Regular sources of the Kerr-Schild class for rotating and nonrotating black hole solutions

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(Received 26 September 2001; published 5 March 2002)

A unified approach to regular interiors of black holes with smooth matter distributions in the core region is given. The approach is based on a class of Kerr-Schild metrics representing minimal deformations of the Kerr-Newman solution, and allows us to give a common treatment for (charged and uncharged) rotating and nonrotating black holes. It is shown that the requirement of smoothness of the source constrains the structure of the core region in many respects: in particular, for Schwarzschild holes a de Sitter core can be selected, which is surrounded by a smooth shell giving a leading contribution to the total mass of the source. In the rotating, noncharged case the source has a similar structure, taking the form of a (anisotropic and rotating) de Sitter–like core surrounded by a rotating elliptic shell. The Kerr singular ring is regularized by anisotropic matter rotating in the equatorial plane, so that the negative sheet of the Kerr geometry is absent. In the charged case the sources take the form of "bags," which can have de Sitter or anti–de Sitter interiors and a smooth domain wall boundary, with a tangential stress providing charge confinement. The Arnowitt-Deser-Misner and Tolman relations are used to calculate the total mass of the sources.

DOI: 10.1103/PhysRevD.65.064039 PACS number(s): 04.70.−s, 04.20.Dw, 04.20.Jb, 97.60.Lf

I. INTRODUCTION

This paper is an attempt at a unification of two research lines on black hole solutions, which have been developed (almost independently) for a long time. A first line of investigation has to do with the problem of the final state in gravitational collapse, and stems from pioneering observations by Gliner [1] and Sakharov [2], who suggested that matter at superhigh densities should have the equation of state $p = -\epsilon$, so that the the stress-energy tensor takes the "lambda term" form

$$T_{ik} = \Lambda g_{ik} \tag{1}$$

at the late stage of collapse. Further, Zel'dovich and Novikov proposed [3] such a stress-energy tensor to arise as the result of gravitational interactions in a vacuum polarization process. These considerations led naturally to the hypothesis that an unlimited increase of spacetime curvature during the collapse process had to be halted by the formation of a core region with a constant, limiting value of the curvature determined by dominant effects of quantum fluctuations. The issue received renewed attention over 20 years later, essentially following the papers by Frolov, Markov and Mukhanov [4,5]. Their model consists of a de Sitter core inside a Schwarzschild black hole, matched with the external solution via a thin transition layer. All investigations along this line have been restricted, so far, to the nonrotating case only [6–17] (see also [18–20] for the Reissner-Nordström case).

Another open line of research is connected with the analysis of the structure of the singularity of the rotating (Kerr and Kerr-Newman) black holes. As is well known, this singularity takes the form of a ring which is a branch line of the space, leading to a two-sheeted topology. Going through the Kerr ring one obtains a second ("negative") sheet of the metric where the values of the mass and charge change their signs while fields change their directions. In this region closed timelike curves exist, so that causality violations occur. This led to approaches which attempted to avoid the two-sheetedness with procedures meant to truncate the negative sheet. A procedure of this kind was first developed by Israel [21], who used the surface of the disk spanned by the singular ring as the surface of truncation. The resulting metric has a finite jump of the first derivative on the disk, thus leading to a distributional matter source located on the surface. In this way the Kerr solution is interpreted as being the field generated by a very exotic stress-energy tensor: a layer of negative mass rotating with superluminal velocities.

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The Israel interpretation was improved by Hamity [22], who noticed that the disk can be considered as being in rigid relativistic rotation. In the co-rotating reference system, the stress-energy tensor takes a diagonal form, with zero energy density and a negative pressure, which, however, grows up to infinity on approaching the singular boundary of the disk.

Another approach was given, for the Kerr-Newman solution, by López [23], who constructed the source in the form of a rigidly rotating ellipsoidal shell (bubble) covering the singular ring. The singular ring is removed, since the interior of the bubble is flat. A continuous matching of the flat interior with the external metric of the Kerr-Newman field can be obtained, however, only by a special choice of the shell "radius" $r_{shell}=r_e=e^2/2m$, where r is the Kerr ellipsoidal radial coordinate.

Actually, r_e is the so called "classical size" of a particle with charge e and mass m, and indeed the problem of Kerr's sources received attention also from the point of view of constructing classical models of the electron, after Carter's remark [24] that the Kerr-Newman solution possesses the same giromagnetic ratio $g=2$ of the Dirac electron.[1] Interestingly enough, as far as models of spinning particles are concerned, the values of charge e and the rotation parameter $a=J/m$ are very high with respect to the mass m so that the horizons of the Kerr-Newman solution are absent and the Kerr singular ring is a naked singularity, visible also to far-away observers.

In spite of the progress in understanding the structure of regular black hole solutions in both the aforementioned approaches, there was still one common drawback connected with the necessity of involving in the models a thin (or at least infinitely thin) transition layer, while smooth models for the black hole (BH) interior would obviously be more satisfactory [9,30,31,15,12,18–20,32,28,29]. In many such attempts, the treatment is based upon the Kerr-Schild class of metrics [15,31,32,12,29], which is obviously connected with the fact that all stationary black holes (that is, all *known* black holes) are particular cases of the Kerr-Schild geometry. On the other hand, the de Sitter core region can be described in Kerr-Schild form, too. Therefore, it looks convenient to describe in this same form the transition region connecting the core and external geometry. A remarkable property of the Kerr-Schild class is that such a description can be performed in a *unique* fashion for charged, uncharged, rotating, and nonrotating BH solutions, by using a smooth function of a radial coordinate $f(r)$ to interpolate between the core and the external field. This is the approach used in the present paper. Sources of BH solutions are constructed as smooth deformations of the electro-vacuum Kerr-Schild metrics retaining the main structure of this geometry, namely the double principal null (PN) congruence. In this way we obtain a class of sources which covers almost all previous models of nonrotating sources [5,6,9,11,14,15,13,12,18–20] generalizing them to the rotating case. It contains smooth analogues of known shell-like models, in particular, the rotating and the nonrotating shell (bubble) models of charged sources [33,23,27]. Several new interesting features appear in this way.

Throughout this work, latin indices run from 0 to 3. We write Einstein's equations in the form $-R_{ab}+(1/2)g_{ab}R=8\pi T_{ab}$, where R_{ab} is the Ricci tensor, $R_{ab}\equiv R^c_{abc}$, and units are chosen so that $G=1$, $c=1$. The Lorentz signature is taken to be $-+++$.

II. GENERALIZED KERR-NEWMAN SOLUTIONS

The Kerr-Newman solution in the Kerr-Schild form is [34]

$$g_{ik}=\eta_{ik}+2he^3_i e^3_k. \tag{2}$$

Here $\eta_{ik}=\mathrm{diag}(-1,1,1,1)$, h is a scalar function (to be specified below), and the null vector field e^3_i is tangent to the Kerr PN congruence. The explicit form of the Kerr PN congruence is not essential for our analysis. We will assume that in the generalization of the Kerr-Newman solution to the interior case, the Kerr PN congruence retains its form and also the properties of being geodesic and shear free.

A simple way to generalize the Kerr-Newman solution—obtaining an interior solution which is still of the Kerr-Schild type—is to replace the factor $f_{KN}=mr-e^2/2$ in the function h with an arbitrary function $f(r)$ [31]. This procedure can be seen as the introduction of a smooth distribution for charges and masses. This distribution is purely "radial" in that it depends only on the r coordinate, which is confocal to the angular coordinate θ of an oblate ellipsoidal coordinate system, and becomes the standard Schwarzschild radial coordinate if the oblateness goes to zero (see, e.g., [35]).

The Kerr-Schild metric takes a convenient form in a basis where the one-form e^3 is normalized in such a way that its time component equals one. In this basis the function h takes the form

$$h=\frac{f(r)}{\Sigma}=\frac{f(r)}{r^2+a^2\cos^2\theta}. \tag{3}$$

For regular $f(r)$, this function can be singular only in the equatorial plane $\theta=\pi/2$, at $r=0$. In this case the behavior near $r=0$ is of the form $h\sim f(r)/r^2$, as in the nonrotating case (when $a=0$). This allows us to apply the same approach to the regularization of the metrics, both for the nonrotating and for the rotating cases.

