Dynamics of a self-gravitating lightlike matter shell: A gauge-invariant Lagrangian and Hamiltonian description

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A complete Lagrangian and Hamiltonian description of the theory of self-gravitating lightlike matter shells is given in terms of gauge-independent geometric quantities. For this purpose the notion of an extrinsic curvature for a null-like hypersurface is discussed and the corresponding Gauss-Codazzi equations are proved. These equations imply Bianchi identities for spacetimes with null-like, singular curvature. The energy-momentum tensor density of a lightlike matter shell is unambiguously defined in terms of an invariant matter Lagrangian density. The Noether identity and Belinfante-Rosenfeld theorem for such a tensor density are proved. Finally, the Hamiltonian dynamics of the interacting "gravity+matter" system is derived from the total Lagrangian, the latter being an invariant scalar density.

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I. INTRODUCTION

A self-gravitating matter shell [1,2] became an important laboratory for testing the global properties of a gravitational field interacting with matter. Models of a thin matter layer allow us to construct useful minisuperspace examples. Toy models of quantum gravity, based on these examples, may give us a deeper insight into a possible future shape of the quantum theory of gravity [3]. Especially interesting are null-like shells, carrying a self-gravitating lightlike matter [4]. Classical equations of motion of such a shell have been derived by Barrabès and Israel in their seminal paper [5].

In the present paper we give a complete Lagrangian and Hamiltonian description of a physical system composed of a gravitational field interacting with a lightlike matter shell. The paper contains two main results, which, in our opinion, improve slightly the existing classical theory of a null-like shell and provide an appropriate background for its quantized version. The first result is the use of fully gauge-invariant, intrinsic geometric objects encoding the physical properties of both the shell (as a null-like surface in spacetime [6]) and the lightlike matter living on the shell. We begin with a description of an "extrinsic curvature" of a null-like hypersurface S in terms of a mixed contravariant-covariant tensor density $Q^a{}_b$—an appropriate null-like analogue of the Arnowitt-Deser-Misner (ADM) momentum (cf. [7]). For a nondegenerate (timelike or spacelike) hypersurface, the extrinsic curvature may be described in many equivalent ways: by tensors or tensor densities, both of them in the contravariant, covariant, or mixed version. In a null-like case, the degenerate metric on S does not allow us to convert tensors into tensor densities and vice versa. Also, we are not allowed to raise covariant indices, whereas lowering the contravariant indices is not an invertible operator and leads to information loss. It turns out that only the mixed tensor density $Q^a{}_b$ has the appropriate null-like limit and enables us to formulate the theory of a null-like shell in full analogy with the nondegenerate case. We prove the Gauss-Codazzi equations for the extrinsic curvature described by this tensor density. In particular, the above notion of an extrinsic curvature may be applied to analyze the structure of nonexpanding horizons [8].

The quantity $Q^a{}_b$ defined in Sec. III enables us to consider spacetimes with singular (distributionlike) curvature confined to a null-like hypersurface, and to prove that the Bianchi identities (understood in the sense of distributions) are necessarily satisfied in this case. Such spacetimes are a natural arena for the theory of a null-like matter shell.

The second main result consists in treating the lightlike matter in a fully dynamical (and not *phenomenological*) way. All the properties of the matter are encoded in a matter Lagrangian, which is an invariant scalar density on S (no invariant scalar Lagrangian exists at all for such matter, because conversion from scalar densities to scalars and vice versa is impossible). The Lagrangian gives rise to a gauge-invariant energy-momentum tensor density $T^a{}_b$, which later—due to Einstein equations—arises as a source of gravity. Both Noether and Belinfante-Rosenfeld identities for the quantity $T^a{}_b$ are proved: they are necessary for the consistence of the theory. We stress that the contravariant symmetric energy-momentum tensor T^{ab} cannot be defined unambiguously, whereas the covariant tensor T_{ab}, obtained by lowering the index with the help of a degenerate metric on S, loses partial information contained in $T^a{}_b$. On the contrary, the mixed contravariant-covariant tensor density $T^a{}_b$ is unambiguously defined and contains—as in the nondegenerate case—the entire dynamical information about the underlying matter.

In Sec. VI we use a method of variation of the total (gravity+matter) Lagrangian proposed in Ref. [9] and derive in this way the Barrabès-Israel equations for gravity, together with the dynamical equations for the matter degrees of freedom. In Sec. VII we show how to organize the gravitational and matter degrees of freedom into a constrained Hamiltonian system, with the ADM mass at infinity playing the role of the total (gravity+matter) Hamiltonian. Finally, the structure of constraints is analyzed in Sec. VIII. To clarify the exposition of geometric and physical ideas some of the technical proofs have been shifted to the Appendixes.

II. INTRINSIC GEOMETRY OF A NULL HYPERSURFACE

A null hypersurface in a Lorentzian spacetime M is a three-dimensional submanifold $S\subset M$ such that the restriction g_{ab} of the spacetime metrics $g_{\mu\nu}$ to S is degenerate. We shall often use adapted coordinates, where coordinate x^3 is constant on S. Space coordinates will be labeled by k, $l=1,2,3$; coordinates on S will be labeled by a, $b=0,1,2$; finally, coordinates on $S_t:=V_t\cap S$ (where V_t is a Cauchy surface corresponding to a constant value of the coordinate $x^0=t$) will be labeled by A, $B=1,2$. Spacetime coordinates will be labeled by Greek characters α, β, μ, ν.

We always assume in the following that x^0 is a timelike coordinate in the four-dimensional sense, i.e., that the following inequality holds: $(dx^0)^2=g^{00}<0$ (see Appendix A for the properties of the metric in a neighborhood of S). We stress that a coordinate defined only on S cannot be called "timelike" or "spacelike" because the metric g_{ab} on S cannot be inverted, and consequently, there is no way to define the square of its differential. Our assumption about the timelike character of x^0 applies, therefore, to the four-dimensional coordinate and not to its three-dimensional restriction to the surface S.

The nondegeneracy of the spacetime metric implies that the metric g_{ab} induced on S from the spacetime metric $g_{\mu\nu}$ has a signature $(0,+,+)$. This means that there is a nonvanishing null-like vector field X^a on S, such that its four-dimensional embedding X^μ to M (in adapted coordinates $X^3=0$) is orthogonal to S. Hence, the covector $X_\nu=X^\mu g_{\mu\nu}=X^a g_{a\nu}$ vanishes on vectors tangent to S, and therefore the following identity holds:

$$X^a g_{ab}\equiv 0. \tag{1}$$

It is easy to prove [10] that integral curves of X^a, after a suitable reparametrization, are geodesic curves of the spacetime metric $g_{\mu\nu}$. Moreover, any null hypersurface S may always be embedded in a one-parameter congruence of null hypersurfaces.

We assume that topologically we have $S=\mathbb{R}^1\times S^2$. Since our considerations are purely local, we fix the orientation of the $\mathbb{R}^1$ component and assume that null-like vectors X describing degeneracy of the metric g_{ab} of S will be always compatible with this orientation. Moreover, we shall always use coordinates such that the coordinate x^0 increases in the direction of X, i.e., the inequality $X(x^0)=X^0>0$ holds. In these coordinates degeneracy fields are of the form $X=f(\partial_0-n^A\partial_A)$, where $f>0$, $n_A=g_{0A}$, and we raise indices with the help of the two-dimensional matrix $\tilde{\tilde{g}}^{AB}$, inverse to g_{AB}.

If by λ we denote the two-dimensional volume form on each surface $x^0=$const,

$$\lambda:=\sqrt{\det g_{AB}}, \tag{2}$$

then, for any degeneracy field X of g_{ab}, the following object

$$v_X:=\frac{\lambda}{X(x^0)}$$

is a scalar density on S. Its definition does not depend on the coordinate system (x^a) used in the above definition. To prove this statement it is sufficient to show that the value of v_X gets multiplied by the determinant of the Jacobi matrix when we pass from one coordinate system to another. This means that $\mathbf{v}_X:=v_X dx^0\wedge dx^1\wedge dx^2$ is a coordinate-independent differential three-form on S. However, v_X depends upon the choice of the field X.

