the standard undeformed result [10].

In order to compute the matrix element of S to second order in λ we must generalize Wick's theorem since the Feynman propagator in this case depends on the number operator. We hope to report on this computation in the near future.

V. TOWARD A MULTIPARTICLE QUANTUM FIELD THEORY

In the last section we constructed the propagator and the matrix element of the S matrix to first order of the coupling constant of a q-deformed spin-0 QFT. We stress that we considered a q-deformed Heisenberg algebra and the quantum mechanics was taken as being the standard one. The advantage of studying a deformed system is that we can make contact with the nondeformed one at all steps of the computation.

In this section we are going to discuss the possibility of constructing a QFT by interpreting each state of a generalized Heisenberg algebra as a particle with a different mass. We consider a Heisenberg algebra, the generators of which are given as

$$A^{\dagger}=S(P)\overline{T}, \tag{98}$$

$$A=TS(P), \tag{99}$$

$$J_0=\sqrt{V(P)+m_q^2}, \tag{100}$$

where $V(x)$ is a general function of x, and T,P are defined in Eqs. (36), (40), respectively, and $S(P)^2=J_0-m_q$. In the special case where $V(P)=P^2$ we have the relativistic square-well algebra discussed in [5].

Let us associate with each point of the discrete $\vec{k}$ space, defined in the previous section, an independent copy of the one-dimensional momentum lattice defined in Sec. III for each point of this $\vec{k}$ lattice, so that $P_{\vec{k}}^{\dagger}=P_{\vec{k}}$ and $T_{\vec{k}}$, $\overline{T_{\vec{k}}}$, and $S_{\vec{k}}$ are defined by means of the previous definitions, Eqs. (35), (36), and (55), through the substitution $P\rightarrow P_{\vec{k}}$. Then we can define for each point of this lattice a Heisenberg algebra as

$$A_{\vec{k}}^{\dagger}=S(P_{\vec{k}})\overline{T_{\vec{k}}}, \tag{101}$$

$$A_{\vec{k}}=T_{\vec{k}}S(P_{\vec{k}}), \tag{102}$$

$$J_0(\vec{k})=\sqrt{\vec{k}^2+V(P_{\vec{k}})+m_q^2}, \tag{103}$$

where $S(P_{\vec{k}})^2=J_0(\vec{k})-\sqrt{\vec{k}^2+m_q^2}$.

The Hilbert space of the associated QFT is spanned by the vectors

$$|0\rangle,\quad A_{\vec{k}}^{\dagger}|0\rangle,\quad A_{\vec{k}}^{\dagger}A_{\vec{k}'}^{\dagger}|0\rangle \quad \text{for} \quad \vec{k}\neq\vec{k}',\quad (A_{\vec{k}}^{\dagger})^2|0\rangle,\ \ldots . \tag{104}$$

Notice that the state $(A_{\vec{k}}^{\dagger})^n|0\rangle$ has a $J_0(\vec{k})$ eigenvalue given by $\sqrt{\vec{k}^2+V(an)+m_q^2}$, which can be interpreted as the energy state of a particle with mass $\sqrt{V(an)+m_q^2}$. Thus, the associated interacting QFT would describe particles with mass spectrum $\sqrt{V(an)+m_q^2}$, with $n=1,2,\ldots$, unified by the generalized Heisenberg algebra under consideration. The possibility of having a QFT unifying a spectrum of particles of different masses is appealing, with potential applications in hadronic phenomenology. There are some points to be understood before developing such a QFT, such as, for instance, Lorentz invariance, which plays an important role in a theory describing relativistic particles. We hope to develop this point and to report on such a QFT in the near future.

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