Stochastic theory of relativistic particles moving in a quantum field: Scalar Abraham-Lorentz-Dirac-Langevin equation, radiation reaction, and vacuum fluctuations

Philip R. Johnson* and B. L. Hu†
Department of Physics, University of Maryland, College Park, Maryland 20742-4111
(Received 30 December 2000; revised manuscript received 14 September 2001; published 28 February 2002)

We apply the open systems concept and the influence functional formalism to establish a stochastic theory of relativistic moving spinless particles in a quantum scalar field. The stochastic regime resting between the quantum and semiclassical regimes captures the statistical mechanical attributes of the full theory. Applying the particle-centric world line quantization formulation to describe charged particles in a scalar quantum field environment, we derive a modified Abraham-Lorentz-Dirac (ALD) equation with time-dependent coefficients and show that it is the correct semiclassical limit for nonlinear particle-field systems without the need of making the dipole or nonrelativistic approximations. Our modified ALD equation is causal and free of runaway solutions. We show this technically, as a consequence of the nonequilibrium open system dynamics, and conceptually, invoking decoherence. Progressing to the stochastic regime, we derive a relativistic ALD-Langevin (ALDL) equation for nonlinearly coupled charges in a scalar quantum field. The ALD and ALDL equations clarify the relation of radiation reaction, dissipation and vacuum fluctuations. This self-consistent treatment serves as a new platform for investigations into problems related to relativistic moving charges.

DOI: 10.1103/PhysRevD.65.065015 PACS number(s): 03.70.+k, 05.40.−a, 12.20.Ds, 32.80.−t

I. INTRODUCTION

This is the second in a series of papers [1,2] exploring the regime of stochastic behavior manifested by relativistic particles moving through quantum fields. It highlights a self-consistent treatment of relativistic, nonlinearly interacting quantum dynamical particle-field systems, and the effects of and the interconnections amongst noise, decoherence, dissipation, fluctuations, and correlation.

In [1] we have set up the basic framework built on the concepts of quantum open systems [3], the model of quantum Brownian motion (QBM) [4], and the methodologies of the influence functional [5] or the closed-time-path [6] coarse-grained effective action [7], and world line quantization [8]. In this paper we apply this framework to spinless relativistic moving particles in a quantum scalar field. The interaction is chosen to be the scalar analog of QED coupling so that we can avoid the complications of photon polarizations and gauge invariance in an electromagnetic field. The results here should describe correctly particle motion when spin and photon polarization are unimportant, and when the particle is sufficiently decohered such that its quantum fluctuations effectively produce stochastic dynamics [4,9–14].

We divide our introduction into four parts: First, we describe the main results from this work. Second, we give a discussion of the pathologies and misconceptions of the Abraham-Lorentz-Dirac equation from the conventional approach, specifically the existence of preaccelerating and runaway solutions, and the misconception that classical radiation reaction and vacuum fluctuations are related by a fluctuation-dissipation relation. Third, we describe the open system concept and the coarse-grained effective action technique our approach is based on. We distinguish between the four regimes: classical, semiclassical, stochastic and quantum and discuss how the processes of fluctuations, noise, decoherence and dissipation are interrelated. Decoherence due to noise is instrumental to the emergence of a classical solution, the presence of a stochastic component in the trajectory, and provides a rationale for the cure of pathologies. This requires a proper treatment of causal and non-Markovian behavior on the basis of self-consistent back reaction. Finally we give a brief summary of previous work and describe their shortcomings which justify a new approach as detailed here.

A. Main results

The main results of this investigation are the following.

(1) First principles derivation of a time-dependent Abraham-Lorentz-Dirac (ALD) equation for the semiclassical limit of relativistic particles in a scalar quantum field without making the dipole or nonrelativistic approximations. Our time-dependent ALD equation is fully causal; its low-energy and late time limit is the ordinary ALD equation.

(2) Consistent resolution of the paradoxes of the ALD equation, including the problems of runaway and acausal (e.g. preaccelerating) solutions, and other pathologies. We show how the non-Markovian nature of the quantum particle open-system enforces causality in the equations of motion. We also discuss the crucial conceptual role that decoherence plays in understanding these problems.

(3) Derivation of multiparticle Abraham-Lorentz-Dirac-Langevin (ALDL) equations describing the quantum stochastic dynamics of relativistic particles. The familiar classical ALD equation is reached as its *noise-averaged* form. The stochastic regime, characterized by balanced noise and dissipation, plays a crucial role in bridging the gap between quan-

*Electronic address: philipj@physics.umd.edu
†Electronic address: hub@physics.umd.edu

 ©2002 The American Physical Society

tum and (emergent) classical behaviors. The *N*-particle-irreducible (NPI) and master effective action [15] provide a route for generalizing our treatment to the *self-consistent* inclusion of higher order quantum corrections.

B. Pathologies and misconceptions

1. Runaway solutions and preacceleration

The classical theory of moving charges interacting with a classical electromagnetic field has controversial difficulties associated with back reaction [16]. The generally accepted classical equation of motion in a covariant form for charged, spinless point particles, including the effect of radiation reaction, is the Abraham-Lorentz-Dirac (ALD) equation [17]:

$$\ddot{z}^{\mu}+(2e^2/3m)(\dot{z}^{\mu}\ddot{z}^2+\dddot{z}^{\mu})=(e/m)\dot{z}_{\nu}F^{\mu\nu}_{ext}(z). \quad (1.1)$$

The time scale $\tau_0=(2e^2/3mc^3)$ determines the relative importance of the radiation reaction term. For electrons, $\tau_0 \sim 10^{-24}$ sec, which is roughly the time it takes light to cross the electron classical radius $r_0\sim 10^{-15}$ m. The ALD equation has been derived in a variety of ways, often involving some regularization procedure leading to a renormalization of the particle's mass. Feynman and Wheeler derived this result from their ''Absorber'' theory which symmetrically treats both advanced and retarded radiation on the same footing [18]. Coleman solved the classical problem using the same configuration-space Pauli-Villars regulated Greens function that we employ in the Appendix [19].

The ALD equation has strange features, whose status are still debated. It is a third order differential equation requiring the specification of extra initial data (e.g. the initial acceleration) in addition to the usual position and velocity required by first order Hamiltonian systems. This leads to the existence of runaway solutions. Physical (e.g. nonrunaway) solutions may be enforced by transforming Eq. (1.1) to a second order integral equation with boundary condition such that the final energy of the particle is finite and consistent with the total work done on it by external forces. But the removal of runaway trajectories comes at a price: the solutions to the integral equation exhibit the acausal phenomena of preacceleration on time scales τ_0. This is the source of lingering questions on whether the classical theory of point particles and fields is causal. This is one major conceptual point we want to clarify. The other is the relation between radiation reaction and vacuum fluctuations, a point of long standing confusion.

2. Radiation reaction and vacuum fluctuations

Radiation reaction (RR) is often regarded as necessarily balanced by vacuum fluctuations (VF) via a fluctuation dissipation relation (FDR). This is a misconception. RR exists already at the classical level, whereas VF is of quantum nature [2,20]. An interesting example involves the special case of uniform acceleration where RR vanishes in the classical limit but vacuum fluctuations of quantum fields have an ubiquitous existence. This well-known classical result of vanishing RR is at first sight surprising because a uniformly accelerated particle does radiate.[1] We will show in this paper that the average radiation reaction force on a uniformly accelerated particle also vanishes in the semiclassical limit. However, at the stochastic level there are quantum-fluctuation-induced variations in the radiation reaction force that act to damp fluctuations of the particle away from its averaged trajectory. It is this dissipative effect which is related to vacuum fluctuations by a generalized set of FDRs [1]. The uniformly accelerated particle example illustrates the distinction between semiclassical RR forces for which a FDR plays no role, and the deviations from the semiclassical RR force of stochastic nature that are governed by a FDR.

To remedy the ALD pathologies and correct the RR versus VF misconceptions we need a brief expose of the conceptual framework and methodology of open systems and initial value functional formulations.

C. Quantum, stochastic, semiclassical and classical

1. Quantum open system and coarse-grained effective action

A closed quantum system can be partitioned into several subsystems according to the relevant physical scales. If one is interested in the details of one such subsystem, call it the distinguished, or relevant *system*, and decides to ignore certain details of the other subsystems, comprising the *environment*, the distinguished subsystem is thereby rendered as an open system. The overall effect of the coarse-grained environment on the open system can be captured by the influence functional technique of Feynman and Vernon [5], or the closely related closed-time-path effective action method of Schwinger and Keldysh [6]. These are initial value formulations. For the model of particle-field interactions under study, this approach yields an exact, nonlocal and nonlinear, coarse-grained effective action (CGEA) for the particle motion [7]. The CGEA may be used to treat the nonequilibrium quantum dynamics of interacting particles.

When the particle trajectory becomes largely well defined (with some degree of stochasticity caused by noise) as a result of effective decoherence due to interactions with the field the CGEA can be meaningfully transcribed into a stochastic effective action, describing stochastic particle motion. In this program of investigation we take a microscopic view, using quantum field theory as the tool to give a first-principles derivation of moving particles interacting with a quantum field from an open-systems perspective. We highlight two regimes between the classical and the quantum: At the *semiclassical* level, where a classical particle is treated self-consistently with back reaction from the quantum field, an equation of motion—the ALD equation—for the *mean* coordinates of the particle trajectory is obtained. At the *sto-*

[1]This is a case where the near and intermediate field dynamics cannot be ignored. Work is done on and by the so-called acceleration field (also known as the Shott field) at different stages of the particle motion such that the total energy content of radiation field, particle, *and* acceleration field is conserved [21]. It is incorrect to demand equality between the particle and (asymptotic) radiation fields alone; they are not the complete system.

chastic level self-consistent back reaction of the *fluctuations* in the quantum field is included in our consideration, and the ALDL equation is obtained.

2. Decoherent histories, semiclassicality and stochasticity

A consequence of coarse-graining the environment (quantum field) is the appearance of noise which is instrumental to the decoherence of the system and the emergence of a classical particle picture. Decoherence or dephasing refers to the loss of phase coherence in the quantum open system arising from the interaction of the system with the environment [9]. When the environment is a quantum field quantum fluctuations (under certain approximations) can act effectively as a classical stochastic source, or noise [22].

Under reasonable physical conditions [11,14] the evolution propagator for the reduced density matrix of the open system is dominated (via the stationary phase approximation) by the particle trajectory giving the extremal solution of the real part of the coarse-grained effective action (CGEA). Because the CGEA is derived by summing over all histories of the quantum field, this extremal solution—the ALD equation—incorporates the average radiation reaction force, and hence gives the self-consistent semiclassical trajectory.

In this emergent picture of quantum to classical transition, there is always some degree of residual stochasticity in the system dynamics [11,14]. Stochastic fluctuations around decoherent semiclassical trajectories are described by the imaginary part of the CGEA. We use this fact below to derive the relativistic ALDL equation. In [1] we show that an approximate FDR may be obtained describing the balance of fluctuations and dissipation about the semiclassical solution. That result applies to the ALDL equation here.

3. Pathology-free modified ALD equation

This view of the emergence of semiclassical solutions as decoherent histories [10,11,14] also suggests a new way to look at the paradoxes of charged particle radiation reaction in the ALD equation. While it has long been felt that the resolution of these problems must lie in quantum theory, this still leaves open the following questions: when, if ever, does the ALD equation appropriately characterize the classical limit of particle back reaction; how does the classical limit emerge; and what imprints do the correlations of the quantum field environment leave.

Since the semiclassical limit describes the equations of motion for the expectation value corresponding to the quantum-averaged particle trajectory *after* sufficient coarse-graining we are led to ask: (1) are the decoherent histories describing particle trajectories the solutions to a (possibly modified) ALD equation, (2) are these solutions unique and runaway free, and (3) are they causal without preacceleration on the coarse-grained scale in which they decohere?

We show below that the semiclassical solutions *are* indeed described by a modified ALD equation with time-dependent coefficients satisfying these criteria. The time-dependent effects act to preserve causality, occurring in the short time after the field begins to dress the initial particle state. At latter times the solutions essentially obey the ordinary ALD equation with no runaway trajectories. Whether the semiclassical solution is unique depends on both the initial particle and field states. If the field begins in a superposition of macroscopically distinct configurations, each configuration may lead to macroscopically distinct particle trajectories. This happens because there will, in that case, no longer be a single extremal solution of the CGEA around which the evolution propagator can be expanded. If these distinct coarse-grained trajectories decohere they are identified as a set of semiclassical solutions. In this paper we assume initial field states that are simple functional Gaussians (like vacuum, squeezed, coherent, or thermal field states) where this problem does not arise. Finally, the particle's initial state may involve a superposition of distinguishable wave packets that, after coarse-graining and decoherence, lead to multiple semiclassical solutions. Thus, only for initially localized particle states in quantum fields that are sufficiently classical (i.e. not involving macroscopically distinguishable superpositions) should we expect to find a unique (single) semiclassical solution. Finally, because the semiclassical limit describes the quantum average of coarse-grained histories, the ALD equation loses its meaning in the finest-grained quantum limit.

D. Prior work in relation to ours

(1) There are many works on *non*relativistic quantum (and semiclassical) radiation reaction (for atoms as well as charges moving in quantum fields) including those by Rohrlich, Moniz-Sharp, Cohen-Tannoudji *et al.*, Milonni, and others [21,23,24]. Kampen, Moniz, Sharp, and others [25] suggested that the problems of causality and runaways can be resolved in both classical and quantum theory by considering extended charge models. It is true that extended objects can give consistent causal dynamics, but this approach, while quite interesting, misses the point that point particles (as implied by local quantum field theories) should obey a good low-energy effective theory that is consistent so long as the high-energy (short distance) structure of the particle is not being probed. Wilson, Weinberg, and others [26] have shown how effective theory description is sufficient to understand low-energy physics because complicated, and often irrelevant, high-energy details of the fine structure are not being probed at the physical energy scales of interest. Therefore one should not need to invoke extended charge theory to understand low-energy particle dynamics and radiation reaction.

(2) By examining the time dependence of operator canonical commutation relations Milonni showed the necessity of electromagnetic field vacuum fluctuations for radiating *nonrelativistic* charges [24]. If a quantum particle is coupled to a *classical* electromagnetic field, radiative losses lead to a contraction in "phase space" of the particle position and momentum violating the commutation relations $[\hat{\mathbf{x}},\hat{\mathbf{p}}]=i\hbar$. The vacuum field balances this dissipation effect preserving the commutation relations as a consequence of a fluctuation-dissipation relation (FDR). Our relativistic treatment (within a particle-centric or world line framework) generalizes these considerations to a particle's spacetime variables. For ex-

ample, the relativistic ALD-Langevin equation derived below describes fluctuations in the particle's time as well as space coordinate. This is expected since relativity requires that a physical (on-shell) particle satisfy $\dot{z}^\mu \dot{z}_\mu = \dot{t}^2(\tau) - \dot{\mathbf{z}}^2(\tau) = 1$, and therefore any fluctuations in spatial variables must be balanced by fluctuations in the time coordinate.

(3) The derivation and use of quantum Langevin equations (QLE's) to describe fluctuations of a system in contact with a quantum environment has a long history [27]. Typically, QLE's are assumed to describe fluctuations in the linear response regime for a system around equilibrium, but their validity need not be so restrictive. Nonequilibrium conditions can be treated with the Feynman-Vernon influence functional as exemplified by Caldeira and Leggett's study of quantum Brownian motion (QBM), which has led to an extensive literature [4], particularly in regard to decoherence issues [9,10]. Barone and Caldeira [13] have applied this method to the question of whether nonrelativistic, dipole coupled electrons decohere in a quantum electromagnetic field. An advantage of Barone and Caldeira's work is that it is not limited to initial factorization of the system and environment states; they use the preparation function method which allows the inclusion of certain kinds of initial particle-field correlations. Despite this, Romero and Paz have pointed out that the preparation function method still suffers from an (implicit) unphysical depiction of instantaneous measurement characterizing the initial state preparation [28]. As a consequence, anomalous short time behavior still manifests on the cutoff time scale (including the decoherence rate having a strong cutoff dependence). This fact indicates that the preparation function does not truly resolve the issue. For this reason, and for simplicity in illustration, we stay with the simpler assumption of initial factorizability. Clearly, going beyond this by correctly describing the kinds of initial states produced by physically realistic preparation procedures is an important problem both in regard to the short time dynamics, and questions like decoherence that are sensitive to the cutoff.

(4) Ford, Lewis, and O'Connell (FOL) have extensively discussed the electromagnetic field as a thermal bath in the linear, dipole coupled regime [29], and pioneered the application of QLE's to nonrelativistic particle motion in QED. They have detailed the conditions for causality in the thermodynamic, equilibrium limit described by the late time linear quantum Langevin equation. A crucial point of their analysis is that particle motion is runaway free and causal in the late time limit as long as the spectral density of the field is cut off below a critical value determined by the classical electron radius. In [30], they suggest a form of the equations of motion that gives fluctuations without dissipation for a free electron, but this result is special to the case where the cutoff is precisely equal to the critical value, which in turn implies that the bare mass of the particle exactly vanishes. In contrast, we take the effective theory point of view which emphasizes the typical insensitivity of low-energy phenomena to unobserved high-energy structure. When one does not assume a special value for the cutoff one generally does find both fluctuations and dissipation, though Ford and Lewis's counter-example shows how careful one must be in making automatic assumptions about the existence and nature of FDRs.

Our results agree with the conclusion that causality requires a cutoff below the critical value, but we find no compelling justification for the claim that the cutoff *should* take exactly that special value. It is certainly interesting that a critical value of the cutoff implies that all higher time derivatives vanish from the equations of motion. However, even without a specially chosen cutoff, the influence of higher derivative terms is strongly suppressed at low energies, as FOL pointed out, and hence the high-energy structure of the theory (like the exact cutoff) is largely irrelevant to the low-energy behavior.