It follows immediately from the above definition that the following object,

$$\Lambda=v_X X,$$

is a well defined (i.e., coordinate-independent) vector density on S. Obviously, it *does not depend* upon any choice of the field X:

$$\Lambda=\lambda(\partial_0-n^A\partial_A). \tag{3}$$

Hence, it is an intrinsic property of the internal geometry g_{ab} of S. The same is true for the divergence $\partial_a\Lambda^a$, which is therefore an invariant, X-independent, scalar density on S. Mathematically (in terms of differential forms), the quantity Λ represents the two-form

$$\mathbf{L}:=\Lambda^a(\partial_a\rfloor dx^0\wedge dx^1\wedge dx^2),$$

whereas the divergence represents its exterior derivative (a three-form): $d\mathbf{L}=(\partial_a\Lambda^a)dx^0\wedge dx^1\wedge dx^2$. In particular, a null surface with vanishing $d\mathbf{L}$ is called a *nonexpanding horizon* [8].

Both objects $\mathbf{L}$ and $\mathbf{v}_X$ may be defined geometrically, without any use of coordinates. For this purpose we note that at each point $x\in S$, the tangent space T_xS may be quotiented with respect to the degeneracy subspace spanned by X. The quotient space carries a nondegenerate Riemannian metric and therefore is equipped with a volume form ω (its coordinate expression would be $\omega=\lambda dx^1\wedge dx^2$). The two-form $\mathbf{L}$ is equal to the pull-back of ω from the quotient space to T_xS. The three-form $\mathbf{v}_X$ may be defined as a product, $\mathbf{v}_X=\alpha\wedge\mathbf{L}$, where α is *any* one-form on S, such that $\langle X,\alpha\rangle\equiv 1$.

The degenerate metric g_{ab} on S does not allow us to define via the compatibility condition $\nabla g=0$, any natural connection, which could apply to generic tensor fields on S. Nevertheless, there is one exception: we are going to show that the degenerate metric defines *uniquely* a certain covariant, first-order differential operator that will be extensively used in our paper. The operator may be applied only to mixed (contravariant-covariant) tensor-density fields $\mathbf{H}^a{}_b$, satisfying the following algebraic identities:

$$\mathbf{H}^a{}_b X^b = 0, \tag{4}$$

$$\mathbf{H}_{ab} = \mathbf{H}_{ba}, \tag{5}$$

where $\mathbf{H}_{ab} := g_{ac}\mathbf{H}^c{}_b$. Its definition cannot be extended to other tensorial fields on S. Fortunately, as will be seen, the extrinsic curvature of a null-like surface and the energy-momentum tensor of a null-like shell are described by tensor densities of this type.

The operator, which we denote by $\overline{\nabla}_a \mathbf{H}^a{}_b$, could be defined by means of the four-dimensional metric connection in the ambient spacetime M in the following way. Given $\mathbf{H}^a{}_b$, take any of its extensions $\mathbf{H}^{\mu\nu}$ to a four-dimensional, symmetric tensor density, "orthogonal" to S, i.e., satisfying $\mathbf{H}^{\perp\nu} = 0$ ($\perp$ denotes the component transversal to S). Define $\overline{\nabla}_a \mathbf{H}^a{}_b$ as the restriction to S of the four-dimensional covariant divergence $\nabla_\mu \mathbf{H}^\mu{}_\nu$. As will be seen in the following, ambiguities that arise when extending the three-dimensional object $\mathbf{H}^a{}_b$ on S to the four-dimensional one finally cancel, and the result is unambiguously defined as a covector density on S. It turns out, however, that this result does not depend upon the spacetime geometry and may be defined intrinsically on S. This is why we first give this intrinsic definition in terms of the degenerate metric.

In case of a nondegenerate metric, the covariant divergence of a symmetric tensor $\mathbf{H}$ density may be calculated by the following formula:

$$\begin{aligned}\nabla_a \mathbf{H}^a{}_b &= \partial_a \mathbf{H}^a{}_b - \mathbf{H}^a{}_c \Gamma^c_{ab} \\ &= \partial_a \mathbf{H}^a{}_b - \tfrac{1}{2}\mathbf{H}^{ac} g_{ac,b},\end{aligned} \tag{6}$$

with $g_{ac,b} := \partial_b g_{ac}$. In the case of our degenerate metric, we want to mimic the last formula, but here raising of indices of $\mathbf{H}^a{}_b$ makes no sense. Nevertheless, formula (6) may be given a unique sense also in the degenerate case, if applied to a tensor density $\mathbf{H}^a{}_b$ satisfying identities (4) and (5). Namely, we take as $\mathbf{H}^{ac}$ *any* symmetric tensor density that reproduces $\mathbf{H}^a{}_b$ when lowering an index:

$$\mathbf{H}^a{}_b = \mathbf{H}^{ac} g_{cb}. \tag{7}$$

It is easily seen that such a tensor density always exists due to identities (4) and (5), but reconstruction of $\mathbf{H}^{ac}$ from $\mathbf{H}^a{}_b$ is not unique, because $\mathbf{H}^{ac} + CX^aX^c$ also satisfies Eq. (7) if $\mathbf{H}^{ac}$ does. Conversely, two such symmetric tensors $\mathbf{H}^{ac}$ satisfying Eq. (7) may differ only by CX^aX^c. This nonuniqueness does not influence the value of Eq. (6), because of the following identity implied by Eq. (1):

$$\begin{aligned}0 &\equiv (X^aX^c g_{ac})_{,b} \\ &= X^aX^c g_{ac,b} + 2X^a g_{ac} X^c{}_b \\ &= X^aX^c g_{ac,b}.\end{aligned} \tag{8}$$

Hence, the following definition makes sense:

$$\overline{\nabla}_a \mathbf{H}^a{}_b := \partial_a \mathbf{H}^a{}_b - \tfrac{1}{2}\mathbf{H}^{ac} g_{ac,b}. \tag{9}$$

The right-hand side does not depend upon any choice of coordinates (i.e., transforms similar to a genuine covector density under a change of coordinates). The proof is straightforward and does not differ from the standard case of formula (6), when the metric g_{ab} is nondegenerate.

To express directly the result in terms of the original tensor density $\mathbf{H}^a{}_b$, we observe that it has five independent components and may be uniquely reconstructed from $\mathbf{H}^0{}_A$ (two independent components) and the symmetric two-dimensional matrix $\mathbf{H}_{AB}$ (three independent components). Indeed, identities (4) and (5) may be rewritten as follows:

$$\mathbf{H}^A{}_B = \tilde{\tilde{g}}^{AC}\mathbf{H}_{CB} - n^A \mathbf{H}^0{}_B, \tag{10}$$

$$\mathbf{H}^0{}_0 = \mathbf{H}^0{}_A n^A, \tag{11}$$

$$\mathbf{H}^B{}_0 = (\tilde{\tilde{g}}^{BC}\mathbf{H}_{CA} - n^B \mathbf{H}^0{}_A) n^A. \tag{12}$$

There is a one-to-one correspondence between $\mathbf{H}^a{}_b$ and $(\mathbf{H}^0{}_A, \mathbf{H}_{AB})$.

To reconstruct $\mathbf{H}^{ab}$ from $\mathbf{H}^a{}_b$ up to an arbitrary additive term CX^aX^b, take the following, coordinate-dependent, symmetric quantity:

$$\mathbf{F}^{AB} := \tilde{\tilde{g}}^{AC}\mathbf{H}_{CD}\tilde{\tilde{g}}^{DB} - n^A \mathbf{H}^0{}_C \tilde{\tilde{g}}^{CB} - n^B \mathbf{H}^0{}_C \tilde{\tilde{g}}^{CA}, \tag{13}$$

$$\mathbf{F}^{0A} := \mathbf{H}^0{}_C \tilde{\tilde{g}}^{CA} =: \mathbf{F}^{A0}, \tag{14}$$

$$\mathbf{F}^{00} := 0. \tag{15}$$

It is easy to observe that any $\mathbf{H}^{ab}$ satisfying Eq. (7) must be of the form

$$\mathbf{H}^{ab} = \mathbf{F}^{ab} + \mathbf{H}^{00} X^a X^b. \tag{16}$$

The nonuniqueness in the reconstruction of $\mathbf{H}^{ab}$ is therefore completely described by the arbitrariness in the choice of the value of $\mathbf{H}^{00}$. Using these results we finally obtain

$$\begin{aligned}\overline{\nabla}_a \mathbf{H}^a{}_b &:= \partial_a \mathbf{H}^a{}_b - \tfrac{1}{2}\mathbf{H}^{ac} g_{ac,b} = \partial_a \mathbf{H}^a{}_b - \tfrac{1}{2}\mathbf{F}^{ac} g_{ac,b} \\ &= \partial_a \mathbf{H}^a{}_b - \tfrac{1}{2}(2\mathbf{H}^0{}_A n^A{}_{,b} - \mathbf{H}_{AC}\tilde{\tilde{g}}^{AC}{}_{,b}).\end{aligned} \tag{17}$$

The operator on the right-hand side of Eq. (17) may thus be called the (three-dimensional) covariant derivative of $\mathbf{H}^a{}_b$ on S with respect to its degenerate metric g_{ab}. We have just proved that it is well defined (i.e., coordinate-independent) for a tensor density $\mathbf{H}^a{}_b$ fulfilling conditions (4) and (5).