By using the ansatz (3) and the machinery of the Debney-Kerr-Schild approach [34], we obtain the following tetrad components for the Ricci tensor:

$$R'_{12}=-2G, \tag{4}$$

$$R'_{34}=D+2G, \tag{5}$$

$$R'_{12}-R'_{34}=-(D+4G), \tag{6}$$

$$R'_{23}=(D+4G)(r,_2-P_{\bar{Y}}), \tag{7}$$

[1]In this connection a series of works followed, on the models of spinning particle based on the Kerr-Newman solution [21,23,25–29].

$$R'_{13}=(D+4G)(r_{,1}-P_Y), \tag{8}$$

$$R'_{33}=-2(D+4G)(r_{,1}-P_Y)(r_{,2}-P_{\bar{Y}}), \tag{9}$$

where

$$D=-\frac{f''}{\Sigma}, \tag{10}$$

$$G=\frac{f'r-f}{\Sigma^2} \tag{11}$$

[here $f'=\partial_r f(r)$, and $P_{\bar{Y}}=\partial_{\bar{Y}}P=2^{-1/2}Y$, $P_Y=\partial_Y P=2^{-1/2}\bar{Y}$; $,_a=e_a^i\partial_i$ are the tetrad derivations, and the corresponding tetrad is given in Appendix A, Eqs. (A32)–(A35)]. This expression allows us to write the stress-energy tensor [Appendix A, Eq. (A37)], which acquires a very transparent form if an orthonormal tetrad $\{u,l,m,n\}$ (C10) connected with Boyer-Lindquist coordinates is used (Appendix C):

$$T_{ik}=(8\pi)^{-1}[(D+2G)g_{ik}-(D+4G)(l_il_k-u_iu_k)]. \tag{12}$$

In the above formula, u^i is a timelike vector field given by

$$u^i=\frac{1}{\sqrt{\Delta\Sigma}}(r^2+a^2,0,0,a),$$

where $\Delta=r^2+a^2-2f(r)$.

In this expression one immediately recognizes that if the matter of the source is thought of as being separated into ellipsoidal layers corresponding to constant values of the coordinate r, each layer rotates with angular velocity $\omega(r)=u^\phi/u^0=a/(a^2+r^2)$. This rotation becomes rigid only in the thin shell approximation $r=r_0$. The linear velocity of the matter with respect to the auxiliary Minkowski space is $v=a\sin\theta/\sqrt{a^2+r^2}$, so that on the equatorial plane $\theta=\pi/2$, for small values of r ($r\ll a$), one has $v\approx 1$, which corresponds to an oblate, relativistically rotating disk.

The energy density ρ of the material satisfies to $T^i_k u^k=-\rho u^i$ and is, therefore, given by

$$\rho=\frac{1}{8\pi}2G. \tag{13}$$

There are only two distinct spacelike eigenvalues, corresponding to the radial and tangential pressures of the nonrotating case, namely,

$$p_{rad}=-\frac{1}{8\pi}2G=-\rho, \tag{14}$$

$$p_{tan}=\frac{1}{8\pi}(D+2G)=\rho+\frac{D}{8\pi}. \tag{15}$$

Singularities can arise only at $r=0$ on the equatorial plane, so that the regularity properties of the stress-energy tensor can be studied together in both the rotating and the nonrotating cases (energy conditions will be dealt with in Sec. V).

III. INTERIORS FOR THE KERR-SCHILD CLASS OF BH SOLUTIONS

A. The nonrotating case

We shall consider in this section the nonrotating case from a unifying viewpoint, before moving to the rotating case.

1. Core region

We follow here a somewhat classical approach, motivated by well known (and expected) properties of the strong field regime of gravitation (see, e.g., [5]). We thus consider the center of the core region as a spacetime of constant curvature. Therefore, we have to use for this region a regularizing function $f(r)=f_0(r)=\alpha r^4$, where $\alpha=8\pi\Lambda/6$. This provides smoothness of the metric up to the second derivative and removes the singularity at $r=0$. The scalar curvature invariant $R=2D=-2f_0''/r^2=-24\alpha$ is constant, as well as the density $\rho=(1/8\pi)\Lambda$.

2. Exterior region

As is obvious, the exterior is determined unambiguously by the Birkhoff-Israel theorem. Thus the function f must coincide with $f_{KN}=mr-e^2/2$.

3. Transition region

We assume a de Sitter-like behavior of the spacetime only near the center (“core” region). Therefore, between the boundary of the source—assumed to lie in the region of trapped surfaces—and the de Sitter core, a “transition region” can exist, which interpolates between the core and the vacuum in such a way that the resulting metric is regular everywhere. Due to the assumed symmetries, different kinds of transitions correspond to different choices of the function $f(r)$. Calculating the second fundamental form of the $r=\text{const}$ surfaces, it is easy to check that, to avoid the presence of singular (shell-like) distributions of matter on the inner (de Sitter transition) and on the outer (transition-vacuum) matching hypersurfaces, the function f must be C^1. Further, one can ask the transition region to be “thin” in the sense that the thickness δ is much smaller than the radial position of the transitional layer r_0 which would correspond to a cusp at the intersection of the plots of $f_0(r)$ and $f_{KN}(r)$. As a result of these assumptions the position of the transition layer can be easily estimated analytically. It is, however, more transparent the use of a graphical representation of the different kinds of situations which may occur.

As a result of these assumptions the position of the transition layer can be easily estimated as a root r_0 of the equation $f_0(r_0)=f_{KN}(r_0)$. We shall see that this relation turns out a necessary condition for consistency of the source models with respect to the Tolman and Arnowitt-Deser-Misner (ADS) mass relations. It is, however, more transparent the use of a graphical representation.

B. Graphical analysis

1. Case $\alpha>0$: de Sitter interior, uncharged source

Figure 1 shows that there is only one intersection between $f_0(r)=\alpha r^4$ and $f_{KN}(r)=mr$. Therefore, the position of the

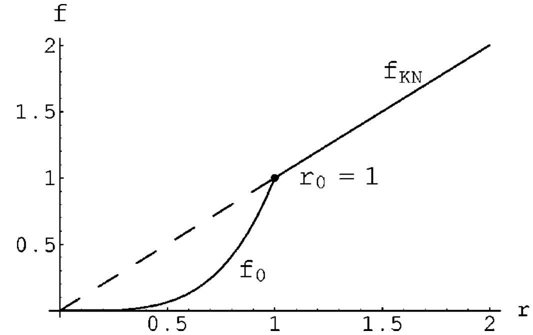

FIG. 1. Position of phase transition r_0 as an intersection of plots $f_0(r)$ and $f_{KN}(r)$. Uncharged source, $\alpha>0$, arbitrary units.

transition layer will be $r_0=(m/\alpha)^{-1/3}$. As seen in the picture, the second derivative of the corresponding interpolating function will be negative at this point, yielding an extra contribution to the positive tangential pressure in the transition region. Solutions of this class can be constructed in such a way that the weak energy condition is satisfied (see Sec. V).

2. Case $\alpha>0$: de Sitter interior, charged source

Figure 2 shows that for charged sources there are two intersections, $r_{0,1}$ and $r_{0,2}$, of the functions $f_0(r)=\alpha r^4$ and $f_{\mathrm{KN}}(r)=mr-e^2/2$. The root $r_{0,2}$ corresponds to a negative second derivative of the function $f(r)$ and leads to a picture similar to that for the uncharged source with the intermediate shell with pressure (Fig. 3). However, the smaller root $r_{0,1}$ corresponds to a source of smaller size and has a positive second derivative leading to an intermediate shell with a tangential stress. Thus, this source resembles a bubble with a de Sitter (dS) interior and a domain wall boundary confining the charge of the source (Fig. 4).

3. Case $\alpha<0$: anti–de Sitter interior, uncharged source

In this case matching the interior and exterior regions turns out to be impossible for the usual black hole solutions. However, there is an exotic case of a black hole solution with negative mass. Graphical analysis shows (Fig. 5) that there is a solution with a positive second derivative of the interpolating function $f(r)$ leading to a shell with tangential stress. This exotic source resembles an AdS bubble with a domain wall boundary and negative total mass, like those occurring in supergravity [36].

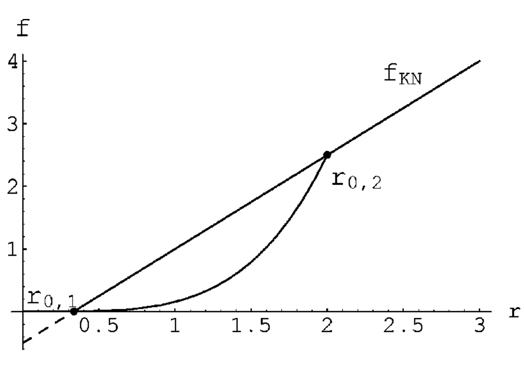

FIG. 2. Two possible positions for a point of phase transition $r_{0,1}$ and $r_{0,2}$ for a charged source and $\alpha>0$.