Ford and O'Connell also propose a relativistic generalization of their modified (i.e. critical cutoff) equations of motion for the average trajectory derived from the nonrelativistic QLE [29]. Our derivation of the stochastic limit goes well beyond this by starting from relativistic quantum field theory, and by yielding a relativistic Langevin equation.

(5) A perturbative expansion agreeing with the ALD equation after a derivative reduction scheme up to order e^3 has been derived from QED field theory by Krivitskii and Tsytovich [31], including the additional forces arising from particle spin. Their work shows that a perturbative form of the ALD equation may be understood from conventional field theory, but the authors have not addressed the role of fluctuations, correlation, decoherence, time-dependent renormalization, nor self-consistent back reaction. Our method should be effectively equivalent (ignoring spin) to summing the Feynman diagrams without electron loops to all orders of e in [31], and hence produces the full third derivative form of the ALD equation.

(6) Low [32] showed that runaway solutions apparently do not occur in spin 1/2 QED; but he does not derive the ALD equation, or address the semiclassical or stochastic limits. While an important result, we emphasize the view that it does not, and should not, matter whether particles are spin 1/2, spin 0, or have some other internal structure in regard to the causality of the low-energy effective theory for center of mass particle motion.

(7) Using the influence functional, Diósi [33] derived a Markovian master equation in nonrelativistic quantum mechanics. Ford [12] has considered the loss of electron coherence from vacuum fluctuations. In contrast, it is our intent to emphasize the non-Markovian, nonlinear, and nonequilibrium regimes with special attention paid to self-consistency. Ford [12] has considered the loss of electron coherence from vacuum fluctuation.

E. Organization and notation

In Sec. II we obtain the influence functional (IF), coarse-grained effective action (CGEA), and stochastic effective action (S_χ) for spinless relativistic particles. We show how to use S_χ to derive both nonlinear semiclassical equations of motion for relativistic particles, and stochastic equations of motion (a Langevin equation) describing the fluctuations in relativistic particle motion induced by quantum fluctuations in a field. In Sec. III we consider the single particle case, and derive the (scalar-field) modified Abraham-Lorentz-Dirac (ALD) equation with time-dependent coefficients as the self-consistent semiclassical limit. Section IV shows how this limit emerges free of pathologies. In Sec. V we derive a Langevin equation—the ALDL equation—taking into account the stochastic fluctuations of relativistic particles about the semiclassical trajectories for both one particle and multiparticle cases. In Sec. VI we give a simple example of these equations for a single free particle in a vacuum scalar field. In Sec. VII we summarize our main results and mention areas of applications, of both theoretical and practical interest.

Greek letters will run from 0 to 3. We use the notation $dx = d^4x = d\mathbf{x}dt$, and $x = (\mathbf{x},t)$. Spatial (3-vectors) appear as bold faced. A bar denotes a semiclassical trajectory (e.g. $\overline{z}$), and a tilda denotes a stochastic trajectory (e.g. $\tilde{z}$). The measures for path (or functional) integrals are denoted

$$Dz = \prod_\tau dz^0(\tau) d\mathbf{z}(\tau)$$

where the parameter τ is discretized, or

$$D\varphi = \prod_x d\varphi(x)$$

where the spacetime coordinate x is discretized. The metric is $g_{\mu\nu} = diag(1,-1,-1,-1)$, and we set $c=1$.

II. SPINLESS RELATIVISTIC PARTICLES MOVING IN A SCALAR FIELD

A. Coarse-grained and stochastic effective action

Relativistic quantum theories are usually focused primarily on quantized fields, the notion of particles following trajectories is somewhat secondary. As explained in greater detail in [1], we employ a "hybrid" model in which the environment is a field, but the system is the collection of particle world lines with spacetime coordinates $z_n^\mu(\tau_n)$ where n indicates the n^{th} particle coordinate, with world line parameter τ_n, charge e_n, and mass m_n. For simplicity, we denote the entire collection of particle world line coordinates $\{z_n^\mu(\tau_n)\}$ by z, stating explicitly when we are only considering the case of a single particle.

The initial quantum state of the particle plus field system at time t_i, described by the density matrix $\hat{\rho}(t_i)$, is assumed to be uncorrelated and expressible as the product of particle and field density matrices, $\hat{\rho}_z$ and $\hat{\rho}_\varphi$, respectively:

$$\hat{\rho}(t_i) = \hat{\rho}_z(t_i) \otimes \hat{\rho}_\varphi(t_i)$$
$$= \int d\varphi_i d\varphi_i' d\mathbf{z}_i d\mathbf{z}_i' \rho_\varphi(\varphi_i,\varphi_i') \rho_z(\mathbf{z}_i,\mathbf{z}_i') |\varphi_i, z_i\rangle\langle\varphi_i', z_i'|. \tag{2.1}$$

We define the basis states by $|\varphi_i, z_i\rangle = |\varphi_i\rangle \otimes |z_i\rangle$, where $|z_i\rangle$ are positive or negative frequency relativistic configuration-space states defined as

$$|z_i\rangle \equiv |\mathbf{z}_i, t_i\rangle$$
$$= \int \frac{d^4p}{(2\pi)^4} \delta^{(4)}(p^2 - m^2)\theta(\pm p^0) e^{(-i/\hbar)p_\mu z^\mu} |\mathbf{p}, p_0\rangle. \tag{2.2}$$

Because we are interested in the semiclassical equations of motion (and stochastic fluctuations around the semiclassical solutions) for well-localized particles, we restrict to the case where

$$\rho_z(\mathbf{z}_i,\mathbf{z}_i') = \Pi_i \delta(\mathbf{z}_i - \mathbf{z}_{i0}) \delta(\mathbf{z}_i' - \mathbf{z}_{i0}), \tag{2.3}$$

with $\mathbf{z}_{i0}$ the initial position of the ith particle. The particle state in Eq. (2.3) should be antisymmetrized (for fermions) or symmetrized (for bosons), but we will ignore this effect of particle statistics at the semiclassical (and stochastic) level. The Lorentz-invariant configuration-space states defined by Eq. (2.2) are the most localizable relativistic one-particle states, nonetheless they still correspond to particles with size characterized by the Compton wavelength $\lambda_c = h/mc$. This can be verified by computing the overlap $\langle \mathbf{z},t|\mathbf{z}',t\rangle \neq \delta(\mathbf{z} - \mathbf{z}')$ between two such states at equal times. As a consequence the semiclassical equations of motion for the mean particle trajectory will not have a resolution finer than λ_c (and it may be considerably coarser).

In the initially uncorrelated (i.e. factorized) case involving highly localized particle initial states, the particle's reduced density matrix at later times is given by

$$\rho_r(z_f,z_f') = \int d\mathbf{z}_i d\mathbf{z}_i' J_r(z_f,z_f';z_i,z_i') \rho_z(\mathbf{z}_i,\mathbf{z}_i')$$
$$= J_r(z_f,z_f';z_{i0},z_{i0}). \tag{2.4}$$

The reduced density-matrix evolution operator is given by

$$J_r(z_f,z_f';z_{i0},z_{i0}) = \int_{z_i,z_i'=z_{i0}}^{z_f,z_f'} Dz Dz' e^{(i/\hbar)(S_z[z]S_z[z'])} F[z,z']$$
$$= \int_{z_i,z_i'=z_{i0}}^{z_f,z_f'} Dz Dz' e^{(i/\hbar)S_{CGEA}[z,z']}. \tag{2.5}$$

$F[z,z']$ is the Feynman-Vernon influence functional [5] for a pair of particle histories $(\{z_n\},\{z_n'\})$. The influence functional for a free, massless scalar field is shown in Sec. II B, Eq. (2.20). The coarse-grained effective action (CGEA) is defined as

$$S_{CGEA}[z,z'] = S_z[z] - S_z[z'] + S_{IF}[z,z'] \tag{2.6}$$

$$S_{IF} = -i\hbar \ln F[z,z']. \tag{2.7}$$

S_{IF} is called the influence action. The CGEA contains the full information about the influence of the (coarse-grained) field on the particle, and hence is a highly nonlocal object.

The measure Dz indicates path integration[2] over the particle world lines $z_n^\mu(\tau)$. Development of the full quantum dynamics requires evaluation of these world line path integrals; because particle-field couplings are nonlinear one may resort to perturbation theory in practice. The treatment of this problem is the topic of our second series of papers. For detailed discussion on the extraction of a semiclassical and stochastic limit from the open-system evolution propagator (2.5) in this context see [1]. In brief, semiclassical trajectories (world lines) exist when J_r is dominated by the stationary phase solution of the world line path integrals, which are the extremal solutions to the real part of S_{CGEA}. Under related conditions to those that imply decoherence of the trajectories (see [1]), the quantum fluctuations of the particles around the stationary phase solution act effectively as a classical stochastic source. In this case, it is also possible to introduce a real stochastic effective action $S_\chi[z,z']$ such that

$$e^{(i/\hbar)S_{CGEA}[z,z']}=\int D\chi P[\chi]e^{(i/\hbar)S_\chi[z,z']}, \tag{2.8}$$

where $P[\chi]$ is a positive definite probability measure for a stochastic field $\chi(\mathbf{x},t)$. When the stochastic regime physically obtains, the stochastic equations of motion derived as the extremal solution of S_χ then encode the same information as the (symmetrized) quantum correlation functions for the particle coordinates. We shall use these facts to obtain both the semiclassical and stochastic limit for relativistic particles below.

While we emphasize the fact that this particle-field model has a well-founded microscopic quantum theory which thus allows one to explore in detail the quantum, stochastic, and semiclassical regimes, we might also view our model in analogy to the treatment of quantum fields in curved spacetime [34]. There, one takes the gravitational field (spacetime) as a classical system coupled to quantum fields. It is important to include the back reaction of quantum fields on the classical spacetime dynamics. The back reaction of their mean yields the semiclassical Einstein equation which forms the basis of semiclassical gravity [35]. The inclusion of fluctuations of the stress energy of the quantum fields and the induced metric fluctuations yields the Einstein-Langevin equations which forms the basis of stochastic semiclassical gravity [36]. In our work here, the particle coordinates are analogous to the gravitational field (metric tensor).

B. The scalar field influence functional and stochastic equations of motion

The free particle action is

$$S_z[z]=\sum_n\int d\tau_n\sqrt{(\dot z_n^\mu)^2}[m_n+V(z_n(\tau_n))]. \tag{2.9}$$

From S_z follow the relativistic equations of motion:

$$m\dot z_n^\mu/\sqrt{\dot z_n^2}=-\partial^\mu V(z_n). \tag{2.10}$$

For generality, we include a possible background potential V, in addition to the quantum scalar field environment treated below. The scalar current is

$$j[z,x]=\sum_n\int d\tau_n e_n u_n(\tau_n)\delta(x-z_n(\tau_n)), \tag{2.11}$$

with

$$u_n(\tau_n)=\sqrt{\dot z_n^\mu(\tau_n)\dot z_{n,\mu}(\tau_n)}. \tag{2.12}$$

In [2], we treat a vector current coupled to the electromagnetic vector potential A_μ. In both cases we assume spinless particles. The inclusion of spin or color is important to making full use of these methods in QED and QCD.

The interaction between the particles and a scalar field is given by the monopole coupling term

$$S_{int}=\int dx\, j[z,x]\varphi(x)=e\sum_n\int d\tau_n u_n(\tau_n)\varphi(z_n(\tau_n)). \tag{2.13}$$

This is the general type of interaction treated in [1]. In the second line we have used the expression for the current (2.11). The free field action is

$$S_\varphi=\frac{1}{2}\int dx\,\partial_\mu\varphi\partial^\mu\varphi. \tag{2.14}$$

Because the interaction is linear in φ, the exact expression for the influence functional may be obtained (for a summary see [1]):

$$F[j^\pm]=\exp\left\{-\frac{i}{\hbar}\int dx\int dx'[2j^-(x)G^R(x,x')j^+(x') -ij^-(x)G^H(x',x')j^-(x')]\right\}, \tag{2.15}$$

where

$$j^-\equiv(j[z,x]-j[z',x]) \tag{2.16}$$

$$j^+\equiv(j[z,x]+j[z',x])/2 \tag{2.17}$$

and

$$G^R(x,x')=\theta(x^0-x^{0'})\langle[\hat\varphi(x),\hat\varphi(x')]\rangle \tag{2.18}$$

$$G^H(x,x')=\langle\{\hat\varphi(x),\hat\varphi(x')\}\rangle. \tag{2.19}$$

G^R and G^H are the scalar field retarded and Hadamard Green's functions, respectively. Substitution of Eq. (2.11) then gives the multiparticle influence functional

[2] The relativistic particle action is reparametrization invariant; hence, the path integral requires gauge fixing to prevent summing over an infinity of gauge-equivalent histories. For simplicity, we assume the gauge $\dot z^\mu\dot z_\mu=1$ making the parameters τ_n proper times. Treatment of gauge fixing becomes important in the full quantized world line theory.

$$F[\{z\},\{z'\}]=\exp\Bigg\{-\frac{e^2}{\hbar}\sum_{nm}\int d\tau_n\int d\tau_m\theta(T-z_n^0)\times\theta(z_n^0-z_m^0)[u_nG^H(z_n,z_m)u_m-u_n'G^H(z_n',z_m)u_m-u_nG^H(z_n,z_m')u_m'+u_n'G^H(z_n',z_m')u_m'+u_nG^R(z_n,z_m)u_m-u_n'G^R(z_n',z_m)u_m+u_nG^R(z_n,z_m')u_m'-u_n'G^R(z_n',z_m')u_m']\Bigg\} \tag{2.20}$$

where $u_n'=u_n(z_n')$, etc. The influence functional may be expressed more compactly by using a matrix notation where $\mathbf{u}_n^T=(u_n,u_n')\equiv(u_n^1,u_n^2)$, giving

$$F[z^a]=\exp\Bigg\{-\frac{e^2}{\hbar}\sum_{nm}\int d\tau_n d\tau_m(\mathbf{u}_n^T\mathbf{G}_{nm}^R\mathbf{u}_m+\mathbf{u}_n^T\mathbf{G}_{nm}^H\mathbf{u}_m)\Bigg\} \tag{2.21}$$

$$=\exp\Bigg\{-\frac{e^2}{\hbar}(\mathbf{u}_n^T\mathbf{G}_{nm}^R\mathbf{u}_m+\mathbf{u}_n^T\mathbf{G}_{nm}^H\mathbf{u}_m)\Bigg\} \tag{2.22}$$

$$=\exp\Bigg\{\frac{i}{\hbar}S_{IF}[z^a]\Bigg\}. \tag{2.23}$$

The superscript T denotes the transpose of the column vector $\mathbf{u}$. In Eq. (2.22), and below, we leave the sum Σ_{nm} and integrations $\int d\tau_n d\tau_m$ implicit for brevity. The matrices $\mathbf{G}_{nm}^H,\mathbf{G}_{nm}^R$ are given by

$$\mathbf{G}_{nm}^H=\theta(T-z_n^0)\,\theta(z_n^0-z_m^0)\times\begin{pmatrix} G_{(11)}^H(z_n^1,z_m^1) & -G_{(12)}^H(z_n^1,z_m^2)\\ -G_{(21)}^H(z_n^2,z_m^1) & G_{(22)}^H(z_n^2,z_m^2)\end{pmatrix} \tag{2.24}$$

and

$$\mathbf{G}_{nm}^R=\theta(T-z_n^0)\,\theta(z_n^0-z_m^0)\times\begin{pmatrix} G_{(11)}^R(z_n^1,z_m^1) & G_{(12)}^R(z_n^1,z_m^2)\\ -G_{(21)}^R(z_n^2,z_m^1) & -G_{(22)}^R(z_n^2,z_m^2)\end{pmatrix}. \tag{2.25}$$

In Eq. (2.23), we have defined the influence action S_{IF}. $G^{H,R}$ are the scalar-field Hadamard or Retarded Green's functions evaluated at various combinations of spacetime points $z_{n,m}^{1,2}$.

After defining the sum and difference variables

$$z^-=(z-z') \tag{2.26}$$

$$z^+=(z+z')/2, \tag{2.27}$$

the stochastic effective action is given by (see [1])

$$S_\chi[z^\pm]=S_A[z^\pm]+\int dx j^-[z^\pm,x)\Bigg\{\chi(x)+2\int dx' G^R(x,x')j^+[z^\pm,x')\Bigg\}=S_A[z^a]+\mathbf{u}_n^T\mathbf{G}_{nm}^R\mathbf{u}_m+\mathbf{u}_n^T\boldsymbol{\varpi}_n, \tag{2.28}$$

where

$$\mathbf{u}_n^T\boldsymbol{\varpi}_n=u_n(z_n)\chi(z_n)-u_n(z_n')\chi(z_n'), \tag{2.29}$$

and $\chi(z_n)$ is the stochastic field evaluated at the spacetime position of the nth particle. The stochastic field has vanishing mean and autocorrelation function given by

$$\langle\chi(x)\chi(x')\rangle_s=\hbar\langle\{\hat{\varphi}(x),\hat{\varphi}(x')\}\rangle=\hbar G^H(x,x'). \tag{2.30}$$

Hence, $\chi(x)'s$ statistics encodes those of the quantum field $\hat{\varphi}(x)$.