Equation (9) suggests yet another definition of the covariant divergence operator. At a given point $x \in S$ choose any coordinate system such that derivatives of the metric components g_{ac} vanish at x, i.e., $g_{ac,b}(x) = 0$. Such a coordinate system may be called *inertial*. The covariant divergence may thus be defined as a partial divergence but calculated in an inertial system: $\overline{\nabla}_a \mathbf{H}^a{}_b := \partial_a \mathbf{H}^a{}_b$. Ambiguities in the choice of an inertial system do not allow us to extend this definition to a genuine covariant derivative $\overline{\nabla}_c \mathbf{H}^a{}_b$. However, it may be easily checked that they are sufficiently mild for an unambiguous definition of the divergence (see the Remark at the end of Sec. V).

The above two equivalent definitions of the operator $\overline{\nabla}$ use only the intrinsic metric of S. We want to prove now that they coincide with the definition given in terms of the four-dimensional spacetime metric connection. For that purpose observe that the only nonuniqueness in the reconstruction of the four-dimensional tensor density of $\mathbf{H}^{\mu\nu}$ is of the type $CX^{\mu}X^{\nu}$. Indeed, any such reconstruction may be obtained from a reconstruction of $\mathbf{H}^{ac}$ by setting $\mathbf{H}^{3\nu}=0$ in a coordinate system adapted to S (i.e., such that the coordinate x^3 remains constant on S). Now, calculate the four-dimensional covariant divergence $\mathbf{H}_{\nu}:=\nabla_{\mu}\mathbf{H}^{\mu}{}_{\nu}$. Because of the geodesic character of integral curves of the field X, the only nonuniqueness that remains after this operation is of the type $\tilde{C}X_{\nu}$. Hence, the restriction $\mathbf{H}_b$ of $\mathbf{H}_{\nu}$ to S is already unique. Because of Eq. (6), it equals

$$\nabla_{\mu}\mathbf{H}^{\mu}{}_{b}=\partial_{\mu}\mathbf{H}^{\mu}{}_{b}-\tfrac{1}{2}\mathbf{H}^{\mu\lambda}g_{\mu\lambda,b}$$
$$=\partial_{a}\mathbf{H}^{a}{}_{b}-\tfrac{1}{2}\mathbf{H}^{ac}g_{ac,b}=\overline{\nabla}_{a}\mathbf{H}^{a}{}_{b}. \qquad (18)$$

III. EXTRINSIC GEOMETRY OF A NULL HYPERSURFACE: GAUSS-CODAZZI EQUATIONS

To describe the exterior geometry of S we begin with covariant derivatives along S of the orthogonal vector X. Consider the tensor $\nabla_a X^{\mu}$. Unlike in the nondegenerate case, there is no unique "normalization" of X, and therefore such an object does depend upon a choice of the field X. The length of X is constant (because vanishes). Hence, the tensor is again orthogonal to S, i.e., the components corresponding to $\mu=3$ vanish identically in adapted coordinates. This means that $\nabla_a X^b$ is a purely three-dimensional tensor situated on S. For our purposes it is useful to use the "ADM-like" version of this object, defined in the following way:

$$Q^{a}{}_{b}(X):=-s\{v_X(\nabla_b X^a-\delta^a_b\nabla_c X^c)+\delta^a_b\partial_c\Lambda^c\}, \qquad (19)$$

where $s:=\operatorname{sgn} g^{03}=\pm 1$. Because of the above convention, the extrinsic curvature $Q^a{}_b(X)$ detects only the *external orientation* of S and does not detect any internal orientation of the field X.

Remark. If S is a *nonexpanding horizon*, the last term in the above definition vanishes.

The last term in Eq. (19) is X independent. It has been introduced in order to correct algebraic properties of the quantity $v_X\ (\nabla_b X^a-\delta^a_b\nabla_c X^c)$: we prove in the Appendix A [see the Remark after Eq. (A26)] that $Q^a{}_b$ satisfies identities (4) and (5) and, therefore its covariant divergence with respect to the degenerate metric g_{ab} on S is uniquely defined. This divergence enters into the Gauss-Codazzi equations that we are going to formulate now. Gauss-Codazzi equations relate the divergence of Q with the transversal component $\mathcal{G}^{\perp}_b$ of the Einstein tensor density $\mathcal{G}^{\mu}{}_{\nu}=\sqrt{|\det g|}(R^{\mu}{}_{\nu}-\delta^{\mu}_{\nu}\frac{1}{2}R)$. The transversal component of such a tensor density is a well-defined three-dimensional object situated on S. In a coordinate system adapted to S, i.e., such that the coordinate x^3 is constant on S, we have $\mathcal{G}^{\perp}{}_b=\mathcal{G}^3{}_b$. Because of the fact that $\mathcal{G}$ is a tensor density, components $\mathcal{G}^3{}_b$ *does not change* with changes of the coordinate x^3, provided it remains constant on S. These components describe, therefore, an intrinsic covector density situated on S.

Proposition 1. The following null-like surface version of the Gauss-Codazzi equation is true:

$$\overline{\nabla}_a Q^a{}_b(X)+s v_X\partial_b\left(\frac{\partial_c\Lambda^c}{v_X}\right)\equiv-\mathcal{G}^{\perp}{}_b. \qquad (20)$$

We remind the reader that the ratio between two scalar densities, $\partial_c\Lambda^c$ and v_X, is a scalar function. Its gradient is a covector field. Finally, multiplied by the density v_X, it produces an intrinsic covector density on S. This proves that the left-hand side also is a well-defined, geometric object situated on S.

To prove consistency of Eq. (20), we must show that the left-hand side does not depend upon a choice of X. For this purpose consider another degeneracy field: fX, where $f>0$ is a function on S. We have

$$-sQ^a{}_b(fX)=v_{fX}(\nabla_b(fX^a)-\delta^a_b\nabla_c(fX^c))+\delta^a_b\partial_c\Lambda^c$$
$$=\frac{1}{f}v_X(f\nabla_b X^a+X^a\partial_b f-\delta^a_b f\nabla_c X^c$$
$$-\delta^a_b X^c\partial_c f)+\delta^a_b\partial_c\Lambda^c$$
$$=-sQ^a{}_b(X)+\Lambda^a\varphi_{,b}-\delta^a_b\Lambda^c\varphi_{,c}, \qquad (21)$$

where $\varphi:=\log f$. It is easy to see that the tensor

$$q^a{}_b(\varphi):=\Lambda^a\varphi_{,b}-\delta^a_b\Lambda^c\varphi_{,c} \qquad (22)$$

satisfies identity (4). Moreover, $q_{ab}=-g_{ab}\Lambda^c\varphi_{,c}$, which proves Eq. (5). On the other hand, we have

$$v_{fX}\partial_b\left(\frac{\partial_c\Lambda^c}{v_{fX}}\right)=v_X\partial_b\left(\frac{\partial_c\Lambda^c}{v_X}\right)+(\partial_c\Lambda^c)\varphi_{,b} \qquad (23)$$

But, using formula (17) we immediately get

$$\overline{\nabla}_a q^a{}_b(\varphi)=(\partial_c\Lambda^c)\varphi_{,b},$$

which proves that the left-hand side of Eq. (20) does not depend upon any choice of the field X. The complete proof of the Gauss-Codazzi equation (20) is given in Appendix A.[1]

[1]In the nondegenerate case, there are four independent Gauss-Codazzi equations: besides: $\mathcal{G}^{\perp}{}_b$, there is an additional equation relating $\mathcal{G}^{\perp}{}_{\perp}$ with the (external and internal) geometry of S. In the degenerate case, the vector orthogonal to S is—at the same time—tangent to it. Hence, $\mathcal{G}^{\perp}{}_{\perp}$ is a combination of quantities $\mathcal{G}^{\perp}{}_b$ and there are only three independent Gauss-Codazzi equations.