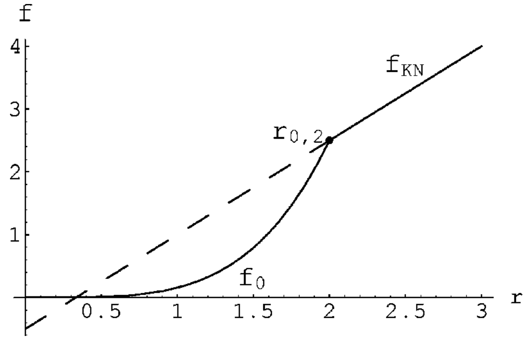

FIG. 3. Choice of $r_{0,2}$ as point of phase transition.

4. Case $\alpha\leqslant 0$: flat and anti–de Sitter interior, charged source

In this case a matching of the interior and the exterior regions can be reached via an intermediate shell, having a positive second derivative and leading to a positive tangential stress of the shell confining the charge of the source. This graphical analysis (Fig. 6) provides a classification of the simplest possible matching of AdS and dS interiors of the source with the exterior black hole solutions via a smooth intermediate layer. Of course, more complicated transition layers can be considered, where the second derivative of the interpolating function $f(r)$ changes sign several times.

C. The rotating case

A remarkable property of the Kerr-Schild class is that the above treatment is easily extensible to the rotating case. In fact, the function f is, also in this case, a function of r only, and the Kerr and Kerr-Newman solutions are obtained with the *same* functions that correspond to the Schwarzschild and to the Reissner-Nordström solutions, respectively. Of course, the definition of the coordinate r is now completely different, since the surfaces $r=$const are ellipsoidal, and described by the equation

$$\frac{x^2+y^2}{r^2+a^2}+\frac{z^2}{r^2}=1. \tag{16}$$

The relations for the metric and stress-energy tensor are characterized by a more complicated form of the function $\Sigma=r^2+a^2\cos^2\theta$. As a result, the components of the metric and stress-energy tensor increase when approaching the equato-

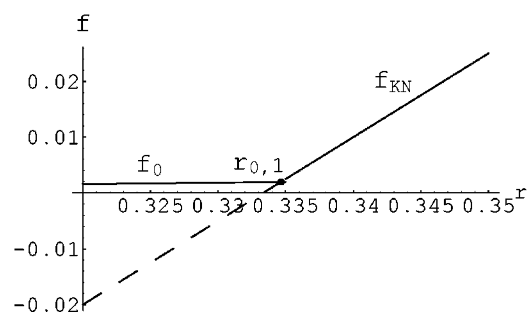

FIG. 4. Strongly scaled section of Fig. 1 corresponding to a phase transition at point $r_{0,1}$.

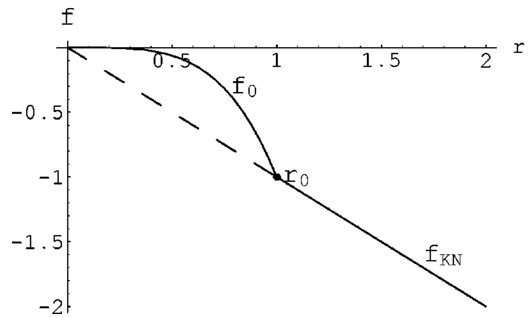

FIG. 5. Nonstable case corresponding to $\alpha<0$ and negative mass.

rial plane $\cos\theta=0$, where they take the same form as in the nonrotating case. The singularity of the Kerr solution can be suppressed by analyzing the metric and the stress-energy tensor near the Kerr singular ring $r\approx\cos\theta\approx0$. As we have already seen, the condition on the behavior of the function f when $r\rightarrow0$ remains the same as for the nonrotating case.

The intermediate shell, which matches the interior of the source and the external geometry, is foliated on the rotating ellipsoidal layers. The thin shell is characterized by an increasing of the tangential stress (or pressure) and can be considered as a rigidly rotating boundary of the disk-like source, similarly as in the López singular shell model [23].

IV. CAUSAL STRUCTURE

The causal structure of nonrotating black hole interiors is well known [16,13]. It closely resembles that of the RN spacetime, with the key difference that the singularity is replaced by the matter-filled region. From the topological point of view, it was shown by Borde that a change of topology occurs, making it possible for regular solutions to exist [16,17].

The case of a rotating black hole interior will now be analyzed. This is best visualized, again, with the use of a two-dimensional plot in the equatorial plane. Horizons of the spacetime, if any, are defined by $\Delta=0$, that is, by the equation $f(r)=r^2+a^2$. Plotting the parabola and taking into account the properties of the function f, one realizes that two different situations may occur.

Case I. The parabola r^2+a^2 and the external function f intersect at one point, which corresponds to a new Cauchy horizon replacing the vacuum one (the event horizon obviously remains in the vacuum region).

Case II. For large enough values of the angular momentum, no intersection exists. The Cauchy horizon remains the original (vacuum) one, and there is no horizon present in the matter filled region.

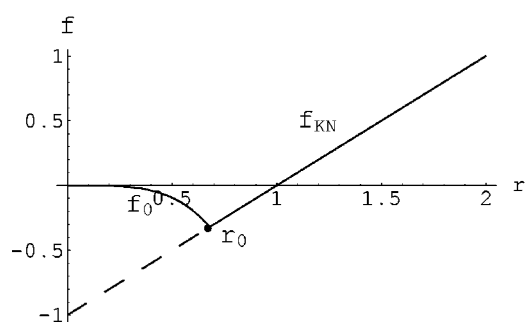

FIG. 6. Class of stable states for charged sources and $\alpha\leqslant0$ (flat or AdS interior and $r_0\leqslant e^2/2m$).

V. LOCAL ENERGY CONDITIONS AND INTERPRETATION

The physical reasonability of a classical solution of the Einstein equations relies on the fulfillment of the energy conditions, which are a common tool for getting insight into some energetic properties of any spacetime. However, and in spite of the fact that a quantum theory of gravity is not available yet, increasing evidence, both experimental and theoretical (see, e.g. [37–39], the recent account in [40] and references therein) shows that quantum effects—specially those associated with the quantum vacuum—may lead to a general violation of some or even *all* of such conditions. Therefore, as we aim at describing nonsingular black holes which are only likely to be understood via quantum effects, we should not expect all the energy conditions to be satisfied (actually, the strong and the dominant energy conditions have to be violated somewhere, already on classical grounds [13]). In any case, energy conditions are a useful tool which may still serve to assess whether a classically dominated field is the source responsible for the spacetime considered, or, on the contrary, that such a possibility is banned and one should seek for a quantum origin of the source. This remark is important because, otherwise, energy conditions might impel us to disregard some spacetimes that may well fit with the current knowledge of the interface between quantum physics and gravitation. Let us begin now with the study of *local* energy conditions. In the following section we will implicitly consider some *averaged*—or "extended"—energy conditions.

Let $\vec{V}$ be any 4-velocity vector field and let $\vec{N}$ be any null vector field. We shall denote $S_{\lambda\mu}T^{\lambda}T^{\mu}$ by S_{TT} for any (symmetric) rank-two tensor field $\mathbf{S}$, and any vector field $\vec{T}$.

(i) Strong energy condition (SEC): a system satisfies the SEC if and only if

$$-R_{VV}\propto T_{VV}+(1/2)T\geqslant0,\quad\forall\vec{V},\tag{17}$$

where $R_{\lambda\mu}$ is the Ricci tensor and $T_{\lambda\mu}$ is the stress-energy tensor (being T its trace). Fulfillment of SEC is a "cornerstone" in singularity theorems and must be violated in the case where the spacetime is nonsingular. It can be shown also that the dominant energy condition ($T^{\mu}_{\nu}V^{\nu}$ non-space-like for any non-space-like V) is violated in this case.

(ii) Weak energy condition (WEC): a system satisfies the WEC if and only if

$$T_{VV}\geqslant0,\quad\forall\vec{V}.\tag{18}$$

These are clearly connected with the sign of the energy density measured by an observer with 4-velocity $\vec{V}$.

(iii) Null energy condition (NEC): a system satisfies NEC if and only if

$$T_{NN} \geqslant 0, \quad \forall \vec{N}. \tag{19}$$

This last energy condition may be viewed as a limiting case of (ii) for ultrarelativistic observers. Basically, it includes some commonly used spacetimes, such as anti–de Sitter.

The physical validity of (i) has been objected to many times and on different grounds, and violation of the SEC is nowadays well understood. On the other hand, (ii) is usually considered as a necessary condition for a gravitational system to be acceptable. While this is certainly right for classical matter, it becomes doubtful in the case when quantum effects play a relevant role.