We may express the stochastic effective action in terms of stochastic variables $\eta_\mu(\tau)$ coupled directly to the particle trajectories by writing the influence functional in terms of a cumulant expansion:

$$F[z^\pm]=\exp\Bigg\{\frac{i}{\hbar}\int d\tau_1 z_1^{\mu-}C_{1,\mu}^{(\eta)}(\tau_1;z^+]-\frac{1}{2\hbar^2}\int\int d\tau_1 d\tau_2 z_1^{\mu-}z_2^{\nu-}C_{2,\mu\nu}^{(\eta)}(\tau_1,\tau_2;z^+]+\cdots\Bigg\}=\exp\Bigg\{\frac{i}{\hbar}\int d\tau_1 z_1^{\mu-}(C_{1,\mu}^{(\eta)}(\tau_1;z^+])\Bigg\}\times\int D\eta_\mu P[\eta_\mu;z^+]e^{(i/\hbar)\int d\tau_1 z_1^{\mu-}\eta_\mu(\tau_1;z^+]}. \tag{2.31}$$

The $C_{n>1}^{(\eta)}$ are then the cumulants of the noise $\eta_\mu(\tau)$ whose probability distribution is $P[\eta_\mu;z^+]$. They are given by

$$C_{n,\mu_1\ldots\mu_n}^{(\eta)}=\left(\frac{\hbar}{i}\right)^n\frac{\delta^n S_{IF}[z^\pm]}{\delta z^{\mu_1-}(\tau_1)\ldots\delta z^{\mu_n-}(\tau_n)}\Big|_{z^-=0}. \tag{2.32}$$

Note that because of the nonlinear nature of the coupling the cumulants $C_{n>2}^{(\eta)}$ do not vanish as is the case for linear theories with Gaussian noise.

The stochastic effective action S_η is then

$$S_\eta[z^\pm,\eta]=S_z[z^\pm]+\int d\tau z^{\mu-}(\tau)(C_{1,\mu}^{(\eta)}(\tau,z^+]+\eta_\mu(\tau)). \tag{2.33}$$

The stochastic equations of motion follow immediately from[3]

$$\left.\frac{\delta S_{\eta}}{\delta z^{\mu -}(\tau)}\right|_{z^{-}=0,z^{+}=z}=0, \tag{2.34}$$

giving

$$\frac{\delta S_{z}[z]}{\delta z^{\mu}(\tau)}+C_{1,\mu}^{(\eta)}(\tau,z]+\eta_{\mu}(\tau)=0. \tag{2.35}$$

The cumulant $C_{1,\mu}^{(\eta)}(\tau,z]$ describes dissipation, or in this context radiation reaction. If Eq. (2.35) were linear, we would interpret it as a Langevin equation for the particle trajectories with additive, but colored noise $\eta(\tau)$. In fact, as it stands Eq. (2.35) is nonlinear, and the noise depends in a complicated way on the trajectories $z^{+}=z$ because the probability distribution $P[\eta;z]$ is itself a functional of the trajectories.

The semiclassical equations of motion are given by

$$\frac{\delta S_{z}[\overline{z}]}{\delta \overline{z}^{\mu}(\tau)}+C_{1,\mu}^{(\eta)}(\tau,\overline{z}]=0, \tag{2.36}$$

where $\overline{z}_{\mu}(\tau)$ is the semiclassical trajectory. These equations of motion will be nonlinear in general. To find linear stochastic equations of motion (a Langevin equation) for fluctuations around the mean trajectory $\overline{z}$, given to lowest order by the solution to the generally nonlinear equations of motion (2.36), we expand Eq. (2.35) to linear order in the deviation variable

$$\tilde{z}^{\mu}\equiv z^{\mu}-\overline{z}^{\mu}. \tag{2.37}$$

The linearized Langevin equations, with a Gaussian noise approximation, becomes

$$\int d\tau\left(\frac{\delta^{2}S_{z}[\overline{z}]}{\delta\overline{z}^{\mu}(\tau)\,\delta\overline{z}^{\nu}(\tau')}+\frac{\delta C_{1,\mu}^{(\eta)}(\tau,\overline{z}]}{\delta\overline{z}^{\nu}(\tau')}\right)\tilde{z}^{\nu}(\tau')=\eta_{\mu}(\tau;\overline{z}), \tag{2.38}$$

with noise correlator

$$\langle\eta_{\mu}(\tau)\eta_{\nu}(\tau')\rangle=C_{2,\mu\nu}^{(\eta)}(\tau,\tau';\overline{z}] \tag{2.39}$$

$$\langle\eta_{\mu}(\tau)\rangle=0. \tag{2.40}$$

[3]One can verify by examining the structure of the effective action that the functional derivative with respect to the difference variable z^{-} (i.e. $\delta/\delta z^{-}$) corresponds to finding the expectation value of z^{+}. Setting $z^{-}=0$, after taking the functional derivative, thus corresponds to the desired expectation value $z^{+}=z$.

We shall apply both Eqs. (2.36) and (2.38) below in deriving the semiclassical and stochastic limit for relativistic particles. One of the important advantages of the world line formulation for relativistic quantum particles is that we can more readily evaluate (perhaps numerically, or exactly for special cases such as uniform acceleration) the nonlinear equations of motion (2.36) to obtain a semiclassical solution that provides the basis for deriving a stochastic equation including fluctuations. The Langevin equations therefore self-consistently depend on the nonlinear semiclassical equations of motion.

This is one aspect that distinguishes our approach (world line formulation) from a purely field theoretic treatment (e.g. Dirac field) of relativistic charged particles. In general, nonlinear semiclassical equations of motion for a relativistic charged particle field are not so easy to solve, and a Langevin equation describing fluctuations of the field around its nonlinear semiclassical solution is a far more complicated object than that obtained in Eq. (2.36). Albeit, a Langevin equation for a relativistic field contains far more information than Eq. (2.38), but when distinct particle trajectories are of interest our approach should be considerably more efficient at extracting the desired (though more limited) information.

C. Langevin integrodifferential equations of motion

We define new sum and difference variables

$$v^{\pm}=(u\pm u')/2 \tag{2.41}$$

with $\mathbf{v}^{T}=(v^{-},v^{+})$. Then the influence action has the form

$$S_{IF}[z^{a}]=\frac{e^{2}}{\hbar}(\mathbf{v}^{T}\mathbf{G}_{v}^{R}\mathbf{v}+i\mathbf{v}^{T}\mathbf{G}_{v}^{H}\mathbf{v}), \tag{2.42}$$

where

$$\mathbf{G}_{v}^{R}=\begin{pmatrix} G_{11}^{R}-G_{12}^{R}+G_{21}^{R}-G_{22}^{R} & G_{11}^{R}+G_{12}^{R}+G_{21}^{R}+G_{22}^{R}\\ G_{11}^{R}-G_{12}^{R}-G_{21}^{R}+G_{22}^{R} & G_{11}^{R}+G_{12}^{R}-G_{21}^{R}-G_{22}^{R}\end{pmatrix} \tag{2.43}$$

and

$$\mathbf{G}_{v}^{H}=\begin{pmatrix} G_{11}^{H}+G_{12}^{H}+G_{21}^{H}+G_{22}^{H} & G_{11}^{H}-G_{12}^{H}+G_{21}^{H}-G_{22}^{H}\\ G_{11}^{H}+G_{12}^{H}-G_{21}^{H}-G_{22}^{H} & G_{11}^{H}-G_{12}^{H}-G_{21}^{H}+G_{22}^{H}\end{pmatrix}. \tag{2.44}$$

The lowest order cumulant in the Langevin equation, $C_{1,\mu}^{(\eta)}$, is found by evaluating $\delta S_{IF}/\delta z^{\mu-}$, and then setting $z^{-}=0$ and $z^{+}=z$. There are two kinds of terms that arise: those where $\delta/\delta z^{\mu-}$ acts on $\mathbf{v}$, and those where it acts on $G^{R,H}$. For the $\delta\mathbf{v}/\delta z^{\mu-}$ terms, setting $z^{-}=0$ collapses the matrices (2.43) and (2.44) to one term each: only the (1,2) term of G_{v}^{R}, and the (1,1) term of G_{v}^{H} survive. But, setting $z^{-}=0$ gives $v^{-}=0$, $v^{+}=u$, and since the (1,1) term of G_{v}^{H} is proportional to two factors of v^{-}, it also vanishes. When

$\delta/\delta z^{\mu -}$ acts on G^R_v, only the (2,2) term survives[4] (because it is the only term not proportional to a factor of v^-). For similar reasons, the only contributing element of G^H is also its (2,2) term. To evaluate $(\delta G^R_{v(22)}/\delta z^{\mu -})|_{z^-=0}$, we note

$$\frac{\delta}{\delta z^{-\mu}}=\frac{1}{2}\left\{\frac{\delta}{\delta z}-\frac{\delta}{\delta z^{\mu'}}\right\}. \tag{2.45}$$

After a little algebra, we are left with

$$(\delta G^R_{v(22)}/\delta z^{\mu -})|_{z^-=0}=\frac{\delta G^R(z(\tau),z(\tau'))}{\delta z^\mu(\tau)}, \tag{2.46}$$

where the derivative only acts on the $z(\tau)$, and not the $z(\tau')$ argument, in G^R. The same algebra in evaluating the $(\delta G^H_{v(22)}/\delta z^{\mu -})|_{z^-=0}$ term shows that all the factors cancel, and therefore G^H does not contribute to the first cumulant at all. Because the imaginary part of $S_{IF}[z^a]$ does not contribute to the first cumulant, the equations of motion of the mean-trajectory are explicitly real, which is an important consequence of using an initial value formulation like the influence functional (or closed-time-path) method.

The first cumulant, describing *radiation reaction*, is therefore given by

$$\begin{aligned}C^{(\eta)}_{1,\mu}[z,\tau)&=\left.\frac{\delta S_{IF}}{\delta z^{\mu -}(\tau)}\right|_{z^-=0}\\&=\int_{\tau_i}^{\tau}d\tau' e^2\left\{\left(\frac{\delta u(z)}{\delta z^\mu(\tau)}\right)G^R(z(\tau),z(\tau'))u(z(\tau'))\right.\\&\quad\left.+u(z(\tau))\left(\frac{\delta G^R(z(\tau),z(\tau'))}{\delta z^\mu(\tau)}\right)u(z(\tau'))\right\}.\end{aligned} \tag{2.47}$$

This expression is explicitly causal both because the proper time integration is only over values $\tau'<\tau$, and because of the explicit occurrence of the retarded Green's function. Contrary to common perception, the radiation reaction force given by $C^{(\eta)}_{1,\mu}$ is not necessarily dissipative in nature. For instance, we shall see that $C^{(\eta)}_{1,\mu}$ vanishes for uniformly accelerated motion, despite the presence of radiation from a uniformly accelerating charge. In other circumstances, $C^{(\eta)}_{1,\mu}[z,\tau)$ may actually provide an anti-damping force for some portions of the particle trajectory.

[4]Note that for linear theories, unlike the case here, $G^{R,H}$ is not a function of the dynamical variables, but is instead (at most) a function of some predetermined kinematical variables that *a priori* specify the trajectory. Hence, there is no contribution from $\delta G^{R,H}/\delta z^{\mu -}$ type terms.

Next we evaluate the second cumulant. After similar manipulations as above, we find that the second cumulant involves only G^H. We note that the action of $\delta j/\delta z^\mu$ on an arbitrary function is given by

$$\begin{aligned}\int dx\frac{\delta j(x)}{\delta z^\mu(\tau)}f(x)&=e\int dx\int d\tau\frac{\delta}{\delta z^\mu}[(\dot z^2)^{1/2}\\&\quad\times\delta(x-z(\tau))]f(x)\\&=e\left\{\frac{d}{d\tau}\left(\frac{\dot z_\mu}{u(\tau)}\right)+\left(\frac{\dot z_\mu}{u(\tau)}\right)\frac{d}{d\tau}\right.\\&\quad\left.-u(\tau)\frac{\partial}{\partial z^\mu}\right\}f(z(\tau)).\end{aligned} \tag{2.48}$$

Because of the constraint $\dot z^2=1^2$ we can set $u=1$ and $\dot u=0$. Also $\dot z^\mu\ddot z_\mu=0$. Then

$$\begin{aligned}\int dx\frac{\delta j(x)}{\delta z^\mu(\tau)}f(x)&=e\{(\ddot z_\mu+\dot z^\nu\dot z_{[\mu}\partial_{\nu]})\}f(z(\tau))\\&\equiv e\vec{w}_\mu(z)f(z(\tau)).\end{aligned} \tag{2.49}$$

This last expression defines the operator $\vec{w}_\mu(z)$. We have used $d/d\tau=\dot z_\nu\partial^\nu$. The operator $\vec{w}_\mu$ satisfies the identity

$$\dot z^\mu\vec{w}_\mu(z)=(\dot z^\mu\ddot z_\mu+\dot z^\mu\dot z^\nu\dot z_{[\mu}\partial_{\nu]})=0, \tag{2.50}$$

as long as the solution z_μ is on-shell. This identity ensures that neither radiation reaction nor noise-induced fluctuations in the particle's trajectory move the particle off-shell (i.e., the stochastic equations of motion preserve the constraint $\dot z^2=1$).

With these definitions, the noise $\eta^\mu(\tau)$ is given by

$$\begin{aligned}\eta_\mu(\tau)&=e\hbar^{1/2}\vec{w}_\mu(z)\chi(z(\tau))\\&=e\hbar^{1/2}\{(\ddot z_\mu+\dot z^\nu\dot z_{[\mu}\partial_{\nu]})\}\chi(z),\end{aligned} \tag{2.51}$$

and the second-order noise correlator by

$$\begin{aligned}C_2^{(\eta)\mu\nu}[z;\tau,\tau')&=\langle\{\eta^\mu(\tau),\eta^\nu(\tau')\}\rangle\\&=e^2\hbar\vec{w}^\mu(z)\vec{w}^\nu(z')\langle\{\chi(z(\tau)),\chi(z(\tau'))\}\rangle\\&=e^2\hbar\vec{w}^\mu(z)\vec{w}^\nu(z')G^H(z(\tau),z(\tau')).\end{aligned} \tag{2.52}$$

The operator $\vec{w}^\mu(z)$ acts only on the z in $G^H(z,z')$; likewise, the operator $\vec{w}^\nu(z')$ acts only on z'.

This scalar field result is reminiscent of electromagnetism, where the Lorentz force from the (antisymmetric) field strength tensor $F^{EM}_{\mu\nu}$ is $f^{EM}_\mu=\dot z^\nu F^{EM}_{\mu\nu}=\dot z^\nu\partial_{[\mu}A_{\nu]}$. The antisymmetry of $F^{EM}_{\mu\nu}$ implies $\dot z^\mu f^{EM}_\mu=0$. We may define a scalar analog of the antisymmetric (second rank) field strength tensor by

$$21F^{\chi}_{\mu\nu} \equiv \dot{z}_{[\mu}\partial_{\nu]}\chi. \tag{2.53}$$

This shows that the second term on the right-hand side of Eq. (2.51) gives the scalar analog of (a stochastic) electromagnetic Lorentz force: $f^{\chi}_{\mu} = \dot{z}^{\nu}F^{\chi}_{\mu\nu}$. The first term on the right-hand side (RHS) of Eq. (2.51), $\ddot{z}_{\mu}\chi(z)$, does not occur in the treatment of the electromagnetic field. In the scalar-field theory, this term may be thought of as a stochastic component to the particle mass.

The stochastic equations of motion are

$$m\ddot{z}_{\mu} = -\partial_{\mu}V(z) + e^2\int^{\tau} d\tau' \vec{w}_{\mu}(z)G^R(z(\tau),z(\tau')) + e\hbar^{1/2}w_{\mu}(z)\chi(z(\tau)). \tag{2.54}$$

The result (2.54) is formally a set of nonlinear stochastic integrodifferential equations for the particle trajectories $z_{\mu}[\eta;\tau)$. Noise is absent in the classical limit found by the prescription $\hbar \to 0$ (this definition of classicality is formal in that the true semiclassical or classical limit requires coarse-graining and decoherence, and is not just a matter of taking the limit $\hbar \to 0$).

The generalization of Eq. (2.54) to multiparticles is now straightforward. If we reinsert the particle number indices, the first cumulant is

$$C^{\eta}_{1(n)\mu}[z,\tau) = \left.\frac{\delta S_{IF}[z]}{\delta z^{\mu-}_{n}(\tau)}\right|_{z^{-}_{m}=0} = \sum_{m}\int d\tau'_m e^2 \vec{w}_{n\mu}(z_n) \times G^R(z_n(\tau_n),z_m(\tau'_m))u(z_m(\tau'_m)), \tag{2.55}$$

the noise term is given by

$$\eta^{\mu}_{n}(\tau) = e\hbar^{1/2}w^{\mu}_{n}(z_n)\vec{\chi}(z_n(\tau_n)), \tag{2.56}$$

and the noise correlator is

$$C^{\eta\mu\nu}_{2(nm)}[z;\tau_n,\tau'_m) = \langle\{\eta^{\mu}_{n}(\tau_n),\eta^{\nu}_{m}(\tau'_m)\}\rangle = e^2\hbar\vec{w}^{\mu}_{n}(z_n)\vec{w}^{\nu}_{m}(z_m)G^H(z_n(\tau_n),z_m(\tau'_m)). \tag{2.57}$$

The nonlinear multiparticle Langevin equations are therefore

$$m\ddot{z}_{n\mu}(\tau) = -\partial_{\mu}V(z_n(\tau)) + e\hbar^{1/2}\vec{w}_{\mu}(z_n)\chi(z_n(\tau)) + e^2\sum_{m}\int_{\tau_i}^{\tau_f} d\tau' \vec{w}_{\mu}(z_n(\tau))G^R(z_n(\tau),z_m(\tau')). \tag{2.58}$$

The $n \neq m$ terms in Eq. (2.58) are particle-particle interaction terms. Because of the appearance of the retarded Green's function, all of these interactions are causal. The $n = m$ terms are the self-interaction (radiation reaction) forces. The $n \neq m$ noise correlator terms represent nonlocal particle-particle correlations: the noise that one particle sees is correlated with the noise that every other particle sees. The nonlocally correlated stochastic field $\chi(x)$ reflects the correlated nature of the quantum vacuum. From the fluctuation-dissipation relations found in [1] the $n=m$ quantum noise is related to the $n=m$ dissipative forces. Under some, but *not* all, circumstances[5] the $n \neq m$ correlation terms are likewise related to the $n \neq m$ propagation (interaction) terms through a multiparticle generalization of the FDR, called a propagation-correlation relation [37].