IV. BIANCHI IDENTITIES FOR SPACETIMES WITH DISTRIBUTION VALUED CURVATURE

In this paper we consider a space-time M with distribution valued curvature tensor in the sense of Taub [11]. This means that the metric tensor, although continuous, is not necessarily smooth in C^1 across S: we assume that the connection coefficients $\Gamma^{\lambda}_{\mu\nu}$ may have only step discontinuities (jumps) across S. Formally, we may calculate the Riemann curvature tensor of such a spacetime, but derivatives of these discontinuities with respect to the variable x^3 produce a $\boldsymbol{\delta}$-like, singular part of R:

$$\mathrm{sing}(R)^{\lambda}{}_{\mu\nu\kappa}=(\delta^3_{\nu}[\Gamma^{\lambda}_{\mu\kappa}]-\delta^3_{\kappa}[\Gamma^{\lambda}_{\mu\nu}])\boldsymbol{\delta}(x^3), \tag{24}$$

where by $\boldsymbol{\delta}$ we denote the Dirac distribution (in order to distinguish it from the Kronecker symbol δ) and by $[f]$ we denote the jump of a discontinuous quantity f between the two sides of S. The formula above is invariant under *smooth* transformations of coordinates. There is, however, no sense in imposing such a smoothness across S. In fact, the smoothness of spacetime is an independent condition on both sides of S. The only reasonable assumption imposed on the differentiable structure of M is that the metric tensor—which is smooth separately on both sides of S—remains continuous[2] across S. Admitting coordinate transformations preserving the above condition, we lose some of the information contained in quantity (24), which now becomes coordinate dependent. It turns out, however, that another part, namely the Einstein tensor density calculated from Eq. (24), preserves its geometric, intrinsic (i.e., coordinate-independent) meaning. In the case of a nondegenerate geometry of S, the following formula was used by many authors [1–3,12,13]:

$$\mathrm{sing}(\mathcal{G})^{\mu\nu}=\mathbf{G}^{\mu\nu}\boldsymbol{\delta}(x^3), \tag{25}$$

where the "transversal-to-S" part of $\mathbf{G}^{\mu\nu}$ vanishes identically:

$$\mathbf{G}^{\perp\nu}\equiv 0, \tag{26}$$

and the "tangent-to-S" part $\mathbf{G}^{ab}$ equals the jump of the ADM extrinsic curvature Q^{ab} of S between the two sides of the surface:

$$\mathbf{G}^{ab}=[Q^{ab}]. \tag{27}$$

This quantity is a purely *three-dimensional*, symmetric tensor density situated on S. When multiplied by the *one-dimensional* density $\boldsymbol{\delta}(x^3)$ in the transversal direction, it produces the *four-dimensional* tensor density $\mathcal{G}$ according to formula (25).

Now, let us come back to the case of our degenerate surface S. One of the goals of the present paper is to prove that formulas (25) and (26) remain valid also in this case. In particular, the latter formula means that the four-dimensional quantity $\mathcal{G}^{\mu\nu}$ reduces in fact to an intrinsic, three-dimensional quantity situated on S. However, formula (27) cannot be true, because—as we have seen—there is no way to define uniquely the object Q^{ab} for the degenerate metric on S. Instead, we are able to prove the following formula:

$$\mathbf{G}^{a}{}_{b}=[Q^{a}{}_{b}(X)], \tag{28}$$

where the bracket denotes the jump of $Q^{a}{}_{b}(X)$ between the two sides of the singular surface. Observe that this quantity does not depend upon any choice of X. Indeed, formula (21) shows that Q changes identically on both sides of S when we change X and, hence, these changes cancel. This proves that the singular part $\mathrm{sing}(\mathcal{G})^{a}{}_{b}$ of the Einstein tensor is well defined.

Remark. Otherwise, as in the nondegenerate case, the contravariant components $\mathbf{G}^{ab}$ in formula (25) do not transform as a tensor density on S. Hence, the quantity defined by these components would be coordinate dependent. According to Eq. (28), $\mathbf{G}$ becomes an intrinsic three-dimensional tensor density on S only after lowering an index, i.e., in the version of $\mathbf{G}^{a}{}_{b}$. This proves that $\mathbf{G}^{\mu\nu}$ may be reconstructed from $\mathbf{G}^{a}{}_{b}$ up to an additive term $CX^{\mu}X^{\nu}$ only. We stress that the dynamics of the shell, which we discuss in the sequel, is unambiguously expressed in terms of the gauge-invariant, intrinsic quantity $\mathbf{G}^{a}{}_{b}$. Proofs of the above facts are given in the Appendix A.

We conclude that the total Einstein tensor of our spacetime is a sum of the regular part[3] reg $(\mathcal{G})$ and the singular part sing $(\mathcal{G})$ above existing on the singularity surface S. Thus

$$\mathcal{G}^{\mu}{}_{\nu}=\mathrm{reg}(\mathcal{G})^{\mu}{}_{\nu}+\mathrm{sing}(\mathcal{G})^{\mu}{}_{\nu}, \tag{29}$$

and the singular part is given *up to an additive term* $CX^{\mu}X_{\nu}\boldsymbol{\delta}(x^3)$. Due to Eq. (8), the following *four-dimensional* covariant divergence is unambiguously defined:

$$0=\nabla_{\mu}\mathcal{G}^{\mu}{}_{c}=\partial_{\mu}\mathcal{G}^{\mu}{}_{c}-\mathcal{G}^{\mu}{}_{\alpha}\Gamma^{\alpha}_{\mu c}=\partial_{\mu}\mathcal{G}^{\mu}{}_{c}-\tfrac{1}{2}\mathcal{G}^{\mu\lambda}g_{\mu\lambda,c}\,. \tag{30}$$

We are going to prove that this quantity vanishes identically. Indeed, the regular part of this divergence vanishes on both sides of S due to Bianchi identities: $\mathrm{reg}(\nabla_{\mu}\mathcal{G}^{\mu}{}_{c})\equiv 0$. As a next step we observe that the singular part is proportional to $\boldsymbol{\delta}(x^3)$, i.e., that the Dirac $\boldsymbol{\delta}$ contained in sing $(\mathcal{G})$ will not be differentiated, when we apply the above covariant derivative to the singular part (25). This is true because $\mathrm{sing}(\mathcal{G})^{3}{}_{\nu}=0$. Hence, only the covariant divergence of $\mathbf{G}$ along S [multiplied by $\boldsymbol{\delta}(x^3)$] remains. Another $\boldsymbol{\delta}$-like term is obtained from $\partial_{\mu}\mathcal{G}^{\mu}{}_{c}$, when applied to the (piecewise continuous)

[2] Many authors insist in relaxing this condition and assuming only the continuity of the three-dimensional intrinsic metric on S. We stress that the (apparently stronger) continuity condition for the four-dimensional metric does not lead to any loss of generality and may be treated as an additional, technical gauge imposed *not upon the physical system* but upon its mathematical parametrization. We discuss thoroughly this issue in a Remark at the end of this section.

[3] The regular part is a smooth tensor density on both sides of the surface S (calculated for the metric g separately) with a possible step discontinuity across S.

regular part of $\mathcal{G}$. This way we obtain the term $[\mathrm{reg}(\mathcal{G})^{\perp}{}_c]\boldsymbol{\delta}(x^3)$. Finally, the total singular part of the Bianchi identities reads

$$\mathrm{sing}(\nabla_\mu \mathcal{G}^\mu{}_c) = ([\mathrm{reg}(\mathcal{G})^{\perp}{}_c] + \overline{\nabla}_a \mathbf{G}^a{}_b)\boldsymbol{\delta}(x^3) \equiv 0 \qquad (31)$$

and vanishes identically due to the Gauss-Codazzi equation (20), when we calculate its jump across S. Hence, we have proved that the Bianchi identity $\nabla_\mu \mathcal{G}^\mu{}_c \equiv 0$ holds universally (in the sense of distributions) for spacetimes with singular, lightlike curvature.