It is useful to represent $\vec{V}$ as

$$\vec{V} = A\vec{u} + B\vec{l} + C\vec{m} + E\vec{n}$$

where $\{\vec{u}, \vec{l}, \vec{m}, \vec{n}\}$ is any basis of the tangential space. In particular we will choose the *orthonormal* basis given in Appendix C, which corresponds to a comoving observer, because it gives raise to simpler expressions in this case. Since $\vec{V}$ is timelike one has

$$A^2 = 1 + B^2 + C^2 + E^2.$$

From Eq. (12) one can get

$$\begin{aligned} 8\pi T_{VV} &= -(D+2G) - (D+4G)(B^2 - A^2) \\ &= 2G + (D+4G)(C^2 + E^2), \end{aligned}$$

and

$$8\pi T = 2D,$$

whence

$$T_{VV} + (1/2)T = (D+2G) + (D+4G)(C^2 + E^2).$$

We can get a more direct expression, since a direct computation shows that

$$D + 4G = -\frac{\Sigma}{r} G'. \tag{20}$$

Finally, this can be written in terms of the pressure, stress and energy density measured by the comoving observer using

$$8\pi\rho = 2G, \quad 8\pi p_{rad} = -\rho, \quad 8\pi p_{tan} = (D+2G).$$

The result is

$$T_{VV} = \rho - \frac{\Sigma}{2r}\rho'(C^2 + E^2),$$

$$T_{VV} + \frac{1}{2}T = p_{tan} - \frac{\Sigma}{2r}\rho'(C^2 + E^2), \tag{21}$$

where $\rho' = \partial_r \rho$. It follows that (i) the SEC is satisfied if and only if $p_{tan} \geqslant 0$ and $\rho' \leqslant 0$ in the region of study. (ii) The WEC is satisfied if and only if $\rho \geqslant 0$ and $\rho' \leqslant 0$ in the region of study.

Notice that ρ' plays an essential role in both cases, besides the more natural quantities p_{tan} and ρ. We can now analyze what happens in each region. The exterior field has $f_{KN}(r) = mr - e^2/2$. Therefore, $\rho = p_{tan} = e^2/\Sigma^2$. Whence, one has $\rho, p_{tan} > 0$ and $\rho' < 0$. SEC and WEC are thus obviously satisfied. For the rest of the analysis, it is worth considering first the nonrotating case, where $\rho = 6\alpha$, $p_{tan} = -6\alpha$ and $\rho' = 0$. Therefore, for $\alpha > 0$ (de Sitter) WEC is satisfied whereas SEC is not and the singularity is avoided. For $\alpha < 0$ (anti–de Sitter) WEC is clearly violated and the singularity is again avoided.

As long as we aim at describing general properties, it is worth keeping the freedom of choice of $f(r)$ in the shell region. The main conclusion is that only for a de Sitter core $(\alpha > 0)$ plus an exterior electromagnetic field satisfying $6\alpha r_1^4/e^2 > 1$ (which includes the uncharged case) may the WEC be satisfied throughout the whole system. The reason is as follows: if $6\alpha r_1^4/e^2 < 1$ then $\rho_0 (=6\alpha)$ is smaller than $\rho_{ext} (=\rho_{KN} = e^2/r_1^4)$ at r_1. Therefore, the function $f(r)$ in the transition region—which is at least C^1 in that region—must be increasing in some open interval.

In fact the restriction $6\alpha r_1^4/e^2 > 1$ is satisfied in most previous attempts [5,6,8,9,11,13,15]. Surprisingly, in the models [19,20], one can easily see that the WEC and SEC are found to be violated. For this it suffices to compute the density ρ in these models. In the models of [19] one has $f(r) = [mr^4/(r^2+e^2)^{3/2}]\exp(-e^2/2mr)$ and therefore (recall that these examples are nonrotating solutions) $8\pi\rho = 2G = 2\,(f'r - f)/r^4 = [e^2/r(r^2+e^2)^{3/2}]\,[6mr/(r^2+e^2) + 1]\exp(-e^2/2mr)]$. For the models in [20] one has $f(r) = mr[1 - \tanh(e^2/2mr)]$ and $8\pi\rho = e^2/[r^4\cosh^2(e^2/2mr)]$. Now, in both cases one has $\rho(r \to 0^+) \to 0^+$ and $\rho(r \to +\infty) \to 0^+$. Consequently, there are open regions where ρ is an increasing function and therefore SEC and WEC are violated. This adds new examples of SEC and/or WEC violations (see, e.g., [40] for a recent review), due in this case to non-linear electrodynamics, and also is a warning about the actual relevance of their fulfillment.

Finally, let us consider the rotating case. In the core one has $\rho = 6\alpha r^4/\Sigma^2$, $p_{tan} = -6\alpha r^2(r^2 + 2a^2\cos^2\theta)/\Sigma^2$ and $\rho' = 12\alpha a^2 r^3 \cos^2\theta/\Sigma^3$. For $\alpha > 0$, it is clear that both SEC and WEC (because $\rho' > 0$) are violated. For $\alpha < 0$, it is clear that WEC is violated, although the SEC is satisfied.

The main conclusion is that the WEC and SEC are—unavoidably—violated in the rotating case, except at the equatorial plane, which follows the pattern of the nonrotating case, already explained above. For the case of NEC, similar computations bring to the conclusion that it is fulfilled if and only if $\rho' < 0$ and therefore, previous considerations show that it is again generically *violated* inside the object.

At first sight, this might be considered as a drawback of these models. However, the whole thing is saying *only* that the models cannot account for a *classical* interior of a Kerr-Newman spacetime, so that the nature of the source should be sought within quantum field theory. But this, in turn, is often considered to be the natural framework to work in (e.g., assuming interior models suggested by supergravity or string theory).

VI. CONTRIBUTIONS TO THE TOTAL MASS COMING FROM DIFFERENT REGIONS OF THE REGULARIZED SOURCES

There are basically two ways to study the energy contributions from different parts of the source. One consists in evaluating the contributions coming from both the energy density and from the pressures of the system, whereas the other takes into account the contribution of the energy density only. The first is called the *Tolman mass* of the system and the latter, the *ADM mass*. In order for these quantities to be well defined it is necessary that the system be stationary and asymptotically flat (see, e.g., [41,35]). Our models do fulfill such a requirement. These masses are computed for an observer at rest with respect to the asymptotically flat region. The tetrad corresponding to such an observer is explicitly given in Appendixes A and B in terms of the Kerr tetrad forms and the Kerr coordinates.

Given the previous conditions, the Tolman mass is defined by

$$M_{\mathrm{Tol}}=\int_{\Omega}dx^3\,\sqrt{-g}(T^1_1+T^2_2+T^3_3-T^0_0) \tag{22}$$

(numerical values of the indexes refer to asymptotically Cartesian components). It explicitly takes into account the contributions to the gravitational mass coming from the energy density and the pressure of the matter forming the source [42,41] (Ω is a region of the source). Now, using the expressions for the Kerr tetrad forms and the stress energy tensor given in Appendixes A and B, we get

$$M_{\mathrm{Tol}}=(8\pi)^{-1}\int_0^{\pi}d\theta\int_0^{2\pi}d\phi\int_{r_a}^{r_b}dr \times 2[D+2G+(D+4G)a^2\sin^2\theta/\Sigma]\Sigma\,\sin\theta. \tag{23}$$

On the other hand, and given the aforementioned conditions, ADM mass is defined by

$$M_{\mathrm{ADM}}=-\int_{\Omega}dx^3\sqrt{-g}\,T^0_0 \tag{24}$$

and can be expressed as follows:

$$M_{\mathrm{ADM}}=(8\pi)^{-1}\int_0^{\pi}d\theta\int_0^{2\pi}d\phi\int_{r_a}^{r_b}dr \times[2G+(D+4G)a^2\sin^2\theta/\Sigma]\Sigma\,\sin\theta. \tag{25}$$

Another way of writting these quantities is

$$M_{\mathrm{Tol}}=-(4\pi)^{-1}\int_{\Omega}dx^3\sqrt{-g}R_{uu}, \tag{26}$$

$$M_{\mathrm{ADM}}=\int_{\Omega}dx^3\sqrt{-g}T_{uu}, \tag{27}$$

where u is the timelike vector of the Kerr tetrad, i.e. $u=e^{4'}-(1/2)e^{3'}$ using the notation of Appendixes A and B. The corresponding observer is at rest with respect to the asymptotically flat region. From these expressions, one sees that the positivity of the Tolman mass is related with an averaged form of the strong energy condition, while the ADM mass is (obviously) connected with the weak energy condition. As we shall see, the fact that our models are nonsingular and therefore that the strong energy condition must be violated (at least close to the core) will be reflected in the negative values of the Tolman mass in some regions.