We have already noted that the first term in Eq. (2.51) has the form of a stochastic contribution to the particle's effective mass, thus allowing us to define the stochastic mass as

$$m_{\chi} = (m + e\hbar^{1/2}\chi(z)). \tag{2.59}$$

Fluctuations of the stochastic mass automatically preserve the mass-shell condition. Likewise, the effective stochastic force F^{χ}_{μ} satisfies

$$\dot{z}_{\mu}F^{\chi}_{\mu} = \dot{z}_{\mu}\dot{z}_{\nu}\dot{z}^{[\nu}\partial^{\mu]}\chi(z) = \dot{z}_{(\mu}\dot{z}_{\nu)}\dot{z}^{[\nu}\partial^{\mu]}\chi(z) = 0. \tag{2.60}$$

This shows that the stochastic fluctuation-forces preserve the constraint $\dot{z}^2 = 1$, as noted above.

III. THE SEMICLASSICAL REGIME: THE SCALAR-FIELD ALD EQUATION

We have emphasized in [1] that the emergence of a Langevin equation (2.54) or (2.58) presupposes decoherence working to suppress large fluctuations away from the semiclassical trajectories found from the coarse-grained effective action. Hence, the Langevin equation describes the dynamics of the fluctuations $\tilde{z}^{\mu} \equiv z^{\mu} - \overline{z^{\mu}}$ around the semiclassical solution $\overline{z^{\mu}}$. The semiclassical limit is therefore the noise average of the Langevin equation. The stochastic regime is characterized by fluctuations around the semiclassical solution which originate from quantum fluctuations in the field but that are rendered effectively classical and stochastic via decoherence. In our second series of papers we explore the stochastic behavior due to higher-order quantum effects that refurbish the particle's quantum nature.

On a conceptual level, we note that the full quantum theory in the world line path integral formulation involves summing over all world lines of the particle joining the initial and final spacetime positions, z_i and z_f, respectively. In summing over particle histories there is no distinction between, say, a runaway trajectory and any other type of

[5]See [37] and Ref. [1], Sec. IV, for a discussion of this point. In brief, the noise correlation between spacelike separated charges does not vanish owing to the nonlocality of quantum theory, but the causal force terms involving G^R do always vanish between spacelike separated points. These two kinds of terms are only connected through a "propagation-correlation" relation (the multiparticle generalization of an FDR) when one particle is in the other's casual future (or past).

trajectory.[6] They are just different possible fine-grained histories included in the path integral. Furthermore, no meaningful sense of causality is associated with individual fine-grained histories. Any particular path in the sum going through the intermediate point z at world line parameter time τ bears no causal relation to it than going through the point z' at some later parameter time τ'. Only physical observables computed from the full sum over histories need be causal. In terms of decoherent histories analysis the semiclassical trajectory is associated with a coarse-grained, decohered particle trajectory. Unlike fine-grained histories, coarse-grained decohered histories should be causal since they can be associated with physical observables—the mean particle trajectories.

More generally, questions of causality, uniqueness, and runaways arise in regard to the solutions to the equations of motion for the hierarchy of correlation functions of the world line coordinates (the expectation value that gives the semiclassical trajectory being just the lowest order example). It is here that an initial value formulation is crucial because only then are the equations of motion guaranteed to be real and causal [6]. In contrast, equations of motion found from the in-out effective action (a transition amplitude formulation) are generally neither real nor casual. Moreover, the equations of motion for correlation functions must be unique and fully determined by the initial state if the theory is complete.

So the pertinent questions regarding radiation reaction in a quantum to stochastic to semiclassical treatment are the following: What are the equations of motion for the mean and higher-order correlation functions? Are *these* equations of motion causal and well defined? Specifically, is the semiclassical (quantum-averaged) solution unique and physical (e.g. causal and runaway free)? How significant are the quantum fluctuations around the semiclassical trajectory? When does decoherence suppress the probability of observing large fluctuations in the motion? When do the quantum fluctuations assume a classical stochastic behavior? For example, quantum theory predicts that there is always a possibility of observing a trajectory very different than the quantum-average one, including trajectories that might look like runaways, but decoherence should suppress this probability to a negligible amplitude in both the semiclassical and stochastic regimes.

It is only by addressing these questions that the true semiclassical motion may be identified, together with the noise associated with quantum fluctuations which is instrumental for decoherence. With this discussion as our guide, we proceed in two steps. First, in Sec. III A we find the semiclassical limit for the equations of motion and evaluate its causal properties. Second, in Sec. III B we describe the stochastic fluctuations around that limit. Since we are dealing with a nonlinear theory, the fluctuations themselves must depend on the semiclassical limit in a self-consistent fashion. These two steps constitute the full back reaction problem for nonlinear particle field interactions in the semiclassical and stochastic regimes. For more general conceptual discussions on decoherent histories and semiclassical domains, see [10,11,14].

A. Divergences and regularization

Even in classical electromagnetic theory, point particles can couple to arbitrarily short wavelength modes of the field leading to ultraviolet (UV) divergences. In the usual treatment from quantum field theory, infrared (IR) divergences also arise if one makes an artificial distinction between soft and virtual photon emissions. However, the ''in-in'' method adopted here avoids these by summing over all final field states without distinguishing between soft and virtual quanta. From another point of view, IR divergences are an artifact of incorrectly neglecting recoil (radiation reaction) on the particle motion.

In this work we take the effective field theory philosophy as our guide. One does not need to know the detailed structure of the correct high-energy theory because low-energy processes are largely insensitive to them [26] beyond the effective renormalization of system parameters and suppressed high-energy corrections. These observations are true for both classical and quantum theories, though the classical case is more trivial since one does not have the subtleties associated with divergent loop terms from intermediate virtual processes. In this paper we have integrated out all of the massless scalar photons (the field). Since the field is quadratic in the action this one-loop (in the field) result is exact, but the resulting scalar Green's functions have singularities requiring regularization.

Following standard procedure, we now regularize the field Green's functions by suppressing their high-frequency components. In Sec. III B we discuss how this follows from modifying the field-environment's spectral density at high energy in a way that is consistent with the assumed initial state. Different regulators may give different high-energy corrections but should yield the same low-energy physics. Nonetheless, the general qualitative behavior of possible high-energy corrections may be interesting. Such is the case in effective theories [26] where one shows the generic form and scaling of corrections to low-energy phenomena from the high-energy sector without knowing the true high-energy theory other than some basic properties (e.g. symmetries). The point of such an analysis here is twofold. First, to discover whether the qualitative behavior is consistent as they pertain to issues like causality and consistency. Second, to calculate the possible form of corrections resulting from some assumed high-energy theory that might be tested in principle. It is in this spirit that we explore the general form of high-energy corrections in this paper. We emphasize that the low-energy limit is described by those terms in the equations of motion that are independent of the cutoff (after renormalization).

In this section we adopt a somewhat *ad hoc* regulated Green's function chosen because it is convenient and simple. In the Appendix we show that the low-energy limit is iden-

[6] In fact, trajectories in the path integral can be even stranger than the runaways that appear in classical theory. Most paths are nondifferentiable (infinitely rough) and may even include those that are faster than light or backward in time, depending on how the world line path integral is gauge fixed and defined.

tical to that found by employing Pauli-Villars regularization, though the detailed high-energy corrections differ as one would expect. That the low-energy limits agree is not surprising. More importantly, we show that the qualitative behavior of the high-energy corrections is also identical, and therefore the conclusions following from the general analysis of causality and consistency carried out below applies equally well to Pauli-Villars regularization (and any other regularization that shares some basic features).

Here, we choose a regulated retarded Green's function given by the Gaussian form

$$G^R_\Lambda(\sigma)=\theta(z^0(\tau)-z^0(\tau'))\theta(\sigma)\frac{\Lambda^2 e^{-\Lambda^4\sigma^2/2}}{\sqrt{2\pi^3}} \tag{3.1}$$

with support inside and on the future light cone of $z(\tau')$. We have defined

$$\sigma(s)=y^\mu(s)y_\mu(s) \tag{3.2}$$

$$y^\mu(s)\equiv\overline{z}^\mu(\tau)-\overline{z}^\mu(\tau') \tag{3.3}$$

$$s\equiv\tau'-\tau, \tag{3.4}$$

and the average acceleration

$$a_\mu\equiv\ddot{\overline{z}}_\mu(\tau),\quad a^2=\ddot{\overline{z}}^\mu\ddot{\overline{z}}_\mu. \tag{3.5}$$

In the limit $\Lambda\to\infty$,

$$\lim_{\Lambda\to\infty} G^R_\Lambda(\sigma)=\delta(\sigma)/2\pi, \tag{3.6}$$

giving back the unregulated Green's function. Expanding the function σ for small s gives

$$\begin{aligned}\sigma(s)&=y^\mu(s)y_\mu(s)\\&=s^2-a^4s^4/12+\mathcal{O}(s^5),\end{aligned} \tag{3.7}$$

and

$$G^R_\Lambda(s)=\frac{\Lambda^2 e^{-\Lambda^4 s^4/2}}{\sqrt{2\pi^3}}(1+\mathcal{O}(s^6)), \tag{3.8}$$

assuming a timelike trajectory $z^\mu(\tau)$.

B. The initial state

A significant and outstanding problem in quantum field theory is the description of physical states for interacting systems. For example, the *S*-matrix theory applied to charged particle scattering in QED assumes bare charge states at $t=\pm\infty$, which are subsequently dressed by interactions with the electromagnetic field. The justification of this approach (say, in the LSZ reduction formula) can be subtle. In any case one is generally limited to the scattering of asymptotic states where each particle is sufficiently separated so that its interactions with each other are negligible at initial and final times. Under these conditions, the particle-field interactions dress (renormalize) the particles before they come close together and experience mutual interactions. The separation of these two processes in time is essential to the standard scattering formalism.

Ideally, one would like to be able to describe initially correlated particle-field states at a finite initial time t_i. This would require a description of the true interacting particle states—a difficult and unsolved problem (unfortunately the preparation function method, while it does describe a limited class of more generally correlated states, does not solve this problem). In this context we address another aspect of the radiation reaction problem: whether the nonequilibrium quantum dynamics of an initially bare particle state evolving, through interactions with the field, into a dressed (correlated) state is causal and runaway free.

In a density matrix approach, the asymptotic scattering assumption is equivalent to having a factorized initial state of the form

$$\hat\rho(-\infty)=\hat\rho_z(-\infty)\otimes\hat\rho_\varphi(-\infty), \tag{3.9}$$

where $\hat\rho_z(-\infty)$ is the density matrix for bare charged particle with each particle initially well separated. We, on the other hand, assume a factorized density matrix of the form

$$\hat\rho(t_i)=\hat\rho_z(t_i)\otimes\hat\rho_\varphi(t_i), \tag{3.10}$$

at a finite time t_i in the past, where each charged particle is initially well separated from the others. Hence, our neglect of initial particle-particle correlations is physically viable, but the neglect of initial particle-field correlations (e.g. starting with bare states at a finite time in the past) results in time-dependent renormalization effects, coming from the self-interaction of the particle with itself via the field. This dominates the initial dynamics. Particle-particle interactions for initially well-separated particles only come later due to the finite speed of light.

This paper therefore demonstrates that the semiclassical particle dynamics, starting from an initial density matrix of the form (3.10) is causal and runaway free, but still leaves open questions regarding the possible role of initial correlations between the particle and field that exist in any truly physical initial state. Since the state (3.10) quickly evolves to a correlated density matrix, and the memory of the initial state is rapidly lost, one expects that the late time dynamics should be independent of these initial state assumptions (in the same way that asymptotic scattering theory is insensitive to the use of bare initial or final particle states). But since the neglected initial particle-field correlations are, by nature, nonlocal, one might still wonder if causality can be violated in some more subtle way. We do not think this is the case, but offer no proof here. Our analysis yields the semiclassical (or stochastic) equations of motion and thus does not address this more complicated problem.

We can also consider the state (3.10) from another perspective. Such a product state, constructed out of the basis $|\mathbf{z},t\rangle\otimes|\varphi(x)\rangle$, is highly excited with respect to the interacting theory's true ground state. If the theory were not ultraviolet regulated, producing such a state would require infinite energy. This is not surprising since it is impossible to take a fully dressed particle state and strip away all arbitrarily high-energy correlations by some finite energy operation. With a cutoff, Eq. (3.10) becomes a finite energy state and is thus physically achievable, though depending on the cutoff Λ it may still require considerable energy to prepare. Of course, ultimately taking the limit $\Lambda\to\infty$ returns us to the well-known problem of bare states versus physical states of the interacting (dressed) theory. However, we shall later see that the limit $\Lambda\to\infty$ is inconsistent with causal and runaway-free semiclassical motion. This is already known (for example, see [29] where it is shown that the cutoff must be less than the critical value determined from the classical electron radius).

We can estimate a reasonable value for the cutoff Λ as follows. Assume that the particle-field state is fully correlated but that at t_i some finite energy preparation (a measurement) is made on the particle localizing it to within λ_c. A measurement trying to localize a relativistic particle more sharply will result in pair production. Assume this state preparation is such as to produce the density matrix

$$\hat{\rho}(t_i)=\begin{cases}\hat{\rho}_z(t_i)\otimes\hat{\rho}_\varphi(t_i), & \tilde{\varphi}(k)\ \text{ modes with } |k|<\Lambda_c\\ \hat{\rho}_{\text{correlated}}, & \tilde{\varphi}(k)\ \text{ modes with } |k|>\Lambda_c\end{cases} \tag{3.11}$$

at time t_i, where $\Lambda_c=\lambda_c^{-1}$, and $\tilde{\varphi}(k)$ denotes the k^{th} Fourier mode of the field. The correlated state $\hat{\rho}_{\text{correlated}}$ may, for instance, be the vacuum of the interacting theory above Λ_c. Such a state would entail the particle uncorrelated with field modes below the characteristic frequency Λ_c determined by a preparation maximally localizing the particle to within λ_c, but with the particle-field state undisturbed with regard to modes above Λ_c.

With an initial state taking the form (3.11), the reduced density matrix evolution operator J_r will separate into two terms:

$$J_r=J_{|k|<\Lambda_c}+J_{|k|>\Lambda_c}, \tag{3.12}$$

associated with the two pieces in Eq. (3.11). We will assume that the correlated piece, representing the contributions from the initial particle-field state at high frequencies, is unaffected by the finite energy state preparation. We also assume that this sector is in equilibrium. Then, it is plausible that the $J_{|k|>\Lambda_c}$ term, representing the high-frequency field modes in equilibrium with the particle state, does not contribute to the low-frequency dynamics of the particle's semiclassical (and stochastic) motion. One anticipates that $J_{|k|>\Lambda_c}$ contributes only to time-independent renormalization and exponentially small stochastic effects [38]. If this conjecture is true, then the low-energy time-dependent semiclassical and stochastic dynamics are determined by the $J_{|k|<\Lambda_c}$ term.

Furthermore, as long as the particle's semiclassical trajectory has a sufficiently large radius of curvature at every point in its history, it will not produce any significant radiation into the $|k|>\Lambda_c$ sector of the field. In this case, the influence functional only needs to be computed with field modes satisfying $|k|<\Lambda_c$. We see this by writing the influence functional in the form

$$F[j,j']=\langle\varphi_f|\varphi_f'\rangle=\langle U_j\varphi_i|U_{j'}\varphi_i'\rangle, \tag{3.13}$$

where $U_{j,j'}$ is the evolution operator for the field in the presence of a classical current j or j'. If the currents associated with j (determined by the particle trajectory) do not produce radiation in $|k|>\Lambda_c$ field modes, then $\langle\tilde{\varphi}_f(k)|\tilde{\varphi}_f'(k)\rangle=1$ for $|k|>\Lambda_c$.

So it is consistent, given our conjecture that $J_{|k|>\Lambda_c}$ does not effect the low-frequency equations of motion, to compute the influence functional (and hence the coarse-grained effective action) by only integrating over field modes with $|k|<\Lambda_c$. This argument implies that we cut off the field spectral density at $|k|=\Lambda_c$ in computing the noise and dissipation kernels. Because the field retarded and Hadamard Green's functions that appear in the equations of motion are found from the noise and dissipation kernels (see, for example [1]), we obtain the regulated Greens functions

$$G^R(x,y)=\frac{1}{\sqrt{2L^3}}\sum_{k<\Lambda_c}\cos|k|(x^0-y^0)\cos\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}) \tag{3.14}$$

and

$$G^H(x,y)=\frac{1}{\sqrt{2L^3}}\sum_{k<\Lambda_c}\sin|k|(x^0-y^0)\cos\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}). \tag{3.15}$$

When the sum over modes is unrestricted we recover the usual free field Green's functions. But, rather than using the regulated expressions in Eq. (3.14), which are not covariant, we instead use a covariant (regulated) Green's function with the same effective behavior of removing the influence of the $|k|>\Lambda_c$ Fourier components of the field. Hence, the use of regulated Green's functions follows directly from there being a cutoff in the spectral density of the relevant modes of the field environment. For particles maximally localized initially, this entails the identification $\Lambda\approx\Lambda_c=\lambda_c^{-1}$. Note that this is a common choice of cutoff in the literature.

Because our semiclassical (and stochastic) equations of motion describe coarse-grained particle motion on a scale expected to be much larger than λ_c, we will ultimately drop terms proportional to inverse powers of Λ_c. Furthermore, we will see below that the late time behavior of the equations of motion is largely insensitive to the details of the initial state. Finally, consistency requires that the noise kernel should also be regulated in terms of the same effective spectral density.