It is worthwhile to notice that the last term in definition (19) of the tensor density Q of S is identical on its both sides. Hence, its jump across S vanishes identically. In this way the singular part of the Einstein tensor density (28) reduces to:

$$\mathbf{G}^a{}_b = [Q^a{}_b] = -s v_X([\nabla_b X^a] - \delta^a_b[\nabla_c X^c]). \qquad (32)$$

Remark. The possibility of defining the singular Einstein tensor and its divergence via the standard formulas of Riemannian geometry (but understood in the sense of distribution) simplifies considerably the mathematical description of the theory. This techniques is based, however, on the continuity assumption for the four-dimensional metric. This is not a geometric or physical condition imposed on the system, but only the coordinate (gauge) condition. Indeed, whenever the three-dimensional, internal metric on S is continuous, also the remaining four components of the total metric can be made continuous by a simple change of coordinates. In this new coordinate system we may use our techniques based on the theory of distributions and derive both the Lagrangian and the Hamiltonian version of the dynamics of the total (''gravity+shell'') system. As will be seen in the following, the dynamics derived this way does not depend upon our gauge condition and is expressed in terms of equations that also apply to general coordinates. As an example of such an equation consider formula (28) which—even if derived here by technique of distributions under more restrictive conditions—remains valid universally. We stress that even in a smooth, vacuum spacetime (no shell at all) one can consider nonsmooth coordinates, for which only the internal metric g_{ab} on a given surface, say $\{x^3 = C\}$, is continuous, whereas the remaining four components $g_{3\mu}$ may have jumps. The entire canonical gravity may be formulated in these coordinates. In particular, the Cauchy surfaces $\{x^0 = \mathrm{const}\}$ would be allowed to be nonsmooth here. Nobody uses such a formulation (even if it is fully legitimate) because of its relative complexity: the additional gauge condition imposing the continuity of the whole four-dimensional metric makes life much easier.

V. ENERGY-MOMENTUM TENSOR OF A LIGHTLIKE MATTER: BELINFANTE-ROSENFELD IDENTITY

The goal of this paper is to describe the interaction between a thin lightlike matter shell and the gravitational field. We derive all the properties of such a matter from its Lagrangian density L. It may depend upon (nonspecified) matter fields z^K living on a null-like surface S, together with their first derivatives $z^K{}_a := \partial_a z^K$ and—of course—the (degenerate) metric tensor g_{ab} of S:

$$L = L(z^K; z^K{}_a; g_{ab}). \qquad (33)$$

We assume that L is an invariant scalar density on S. Similarly as in the standard case of canonical field theory, invariance of the Lagrangian with respect to reparametrizations of S implies important properties of the theory: the Belinfante-Rosenfeld identity and the Noether theorem, which will be discussed in this section. To get rid of some technicalities, we assume in this paper that the matter fields z^K are ''spacetime scalars,'' like, e.g., material variables of any thermomechanical theory of continuous media (see, e.g., Refs. [12,14]). This means that the Lie derivative $\mathcal{L}_Y z$ of these fields with respect to a vector field Y on S coincides with the partial derivative:

$$(\mathcal{L}_Y z)^K = z^K{}_a Y^a.$$

The following lemma characterizes Lagrangians that fulfill the invariance condition.

Lemma V.1. The Lagrangian density (33) concentrated on a null hypersurface S is invariant if and only if it is of the form

$$L = v_X f(z; \mathcal{L}_X z; g), \qquad (34)$$

where X is any degeneracy field of the metric g_{ab} on S and $f(\cdot;\cdot;\cdot)$ is a scalar function and is homogeneous of degree 1 with respect to its second variable.

Proof of the Lemma and examples of invariant Lagrangians for different lightlike matter fields are given in Appendix C.

Remark. Because of the homogeneity of f with respect to $\mathcal{L}_X z$, the above quantity does not depend upon a choice of the degeneracy field X.

Dynamical properties of such a matter are described by its canonical energy-momentum tensor density, defined in a standard way:

$$T^a{}_b := \frac{\partial L}{\partial z^K{}_a} z^K{}_b - \delta^a{}_b L. \qquad (35)$$

It is ''symmetric'' in the following sense:

Proposition 2. The Canonical energy-momentum tensor-density $T^a{}_b$ constructed from an invariant Lagrangian density fulfills identities (4) and (5), i.e., the following holds:

$$T^a{}_b X^b = 0 \quad \text{and} \quad T_{ab} = T_{ba}. \qquad (36)$$

Proof. For a Lagrangian density of the form (34) we have

$$T^a{}_b = \frac{\partial L}{\partial z^K{}_a} z^K{}_b - \delta^a{}_b L$$
$$= v_X\left(X^a \frac{\partial f}{\partial (z^K{}_d X^d)} z^K{}_b - \delta^a{}_b f \right), \qquad (37)$$

whence

$$T_{ab} = T^c{}_b g_{ca} = -v_X f g_{ab} = T_{ba}. \qquad (38)$$

Homogeneity of f with respect to the argument $(z^K{}_d X^d)$ implies

$$T^a{}_b X^b = v_X X^a \left(\frac{\partial f}{\partial (z^K{}_d X^d)} (z^K{}_b X^b) - f \right) = 0. \quad (39)$$

■

In the case of a nondegenerate geometry of S, one considers also the symmetric energy-momentum tensor density τ^{ab}, defined as follows:

$$\tau^{ab} := 2 \frac{\partial L}{\partial g_{ab}}. \quad (40)$$

In our case the degenerate metric fulfills the constraint $\det g_{ab} \equiv 0$. Hence, the above quantity *is not* uniquely defined. However, we may define it, but only *up to an additive term* equal to the annihilator of this constraint. It is easy to see that the annihilator is of the form $CX^a X^b$. Hence, ambiguity in the definition of the symmetric energy-momentum tensor is precisely equal to ambiguity in the definition of T^{ab}, if we want to reconstruct it from the well-defined object $T^a{}_b$. This ambiguity is canceled when we lower an index. We shall prove in the next theorem, that for field configurations satisfying field equations, both the canonical and the symmetric tensors coincide.[4] This is an analog of the standard Belinfante-Rosenfeld identity [16]. Moreover, the Noether theorem (vanishing of the divergence of T) is true. We summarize these facts in the following.

Proposition 3. If L is an invariant Lagrangian and if the field configuration z^K satisfies Euler-Lagrange equations derived from L,

$$\frac{\partial L}{\partial z^K} - \partial_a \frac{\partial L}{\partial z^K{}_a} = 0, \quad (41)$$

then the following statements are true.

(1) Belinfante-Rosenfeld identity: the canonical energy-momentum tensor $T^a{}_b$ coincides with (minus—because of the convention used) symmetric energy-momentum tensor τ^{ab}:

$$T^a{}_b = -\tau^{ac} g_{cb}, \quad (42)$$

(2) Noether theorem:

$$\overline{\nabla}_a T^a{}_b = 0. \quad (43)$$

Proof. Invariance of the Lagrangian with respect to spacetime diffeomorphisms generated by a vector field Y on S means that transporting the arguments $(z;\partial z;g)$ of L along Y gives the same result as transporting directly the value of the scalar density L on S:

$$\frac{\partial L}{\partial z^K} (\mathcal{L}_Y z)^K + \frac{\partial L}{\partial z^K{}_a} (\mathcal{L}_Y z)^K{}_a + \frac{\partial L}{\partial g_{ac}} (\mathcal{L}_Y g)_{ac} = \mathcal{L}_Y L. \quad (44)$$

Take for simplicity $Y = \partial/\partial x^b$ (or $Y^a = \delta^a{}_b$). Hence, we have $(\mathcal{L}_Y z)^K{}_a = z^K{}_{ba} = z^K{}_{ab}$. Applying this and rearranging terms in the above expression, we obtain

$$\left(\frac{\partial L}{\partial z^K} - \partial_a \frac{\partial L}{\partial z^K{}_a} \right) z^K{}_b + \partial_a \left(\frac{\partial L}{\partial z^K{}_a} z^K{}_b - \delta^a{}_b L \right) + \frac{\partial L}{\partial g_{ac}} g_{ac,b} = 0. \quad (45)$$