We will describe the main properties of the sources coming from an analysis of the Tolman and ADM masses for each of the three regions of the model: interior (core), transition (shell) and exterior region. We will also estimate their relative contributions.

A. Noncharged, nonrotating sources

It is instructive to consider first the simplest case of regularized sources for nonrotating, neutral black hole solutions. In this case the treatment is very transparent and, moreover, it exhibits the main peculiarities common to all the models. In particular, there is an unexpectedly large contribution to the total mass coming from the thin shell on the boundary of the source.

Let us start with the Tolman mass. Setting $a=e=0$ in the relation (23), we get

$$M=(8\pi)^{-1}\int d\theta\int d\phi\int dr2(D+2G)r^2\sin\theta \tag{28}$$

which, in terms of the function $f(r)$, takes the form

$$(8\pi)^{-1}\int_0^{\pi}d\theta\int_0^{2\pi}d\phi\int_{r_a}^{r_b}dr2(2f'r-2f-r^2f'')r^{-2}\sin\theta \times\int_{r_a}^{r_b}(2f'r-2f-r^2f'')r^{-2}dr =[2(f/r)-f']_{r_a}^{r_b}. \tag{29}$$

In this case the model is composed of three regions

$$f(r)=\begin{cases} f_{\mathrm{int}}=\alpha r^4, & 0\leqslant r_0,\\ f_{\mathrm{shell}}(r), & r_0\leqslant r\leqslant r_0(1+\delta),\\ f_{\mathrm{ext}}=mr, & r_0(1+\delta)\leqslant r.\end{cases} \tag{30}$$

For the interior region we obtain

$$M_{\mathrm{int}}=-[2\alpha r^3]_0^{r_0}=-2\alpha r_0^3. \tag{31}$$

Consider the case of positive α (de Sitter core) and assume the shell to be thin, i.e. $\delta\ll 1$, the position of the shell is determined, to first order in δ, by equation $mr_0\approx\alpha r_0^4$, i.e. $m\approx\alpha r_0^3$, and consequently

$$M_{int} \approx -2m. \tag{32}$$

Therefore, we obtain a remarkable result: the de Sitter core gives a *negative* contribution $-2m$ to the total mass of the source. Since the total mass is determined by the parameter m, this means that the shell on the boundary of the core has to give a contribution $+3m$ to the total mass. Indeed, calculating Eq. (28) for the region with $f(r)=f_{\rm shell}$, and assuming again the shell to be thin, we obtain

$$M_{\rm shell} = -f'_{ext}|_{r=r_0(1+\delta)} + f'_{int}|_{r=r_0} + \mathcal{O}(\delta) \approx 3m, \tag{33}$$

providing the balance of the total mass $M_{\rm int}+M_{\rm shell}=m$ (the exterior contribution being zero in this case). One should note that here the shell is assumed to be sufficiently thin but not necessarily infinitely thin. Besides, the exact form of the function $f_{\rm shell}$, and consequently the matter distribution on the shell, are not essential, giving a contribution of order δ. Irrespectively of the value of δ, the contribution from the internal de Sitter core is always negative. We could say that the energetic properties of the core account for the avoidance of the singularity.

Let us now compare the result with the ADM mass ($a=e=0$)

$$M_{\rm ADM} = (8\pi)^{-1}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_{r_a}^{r_b} dr\, 2Gr^2 \sin\theta = [f/r]_{r_a}^{r_b}. \tag{34}$$

It is clear that the ADM mass does not take into account the pressure components of the stress-energy tensor, and consequently, it ''does not feel'' the shell region, i.e. $M_{\rm ADM,shell}=\mathcal{O}(\delta)$. However, the ADM mass obviously yields the right value for the total mass of the source $M_{\rm ADM,int+shell}=m$. The solution of this apparent contradiction can be found in that, at least for the regularized sources with a thin shell, the contribution of the pressure components to the total mass can be viewed as representing a gravitational ''dipole'' $(-3m, +3m)$ which forms the bubble.

B. General case. Charged and rotating source

In the general charged and/or rotating case, it is worth writing all the expressions in terms of the function f. For the Tolman mass, we get

$$M_{\rm Tol} = \int_0^1 dx \int_{y_a}^{y_b} dy \times \frac{2(yf'_y - f)(y^2+2-x^2) - f''_y(x^2+y^2)(1+y^2)}{a(x^2+y^2)^2}, \tag{35}$$

where we have put $x \equiv \cos\theta$ and $y \equiv r/a$ $[f'_y \equiv df(y)/dy,\ f''_y \equiv df'_y/dy]$. This change makes the integrand dimensionless and allows one to have control on the limiting cases $a \ll r$ or $a \gg r$. After some integrations, we get the remarkable result that

$$M_{\rm Tol} = \left\{ \frac{f(r)}{r} + \frac{(a^2+r^2)}{ar} \arctan(a/r)[f(r)/r - f'(r)] \right\}_{y_a}^{y_b}. \tag{36}$$

Hence, the Tolman mass of a layer of the source may be obtained as the difference between the boundary values of a suitable *potential function* (the function between curly brackets above) for any fixed value of the angular momentum (including of course the nonrotating case).[2]

Let us now consider the contributions to the Tolman mass coming from each region (in the sequel we omit the tag ''Tol'' in the masses). We have

$$f(r) = \begin{cases} \alpha r^4, & 0 \leq r_0, \\ f_{\rm shell}(r), & r_0 \leq r \leq r_1, \\ mr - e^2/2, & r_1 \leq r. \end{cases} \tag{37}$$

For the core region, we get

$$M_{\rm core} = \alpha r_0^3 \left[1 - 3\left(\frac{a}{r_0} + \frac{r_0}{a} \right) \arctan\left(\frac{a}{r_0} \right) \right]. \tag{38}$$

In the limit $a/r_0 \ll 1$, e.g. low rotating black holes and fixed r_0, including the nonrotating limit, one gets

$$M_{\rm core} = \begin{cases} -2\alpha r_0^3 \left[1 + (a^2/r_0^2) - (a^4/5r_0^4) + \mathcal{O}\left(\frac{a^6}{r_0^6} \right) \right], \\ -2\alpha r_0^3, \quad a=0. \end{cases} \tag{39}$$

In the limit $r_0/a \ll 1$, e.g. rapidly rotating black holes or very small core ''radius'' r_0 with respect to the size of the removed singular ring—a typical case for parameters of spinning particles—we have

$$M_{\rm core} = -(3\pi\alpha r_0^2 a/2)\left[1 - (8r_0/3\pi a) + (r_0^2/a^2) + \mathcal{O}\left(\frac{r_0^3}{a^3} \right) \right]. \tag{40}$$

As in the case considered previously, for a de Sitter core the contribution of the core itself to the Tolman mass of the object is negative, for any value of r_0 and a, and satisfies $M_{\rm core} \leq -2\alpha r_0^3$. For an anti–de Sitter core the situation is opposite, that is, $M_{\rm core} \geq 2|\alpha| r_0^3$.

The Tolman mass of the exterior region comes from the electromagnetic field and is

$$M_{\rm exterior} = \frac{e^2}{2r_1}\left[1 + \left(\frac{r_1}{a} + \frac{a}{r_1} \right) \arctan\left(\frac{a}{r_1} \right) \right]. \tag{41}$$

Now the limits $a \ll r_1$ (including the nonrotating case) and $r_1 \ll a$ yield

[2]From this result it is easy to see that our matching condition $f(r) \in C^1$ turns out to be necessary and sufficient for the consistency of the Tolman mass.

$$M_{\rm exterior}=\begin{cases}(e^2/r_1)\left[1+(a^2/3r_1^2)-(a^4/15r_1^4)+\mathcal{O}\left(\dfrac{a^6}{r_1^6}\right)\right], & a\ll r_1\\ (e^2/r_1), & a=0\\ (\pi e^2a/4r_1^2)\left[1+(r_1^2/a^2)-(4r_1^3/3\pi a^3)+\mathcal{O}\left(\dfrac{r_1^5}{a^5}\right)\right], & r_1\ll a.\end{cases}\tag{42}$$

We now have to consider the contribution from the shell.

1. Thin shell

The first situation we will deal with is the thin shell approach. The thickness of the shell δ must satisfy the condition $\delta/r_0\equiv r_1/r_0-1\ll1$ (but not necessarily $\delta=0$). In this case only the continuity of the interpolating function $f(r)$ is necessary, when dismissing all terms of order δ or higher. The (only) matching condition is then

$$f_{\rm core}(r_0)=f_{\rm exterior}(r_0).\tag{43}$$

This approach allows us to recover from our models the previous ones involving the assumption of singular distributions both for the stationary and for the static cases.