C. Single-particle scalar-ALD equation

The semiclassical limit is given by the equations of motion for the average trajectory. The regulated semiclassical equations of motion are then

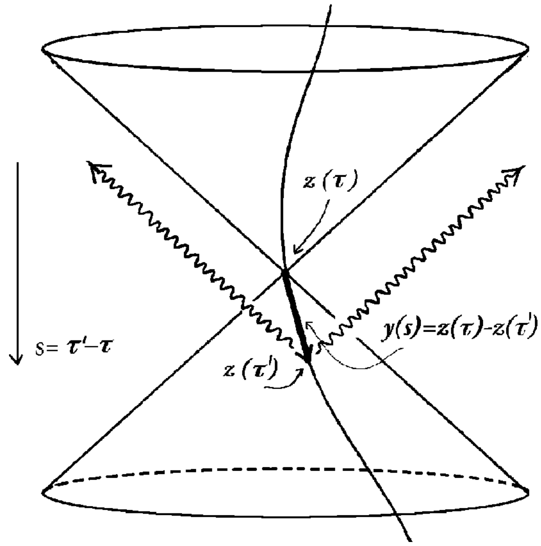

FIG. 1. The trajectory and the light cone of a particle at $z(\tau)$. The radiation reaction force on a particle at $z(\tau)$ depends on the particle trajectory in the interior of its past light cone. For a massless field, the radiation reaction force is local except that the regulated Green's function G^R_Λ smears out the radiation reaction over the past trajectory expanded in a Taylor series around $z(\tau)$, as in Eq. (3.8).

$$m_0 a_\mu(\tau) - \partial_\mu V(\overline{z}) = C^{(\eta)}_{1\mu} = e^2 \int_{\tau_i}^{\tau} d\tau' [a_\mu(\tau) + \dot{\overline{z}}^\nu(\tau)\dot{\overline{z}}_{[\mu}(\tau)\partial_{\nu]}] G^R_\Lambda(\overline{z}(\tau), \overline{z}(\tau')) \tag{3.16}$$

where the ∂_ν acts only on the $\overline{z}(\tau)$ in G^R_Λ, and not the $\overline{z}(\tau')$ term. We add the subscript to m_0 because it is the particle's bare mass, though it is assumed to already include any renormalization from the $|k| > \Lambda_c$ sector of the field. Equation (3.16) describes the full non-Markovian semiclassical particle dynamics, which depend on the past particle history for proper times in the range $[\tau, \tau_i]$.

As an integral equation, Eq. (3.16) depends on the entire history of $\overline{z}(\tau)$ between τ_i and τ. For this reason, the transformation from Eq. (3.16) to a local differential equation of motion requires every derivative of $\overline{z}(\tau)$ [e.g., $d^n\overline{z}(\tau)/d\tau^n$ for all n]. This is the origin of higher time derivatives in the local equations of motion for radiation reaction (classically as well as semiclassically). By integrating out the field variables, we have removed an infinite set of nonlocal (field) degrees of freedom in favor of nonlocal kernels whose Taylor expansions give higher-derivative terms.

We may now transform the integral equation (3.16) to a local differential equation by expanding the functions $y^\mu(s)$ around $s=0$. In Fig. 1 we show a semiclassical trajectory with respect to the light cone at $\overline{z}(\tau)$. Taking τ as fixed, we change integration variables using $ds = d\tau'$. Next, we need the expansions

$$-y^\mu(s) = \dot{\overline{z}}^\mu(\tau)s + a^\mu(\tau)s^2/2 + \dot{a}^\mu(\tau)s^3/6 + \mathcal{O}(s^4), \tag{3.17}$$

$$-\dot{y}^\mu(s) = \dot{\overline{z}}^\mu(\tau) + a^\mu(\tau)s + \dot{a}^\mu(\tau)s^2/2 + \mathcal{O}(s^3), \tag{3.18}$$

$$\sqrt{\sigma(s)} = s - a^2 s^3/24 + \mathcal{O}(s^4), \tag{3.19}$$

and use $\dot{\overline{z}}^2 = 1$, $\dot{\overline{z}}^\mu a_\mu = 0$, and $a^2 = -\dot{\overline{z}}^\mu \dot{a}_\mu$ to simplify. Both $d\overline{z}/d\tau$ and dy/ds are denoted by overdots, $\dot{z}$ and $\dot{y}$, respectively. Using

$$\frac{\partial\sigma(s)}{\partial z^\mu} = 2y^\mu(s) \tag{3.20}$$

$$\frac{d\sigma(s)}{ds} = 2y^\mu \dot{y}_\mu = 2s - a^2 s^3/2 + \mathcal{O}(s^4) \tag{3.21}$$

allows the gradient operator to be expressed as

$$\partial_\mu = \frac{\partial\sigma}{\partial z^\mu}\frac{d}{d\sigma} = 2y_\mu\left(\frac{d\sigma}{ds}\right)^{-1}\frac{d}{ds} = \frac{y_\mu}{y^\nu \dot{y}_\nu}\frac{d}{ds}. \tag{3.22}$$

We define

$$r = \tau - \tau_i \tag{3.23}$$

to be the total elapsed proper time for the particle since the initial time τ_i. Recall that τ_i is defined by $z^0(\tau_i) = t_i$ where t_i is the initial time at which the state $\hat{\rho}(t_i) = \hat{\rho}_z(t_i) \otimes \hat{\rho}_\varphi(t_i)$ is defined.

These definitions and relations let us write the rhs of Eq. (3.16) as

$$e^2 \int_0^r ds \left\{ u^{(1)}_\mu G^R_\Lambda(s) + u^{(1)}_\mu \frac{s}{2}\frac{d}{ds} G^R_\Lambda(s) + u^{(2)}_\mu \frac{s^2}{6}\frac{d}{ds} G^R_\Lambda(s) + \mathcal{O}(s^3) \right\}, \tag{3.24}$$

where

$$u^{(1)}_\mu = a_\mu(\tau) \tag{3.25}$$

$$u^{(2)}_\mu = (\dot{\overline{z}}_\mu a^2 + \dot{a}_\mu) \tag{3.26}$$

and $\mathcal{O}(s^n)$ terms involve up to the $(n+1)$th time derivative of $z(\tau)$. The integrals over s give time-dependent coefficients defined as

$$h^{(0)}(r) = \int_0^r ds\, G^R_\Lambda(s) = \frac{\Lambda}{4\pi}\kappa\left(1 - \frac{\Gamma(1/4, \Lambda^4 r^4/2)}{\Gamma(1/4)}\right) \tag{3.27}$$

and

$$g^{(n)}(r) = \int_0^r ds \frac{s^n}{(n+1)!}\frac{d}{ds}G^R_\Lambda(s) = \frac{32^{-(n-2)/4}}{\pi^{3/2}(n+1)!\,L^{n-2}}\gamma\left(1 + \frac{n}{4}, L^4 r^4/2\right), \tag{3.28}$$

where $\Gamma(x)$ is the gamma function, $\Gamma(x,y)$ is the incomplete gamma function, and $\gamma(x,y)=\Gamma(x)-\Gamma(x,y)$. The constant $\kappa=\Gamma(1/4)/(2^{1/4}\pi^{1/2})\simeq 1.72$ depends on the details of the high-energy cutoff, but is of order one for reasonable cutoffs. For completeness we should also consider higher order terms from the expansion of the regulated Green's function in Eq. (3.8), which start at $\mathcal{O}(s^6)$. These terms involve additional integrals of the form

$$h^{(n)}(r)=\int_0^r ds\frac{s^n}{n!}G_\Lambda^R(s), \quad n\geqslant 6. \tag{3.29}$$

In this analysis we are concerned with the qualitative behavior of these time-dependent coefficients because that behavior is related to the issue of causality. The exact quantitative behavior is dependent on the regularization details but is unobservable at low energy (with large cutoff), as we will show. We discuss this behavior in detail in Sec. IV A, but here we briefly note that the coefficients $g^{(n)}$ and $h^{(n)}$ are bounded for all r, including the late time limit $r\rightarrow\infty$. Also,

$$\lim_{n\rightarrow\infty} h^{(n)}(r)=\lim_{n\rightarrow\infty} g^{(n)}(r)=0. \tag{3.30}$$

Finally, these coefficients scale with Λ as $g^{(n)}(r)\sim\Lambda^{2-n}$ and $h^{(n)}(r)\sim\Lambda^{1-n}$. As a consequence, when Λ is much larger than the other scales of the problem (such as, for example, the inverse radius of curvature of the particle's semiclassical trajectory) the $g^{(n>2)}$ and $h^{(n>1)}$ terms will be suppressed. Assuming this is the case, we will drop such terms in the final equations of motion. However, we keep these terms at this stage so that we may study the qualitative behavior of any higher-derivative effects.

The resulting local equations of motion are

$$\begin{aligned} m_0 a_\mu(\tau)-\partial_\mu V(z) &= C_{1\mu}^{(\eta)} \\ &= e^2 h^{(0)}(r)u_\mu^{(1)}+e^2 g^{(1)}(r)u_\mu^{(1)} \\ &\quad + e^2 g^{(2)}(r)u_\mu^{(2)}+\ \dots . \end{aligned} \tag{3.31}$$

In the Appendix an explicit form of the higher derivative terms is derived, though these are suppressed by the cutoff as we have noted above.

From Eq. (3.25) we deduce that the $u^{(1)}$ terms give time-dependent mass renormalization. Accordingly we define the renormalized mass as

$$\begin{aligned} m(r) &= m_0-e^2h^{(0)}(r)-e^2g^{(1)}(r) \\ &= m_0+\delta m(r). \end{aligned} \tag{3.32}$$

Similarly, the $u^{(2)}$ term is the usual third derivative radiation reaction force from the ALD equation. Thus, the equations of motion may be written as

$$m(r)a_\mu(\tau)-\partial_\mu V(z)=f_\mu^{R.R.}(r) \tag{3.33}$$

where

$$f_\mu^{R.R.}(r)=e^2g^{(2)}(r)(\dot{z}_\mu a^2+\dot{a}_\mu) \tag{3.34}$$

plus higher-derivative terms suppressed by the cutoff.

These semiclassical equations of motion are one of the main results of this paper. They are almost of the ALD form, except for the time-dependent renormalization of the effective mass, the time-dependent radiation reaction force, and the possible presence of higher than third-time derivative terms. The mass renormalization is to be expected, indeed it occurs in the classical derivation as well. It is a consequence of the initial-value formulation with the initial state given at a finite time in the past that these renormalization effects are time dependent. Notice that this result also demonstrates our earlier claim, made in the Introduction, that radiation reaction (RR) vanishes in the semiclassical limit for uniform acceleration. This follows since $\dot{a}_\mu=0\Rightarrow a^2=0$.

IV. RENORMALIZATION, CAUSALITY, AND RUNAWAYS

A. Time-dependent mass renormalization

The time dependence of the effective mass $m(r)$, and particularly the "radiation reaction" coefficient $g^{(2)}(r)$, play an important role in demonstrating the consistency of the semiclassical limit. The initial particle at time t_i $(r=0)$ is assumed, by our choice of initial state, to be fully uncorrelated with the field. This means the particle's initial mass is just the mass m_0. Interactions between the particle and field then "redress" the particle state, one of the consequences being a renormalization of its effective mass. For a particle coupled to a scalar field there are two types of mass renormalization effects, one coming from the $\ddot{z}_\mu\varphi$ interaction, and the other coming from the $\dot{z}^\mu\dot{z}_{[\nu}\partial_{\mu]}\varphi$ interaction. We commented in Sec. II C that the latter interaction is essentially a scalar field version of the electromagnetic coupling, if one defines the scalar field analog of the field strength tensor as shown in Eq. (2.53). We find that this interaction contributes a mass shift at late times of

$$\delta m_1=\kappa(e^2\Lambda/8\pi). \tag{4.1}$$

In the Appendix using the Pauli-Villars regulator one finds this mass shift but with $\kappa\rightarrow c_0=1$. As expected, the precise value of the (unobserved) mass shift is regulator dependent. In [2] it is shown that δm_1 is just one-half the mass shift that is found when the particle moves in the electromagnetic field. This is consistent with the interpretation of the electromagnetic field as equivalent to two scalar fields (one for each polarization). Therefore one expects, and finds, twice the mass renormalization in the electromagnetic case: $\delta m_{EM}=2\,\delta m_1$. (For the same reason we see below that the radiation reaction force, and stochastic noise, are each also reduced by half compared to the electromagnetic case.)

The scalar field, unlike the electromagnetic field, also has a $\ddot{z}_\mu\varphi$ interaction that gives a negative mass shift $\delta m_2=-2\,\delta m_1$. The total mass shift is thus negative. At late times, the renormalized mass is

$$\begin{aligned} m &\equiv \lim_{r\rightarrow\infty} m(r) \\ &= m_0+\delta m_1+\delta m_2 \\ &= m_0-\kappa(e^2\Lambda/8\pi). \end{aligned} \tag{4.2}$$

For a fixed m_0, if the cutoff exceeds $\Lambda > 16\pi m_0/\kappa e^2$, the renormalized mass is negative and the equations of motion are unstable. It is well known that environments with overly large cutoffs can qualitative change the system dynamics. We therefore must assume that the bare mass and cutoff are consistent with $m>0$, which in turn gives a bound on the cutoff:

$$\Lambda < 16\pi m_0/\kappa e^2. \tag{4.3}$$

Thus for the scalar field case the field interaction *reduces* the effective particle mass, whereas in the electromagnetic (vector) field case the field interaction *increases* the effective particle mass (see [2]). This has an interesting consequence if we follow FOL's [29,30] choice of picking the critical cutoff $\Lambda = 16\pi m_0/\kappa e^2$. For the scalar field this would imply that $m=0$. This is a kind of "critical" case balanced between stable and unstable particle motion. While there may be special instances where such behavior is of interest, it does not seem to represent the generic behavior of typical (massive) particles interacting with scalar field degrees of freedom. We will defer further comparison with EM quantum Langevin equations to paper II [2] where the electromagnetic field is treated and the mass renormalization has the more familiar sign.

Consider further the role of causality in the equations of motion. In the preceding paragraph we see that for the long-term motion to be stable (i.e. positive effective particle mass) there is an upper bound on the field cutoff. This is well known, as detailed in [29,30], for example. There, the QLE is assumed to be of the form

$$m\ddot{x} + \int_{-\infty}^{t} dt' \mu(t-t')\dot{x}(t') + V'(x) = F(t), \tag{4.4}$$

where the time integration runs from $-\infty$ to t. Causality is related to the analyticity of $\tilde{\mu}(z)$ [the Fourier transform of $\mu(t-t')$] in the upper half plane of the complex variable z. The traditional QLE of the type (4.4) with the lower integration limit set at $t_i = -\infty$ effectively makes the assumption that the particle has been in contact with its environment far longer than the environment's memory time. In our analysis the lower limit in Eq. (3.16) is τ_i, not $-\infty$. It is the (nonequilibrium) dynamics at early times immediately after the initial particle-field state is prepared where we have something new to say about how causality is preserved, and runaways are avoided.

The proper time dependence of the mass renormalization is shown in Fig. 2. The horizontal axis marks the proper time in units of the cutoff time scale $1/\Lambda$. The vertical scale is arbitrary, depending on the particle bare mass. Notice one unusual feature: the mass shift is not monotonic with time, but instead first overshoots its final asymptotic value. This occurs because there are two competing mass renormalizing interactions that have slightly different time scales. In any case, the time-dependent mass shift is a rapid effect with the final dressed mass m reached within a few $1/\Lambda$. If we set $\Lambda \approx \Lambda_c = \lambda_c^{-1}$, this occurs on the time scale required for light to cross the particle's Compton radius.

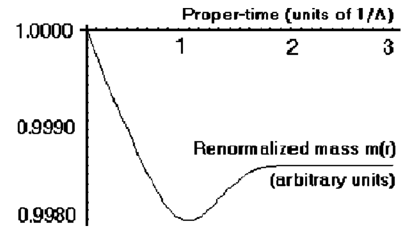

FIG. 2. The time-dependent renormalized mass $m(r)$ of the particle plotted against the proper time, $r=\tau-\tau_i$, elapsed since the initial time t_i at which the factorized initial state of the particle plus field is specified. The actual mass shift depends on the cutoff Λ; the vertical mass units are arbitrary since they depend on the unknown particle bare mass m_0. The renormalization time scale is $\tau_{ren} \sim 1/\Lambda$.

B. Nonequilibrium radiation reaction

The coefficient $g^{(2)}(r)$ determines how quickly the particle is able to build its own self-field after assuming an initially factorized state at t_i. This in turn controls the back reaction on the particle motion, ensuring that it is causal. We find, as is typical of effective theories, that the equations of motion involve higher derivative terms (i.e. $d^n z/d\tau^n$ for $n > 3$) beyond the usual ALD third derivative form (see the Appendix for more detail). These terms are suppressed at low energies (when Λ exceeds the other relevant scales) but there is no reason they must be ruled out absolutely.

Nothing in principle prevents additional higher derivative terms in the interaction Lagrangian as long as they respect the fundamental symmetries (e.g. Lorentz and reparametrization invariance) of our particle-field model. The coupling term we assumed, $e\sqrt{\dot{z}^2}\varphi(z)$, is the only one with a dimensionless coupling constant e, but one could as well have a reparametrization invariant term like

$$\delta\mathcal{L}_{int} = \int d\tau\, e_1 \sqrt{\ddot{z}^{\mu}\ddot{z}_{\mu}/\dot{z}^2}\,\varphi(z), \tag{4.5}$$

where e_1 has mass dimension M^{-1}. In fact, there could be an infinite set of such reparametrization and Lorentz invariant interaction terms in the effective Lagrangian. Our results suggest that even if these terms are not originally present, they may arise as high-energy corrections. Indeed, since one cannot take the limit $\Lambda \to \infty$ without making the renormalized mass negative, we conjecture that any consistent high-energy theory may entail suppressed higher-derivative corrections for the semiclassical motion. If we go further by assuming a cutoff at the Compton wavelength as implied by an initial state of the form (3.11) together with our arguments in Sec. III B, then we expect that additional quantum corrections from the particle will modify the semiclassical equations of motion with higher derivative terms suppressed by Λ_c. But we gain the additional insight that even if such terms are present one expects them not to affect the low energy behavior. This suggests that the ALD equations of motion may be the low-energy semiclassical limit in general for reparametrization and Lorentz invariant particle models. A detailed proof of this conjecture from an effective theory methodology would be interesting.