Because of the Euler-Lagrangian equations (41) and to the definitions (35) and (40) of both the energy-momentum tensors, the formula above reduces to the following statement:

$$\partial_a T^a{}_b + \tfrac{1}{2} \tau^{ac} g_{ac,b} = 0. \quad (46)$$

Our proof of this formula is valid in any coordinate system. In particular, we may use such a system for which all partial derivatives of the metric vanish at a given point $x \in S$. In this particular coordinate system we have

$$\overline{\nabla}_a T^a{}_b(x) = \partial_a T^a{}_b(x) = 0.$$

But $\overline{\nabla}_a T^a{}_b(x) = 0$ is a coordinate-independent statement: once proved in one coordinate system, it remains valid in any other system. Repeating this for all points $x \in S$ separately, we prove the Noether theorem (43). Subtracting now Eq. (46) from Eq. (43) we obtain the following identity:

$$T^{ab} g_{ab,c} = -\tau^{ab} g_{ab,c},$$

which must be true in any coordinate system. Here, both T^{ab} and τ^{ab} are defined only up to an additive term of the form $CX^a X^b$, which vanishes when multiplied by $g_{ab,c}$. In the standard Riemannian or Lorentzian geometry of a nondegenerate metric, the derivatives $g_{ab,c}$ may be freely chosen at each point separately, which immediately implies the Belinfante-Rosenfeld identity $T = -\tau$. In our case, the freedom in the choice of these derivatives is restricted by the constraint. This is the only restriction. Hence, the Belinfante-Rosenfeld identity is true only up to the annihilator of these constraints, i.e., only in the form of Eq. (42). ■

Remark. In a nondegenerate geometry, the vanishing of derivatives of the metric tensor at a point x uniquely defines a local "inertial system" at x. If two coordinate systems, say (x^a) and (y^a), fulfill this condition at x, then second derivatives of x^a with respect to y^b *vanish* identically at this point. The covariant derivative may thus be defined as a partial derivative, but calculated with respect to an inertial system, i.e., to any coordinate system of this class. In our degenerate case, vanishing of derivatives of the metric does not fix uniquely the inertial system. There are different coordinate systems (x^a) and (y^a), for which $g_{ab,c}$ vanishes at x, but we have

[4]In our convention, energy is described by the formula $H = T^0{}_0 = p_K{}^0 \dot{z}^K - L \geqslant 0$, analogous to $H = p\dot{q} - L$ in mechanics and well adapted for Hamiltonian purposes. This convention differs from the one used in Ref. [15], where energy is given by T_{00}. To keep standard conventions for Einstein equations, we take the standard definition of the *symmetric* energy-momentum tensor $\tau^a{}_b$. This is why the Belinfante-Rosenfeld theorem takes form $\tau^a{}_b = -T^a{}_b$.

$$\frac{\partial^2 y^a}{\partial x^b \partial x^c}(x) \neq 0.$$

This is why any attempt to define a covariant derivative for an arbitrary tensor on S fails. This ambiguity is, however, canceled by algebraic properties of our energy-momentum tensor, namely, by identities (4) and (5). This enables us to define unambiguously the covariant divergence of "energy-momentum-like" tensor densities using formula (9).

VI. DYNAMICS OF THE TOTAL GRAVITY+SHELL SYSTEM: LAGRANGIAN VERSION

In this paper we consider dynamics of a lightlike matter-shell discussed in the preceding section, interacting with gravitational field. We present here a method of derivation of the dynamical equations of the system, which applies also to a massive shell and follows the ideas of Ref. [12].

The dynamics of the gravity+shell system will be derived from the action principle $\delta\mathcal{A}=0$, where

$$\mathcal{A}=\mathcal{A}^{\rm reg}_{\rm grav}+\mathcal{A}^{\rm sing}_{\rm grav}+\mathcal{A}_{\rm matter} \tag{47}$$

is the sum of the gravitational action and the matter action. Gravitational action, defined as an integral of the Hilbert Lagrangian, splits into regular and singular parts, according to decomposition of the curvature:

$$L_{\rm grav}=\frac{1}{16\pi}\sqrt{|g|}R=\frac{1}{16\pi}\sqrt{|g|}(\mathrm{reg}(R)+\mathrm{sing}(R)) = L^{\rm reg}_{\rm grav}+L^{\rm sing}_{\rm grav}. \tag{48}$$

Using formulas (25)–(28), we express the singular part of R in terms of the singular part of the Einstein tensor:

$$\sqrt{|g|}\,\mathrm{sing}(R)=-\mathrm{sing}(\mathcal{G})=-\mathbf{G}^{\mu\nu}g_{\mu\nu}\boldsymbol{\delta}(x^3). \tag{49}$$

As analyzed in Sec. IV, an additive, coordinate-dependent ambiguity $CX^\mu X^\nu$ in the definition of $\mathbf{G}^{\mu\nu}$ is irrelevant, because it is canceled when contracted with $g_{\mu\nu}$:

$$\mathbf{G}^{\mu\nu}g_{\mu\nu}=\mathbf{G}^{ab}g_{ab}=\mathbf{G}^a{}_a\,.$$

For the matter Lagrangian $L_{\rm matter}$, we assume that it has properties discussed in the preceding section. Finally, the total action is the sum of three integrals:

$$\mathcal{A}=\int_D L^{\rm reg}_{\rm grav}+\int_D L^{\rm sing}_{\rm grav}+\int_{D\cap S}L_{\rm matter}, \tag{50}$$

where D is a four-dimensional region with boundary in spacetime M, which is possibly cut by a lightlike three-dimensional surface S (actually, because of the Dirac δ factor, the second term reduce to integration over $D\cap S$). Variation is taken with respect to the spacetime metric tensor $g_{\mu\nu}$ and to the matter fields z^K situated on S. The lightlike character of the matter considered here implies the lightlike character of S (i.e., degeneracy of the induced metric $\det g_{ab}=0$) as an additional constraint imposed on g.

We begin with varying the regular part $L^{\rm reg}_{\rm grav}$ of the gravitational action. There are many ways to calculate variation of the Hilbert Lagrangian. Here, we use a method proposed by one of us [9]. It is based on the following, simple observation:

$$\delta\left(\frac{1}{16\pi}\sqrt{|g|}g^{\mu\nu}R_{\mu\nu}\right)=-\frac{1}{16\pi}\mathcal{G}^{\mu\nu}\delta g_{\mu\nu} +\frac{1}{16\pi}\sqrt{|g|}g^{\mu\nu}\delta R_{\mu\nu}, \tag{51}$$

where

$$\mathcal{G}^{\mu\nu}:=\sqrt{|g|}(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R). \tag{52}$$

It is a matter of a simple algebra to show that the last term of Eq. (51) is a complete divergence. Namely, the following formula may be checked by inspection:

$$\pi^{\mu\nu}\delta R_{\mu\nu}=\partial_\kappa(\pi_\lambda^{\mu\nu\kappa}\delta\Gamma^\lambda_{\mu\nu}), \tag{53}$$

where we denote

$$\pi^{\mu\nu}:=\frac{1}{16\pi}\sqrt{|g|}g^{\mu\nu}, \tag{54}$$

$$\pi_\lambda{}^{\mu\nu\kappa}:=\pi^{\mu\nu}\delta^\kappa_\lambda-\pi^{\kappa(\nu}\delta^{\mu)}_\lambda, \tag{55}$$

and $\Gamma^\lambda_{\mu\nu}$ *are not* independent quantities but the Christoffel symbols, i.e., combinations of the metric components $g_{\mu\nu}$ and their derivatives. In the above calculations we use that fact that the covariant derivative $\nabla\pi$ of π with respect to Γ vanishes identically, i.e., that the following identity holds:

$$\partial_\kappa\pi_\lambda{}^{\mu\nu\kappa}\equiv\pi_\alpha{}^{\mu\nu\kappa}\Gamma^\alpha_{\lambda\kappa}-\pi_\lambda{}^{\alpha\nu\kappa}\Gamma^\mu_{\alpha\kappa}-\pi_\lambda{}^{\mu\alpha\kappa}\Gamma^\nu_{\alpha\kappa}. \tag{56}$$

Hence, for the regular part of the curvature we obtain

$$\delta\left(\frac{1}{16\pi}\sqrt{|g|}R\right)=-\frac{1}{16\pi}\mathcal{G}^{\mu\nu}\delta g_{\mu\nu}+\partial_\kappa(\pi_\lambda{}^{\mu\nu\kappa}\delta\Gamma^\lambda_{\mu\nu}). \tag{57}$$

We shall integrate the above equation over both parts D^+ and D^- of D, resulting from cutting D with the surface S. In this way we obtain

$$\delta L^{\rm reg}_{\rm grav}=-\frac{1}{16\pi}\mathrm{reg}(\mathcal{G})^{\mu\nu}\delta g_{\mu\nu}+\mathrm{reg}(\partial_\kappa(\pi_\lambda{}^{\mu\nu\kappa}\delta\Gamma^\lambda_{\mu\nu})). \tag{58}$$