The Tolman mass is, in this case,

$$\begin{aligned}M_{\rm thin\ shell}&=-[f_0'][1+(r_0^2/a^2)]\int_0^1\frac{dx}{x^2+(r_0/a)^2}\\&=-[f_0']\left(\frac{a}{r_0}+\frac{r_0}{a}\right)\arctan\left(\frac{a}{r_0}\right),\end{aligned}\tag{44}$$

where $[f_0']\equiv f'_{\rm exterior}(r_0)-f'_{\rm core}(r_0)=m-4\alpha r_0^3=-3m+2e^2/r_0$ and we have used the matching condition (43). First, we note that the *total* Tolman mass of the model is *m*, as expected. That is, $(\chi\equiv e^2/2mr_0)$,

$$\begin{aligned}&M_{\rm core}+M_{\rm thin\ shell}+M_{\rm exterior}\\&=m(1-\chi)\left[1-3\left(\frac{a}{r_0}+\frac{r_0}{a}\right)\arctan\left(\frac{a}{r_0}\right)\right]\\&\quad+m(3-4\chi)\left(\frac{a}{r_0}+\frac{r_0}{a}\right)\arctan\left(\frac{a}{r_0}\right)\\&\quad+m\chi\left[1+\left(\frac{r_1}{a}+\frac{a}{r_1}\right)\arctan\left(\frac{a}{r_1}\right)\right]\\&=m.\end{aligned}\tag{45}$$

We next consider the relative contribution of each part. These will clearly depend on the ratio χ.

2. Astrophysical sources

One can see that for the case of astrophysical (neutral or weakly charged) sources, where $\chi=0$ or $\chi\ll1$ the thin shell gives the major contribution to the total mass $1\leqslant|M_{\rm thin\ shell}/M_{\rm core}|\leqslant3/2$, for any value of r_0 and a. In this case the core gives a negative contribution to the Tolman mass, whereas the shell yields a positive one—bigger in absolute value than the core mass.

3. Strongly charged sources: particle-like solutions

Let us first note that, classically, the electromagnetic mass of a charged sphere of radius r_0 is given by $M_{\rm cl-em}=e^2/2r_0$. Therefore, the parameter χ can be expressed as $\chi=M_{\rm cl-em}/m$. The case $\chi\sim1$ corresponds to strongly charged sources, where an essential part of the mass is thought to be of electromagnetic origin, as for example in classical models of the electron. The relation between the core and the shell depends on the value of χ in this case.[3] In particular, for $\chi=1$ the core does not yield any contribution to the mass. This result comes from Eq. (43), since $\alpha r_0^3=m-M_{\rm cl-em}=0$. Therefore, $\alpha=0$ and the core is flat in this case. On the other hand, the thin shell contribution to the mass is negative

$$M_{\rm thin\ \ shell}=-m\left(\frac{a}{r_0}+\frac{r_0}{a}\right)\arctan\left(\frac{a}{r_0}\right).\tag{46}$$

For nonrotating sources, setting $a=0$ one obtains $M_{\rm thin\ shell}=-m$ and $M_{\rm exterior}=2m$. In this case one can show that $M_{\rm exterior}$ splits into a pure electromagnetic contribution $M_{\rm cl-em}=m$ and a gravitational contribution to the mass coming from the electromagnetic field $M_{\rm grav-em}=m$, and providing the balance[4] $M_{\rm thin\ shell}+M_{\rm clem}+M_{\rm grav-em}=m$. In the limit of a singular shell, this case corresponds to the classical model of a charged particle considered by Cohen and Cohen [33], which is a modification of the known Dirac classical model for an extended electron [43].

For $\chi>1$, it follows from Eq. (43) that $\alpha r_0^3=m-M_{\rm clem}=m(1-\chi)$, and consequently, $\alpha<0$. Thus, there must be an anti–de Sitter space in the core. Graphical analysis immediately shows that in this case the characteristic radius r_0 is smaller than the classical one. The relation between the contributions of the core and of the shell is found to be $4/3\leqslant|M_{\rm thin\ shell}/M_{\rm core}|\leqslant2$, where now the core yields a positive contribution to the total Tolman mass, and the shell a negative one. Notice that, except for the case of $\chi=3/4$, the Tol-

[3]This situation occurs also in the case of [18–20], in which solutions for nonrotating regularized black holes are given.

[4]Similarly to the models in [23,27].

man mass of the object undergoes a sudden change when passing the shell. This is because, except for that case, $M_{\text{thin shell}} \neq 0$ and $\delta \approx 0$.

C. Comparison with the ADM expression for the total mass

From expressions (25), one can see that, contrary to what happens in the nonrotating case, the term D does give a contribution, due to a Lorentz effect associated with the rotation of the source. Performing the integrations in analogy with previous calculations, one can obtain the result that the ADM mass may also be interpreted as coming from a *potential function*. The expression is

$$M_{\text{ADM}} = \frac{1}{2}\left\{ \frac{f}{r} + f' + \frac{(a^2+r^2)}{ar}\arctan(a/r)[f/r - f'] \right\}_{r_a}^{r_b}. \tag{47}$$

Therefore, the ADM mass of the core region is

$$M_{\text{ADM, core}} = \alpha r_0^3 \left[3 - \frac{1}{2}\left(\frac{a}{r_0} + \frac{r_0}{a} \right) \arctan\left(\frac{a}{r_0} \right) \right]. \tag{48}$$

The ADM mass for the exterior region comes from the pure electromagnetic part and is twice smaller than the Tolman one (since it does not take into account the gravitational contribution of the electromagnetic pressure). The expression is therefore

$$M_{\text{exterior}}^{\text{ADM}} = \frac{e^2}{4r_1}\left[1 + \left(\frac{r_1}{a} + \frac{a}{r_1} \right) \arctan\left(\frac{a}{r_1} \right) \right]. \tag{49}$$

Finally, the ADM contribution from the thin shell is

$$M_{\text{ADM, thin shell}} = ([f_0']/2)\left[-1 - \left(\frac{r_1}{a} + \frac{a}{r_1} \right) \arctan\left(\frac{a}{r_1} \right) \right]. \tag{50}$$

One can see that in the case when the position of the thin shell is determined by the matching condition (43), the balance of the total ADM mass is also attained (the general case being also fulfilled from expression (47) and $f \in C^1$), i.e.

$$M_{\text{core}}^{\text{ADM}} + M_{\text{thin shell}}^{\text{ADM}} + M_{\text{ext}}^{\text{ADM}} = m.$$

Finally, let us notice that the ADM and Tolman masses are related through a simple expression, namely,

$$2M_{\text{ADM}} - M_{\text{Tol}} = [f'(r)]_{r_a}^{r_b}. \tag{51}$$

VII. CONCLUSIONS

We have here considered a wide class of smooth sources for black holes, which includes virtually all the models considered previously in the literature and extends them to the rotating case, thus obtaining a unified framework for arbitrary values of the charge and angular momentum. For nonrotating BH solutions the sources contain a core region representing a spacetime with a constant curvature (Λ term) and a thin (but finite) transitional region (spherical shell) connecting the source with an external black hole geometry. In the case of rotating BHs this shell acquires an ellipsoidal form (a disk of radius $\sqrt{a^2+r_0^2}$ and thickness r_0). For a thin shell, the rotation can be considered as rigid, with angular velocity given by $\omega = a/(a^2+r_0^2)$. For large angular momentum, $a \gg r_0$, the rotation is relativistic and disklike sources are highly oblate. The curvature is not strictly constant in the interior of the disk and it is concentrated in a tube-like neighborhood of the former Kerr singular ring near the border of the disk (the expressions for the curvature and stress-energy tensor for this general case are given in Sec. II, and a complete description is given in the Appendixes). This class of sources includes previous models like [5,6,9,15,18–20] generalizing them to the rotating case, and contains the smooth analogs of known shell-like rotating and nonrotating models as well. In particular, for a special choice of parameters, $r_0 = e^2/2m$, the model acquires a flat interior and turns into the López model of the Kerr-Newman source [23]. However, we have shown that, in the general case, the bubble interior can have both a positive or a negative scalar curvature. The matching condition (43) permits up to connect the parameters of the sources r_0, e, m and $\alpha = \Lambda/6$, and to select the sources which are consistent with respect to the mass-energy balance. Indeed, the balance relation $\alpha r_0^3 + e^2/2r_0 = m$ shows that uncharged black hole solutions have positive α, corresponding to the de Sitter interior of the core region, while otherwise for charged, "small" black holes with $e^2/2r_0 > m$, the sources acquire a negative α, yielding an anti–de Sitter spacetime in the core. Such anti–de Sitter regions are also predicted for strong gravitational fields in supergravity.