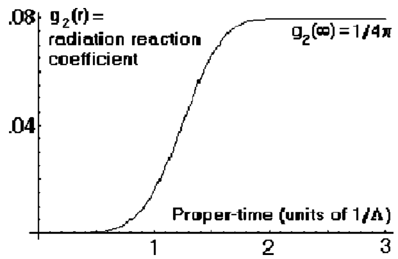

FIG. 3. The time-dependent coefficient $g_2(r)$ that determines the radiation reaction (RR) plotted against the elapsed proper time, $r=\tau-\tau_i$, since the initial time t_i at which the factorized initial state of the particle plus field is specified. The radiation reaction vanishes at $r=0$, but quickly builds to the asymptotic value familiar from the Abraham-Lorentz-Dirac equation on a time scale $1/\Lambda$.

The time scales and relative contributions of these higher-derivative forces are determined by the coefficients $g^{(n)}(r)$. For a fixed cutoff Λ, the late time behavior of the $g^{(n)}(r)$ scale as

$$\lim_{r\to\infty} g^{(n)}(r)=\frac{2^{n/2}\Gamma(1+n/2)}{(2\pi)^{3/2}}\Lambda^{2-n}. \tag{4.6}$$

Therefore, the $g^{(2)}$ term has the late time limit

$$\lim_{r\to\infty} g^{(2)}(r)=\frac{1}{4\pi}, \tag{4.7}$$

which is independent of Λ. Borrowing terminology from effective field theory, this makes the $n=2$ term renormalizable and implies that it corresponds to a ‘‘marginal’’ coupling or interaction. In the same terminology the mass renormalization, scaling as Λ, corresponds to a relevant (renormalizable) coupling. In this vein, the higher-derivative corrections proportional to $g^{(n>2)}$, suppressed by powers of the cutoff scale Λ, correspond to irrelevant couplings.

In Fig. 3 we show the proper time dependence of $g^{(2)}(r)$, with the same horizontal time scale as in Fig. 2. It is evident that the radiation reaction force approaches the ALD form quickly, essentially on the cutoff time. While the radiation reaction force is non-Markovian, the memory time of the field environment is short. The non-Markovian evolution is characterized by the dependence of the coefficients $m(r)$ and $g^{(2)}(r)$ on the initial time τ_i, but become effectively independent of τ_i after $\tau_\Lambda\sim 1/\Lambda$. This parallels the well-known behavior for quantum Brownian motion models found in [4]. The effectively transient nature of the short-time behavior helps justify our use of an initially uncorrelated (factorizable) particle and field state.

Also notice the role played by the particle's elapsed proper time, $r=\tau-\tau_i$. For particles with small average velocities[7] the particle's proper time is roughly the background Minkowski time coordinate t. In this case, the time dependence of the renormalization effects are approximately the same as one would find in the nonrelativistic situation, since $r=(\tau-\tau_i)\approx(t-t_i)$. For rapidly moving particles time dilation can significantly lengthen the observed time scales with respect to the background Minkowski time. This indicates that highly relativistic particles will take longer (with respect to background time t) to equilibrate with the quantum field environment than do more slowly moving particles. Finally, we note that the radiation reaction force is exactly half that found for the electromagnetic field.

C. Causality and early-time behavior

We have shown the emergence at low energies and late times of the ALD radiation reaction force (neglecting other particle structure, such as spin, which gives rise to additional low-energy corrections). The classical ALD equation is plagued with pathologies like acausal and runaway solutions. We now show how the equations of motions in the form [Eqs. (3.33) and (3.34), or more generally in Eq. (A13) involving higher derivatives] preserve the causal nature of the solutions with radiation reaction. To address these questions we now examine the early-time behavior.

As differential equations, Eqs. (3.33) and (3.34) may seem to be unphysical at first sight because of the apparent need to specify initial data

$$\overline{z}^{(n)}_\mu(\tau_i)=d^n\overline{z}_\mu(\tau_i)/d\tau^n \tag{4.8}$$

for $n\geqslant 2$. But the coefficients $g^{(n)}(r)$ satisfy the crucial property that

$$g^{(n)}(0)=0, \tag{4.9}$$

for all n. We see this graphically for $g^{(2)}(r)$ in Fig. 3, and analytically for all $g^{(n)}(r)$ in Eq. (3.28). Consequently, the particle self-force at $\tau=\tau_i$ identically vanishes (to all orders in n), and only smoothly rebuilds as the particle's self-field is reconstituted. Therefore, the initial data at τ_i is fully (and uniquely) determined by

$$ma_\mu(\tau_i)=-\partial_\mu V(\overline{z}(\tau_i)), \tag{4.10}$$

which only requires the ordinary Newtonian initial data. The initial values for the higher derivative terms (e.g. $n\geqslant 2$) in Eq. (3.34) are not independent but are determined iteratively from

$$m\frac{d^n\overline{z}_\mu(\tau_i)}{d\tau_i^n}=-\frac{d^{n-2}}{d\tau_i^{n-2}}\partial_\mu V(\overline{z}(\tau_i)). \tag{4.11}$$

Thus, given the (Newtonian) initial data the equations of motion are determined uniquely for all later times, to any order in n. In the classical ALD equations, one finds runaways even in the case of vanishing external potential, $V=0$. In our case, because $g^{(n)}(0)=0$ for all n, $V=0$ implies that $\overline{z}^{(n)}_\mu(\tau_i)=0$ for all $n\geqslant 2$. With these initial conditions (and $V=0$), the equations of motion (3.33), (3.34) are

$$a_\mu(\tau)=0 \tag{4.12}$$

[7] The small velocity approximation is relative to a choice of reference frame. In [1] we discuss how the choice of an initially factorized state at some time t_i picks out a special frame.

with unique solutions $\overline{z}_\mu(\tau)=\overline{z}_\mu(\tau_i)+(\tau-\tau_i)\dot{\overline{z}}_\mu(\tau_i)$. Runaways do not arise. We find only the physically expected inertial solution.

Finally, we note that the vanishing of the time-dependent coefficients in the modified ALD equation at the initial time t_i is a consequence of the factorized initial state. The initial behavior of our equations then describes the time-dependent ''redressing'' of the particle by the low-energy modes of the field. This happens smoothly in such a way that causality is not violated. If it were technically feasible to treat a more generally correlated initial state we can probe new interesting behavior arising from the particle's correlations with the longer wavelengths of the field. We expect that time-dependent (nonequilibrium quantum theory) effects similar to those illustrated in this paper will also act to preserve causality of the semiclassical equations of motion in this more complicated yet realistic circumstance. Of course, for completely arbitrary initially correlated particle-field states, one must carefully address the question of when and how a semiclassical equations of motion obtains, and what it means.

V. THE STOCHASTIC REGIME AND THE ALD-LANGEVIN EQUATION

A. Single particle stochastic limit

The nonlinear Langevin equation in Eq. (2.54) shows a complex relationship between noise and radiation reaction. A nonlinear Langevin equation similar to these have been found for stochastic gravity by Hu and Matacz [35] describing the stochastic behavior of the gravitational field in response to quantum fluctuations of the stress-energy tensor. We now apply the results in Eq. (2.38) giving the linearized Langevin equation for fluctuations around the mean trajectory. The free kinetic term is given by

$$\begin{aligned}\int d\tau' \tilde{z}^\nu(\tau')\left(\frac{\delta^2 S_z[\overline{z}]}{\delta\overline{z}^\mu(\tau)\,\delta\overline{z}^\nu(\tau')}\right) &= m(r)\int d\tau' \tilde{z}^\nu(\tau') g_{\mu\nu}\frac{d^2}{d\tau^2}\delta(\tau-\tau')\\ &= m(r)\frac{d^2\tilde{z}_\mu(\tau)}{d\tau^2}, \end{aligned}\tag{5.1}$$

where we have included the time-dependent mass renormalization effect in the kinetic term. The external potential term is

$$\int d\tau' \tilde{z}^\nu(\tau')\left(\frac{\delta^2 V(\overline{z})}{\delta\overline{z}^\mu(\tau)\,\delta\overline{z}^\nu(\tau')}\right)=\tilde{z}_\mu(\tau)\frac{\partial^2 V(\overline{z})}{\partial\overline{z}^{\mu 2}}, \tag{5.2}$$

hence, the second derivative of V acts as a force linearly coupled to $\tilde{z}$. The dissipative term for $\tilde{z}$ involves the (functional) derivative of the first cumulant [see Eq. (3.31)] with the mass renormalization $(n=1)$ piece removed, as it has already been included in the kinetic term (5.1) above. We therefore need the (functional) derivative with respect to $\overline{z}_\mu$ of the radiation reaction term

$$f_\mu^{R.R.}=e^2 g^{(2)}(r)(\dot{\overline{z}}_\mu a^2+\dot{a}_\mu). \tag{5.3}$$

To $\mathcal{O}(\tilde{z})$, we have

$$\mathcal{F}_\mu=\int d\tau' \tilde{z}_\nu(\tau')\frac{\delta f_\mu^{R.R.}(\overline{z})}{\delta\overline{z}_\nu(\tau')}. \tag{5.4}$$

$\mathcal{F}_\mu$ is the dissipation force that appears in the linearized Langevin equation. While it is derived from the radiation reaction force $f_\mu^{R.R.}$, it is important to distinguish between the role of $f_\mu^{R.R.}$ at the semiclassical level and $\mathcal{F}_\mu$ at the stochastic level.

We note that $\mathcal{F}_\mu$ is a function of the derivatives $\tilde{z}_\mu^{(1,2,3)}$, therefore we have

$$\begin{aligned}\mathcal{F}_\mu &= \frac{e^2}{2}g^{(2)}(r)[a^2(g_{\mu\nu}-\dot{\overline{z}}_\mu\dot{a}_\nu)\dot{\tilde{z}}^\nu]+(g_{\mu\nu}-\dot{\overline{z}}_\mu\dot{\overline{z}}_\nu)\dddot{\tilde{z}}^\nu]\\ &= \frac{e^2}{2}g^{(2)}(r)(S_{\mu\nu}\dot{\tilde{z}}^\nu+R_{\mu\nu}\dddot{\tilde{z}}^\nu), \end{aligned}\tag{5.5}$$

where

$$R_{\mu\nu}(\overline{z})\equiv(g_{\mu\nu}-\dot{\overline{z}}_\mu\dot{\overline{z}}_\nu) \tag{5.6}$$

$$S_{\mu\nu}(\overline{z})\equiv a^2(g_{\mu\nu}-\dot{\overline{z}}_\mu\dddot{\overline{z}}_\nu). \tag{5.7}$$

The late time $(\Lambda r\gg 1)$ single particle Langevin equations are thus

$$\eta(\tau)=m\ddot{\tilde{z}}_\mu(\tau)+\tilde{z}_\nu\frac{\partial V(\overline{z})}{\partial\overline{z}^\mu\partial\overline{z}_\nu}-\frac{e^2}{8\pi}(S_{\mu\nu}\dot{\tilde{z}}^\nu+R_{\mu\nu}\dddot{\tilde{z}}^\nu). \tag{5.8}$$

Equation (5.8) is the second main result of this paper. They are to be used together with the semiclassical equations of motion in Eq. (3.34) which self-consistently determines $S_{\mu\nu}$ and $R_{\mu\nu}$. Just as was the case for the semiclassical equations of motion, the vanishing of $g^{(2)}(r)$ for $r=0$ implies that the initial data for the stochastic equations of motion are just the ordinary kind involving no higher than first order derivatives.

The noise $\eta_\mu(\tau)$ is given by

$$\eta_\mu(\tau)=\vec{w}_\mu\chi(\overline{z}(\tau))=e(\ddot{\overline{z}}_\mu(\tau)+\dot{\overline{z}}^\nu\dot{\overline{z}}_{[\nu}\partial_{\mu]})\chi(\overline{z}(\tau)), \tag{5.9}$$

where the stochastic field is evaluated using the semiclassical solution $\overline{z}$. The correlator $\langle\eta_\mu(\tau)\eta_\nu(\tau')\rangle$ [see Eq. (5.11)] is then found using the field correlator:

$$\langle\{\chi(\overline{z}(\tau)),\chi(\overline{z}(\tau'))\}\rangle=\hbar G^H(\overline{z}(\tau),\overline{z}(\tau')). \tag{5.10}$$

Equation (5.9) shows that the noise experienced by the particle depends not just on the stochastic properties of the quantum field, but on the mean-solution $\overline{z}(\tau)$. For instance, the $a_\mu\chi(z)=\ddot{\bar{z}}_\mu\chi(z)$ term in Eq. (5.9) gives noise that is proportional to the average particle acceleration. The second term in Eq. (5.9) depends on the antisymmetrized combination $\dot{\bar{z}}_{[\mu}\partial_{\nu]}\chi(\bar{z})$. It follows immediately from $\langle\chi\rangle=0$ that $\langle\eta(\tau)\rangle=0$.

From Eq. (5.11) we see that the noise correlator is

$$\begin{aligned}\langle\eta_\mu(\tau)\eta_\nu(\tau')\rangle&=\vec{w}_\mu(\tau)\vec{w}_\nu(\tau')\langle\chi(\overline{z(\tau)})\chi(\overline{z(\tau')})\rangle\\&=(e^2\hbar/c^4)(\ddot{\bar{z}}_\mu+\dot{\bar{z}}^\lambda\dot{\bar{z}}_{[\mu}\partial^z_{\lambda]})(\ddot{\bar{z}}'_\nu\\&\quad+\dot{\bar{z}}^{\rho\prime}\dot{\bar{z}}'_{[\nu}\partial^{z'}_{\rho]})G^H(\overline{z,z'})\\&=(4\pi\lambda_c r_c)\left(\ddot{\bar{z}}_\mu+\dot{\bar{z}}^\lambda\frac{\dot{\bar{z}}_{[\mu}y_{\lambda]}}{y^2}\frac{d}{d\sigma}\right)\\&\quad\times\left(\ddot{\bar{z}}'_\nu+\dot{\bar{z}}^{\rho\prime}\frac{\dot{\bar{z}}'_{[\nu}y_{\lambda]}}{y^2}\frac{d}{d\sigma}\right)G^H(\sigma).\end{aligned}\tag{5.11}$$

The noise (when the field is initially in its vacuum state) is nonlocal (colored), reflecting the highly correlated nature of the quantum vacuum. Only in the high temperature limit does the noise become approximately local (white). Notice that the noise correlator inherits an implicit dependence on the initial conditions at τ_i through the time-dependent equations of motion for $\bar{z}$. But when $r\Lambda\gg1$, the equations of motion become effectively independent of the initial time. The noise given by Eq. (5.11) is not stationary except for special cases of the solution $\overline{z(\tau)}$.

Finally, we understand the precise meaning of FDRs for nonlinear particle-field systems. Clearly, vacuum fluctuations (VF) and radiation reaction (RR) cannot be directly related, as the vanishing of RR in the semiclassical limit demonstrates. It is the dissipative force $\mathcal{F}_\mu$ that is related to vacuum fluctuations through a fluctuation-dissipation relation [1].

B. Multiparticle stochastic limit and stochastic Ward identities

It is a straightforward generalization to construct multiparticle Langevin equations. The two additional features are particle-particle interactions, and particle-particle correlations. The semiclassical limit is modified by the addition of the terms

$$e^2\sum_{m\neq n}\int_{\tau_i}^{\tau_f}d\tau_m\vec{w}^\mu_n[\bar{z}_n(\tau_n)G^R_\Lambda(\bar{z}_n(\tau_n),\bar{z}_m(\tau_m))].\tag{5.12}$$

The use of the regulated G^R_Λ is essential for consistency between the radiation that is emitted during the regime of time-dependent renormalization and radiation reaction; it ensures agreement between the work done by radiation reaction and the radiant energy. A field cutoff Λ implies that the radiation wave front emitted at τ_i is smoothed on a time scale Λ^{-1}.

We note one significant difference between the single particle and multiparticle theories. For a single particle, the dissipation is local (when the field is massless), and the semiclassical limit (obtaining when $r\Lambda\gg1$) is essentially Markovian. The multiparticle theory is non-Markovian even in the semiclassical limit because of multiparticle interactions. Particle A may indirectly depend on its own past state of motion through the emission of radiation interacting with another particle B, which in turn, emits radiation that reinfluences particle A at some point in the future. The nonlocal field degrees of freedom stores information (in the form of radiation) about the particle's past. Only for a single timelike particle in flat space without boundaries is this information permanently lost insofar as the particle's future motion is concerned. These well-known facts make the multiparticle behavior extremely complicated.