Now, we are going to prove that the analogous formula is valid also for the singular part of the gravitational Lagrangian, i.e., that the following formula holds:

$$\delta L^{\rm sing}_{\rm grav}=-\frac{1}{16\pi}\mathrm{sing}(\mathcal{G})^{\mu\nu}\delta g_{\mu\nu}+\mathrm{sing}(\partial_\kappa(\pi_\lambda{}^{\mu\nu\kappa}\delta\Gamma^\lambda_{\mu\nu})). \tag{59}$$

To prove this formula, we calculate the singular part of the divergence $\partial_\kappa(\pi_\lambda{}^{\mu\nu\kappa}\delta\Gamma^\lambda_{\mu\nu})$. Because all these quantities are invariant, geometric objects ($\delta\Gamma$ is a tensor), we may calculate them in an arbitrary coordinate system. Hence, we may use our adapted coordinate system described in previous sections, where the coordinate x^3 is constant on S. This way, using Eq. (55), we obtain:

$$\begin{aligned}\mathrm{sing}(\partial_\kappa(\pi_\lambda{}^{\mu\nu\kappa}\delta\Gamma^\lambda_{\mu\nu})) &= \boldsymbol{\delta}(x^3)\,\pi_\lambda{}^{\mu\nu\perp}\,\delta[\Gamma^\lambda_{\mu\nu}]\\ &= \boldsymbol{\delta}(x^3)\,\pi_\lambda{}^{\mu\nu3}\,\delta[\Gamma^\lambda_{\mu\nu}]\\ &= \boldsymbol{\delta}(x^3)\,\pi^{\mu\nu}\,\delta[A^3_{\mu\nu}], \end{aligned}\tag{60}$$

where by A we denote

$$A^\lambda_{\mu\nu} := \Gamma^\lambda_{\mu\nu} - \delta^\lambda_{(\mu}\Gamma^\kappa_{\nu)\kappa}. \tag{61}$$

(Do not try to attribute any sophisticated geometric interpretation to $A^\lambda_{\mu\nu}$; it is merely a combination of the Christoffel symbols, which arises frequently in our calculations. It has been introduced for technical reasons only.) The following combination of the connection coefficients will also be useful in the following:

$$\tilde{Q}^{\mu\nu} := \sqrt{|g|}\,(g^{\mu\alpha}g^{\nu\beta} - \tfrac{1}{2}g^{\mu\nu}g^{\alpha\beta})A^3_{\alpha\beta}. \tag{62}$$

It may be immediately checked that

$$\pi^{\mu\nu}\delta A^3_{\mu\nu} = -\frac{1}{16\pi}g_{\mu\nu}\delta\tilde{Q}^{\mu\nu}. \tag{63}$$

In Appendix A we analyze in detail the structure of quantity $\tilde{Q}$. As a combination of the connection coefficients, it *does not* define any tensor density. But it differs from the external curvature $Q(X)$ of S introduced in Sec. III, only by terms containing metric components and their derivatives *along* S. Jumps of these terms across S vanish identically. Hence, the following is true:

$$[\tilde{Q}^{\mu\nu}]\,\boldsymbol{\delta}(x^3) = [Q^{\mu\nu}]\,\boldsymbol{\delta}(x^3) = \mathrm{sing}(\mathcal{G})^{\mu\nu}. \tag{64}$$

Consequently, formulas (60), (63), and (49) imply

$$\begin{aligned}\boldsymbol{\delta}(x^3)\,\pi_\lambda{}^{\mu\nu\perp}\,\delta[\Gamma^\lambda_{\mu\nu}] &= -\frac{1}{16\pi}g_{\mu\nu}\delta\,\mathrm{sing}(\mathcal{G})^{\mu\nu}\\ &= L^{\mathrm{sing}}_{\mathrm{grav}} + \frac{1}{16\pi}\mathrm{sing}(\mathcal{G})^{\mu\nu}\delta g_{\mu\nu},\end{aligned}\tag{65}$$

which ends the proof of Eq. (59). Summing up Eqs. (58) and (59), we obtain

$$\delta L_{\mathrm{grav}} = -\frac{1}{16\pi}\mathcal{G}^{\mu\nu}\delta g_{\mu\nu} + \partial_\kappa(\pi_\lambda{}^{\mu\nu\kappa}\delta\Gamma^\lambda_{\mu\nu}), \tag{66}$$

where both terms are composed of the regular and singular parts.[5]

Now, we calculate the variation of the matter part L_{matter} of the action on S,

$$\begin{aligned}\delta L_{\mathrm{matter}} &= \frac{\partial L_{\mathrm{matter}}}{\partial g_{ab}}\delta g_{ab} + \frac{\partial L_{\mathrm{matter}}}{\partial z^K}\delta z^K + \frac{\partial L_{\mathrm{matter}}}{\partial z^K{}_a}\partial_a\delta z^K\\ &= \frac{1}{2}\tau^{ab}\delta g_{ab} + \left(\frac{\partial L_{\mathrm{matter}}}{\partial z^K} - \partial_a\frac{\partial L_{\mathrm{matter}}}{\partial z^K{}_a}\right)\delta z^K\\ &\quad + \partial_a(p_K{}^a\delta z^K),\end{aligned}\tag{67}$$

where we used definition (40) and have introduced the momentum canonically conjugate to the matter variable z^K:

$$p_K{}^a := \frac{\partial L_{\mathrm{matter}}}{\partial z^K{}_a}. \tag{68}$$

Finally, we obtain the following formula for the variation of the total (matter+gravity) Lagrangian:

$$\begin{aligned}\delta L = &-\frac{1}{16\pi}\mathrm{reg}(\mathcal{G})^{\mu\nu}\delta g_{\mu\nu} + \boldsymbol{\delta}(x^3)\left(\frac{\partial L_{\mathrm{matter}}}{\partial z^K} - \partial_a\frac{\partial L_{\mathrm{matter}}}{\partial z^K{}_a}\right)\\ &\times\delta z^K - \boldsymbol{\delta}(x^3)\frac{1}{16\pi}(\mathbf{G}^{ab} - 8\pi\tau^{ab})\delta g_{ab}\\ &+ \partial_\kappa(\pi_\lambda{}^{\mu\nu\kappa}\delta\Gamma^\lambda_{\mu\nu}) + \boldsymbol{\delta}(x^3)\partial_a(p_K{}^a\delta z^K).\end{aligned}\tag{69}$$

In this section we assume that both $\delta g_{\mu\nu}$ and δz^K vanish in a neighborhood of the boundary ∂D of the spacetime region D (this assumption will be later relaxed, when deriving Hamiltonian structure of the theory). Hence, the last two boundary terms of the above formula vanish when integrated over D. Vanishing of the variation $\delta\mathcal{A}=0$ with fixed boundary values implies, therefore, the Euler-Lagrange equations (41) for the matter field z^K, together with the Einstein equations for gravitational field. The regular part of Einstein equations,

$$\mathrm{reg}(\mathcal{G})^{\mu\nu} = 0,$$

must be satisfied outside of S and the singular part must be fulfilled on S. To avoid irrelevant ambiguities of the type CX^aX^b, we write it in the following form, equivalent to the Barrabès-Israel equation:

$$\mathbf{G}^a{}_b = 8\pi\tau^a{}_b. \tag{70}$$

Summing up singular and regular parts of the above quantities we may write the total Einstein equations in the following way:

[5]In Ref. [9] the formula (66) was proved for regular spacetimes. In Ref. [12] its validity was extended to spacetimes with a three-dimensional, nondegenerate curvature singularity. Here, we have shown that it is also valid for a lightlike curvature singularity.

$$\delta L = \frac{1}{16\pi}(\mathcal{G}^{\mu\nu} - 8\pi\mathcal{T}^{\mu\nu})\delta g_{\mu\nu} + \boldsymbol{\delta}(x^3)\left(\frac{\partial L_{\text{matter}}}{\partial z^K} - \partial_a \frac{\partial L_{\text{matter}}}{\partial z^K{}_a}\right)\delta z^K + \partial_\kappa(\pi_\lambda{}^{\mu\nu\kappa}\delta\Gamma^\lambda_{\mu\nu}) + \boldsymbol{\delta}(x^3)\partial_a(p_K{}^a\delta z^K). \tag{71}$$

Here, we have defined the four-dimensional energy-momentum tensor, $\mathcal{T}^{\mu\nu} := \boldsymbol{\delta}(x^3)\tau^{\mu\nu}$ with $\tau^{3\nu} \equiv 0$. Since τ^{ab} was defined up to an additive term CX^aX^b, this ambiguity remains and $\mathcal{T}^{\mu\nu}$ is defined up to $CX^\mu X^\nu \boldsymbol{\delta}(x^3)$, similarly as the quantity $\mathcal{G}^{\mu\nu}$. This ambiguity is annihilated when contracted with $\delta g_{\mu\nu}$.