Analysis and comparison of the Tolman and ADM mass relations allows one to determine contributions to the total mass going from diverse regions of the source. Both expressions give the correct result at spacelike infinity. However, the ADM expression M_{ADM} does not take into account the gravitational contribution to the total mass coming from the strong tension (or pressure) of the thin transition shell. This contribution can be estimated by using the Tolman relation and shows a new, interesting feature of these models. Indeed, we prove that the contribution to the total mass coming from the thin shell can be extremely large, but it is anyway canceled by the contribution from the core. Therefore, it represents a very strong gravitational "polarization" of the spacetime in the form of a bubble with a sharp and very strong boundary. This phenomenon could very likely have observational manifestations.

In spite of the successful description here presented of virtually all the nonrotating models and of their extension to rotating (i.e. Kerr-Newman) models, the Kerr-Schild class of metrics might turn out to be too restrictive in order to be able to describe some self-consistent field models. In particular, the Casimir effect for a superdense state in the core can be essential [37,38,28]. To construct a field matter model leading to a tangential stress of the shell, scalar fields can be involved [29] in the formation of an object similar to a domain wall boundary of the bubble. However, the Kerr-Schild class (in four dimensions) is hardly compatible with simple models of classical scalar fields, which can be seen from the relation $T^{KS}_{ik}e^{3i}e^{3k}=T^{KS}_{44}=0$ (which is one of the conditions in the derivation of the Kerr-Schild class of metrics [34].) This argument can also be expressed in terms of quantum corrections and the conformal anomaly for scalar fields [15]. This means that either the Kerr-Schild class of the (BH interior) source models has to be modified, to take into account these and other features of the "desired" source, or this has to be done with the field model. In particular, the following close generalizations can be suggested for future work in this field: (i) conformal Kerr-Schild metrics (one of whose representatives is the Nariai solution [44]); (ii) inclusion of dilaton and axion fields; (iii) field models in supergravity [29] and low energy string theory [45]; and (iv) extension to higher dimensions, in particular based on the AdS/conformal field theory (CFT) correspondence ([46], see also the recent review [47]).

Note added. After this paper was finished, we were informed that the advantages of the Kerr coordinates in the analysis of rotating black holes had also been considered in the papers by J. Ibanez, P. Papadopoulos and J. Font [48], and that a close problem was investigated, in the nonrotating case, in the paper by A. DeBenedictis *et al.* [49]. We are thankful to J. Font and A. DeBenedictis for bringing these works to our attention and for their comments.

ACKNOWLEDGMENTS

One of us (A.B.) would like to thank G. Alekseev for stimulating discussion. This investigation has been supported by DGI/SGPI (Spain), project BFM2000-0810, by CIRIT (Generalitat de Catalunya), contract 1999SGR-00257, and by the program INFN (Italy)–DGICYT (Spain).

APPENDIX A: STRESS-ENERGY TENSOR FOR THE GENERALIZED KERR-SCHILD METRICS

Starting from the following general form of the function h:

$$h=f(r)/(\Sigma P^2), \tag{A1}$$

and the Kerr-Schild null tetrad

$$e^1=d\zeta-Ydv \tag{A2}$$

$$e^2=d\bar{\zeta}-\bar{Y}dv \tag{A3}$$

$$e^3=du+\bar{Y}d\zeta+Yd\bar{\zeta}-Y\bar{Y}dv \tag{A4}$$

$$e^4=dv+he^3, \tag{A5}$$

we obtain, using the machinery of the Kerr-Schild formalism [34], the following tetrad components of the Ricci tensor[5]

$$R_{12}=-2G \tag{A6}$$

$$R_{34}=D+2G \tag{A7}$$

$$R_{12}-R_{34}=-(D+4G) \tag{A8}$$

$$R_{23}=(D+4G)r,_2/P \tag{A9}$$

$$R_{13}=(D+4G)r,_1/P \tag{A10}$$

$$R_{33}=-2r,_1r,_2(D+4G)/P^2, \tag{A11}$$

where

$$D=-f''/(r^2+a^2\cos^2\theta), \tag{A12}$$

$$G=(f'r-f)/(r^2+a^2\cos^2\theta)^2. \tag{A13}$$

This expression (A11) can be transformed into another, more convenient tetrad [Eqs. (6.1) of Debney, Kerr, and Schild (DKS) [34]] $e^{a'}$, which is connected with the Kerr angular coordinates (r,t,θ,ϕ) determined by the relations

$$x+iy=(r+ia)e^{i\phi}\sin\theta, \tag{A14}$$

$$z=r\cos\theta, \tag{A15}$$

$$\rho=r+t. \tag{A16}$$

The reverse of the [(6.1) of the DKS paper] relations are

$$e^1=e^{1'}+P_{\bar{Y}}e^{3'}, \tag{A17}$$

$$e^2=e^{2'}+P_Ye^{3'}, \tag{A18}$$

$$e^3=Pe^{3'}, \tag{A19}$$

$$e^4=P^{-1}[e^{4'}-P_Ye^{1'}-P_{\bar{Y}}e^{2'}-e^{3'}P_YP_{\bar{Y}}]. \tag{A20}$$

They yield a new expression for the Ricci tensor

$$R'_{12}=R_{12}, \tag{A21}$$

$$R'_{34}=R_{34}, \tag{A22}$$

$$R'_{23}=PR_{23}+P_{\bar{Y}}(R_{12}-R_{34}), \tag{A23}$$

$$R'_{13}=PR_{13}+P_Y(R_{12}-R_{34}), \tag{A24}$$

$$R'_{33}=P^2R_{33}+2P(P_YR_{23}+P_{\bar{Y}}R_{13})+2P_YP_{\bar{Y}}(R_{12}-R_{34}). \tag{A25}$$

As a result,

$$R'_{12}=-2G, \tag{A26}$$

$$R'_{34}=D+2G, \tag{A27}$$

$$R'_{12}-R'_{34}=-(D+4G), \tag{A28}$$

$$R'_{23}=(D+4G)(r,_2-P_{\bar{Y}}), \tag{A29}$$

$$R'_{13}=(D+4G)(r,_1-P_Y), \tag{A30}$$

[5]These calculations are very tedious and use the extra relations given in Appendix B.

$$R'_{33}=-2(D+4G)(r,_1-P_Y)(r,_2-P_{\bar{Y}}). \quad (A31)$$

The primed tetrad [Eqs. (6.5)–(6.6) of the DKS paper] takes the form

$$e^{1'}=2^{-1/2}e^{i\phi}(r+ia\cos\theta)(d\theta+i\sin\theta d\phi), \quad (A32)$$

$$e^{2'}=2^{-1/2}e^{-i\phi}(r-ia\cos\theta)(d\theta-i\sin\theta d\phi), \quad (A33)$$

$$e^{3'}=dr+dt-a\sin^2\theta d\phi, \quad (A34)$$

$$e^{4'}=dr-a\sin^2\theta d\phi+\frac{1}{2}(2h-1)e^{3'}. \quad (A35)$$

Dropping the primes, we obtain the following expressions for the energy-momentum tensor:

$$8\pi T_{ab}=-R_{ab}+\frac{1}{2}g_{ab}R, \quad (A36)$$

$$8\pi T_{ik}=(D+2G)g_{ik} -(D+4G)(e^3_i\tilde{e}^4_k+\tilde{e}^4_i e^3_k), \quad (A37)$$

where

$$\tilde{e}^4=e^4+(r,_1-P_Y)e^1+(r,_2-P_{\bar{Y}})e^2 -(r,_1-P_Y)(r,_2-P_{\bar{Y}})e^3 =\frac{1}{2}[dr-(dt-a\sin^2\theta d\phi)]+\frac{2f-a^2\sin^2\theta}{2\Sigma}e^3. \quad (A38)$$

This expression for the energy-momentum tensor coincides with the result obtained by Gürses and Gürsey.