We may evaluate the integral in Eq. (5.12) using the identity

$$\delta((\bar{z}_n-\bar{z}_m)^2)=\delta(\tau-\tau^*)/2(\bar{z}_n-\bar{z}_m)^\nu\dot{\bar{z}}_{m\nu},\tag{5.13}$$

where τ^* is the time when particle m crosses particle n's past light cone. (The regulated Green's function would spread this out over approximately a time Λ^{-1}.) Defining $\bar{z}_{nm}\equiv\bar{z}_n-\bar{z}_m$, we use

$$\begin{aligned}\int d\tau_m\partial^\lambda_m G^R(\bar{z}_n,\bar{z}_m)&=-\int d\tau_m\frac{dG^R}{ds}\frac{(\bar{z}_{nm})^\lambda}{(\bar{z}_{nm})^\alpha\dot{\bar{z}}_{m\alpha}}\\&=-\left[\frac{G^R(\bar{z}_{nm})^\lambda}{(\bar{z}_{nm})^\alpha\dot{\bar{z}}_{m\alpha}}\right]\\&\quad+\int dsG^R\frac{d}{ds}\left[\frac{(\bar{z}_{nm})^\lambda}{(\bar{z}_{nm})^\alpha\dot{\bar{z}}_{m\alpha}}\right]\\&=\frac{1}{2(\bar{z}_{nm})^\nu\dot{\bar{z}}_{m\nu}}\frac{d}{ds}\left[\frac{(\bar{z}_{nm})^\lambda}{(\bar{z}_{nm})^\alpha\dot{\bar{z}}_{m\alpha}}\right].\end{aligned}\tag{5.14}$$

The particle-particle interaction terms are then given by

$$\begin{aligned}&e^2\sum_{m\neq n}(a^\mu_n+\dot{\bar{z}}_\beta\dot{\bar{z}}^{[\mu}\partial^{\beta]})G^R(\bar{z}_n,\bar{z}_m)\\&=e^2\sum_{m\neq n}\left(\frac{a^\mu_n}{[(\bar{z}_{nm})^\nu\dot{\bar{z}}_{m\nu}]}+\frac{\dot{\bar{z}}^\mu_n\dot{\bar{z}}_{n\lambda}-\delta^\mu_\lambda}{2(\bar{z}_{nm})^\nu\dot{\bar{z}}_{m\nu}}\right.\\&\quad\left.\times\frac{d}{d\tau_m}\left[\frac{(\bar{z}_{nm})^\lambda}{(\bar{z}_{nm})^\alpha\dot{\bar{z}}_{m\alpha}}\right]_{\tau_m=\tau^*}\right),\end{aligned}\tag{5.15}$$

where, as before, the $\vec{w}^\mu_n(\bar{z}_n)$ acts only on the $\bar{z}_n$, and not the $\bar{z}_m(\tau^*)$. Equation (5.15) is just the scalar analog of the Liénard-Wiechert forces. They include both the near-field and far-field effects.

The long range particle-particle terms in the Langevin equations are found using Eq. (2.58), so we have the additional Langevin term for the nth particle, given by

$$e^2\sum_{m\neq n}\int d\tau_n \tilde{z}_n^\nu \partial_\nu^n\Bigg\{\frac{a_n^\mu}{[(\overline{z_{nm}})^\alpha \dot{\overline{z}}_{m\alpha}]} - \frac{R_{n,\mu\lambda}}{2(\overline{z_{nm}})^\alpha \dot{\overline{z}}_{m\alpha}}\frac{d}{d\tau_m}\left[\frac{(\overline{z_{nm}})^\lambda}{(\overline{z_{nm}})^\beta \dot{\overline{z}}_{m\beta}}\right]\Bigg\}_{\tau_m=\tau^*}. \tag{5.16}$$

Equation (5.16) contains no third (or higher) derivative terms. The multiparticle noise correlator is

$$\langle\{\eta_n^\mu(\tau_n)\eta_m^\nu(\tau_m)\}\rangle = \vec{w}_n^\mu(\tau_n)\vec{w}_m^\nu(\tau_m)\langle\{\chi(\overline{z_n})\chi(\overline{z_m})\}\rangle. \tag{5.17}$$

In general, particle-particle correlations between spacelike separated points will not vanish as a consequence of the nonlocal correlations implicitly encoded by G^H. For instance, when there are two oppositely charged particles which never enter each other's causal future, there will be no particle-particle interactions mediated by G^R; however, the particles will still be correlated through the field vacuum via G^H. This shows that it will not be possible to find a generalized multiparticle fluctuation-dissipation relation under all circumstances [37].

We now briefly consider general properties of the stochastic equations of motion. Notice that the noise satisfies the identity

$$\dot{z}_n^\mu(\tau)\eta_{n\mu}(\tau) = \dot{z}_n^\mu(\tau)\vec{w}_{n\mu}(z_n)\chi(z_n) = 0, \tag{5.18}$$

which follows as a consequence of $\dot{z}_n^\mu \vec{w}_{n\mu}=0$, for any on-shell solutions $\dot{z}_n^\mu$. This is an essential property since the particle fluctuations are real, not virtual. We may use Eq. (5.18) to prove what might be called (by analogy with QED) stochastic Ward-Takahashi identities [39]. The n-point correlation functions for the particle-noise are

$$\begin{aligned}&\langle\{\eta_{\mu_1}(\tau_1)\dots\eta_{\mu_i}(\tau_i)\dots\eta_{\mu_n}(\tau_n)\}\rangle\\ &= \vec{w}_{\mu_1}(z_1)\dots\vec{w}_{\mu_i}(z_i)\dots\vec{w}_{\mu_n}(z_n)\\ &\quad\times\langle\{\chi(z_1)\dots\chi(z_i)\dots\chi(z_n)\}\rangle,\end{aligned} \tag{5.19}$$

with each $\vec{w}_{\mu_i}(z_i)$ acting on the corresponding $\chi(z_i)$. These correlation functions may both involve different times along the world line of one particle (i.e. self-particle noise), and correlations between different particles. From Eqs. (5.19) and (5.18) follow "Ward" identities:

$$\dot{z}_i^{\mu_i}\langle\eta_{\mu_1}(\tau_1)\dots\eta_{\mu_i}(\tau_i)\dots\eta_{\mu_n}(\tau_n)\rangle = 0, \quad \text{for all } i \text{ and } n. \tag{5.20}$$

The contraction of on-shell momenta ($\dot{z}^\mu$) with the stochastic correlation functions always vanishes. These identities are fundamental to the consistency of the relativistic Langevin equations.

VI. EXAMPLE: FREE PARTICLES IN THE SCALAR FIELD VACUUM

As a concrete example, we find the Langevin equations for $V(z)=0$ and the scalar field initially in the vacuum state. Using the stochastic equations of motion we can address the question of whether a free particle will experience Brownian motion induced by the vacuum fluctuations of the scalar field. When $V(z)=0$, we immediately see from the semiclassical equations that $\ddot{\overline{z}}^\mu(\tau)=a_0^\mu=$const: the radiation reaction force identically vanishes. The fact that $g_n(0)=0$ uniquely fixes $a_0^\mu=0$ due to the initial data boundary conditions for the higher derivative terms (4.11); hence, there are no runaway solutions characterized by a nonzero constant acceleration. Writing $\dot{\overline{z}}^\mu(\tau)=v^\mu$ and $\overline{z}^\mu(\tau)=sv^\mu$, where v^μ is a constant spacetime velocity vector satisfying the (mass-shell) constraint $v^2=1$, we conclude that an inertial particle moving in accordance with the semiclassical solution neither radiates nor experiences radiation reaction. Using $\sigma^2=v^2s^2=s^2=(\tau-\tau')^2$, we find from Eq. (5.8) the linearized Langevin equation

$$m\ddot{\tilde{z}}_\mu(\tau)+\frac{e^2}{2}g^{(2)}(r)R_{\mu\nu}\ddot{\tilde{z}}^\nu = \eta_\mu(\tau). \tag{6.1}$$

The noise correlator is found from Eq. (5.11) to be

$$\begin{aligned}&\langle\eta_\eta(\tau)\eta_\nu(\tau')\rangle\\ &= e^2\hbar\left(a_\mu - \frac{d}{d\tau}\left[\dot{\overline{z}}^\lambda\frac{\dot{\overline{z}}_{[\mu}y_{\lambda]}}{y^\alpha\dot{y}_\alpha}\left(\frac{d\sigma^2}{d\tau}\right)^{-1}\right]\right)\\ &\quad\times(4\pi^2\sigma^2)^{-1}\left(a'_\nu + \frac{d}{d\tau'}\left[\dot{\overline{z}}^{\rho\prime}\frac{\dot{\overline{z}}'_{[\nu}y_{\lambda]}}{y^\beta\dot{y}_\beta}\left(\frac{d\sigma^2}{d\tau'}\right)^{-1}\right]\right)\end{aligned} \tag{6.2}$$

$$\begin{aligned}&= e^2\hbar\left(\frac{d}{d\tau}\left[v^\lambda\frac{v_{[\mu}v_{\lambda]}s}{2v^\alpha v_\alpha s^2}\right]\right)\frac{1}{4\pi^2s^2}\\ &\quad\times\left(\frac{d}{d\tau'}\left[v^\rho\frac{v_{[\nu}v_{\lambda]s}}{-2v^\beta v_\beta s^2}\right]\right),\end{aligned} \tag{6.3}$$

where we have substituted the vacuum Hadamard function for the field-noise correlator $\langle\chi\chi'\rangle$. The antisymmetric combination $v_{[\mu}v_{\nu]}$ vanishes, and hence

$$\langle\eta_\eta(\tau)\eta_\nu(\tau')\rangle = 0 \quad \text{for } \ddot{\overline{z}}^\mu=0. \tag{6.4}$$

The origin of this at first surprising result is not hard to find. First, the stochastic mass terms $\ddot{\overline{z}}\chi$ vanish since the average acceleration is zero, but why should the other terms in the correlator vanish? From Eq. (6.3), we see that the second term in the particle-noise correlator involves $\dot{\overline{z}}^\lambda\dot{\overline{z}}_{[\mu}\partial^z_{\lambda]}G^H(\overline{z},\overline{z'})$. But the gradient of the vacuum Hadamard kernel (which is a function of σ) satisfies $\partial_\mu G^H(\sigma)\propto y_\mu$, and is therefore always in the direction of the spacetime vector connecting $\overline{z}(\tau)$ and $\overline{z}(\tau')$. However, for the inertial particle, $y_\mu \propto v_\mu$, and thus the antisymmetrized term, $\dot{\bar{z}}_{[\mu}\partial_{\nu]}G^H(\bar{\sigma}) \propto v_{[\mu}v_{\nu]}G^H$, always vanishes. It therefore follows that a scalar field with the coupling that we have assumed does not induce stochastic fluctuations in a free particles trajectory at this linearized Langevin level. However, there are noise fluctuations when the field is not in the vacuum state (e.g. a thermal quantum field), and/or when the particle is subject to external forces that make its average acceleration nonzero (e.g. an accelerated particle).

We therefore find that the free particle fluctuations (in this special case) obey the equations of motion for $\tilde{z}_\mu$,

$$m\ddot{\tilde{z}}_\mu(\tau) + \frac{e^2}{2} g^{(2)}(r) R_{\mu\nu} \ddot{\tilde{z}}^\nu = 0. \tag{6.5}$$

Equation (6.1) has the unique solution

$$\ddot{\tilde{z}}_\mu(\tau) = 0 \tag{6.6}$$

after recalling that $g^{(2)}(0) = 0 \Rightarrow \ddot{\tilde{z}}^\mu(0) = 0$. The same initial time behavior that gives unique (runaway-free) solutions for $\bar{z}$ (the semiclassical solution) apply to $\tilde{z}$ (the stochastic fluctuations).

Let us comment on the difference of our result from that of [40]. First, the dissipation term in our equations of motion is relativistically invariant for motion in the vacuum, and vanishes for inertial motion. The equations of motion in [40] are not relativistically invariant; the authors of [40] conclude that a particle moving through the scalar vacuum will experience a dissipation force proportional to its velocity which is in direct contradiction with experience. Their result comes from an incorrect treatment of the retarded Green's function in $1+1$ spacetime dimensions.

VII. DISCUSSION

We now summarize the main results of this paper, what follows from here, and point out areas for potential applications of theoretical and practical values.

Perhaps the most significant difference between this and earlier work in terms of approach is the use of a particle-centric and initial value (causal) formulation of relativistic quantum field theory, in terms of world-line quantization and influence functional formalisms, with focus on the coarse-grained and stochastic effective actions and their derived stochastic equations of motion. This is a general approach whose range of applicability extends from the full quantum to the classical regime, and should not be viewed as an approximation scheme valid only for the semiclassical. There exists a great variety of physical problems where a particle-centric formulation is more adept than a field-centric formulation. The initial value technique with full back reaction ensures the self-consistency and causal behavior of the semiclassical and stochastic equations of motion. Our ALD and ALD-Langevin equations for relativistic particles in a scalar quantum field are pathology free.

For further development, in [2] we shall extend these results to spinless particles moving in the quantum electromagnetic field, where we need to deal with the issues of gauge invariance. This is a problem of considerable practical relevance. In a second series of papers [41], we use the same conceptual framework and methodology but go beyond the semiclassical and stochastic regimes to incorporate the full range of quantum phenomena, addressing questions such as the role of dissipation and correlation in charged particle pair-creation, and other quantum relativistic processes. Theoretically, by interpreting a particle's spacetime properties (e.g. its location in time and space) as effectively microscopic "clocks and position markers" (e.g. events) we see that a quantum-covariant description of particle world line motion is not just particle dynamics, it is the quantum dynamics of spacetime events. For example, the *time* of the particle is treated as a quantum variable on equal footing with position, and hence a question like localization applies not just to localization in space, but localization in time. It would be natural to apply this approach to subjects like "time of flight," and questions regarding decoherence of time (variables), together with space variables [e.g. the emergence of (local) time for particles]. At the level of this paper we have derived a particle's time as a stochastic variable.

In conclusion, the self-consistent treatment of back reaction from quantum field activities on a charged particle is an essential yet often neglected factor in many problems dealing with charged particle motion. Our approach is appropriate for any situation where particle motion (as opposed to field properties) is the center of attention. The ALD-Langevin equations for charged particle motion in the quantum electromagnetic field derived in [2] is of particular importance for quantum beam dynamics and heavy-ion physics. Our methods can be applied to relativistic charged particle motion not only in charged particle beams as in accelerators [2,20] and free-electron lasers as in ion optics, but also in strong fields such as particles moving in matter (crystals) or in plasma media as in astrophysical contexts [42].

ACKNOWLEDGMENTS

This work is supported by NSF grant PHY98-00967 and DOE grant DEFG0296ER40949.

APPENDIX: PAULI-VILLARS REGULATOR AND HIGHER DERIVATIVE TERMS IN THE EQUATIONS OF MOTION

The regulated Green's function (3.1) is somewhat *ad hoc*, though perfectly acceptable from an effective theory point of view since it only implies a modification of the unobserved high-energy modes of the field. The strength with which Eq. (3.1) falls off as a function of σ made the analysis straightforward. Alternatively, we could have employed a hard cutoff in the spectrum of the field modes [e.g. Eq. (3.14)], but this conflicts with the desire to produce a covariant formulation.

In this appendix we show that Pauli-Villars regularization gives the same semiclassical limit and qualitative behavior as the Gaussian regulator choice (3.1). This shows how the low-energy physics is insensitive to the choice of the regulator. Applying Pauli-Villars regularization is more subtle, however, as we will show now. The problem is that the Pauli-Villars regulated Green's function does not really fall off that fast in configuration space, and hence the integrals involving σ^n with $n>1$, corresponding to the effect of higher derivative terms (beyond the $\dot{a}$ term), do not converge. While dimensional analysis shows that these terms are suppressed by inverse powers of the cutoff Λ, and thus are presumably negligible at low energies, one might still wonder if there is some long-term inconsistency resulting from Pauli-Villars regularization.

In configuration space, the Pauli-Villars regulated retarded Green's function is

$$G_\Lambda^{P.V.}(\sigma)=\frac{\Lambda J_1(\Lambda\sigma)}{4\pi\sigma} \tag{A1}$$

in the forward light cone. Our goal is to evaluate the integral term in Eq. (3.16). In terms of the variables s and r this involves

$$e^2\int_0^r ds[a_\mu(\tau)+\dot{\bar{z}}^\nu(\tau)\dot{\bar{z}}_{[\mu}(\tau)\partial_{\nu]}]G_\Lambda^{P.V.}(\sigma). \tag{A2}$$

It follows from the semiclassical solution being a timelike trajectory that $\sigma=\sigma(s)$ can be inverted as a function of s. To change variables from s to σ in Eq. (A2) we need $d\sigma/ds$. The Taylor series for $\sigma(s)$ has coefficients involving time derivatives of $\bar{z}^\mu(\tau)$ [e.g. Eq. (3.19)]. Inversion of this series gives

$$\begin{aligned}ds(\sigma)/d\sigma&=\sum_{n=0} s_n\sigma^n/n!\\&=1+a^2\sigma^2/8+a^\mu\dot{a}_\mu\sigma^3/6+\cdots.\end{aligned} \tag{A3}$$

In general, the s_n coefficient is a scalar involving $n+2$ time derivatives.

Next, the gradient operator may be written

$$\partial_\nu=-\frac{y_\nu}{\sigma}\frac{d}{d\sigma}, \tag{A4}$$

and the Taylor series for y_ν/σ gives

$$\begin{aligned}-y_\nu/\sigma&=\sum_{m=0} u_\nu^{(m)}\sigma^m\\&=\dot{\bar{z}}_\nu-\sigma a_\nu/2+\sigma^2(\dot{\bar{z}}_\nu a[2]+4\dot{a}_\nu)/24-\sigma^3(a^2a_v\\&\quad-\dot{\bar{z}}_\nu a^\mu\dot{a}_\mu+\ddot{a}_\nu)/24+\ \ldots\ .\end{aligned} \tag{A5}$$

The $u_\nu^{(m)}$ are also scalars constructed involving terms with $m+2$ and $m+3$ time derivatives. Changing integration variables and rearranging we then have

$$\begin{aligned}e^2\sum_{n=0}^{\infty}&\Bigg\{s_na_\mu\left(\int_0^{\sigma_r}d\sigma\frac{\sigma^n}{n!}G_\Lambda(\sigma)\right)\\&+\sum_{m=0}^{n}\dot{\bar{z}}^\nu\dot{\bar{z}}_{[\mu}u_{\nu]}^{(m+1)}s_{(n-m)}l_{n,m}\\&\times\left(\int_0^{\sigma_r}d\sigma\frac{\sigma^{n+1}}{(n+1)!}\frac{d}{d\sigma}G_\Lambda(\sigma)\right)\Bigg\},\end{aligned} \tag{A6}$$

where

$$l_{n,m}=\frac{(n+1)!}{(n-m)!(m+1)!}. \tag{A7}$$

The upper limit of integration is $\sigma_r\equiv\sigma(r)$, with $\lim_{r\to\infty}\sigma(r)\to\infty$.

Performing the σ integrals with the Gaussian regulator form (3.1) gives our earlier results generalized to all orders. The Pauli-Villars regulator $[G_\Lambda^{P.V.}(\sigma)]$ gives integrals that do not converge for $\sigma^{n>1}$. But one may show that $G_\Lambda^{P.V}(\sigma)\times f(\alpha\sigma)$ does converge even in the limit $\alpha\to0$, and the result is independent of the choice of function $f(\alpha\sigma)$ so long as f is sufficiently regular, goes to zero exponentially with $\sigma\to\infty$, and goes to 1 as $\alpha\to0$. The results for the integrals in Eq. (A6) after this extra regulator (followed by the limit $\alpha\to0$) are

$$\int_0^{\sigma_r}d\sigma\frac{\sigma^n}{n!}G_\Lambda(\sigma)=\frac{\Lambda^{1-n}}{4\pi}c_nh_{P.V.}^{(n)}(\Lambda\sigma_r), \tag{A8}$$

$$\int_0^{\sigma_r}d\sigma\frac{\sigma^{n+1}}{(n+1)!}\frac{d}{d\sigma}G_\Lambda(\sigma)=-\frac{\Lambda^{1-n}}{4\pi}c_ng_{P.V.}^{(n)}(\Lambda\sigma_r) \tag{A9}$$

with

$$h_{P.V.}^{(n)}(\Lambda\sigma_r)=\frac{(\Lambda\sigma_r)^{n+1}\Gamma\left(\frac{3-n}{2}\right)}{2^{n+1}\Gamma\left(\frac{3+n}{2}\right)}\times{}_1F_2\left(\left\{\frac{1+n}{2}\right\},\left\{2,\frac{3+n}{2}\right\},-\frac{\Lambda^2\sigma_r^2}{4}\right), \tag{A10}$$

$$g_{P.V.}^{(n)}(\Lambda\sigma_r)=\frac{(\Lambda\sigma_r)^{n+3}\Gamma\left(\frac{3-n}{2}\right)}{2^{n+4}\Gamma\left(\frac{5+n}{2}\right)}\times{}_1F_2\left(\left\{\frac{3+n}{2}\right\},\left\{3,\frac{5+n}{2}\right\},-\frac{\Lambda^2\sigma_r^2}{4}\right), \tag{A11}$$

where ${}_pF_p$ are the hypergeometric functions,

$$c_n = \frac{\sqrt{\pi}}{n\Gamma\left(\frac{3-n}{2}\right)\Gamma\left(\frac{n}{2}\right)}, \tag{A12}$$

and $c_0 = 1$.

The time-dependent back reaction term is then

$$\frac{e^2}{4\pi}\left\{\sum_{n=0}^{\infty}\Lambda^{1-n}c_n\left[s_n h^{(n)}_{P.V.}(\Lambda\sigma_r)a_\mu - \sum_{m=0}^{n} s_{(n-m)} l_{n,m} g^{(n)}_{P.V.}(\Lambda\sigma_r)\dot{z}^{\nu}\dot{z}_{[\mu}u^{(m+1)}_{\nu]}\right]\right\}. \tag{A13}$$

Of course, the detailed behavior of the time-dependent coefficients depends on the choice of regulator (the Pauli-Villars and Gaussian regulator results are not the same). But what is important is that they have the same crucial qualitative behaviors in common. The $\sigma_r \to \infty$ (i.e. late time) result is given by $h^{(n)}_{P.V.} \to g^{(n)}_{P.V.} \to 1$, and thus at low energies only the $n = 0,1$ terms contribute. These produce mass renormalization and the standard ALD result, as may be easily verified from Eq. (A13). Uniqueness and causality of the equations of motion follow from the fact that $h^{(n)}_{P.V.} = g^{(n)}_{P.V.} = 0$ for all n. Hence, Pauli-Villars leads to the same conclusions that we drew earlier based on the Gaussian regulated Green's function. Moreover, it is clear that any reasonably smooth regulator should share these general features.

Finally, we could derive the higher-derivative terms (those suppressed by inverse powers of Λ) in the Langevin equation. We would do this to demonstrate that these higher-derivative corrections do not spoil the consistency of the stochastic equations of motion with acausal or runaway effects. The argument is the same as that for the semiclassical equations of motion, and it is straightforward after some algebra to see that the vanishing of the time-dependent coefficients $h^{(n)}(\Lambda\sigma_r)$ and $g^{(n)}(\Lambda\sigma_r)$ at $\Lambda\sigma_r = 0$ also removes runaways and acausal solutions.

[1] P. R. Johnson and B. L. Hu, ‘‘Stochastic Theory of Relativistic Particles Moving in a Quantum Field: Influence Functional and Langevin Equation,’’ quant-ph/0012137.

[2] P. R. Johnson and B. L. Hu, ‘‘Stochastic Theory of Relativistic Particles Moving in a Quantum Field: Electromagnetic Abraham-Lorentz-Dirac-Langevin Equation,’’ report, 2001; a summary of this series may be found in P. R. Johnson and B. L. Hu, in *Proceedings of the Second Conference on Quantum Aspects of Beam Physics, 2000*, edited by Pisin Chen (World Scientific, Singapore, 2001).

[3] E. B. Davies, *The Quantum Theory of Open Systems* (Academic, London, 1976); U. Weiss, *Quantum Dissipative Systems* (World Scientific, Singapore, 1993).

[4] A. O. Caldeira and A. J. Leggett, Physica A **121**, 587 (1983); Ann. Phys. (N.Y.) **149**, 374 (1983); V. Hakim and V. Ambegaokar, Phys. Rev. A **32**, 423 (1985); F. Haake and R. Reibold, *ibid.* **32**, 2462 (1985); V. Hakim and V. Ambegaokar, *ibid.* **32**, 423 (1985); A. J. Leggett *et al.*, Rev. Mod. Phys. **59**, 1 (1987); H. Grabert, P. Schramm, and G. L. Ingold, Phys. Rep. **168**, 115 (1988); B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D **45**, 2843 (1992); **47**, 1576 (1993).

[5] R. Feynman and F. Vernon, Ann. Phys. (N.Y.) **24**, 118 (1963); R. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[6] J. Schwinger, J. Math. Phys. **2**, 407 (1961); L. V. Keldysh, Zh. Éksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)]; G. Zhou, Z. B. Su, B. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985); Z. B. Su, L. Y. Chen, X. T. Yu, and K. C. Chou, Phys. Rev. B **37**, 9810 (1988); B. S. DeWitt, in *Quantum Concepts in Space and Time*, edited by R. Penrose and C. J. Isham (Clarendon, Oxford, 1986); R. D. Jordan, Phys. Rev. D **33**, 444 (1986); E. Calzetta and B. L. Hu, *ibid.* **35**, 495 (1987); A. Campos and E. Verdaguer, *ibid.* **49**, 1861 (1994).

[7] B. L. Hu, in *Relativity and Gravitation: Classical and Quantum*, Proceedings SILARG VII, Cocoyoc, Mexico, 1990, edited by J. C. D'Olivo *et al.* (World Scientific, Singapore, 1991).

[8] R. P. Feynman, Phys. Rev. **80**, 440 (1950); J. Schwinger, *ibid.* **82**, 664 (1951); B. S. DeWitt, Phys. Rep., Phys. Lett. **19C**, 297 (1975); D. J. O'Conner, C. R. Stephens, and B. L. Hu, Ann. Phys. (N.Y.) **190**, 310 (1989); A. O. Barut and I. H. Duru, Phys. Rep. **172**, 1 (1989); D. G. C. McKeon, Ann. Phys. (N.Y.) **224**, 139 (1993); M. J. Strassler, Nucl. Phys. **B385**, 145 (1992); H. Kleinert, *Path Integrals in Quantum Mechanics, Statistical and Polymer Physics* (World Scientific, London, 1995); M. J. Schmidt and C. Schubert, Phys. Lett. B **331**, 69 (1994); C. Schubert, hep-th/0101036; J. Jalilian-Marian *et al.*, Phys. Rev. D **62**, 045020 (2000); R. Venugopalan and J. Wirstam, *ibid.* **63**, 125022 (2001).

[9] W. H. Zurek, Phys. Rev. D **24**, 1516 (1981); **26**, 1862 (1982); in *Frontiers of Nonequilibrium Statistical Physics*, edited by G. T. Moore and M. O. Scully (Plenum, New York, 1986); E. Joos and H. D. Zeh, Z. Phys. B: Condens. Matter **59**, 223 (1985); A. O. Caldeira and A. J. Leggett, Phys. Rev. A **31**, 1059 (1985); W. G. Unruh and W. H. Zurek, Phys. Rev. D **40**, 1071 (1989); W. H. Zurek, Prog. Theor. Phys. **89**, 281 (1993); D. Giulini *et al.*, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer-Verlag, Berlin, 1996); C. Kiefer, Phys. Rev. D **46**, 1658 (1992).

[10] R. B. Griffiths, J. Stat. Phys. **36**, 219 (1984); R. Omnès, *ibid.* **53**, 893 (1988); **53**, 933 (1988); **53**, 957 (1988); Ann. Phys. (N.Y.) **201**, 354 (1990); Rev. Mod. Phys. **64**, 339 (1992); M. Gell-Mann and J. B. Hartle, in *Complexity, Entropy and the Physics of Information*, edited by W. H. Zurek (Addison-Wesley, Reading, MA, 1990); Phys. Rev. D **47**, 3345 (1993); J. B. Hartle, in *Directions in General Relativity*, edited by B. L. Hu, M. P. Ryan, and C. V. Vishveswara (Cambridge University Press, Cambridge, England, 1993), Vol. 1; H. F. Dowker and J. J. Halliwell, Phys. Rev. D **46**, 1580 (1992); T. A. Brun, *ibid.* **47**, 3383 (1993); J. P. Paz and W. H. Zurek, *ibid.* **48**, 2728 (1993); J. Twamley, *ibid.* **48**, 5730 (1993).

[11] M. Gell-Mann and J. B. Hartle, Phys. Rev. D **47**, 3345 (1993); B. L. Hu, in *Proceedings of the Second International Workshop on Thermal Fields and Their Applications*, edited by H. Ezawa *et al.* (North-Holland, Amsterdam, 1991); in *Physical Origins of Time Asymmetry*, edited by J. Perez-Mercader, J. J. Halliwell, and W. H. Zurek (Cambridge University Press, Cambridge, England, 1994), Chap. 34.
[12] L. H. Ford, Phys. Rev. D **47**, 5571 (1993); Phys. Rev. A **56**, 1812 (1997).
[13] P. M. V. B. Barone and A. O. Caldeira, Phys. Rev. A **43**, 57 (1991).
[14] J. B. Hartle, R. Laflamme, and D. Marolf, Phys. Rev. D **51**, 7007 (1995); T. A. Brun and J. J. Halliwell, *ibid.* **54**, 2899 (1996); F. Dowker and A. Kent, Phys. Rev. Lett. **75**, 3038 (1995); J. Stat. Phys. **82**, 1575 (1996); J. J. Halliwell, Phys. Rev. D **58**, 105015 (1998); Phys. Rev. Lett. **83**, 2481 (1999); T. A. Brun and J. B. Hartle, Phys. Rev. D **60**, 123503 (1999); A. Kent, Phys. Rev. A **54**, 4670 (1996); Phys. Rev. Lett. **78**, 2874 (1997); **81**, 1982 (1998).
[15] E. Calzetta and B. L. Hu, in *Directions in General Relativity, Vol. II: Brill Festschrift*, edited by B. L. Hu and T. A. Jacobson (Cambridge University Press, Cambridge, England, 1993); in *Heat Kernel Techniques and Quantum Gravity*, Vol. 4 of Discourses in Mathematics and Its Applications, Winnipeg, 1994, edited by S. A. Fulling (Texas A&M University Press, College Station, TX, 1995), hep-th/9501040; C. Anastopoulos, Phys. Rev. D **54**, 1600 (1996); E. Calzetta and B. L. Hu, *ibid.* **61**, 025012 (2000).
[16] P. W. Milonni, *The Quantum Vacuum* (Academic, San Diego, 1994); F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1965); G. N. Plass, Rev. Mod. Phys. **33**, 37 (1961).
[17] M. Abraham and R. Becker, *Electricity and Magnetism* (Blackie, London, 1937); H. A. Lorentz, *The Theory of Electrons* (Dover, New York, 1952), pp. 49 and 253; P. A. M. Dirac, Proc. R. Soc. London **A167**, 148 (1938).
[18] J. Wheeler and R. Feynman, Rev. Mod. Phys. **17**, 157 (1945).
[19] S. Coleman, in *Electromagnetism: Paths to Research*, edited by D. Teplitz (Plenum, New York, 1982).
[20] B. L. Hu and P. R. Johnson, in *Proceedings of the Second Conference on Quantum Aspects of Beam Physics*, edited by P. Chen (World Scientific, Singapore, 2001).
[21] F. Rohrlich, Am. J. Phys. **28**, 639 (1969).
[22] B. L. Hu and A. Matacz, Phys. Rev. D **49**, 6612 (1994); E. Calzetta and B. L. Hu, *ibid.* **49**, 6636 (1994).
[23] E. J. Moniz and D. H. Sharp, Phys. Rev. D **15**, 2850 (1977); C. Cohen-Tannoudji, *Atom-Photon Interactions* (Wiley, New York, 1992).
[24] P. W. Milonni, Phys. Lett. **82A**, 225 (1981).
[25] N. V. Kampen, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **26**, No. 15, 1 (1951); E. Moniz and D. Sharp, Phys. Rev. D **10**, 1133 (1974); H. Levine, E. Moniz, and D. Sharp, Am. J. Phys. **45**, 75 (1977).
[26] S. Weinberg, *The Quantum Theory of Fields (*Cambridge University Press, Cambridge, England, 1995), Vols. I–III; K. Wilson, Rev. Mod. Phys. **55**, 583 (1983).
[27] H. Callen and T. Welton, Phys. Rev. **83**, 34 (1951); H. Mori, Prog. Theor. Phys. **33**, 423 (1965); G. W. Ford, M. Kac, and P. Masur, J. Math. Phys. **6**, 504 (1965); H. Haken, Rev. Mod. Phys. **80**, 67 (1975); H. Dekker, Phys. Rep. **80**, 1 (1981).
[28] L. D. Romero and J. P. Paz, Phys. Rev. A **55**, 4070 (1997).
[29] G. W. Ford, J. T. Lewis, and R. F. O'Connell, Phys. Rev. Lett. **55**, 2273 (1985); Phys. Rev. A **37**, 4419 (1988); G. W. Ford and R. F. O'Connell, Phys. Lett. A **157**, 217 (1991); *ibid.* **174**, 182 (1993).
[30] G. W. Ford and R. F. O'Connell, Phys. Rev. A **57**, 3112 (1998).
[31] V. Krivitskii and V. Tsytovich, Sov. Phys. Usp. **34**, 250 (1991).[Usp. Fiz. Nauk **161**, 125 (1991)].
[32] F. E. Low, Ann. Phys. (N.Y.) **266**, 274 (1998).
[33] L. Diósi, Found. Phys. **20**, 63 (1990).
[34] C. Keifer, T. Padmanabhan, and T. P. Singh, Class. Quantum Grav. **8**, L185 (1991); N.D. Birrell and P.C. Davis, *Quantum Fields in Curved Space* (Cambridge University Press, New York, 1982); S.A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge University Press, Cambridge, 1989); R.M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago, Chicago, 1994).
[35] Ya. Zel'dovich and A. Starobinsky, Zh. Eksp. Teor. Fiz. **61**, 2161 (1971)[Sov. Phys. JETP **34**, 1159 (1971)]; B. L. Hu and L. Parker, Phys. Rev. D **17**, 933 (1978); F. V. Fischetti, J. B. Hartle, and B. L. Hu, *ibid.* **20**, 1757 (1979); J. B. Hartle and B. L. Hu, *ibid.* **20**, 1772 (1979); P. A. Anderson, *ibid.* **28** 271 (1983); **29**, 615 (1984); E. Calzetta and B. L. Hu, *ibid.* **35**, 495 (1987); A. Campos and E. Verdaguer, *ibid.* **49**, 1861 (1994).
[36] B. L. Hu and A. Matacz, Phys. Rev. D **51**, 1577 (1995); A. Campos and E. Verdaguer, *ibid.* **53**, 1927 (1996); R. Martin and E. Verdaguer, *ibid.* **60**, 084008 (1999); B. L. Hu, Int. J. Theor. Phys. **38**, 2987 (1999).
[37] A. Raval, B. L. Hu, and J. Anglin, Phys. Rev. D **53**, 7003 (1996).
[38] E. Calzetta and B. L. Hu, Phys. Rev. D **55**, 3536 (1997).
[39] J. Ward, Phys. Rev. **78**, 182 (1950); Y. Takahashi, Nuovo Cimento **6**, 371 (1957).
[40] G. Gour and L. Sriramkumar, Found. Phys. **29**, 1917 (1999).
[41] P. R. Johnson and B. L. Hu, "Stochastic Theory of Quantum Relativistic Particles Moving in a Quantum Field," report, 2001.
[42] *Quantum Aspects of Beam Physics, Advanced IFCA Beam Dynamics Workshop*, edited by P. Chen (World Scientific, Singapore, 1998).