APPENDIX B: TETRAD FORMS AND REPRESENTATION OF THE KERR-SCHILD CLASS OF METRICS IN THE KERR AND BOYER-LINDQUIST ANGULAR COORDINATES

The Kerr-Schild class of metrics has the form

$$g_{ik}=\eta_{ik}+2hk_ik_k, \quad (B1)$$

where $\eta_{ik}=\mathrm{diag}(-1,1,1,1)$ is the metric of an auxiliary Minkowski space with Cartesian coordinates t,x,y,z. and $h=f(r)/(r^2+a^2\cos^2\theta)$. The Kerr angular coordinates (r,t,θ,ϕ) are determined by the relations

$$x+iy=(r+ia)e^{i\phi}\sin\theta, \quad (B2)$$

$$z=r\cos\theta, \quad (B3)$$

$$\rho=r+t. \quad (B4)$$

In these coordinates, the metric tensor has the form

$$g_{(Kerr)ik}=\begin{pmatrix} 2h-1 & 2h & 0 & -2ha\sin^2\theta \\ 2h & 1+2h & 0 & -(1+2h)a\sin^2\theta \\ 0 & 0 & \Sigma & 0 \\ -2ha\sin^2\theta & -(1+2h)a\sin^2\theta & 0 & (r^2+a^2+2ha^2\sin^2\theta)\sin^2\theta \end{pmatrix}, \quad (B5)$$

where $\Sigma=r^2+a^2\cos^2\theta$. The determinant is $\det g_{Kerr}=-\Sigma^2\sin^2\theta$, and the contravariant form of the metric is

$$g^{ik}_{(Kerr)}=\begin{pmatrix} -(1+2h) & 2h & 0 & 0 \\ 2h & \Delta/\Sigma & 0 & a/\Sigma \\ 0 & 0 & 1/\Sigma & 0 \\ 0 & a/\Sigma & 0 & (\Sigma\sin^2\theta)^{-1} \end{pmatrix}, \quad (B6)$$

where $\Delta=r^2+a^2-2f(r)$.

The Kerr null tetrad [Eqs. (6.5)–(6.6) of the DKS paper] has the form

$$e^{1'}=2^{-1/2}e^{i\phi}(r+ia\cos\theta)(d\theta+i\sin\theta d\phi), \quad (B7)$$

$$e^{2'}=2^{-1/2}e^{-i\phi}(r-ia\cos\theta)(d\theta-i\sin\theta d\phi), \quad (B8)$$

$$e^{3'}=dr+dt-a\sin^2\theta d\phi, \quad (B9)$$

$$e^{4'}=dr-a\sin^2\theta d\phi+\frac{1}{2}(2h-1)e^{3'}. \quad (B10)$$

The contravariant components are

$$e^{1'i}=\frac{e^{i\phi}}{\sqrt{2}(r-ia\cos\theta)}(0,ia\sin\theta,1,1/\sin\theta), \quad (B11)$$

$$e^{2'i}=\frac{e^{-i\phi}}{\sqrt{2}(r+ia\cos\theta)}(0,-ia\sin\theta,1,1/\sin\theta),\tag{B12}$$

$$e^{3'i}=(-1,1,0,0),\tag{B13}$$

$$e^{4'i}=\frac{1}{2}(1+2h,1-2h,0,0).\tag{B14}$$

One can see that the expression for the stress-energy tensor is simplified by the introduction of the null vector $\tilde{e}^4=e^{4'}-Ce^{1'}-\bar{C}e^{2'}-C\bar{C}e^{3'}$, where $C=r_{,1}-P_Y$. The vector $\tilde{e}^4$ belongs to the null tetrad obtained from $e^{a'}$ by a "null rotation" [34] leaving e^3 unchanged $\tilde{e}^3=e^{3'}$. The corresponding null tetrad is completed as follows: $\tilde{e}^1=e^{1'}+\bar{C}e^{3'}$ and $\tilde{e}^2=e^{2'}+Ce^{3'}$. This tetrad is connected with the Boyer-Lindquist representation of the Kerr geometry, in which a symmetry between the null vectors $\tilde{e}^3$ and $\tilde{e}^4$ appears.

APPENDIX C: KERR-SCHILD METRICS IN BOYER-LINDQUIST COORDINATES

Boyer-Lindquist coordinates $\bar{t},r,\theta,\bar{\phi}$ are connected with the Kerr angular coordinates t,r,θ,ϕ by the relations $dt=d\bar{t}+(2f/\Delta)dr$ and $d\phi=d\bar{\phi}+(a/\Delta)dr$, where $\Delta=r^2+a^2-2f(r)$. In Boyer-Lindquist coordinates, the tetrad $\tilde{e}^a$ takes the form

$$\tilde{e}^1=2^{-1/2}e^{i\phi}(r+ia\cos\theta)\left(d\theta+i\sin\theta\frac{r^2+a^2}{\Sigma}d\bar{\phi}-\frac{ia\sin\theta}{\Sigma}d\bar{t}\right),\tag{C1}$$

$$\tilde{e}^2=2^{-1/2}e^{-i\phi}(r-ia\cos\theta)\left(d\theta-i\sin\theta\frac{r^2+a^2}{\Sigma}d\bar{\phi}+\frac{ia\sin\theta}{\Sigma}d\bar{t}\right),\tag{C2}$$

$$\tilde{e}^3=\frac{\Sigma}{\Delta}dr+(d\bar{t}-a\sin^2\theta d\bar{\phi}),\tag{C3}$$

$$\tilde{e}^4=\frac{\Delta}{2\Sigma}\left[\frac{\Sigma}{\Delta}dr-(d\bar{t}-a\sin^2\theta d\bar{\phi})\right].\tag{C4}$$

In Boyer-Lindquist coordinates, the metric tensor takes the form (bars are omitted everywhere)

$$g_{(BL)ik}=\begin{pmatrix} 2f/\Sigma-1 & 0 & 0 & -2af\sin^2\theta/\Sigma \\ 0 & \Sigma/\Delta & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ -2af\sin^2\theta/\Sigma & 0 & 0 & \left(r^2+a^2+\dfrac{2fa^2\sin^2\theta}{\Sigma}\right)\sin^2\theta \end{pmatrix}.\tag{C5}$$

The determinant is $\det g_{BL}=-\Sigma^2\sin^2\theta$.

The contravariant form of the metric is

$$g^{(BL)ik}=\begin{pmatrix} -\dfrac{r^2+a^2+(2f/\Sigma)a^2\sin^2\theta}{\Delta} & 0 & 0 & -\dfrac{2af}{\Sigma\Delta} \\ 0 & \Delta/\Sigma & 0 & 0 \\ 0 & 0 & 1/\Sigma & 0 \\ -\dfrac{2af}{\Sigma\Delta} & 0 & 0 & \dfrac{1-2f/\Sigma}{\Delta\sin^2\theta} \end{pmatrix},\tag{C6}$$

The orthonormal tetrad in the Boyer-Lindquist coordinates has the form

$$u=-\sqrt{\frac{\Delta}{\Sigma}}(dt-a\sin^2\theta d\phi),\tag{C7}$$

$$l=\sqrt{\frac{\Sigma}{\Delta}}dr,\tag{C8}$$

$$n=\sqrt{\Sigma}d\theta,\tag{C9}$$

$$m=\frac{\sin\theta}{\sqrt{\Sigma}}[a\,dt-(r^2+a^2)d\phi], \tag{C10}$$

where u is the unit timelike vector and m the radial one. The corresponding contravariant components are

$$u^i=\frac{1}{\sqrt{\Delta\Sigma}}(r^2+a^2,0,0,a), \tag{C11}$$

$$l^i=\sqrt{\frac{\Delta}{\Sigma}}(0,1,0,0), \tag{C12}$$

$$n^i=\frac{1}{\sqrt{\Sigma}}(0,0,1,0), \tag{C13}$$

$$m^i=\frac{-1}{\sqrt{\Sigma}\sin\theta}(a\sin^2\theta,0,0,1). \tag{C14}$$

The null vector forms $\tilde{e}^3$ and $\tilde{e}^4$ can be expressed via u and l as follows $\tilde{e}^3=\sqrt{\Sigma/\Delta}(l-u)$, and $\tilde{e}^4=\frac{1}{2}\sqrt{\Delta/\Sigma}(l+u)$.

Some useful relations

The following relations are useful for the transition from the Kerr to the BL coordinate system:

$$r,_2-P_{\bar{Y}}=-ia(\cos\theta),_2=\frac{iae^{i\phi}\sin\theta}{\sqrt{2}(r-ia\cos\theta)},$$

$$r,_1-P_Y=ia(\cos\theta),_1=\frac{-iae^{-i\phi}\sin\theta}{\sqrt{2}(r+ia\cos\theta)},$$

$$(r,_1-P_Y)e^1+(r,_2-P_{\bar{Y}})e^2=a\sin^2\theta d\phi,$$

$$(r,_1-P_Y)(r,_2-P_{\bar{Y}})=a^2\sin^2\theta/(2\Sigma).$$

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