VII. DYNAMICS OF THE GRAVITY+SHELL TOTAL SYSTEM: HAMILTONIAN DESCRIPTION

Field equations of the theory (Euler-Lagrange equations for matter and Einstein equations—both singular and regular—for gravity) may thus be written in the following way:[6]

$$\delta L = \partial_\kappa(\pi_\lambda{}^{\mu\nu\kappa}\delta\Gamma^\lambda_{\mu\nu}) + \boldsymbol{\delta}(x^3)\partial_a(p_K{}^a\delta z^K). \tag{72}$$

Indeed, field equations are equivalent to the fact that the volume terms (71) in the variation of the Lagrangian must vanish identically. Hence, the entire dynamics of the theory of the matter+gravity system is equivalent to the requirement that variation of the Lagrangian is equal to boundary terms only. Similarly, as in Eq. (60), we may use definition of $\pi_\lambda{}^{\mu\nu\kappa}$ and express it in terms of the contravariant density of metric $\pi^{\mu\nu}$. In this way we obtain

$$\pi_\lambda{}^{\mu\nu\kappa}\delta\Gamma^\lambda_{\mu\nu} = \pi^{\mu\nu}\delta A^\kappa_{\mu\nu}. \tag{73}$$

Hence, field equations may be written in the following way:

$$\delta L = \partial_\kappa(\pi^{\mu\nu}\delta A^\kappa_{\mu\nu}) + \boldsymbol{\delta}(x^3)\partial_a(p_K{}^a\delta z^K). \tag{74}$$

As soon as we choose a $(3+1)$ decomposition of the spacetime M, our field theory will be converted into a Hamiltonian system with the space of Cauchy data on each of the three-dimensional surfaces playing role of an infinite-dimensional phase space. Let us choose a coordinate system adapted to this $(3+1)$ decomposition. This means that the time variable $t = x^0$ is constant on three-dimensional surfaces of this foliation. We assume that these surfaces are spacelike. To obtain the Hamiltonian formulation of our theory we shall simply integrate Eq. (72) [or—equivalently—Eq. (74)] over such a Cauchy surface $\Sigma_t \subset M$ and then perform a Legendre transformation between time derivatives and corresponding momenta.

[6]Formula (72) is analogous to the formula $dL(q,\dot q) = (p\,dq)^\cdot = \dot p\,dq + p\,d\dot q$ in mechanics, which contains both the dynamical equation $\dot p = \partial L/\partial q$ and the definition of the canonical momentum $p = \partial L/\partial \dot q$. For detailed analysis of this structure see Ref. [9].

In the present paper we consider the case of an asymptotically flat spacetime and assume that also leaves Σ_t of our $(3+1)$ decomposition are asymptotically flat at infinity. To keep control over two-dimensional surface integrals at spatial infinity, we first consider the dynamics of our matter +gravity system in a finite world tube $\mathcal{U}$, whose boundary carries a nondegenerate metric of signature $(-,+,+)$. At the end of our calculations, we shift the boundary $\partial\mathcal{U}$ of the tube to space infinity. We assume that the tube contains the surface S together with our lightlike matter traveling over it.

Denoting by $V := \mathcal{U} \cap \Sigma_t$ the portion of Σ_t which is contained in the tube $\mathcal{U}$, we thus integrate Eq. (74) over the finite volume $V \subset \Sigma_t$ and keep surface integrals on the boundary ∂V of V. They will produce the ADM mass as the Hamiltonian of the total matter+gravity system at the end of our calculations when we pass to infinity with $\partial V = \Sigma_t \cap \partial\mathcal{U}$. Because our approach is geometric and does not depend upon the choice of coordinate system, we may further simplify our calculations using the coordinate x^3 adapted to both S and to the boundary $\partial\mathcal{U}$ of the tube. We thus assume that x^3 is constant on both these surfaces.

It is worthwhile to stress at this point that there is no contradiction in the fact that the surface $\{x^3 = C\}$ has different geometric character for different values of the parameter C: it is null-like for $C = 0$ and timelike for $C \to \infty$ (cf. properties of the coordinate $\xi := r^2 - t^2$ in Minkowski space).

Integrating Eq. (74) over the volume V we thus obtain

$$\delta\int_V L = \int_V \partial_\kappa(\pi^{\mu\nu}\delta A^\kappa_{\mu\nu}) + \int_V \boldsymbol{\delta}(x^3)\partial_a(p_K{}^a\delta z^K) = \int_V (\pi^{\mu\nu}\delta A^0_{\mu\nu})^\cdot + \int_{\partial V}\pi^{\mu\nu}\delta A^\perp_{\mu\nu} + \int_{V\cap S}(p_K{}^0\delta z^K), \tag{75}$$

where by a dot we denote the time derivative. In the above formula we have skipped the two-dimensional divergencies that vanish when integrated over surfaces ∂V and $V \cap S$.

To further simplify our formalism, we denote by $p_K := p_K{}^0$ the timelike component of the momentum canonically conjugate to the field variable z^K and perform the Legendre transformation:

$$(p_K\delta z^K)^\cdot = \dot p_K\delta z^K - \dot z^K\delta p_K + \delta(p_K\dot z^K). \tag{76}$$

The last term, put on the left-hand side of Eq. (75), satisfies the matter Lagrangian and produces the matter Hamiltonian (with minus sign), according to the formula

$$L_{\text{matter}} - p_K\dot z^K = L_{\text{matter}} - p_K{}^0 z^K{}_0 = -T^0{}_0 = \tau^0{}_0. \tag{77}$$

To perform also Legendre transformation in gravitational degrees of freedom we follow here method proposed by one of us [9]. For this purpose we first observe that, due to metricity of the connection Γ, the gravitational counterpart $\pi^{\mu\nu}\delta A^0_{\mu\nu}$ of the canonical one-form $p_K\delta z^K$ reduces as follows:

$$\pi^{\mu\nu}\delta A^0_{\mu\nu} = -\frac{1}{16\pi}g_{kl}\delta P^{kl} + \partial_k\left(\pi^{00}\delta\left(\frac{\pi^{0k}}{\pi^{00}}\right)\right), \tag{78}$$

where P^{kl} denotes the external curvature of Σ written in the ADM form. Similarly, the boundary term $\pi^{\mu\nu}\delta A^{\perp}_{\mu\nu} = \pi^{\mu\nu}\delta A^{3}_{\mu\nu}$ reduces as follows:

$$\pi^{\mu\nu}\delta A^{3}_{\mu\nu} = -\frac{1}{16\pi} g_{ab}\,\delta\mathcal{Q}^{ab} + \partial_a\left(\pi^{33}\delta\left(\frac{\pi^{3a}}{\pi^{33}}\right)\right), \quad (79)$$

where $\mathcal{Q}^{ab}$ denotes the external curvature of the tube $\partial\mathcal{U}$ written in the ADM form. A simple proof of these formulas is given in Appendix D 1.

Using these results and skipping the two-dimensional divergencies that vanish after integration, we may rewrite gravitational part of Eq. (75) in the following way:

$$\int_V (\pi^{\mu\nu}\delta A^{0}_{\mu\nu})^{\cdot} + \int_{\partial V} \pi^{\mu\nu}\delta A^{\perp}_{\mu\nu}$$
$$= -\frac{1}{16\pi}\int_V (g_{kl}\,\delta P^{kl})^{\cdot} - \frac{1}{16\pi}\int_{\partial V} g_{ab}\,\delta\mathcal{Q}^{ab}$$
$$+ \int_{\partial V}\left(\pi^{00}\delta\left(\frac{\pi^{03}}{\pi^{00}}\right) + \pi^{33}\delta\left(\frac{\pi^{30}}{\pi^{33}}\right)\right)^{\cdot}. \quad (80)$$

The last integral may be rewritten in terms of the hyperbolic angle α between surfaces Σ and $\partial\mathcal{U}$, defined as $\alpha = \operatorname{arcsinh}(q)$, where

$$q = \frac{g^{30}}{\sqrt{|g^{00}g^{33}|}}, \quad (81)$$

and the two-dimensional volume form $\lambda = \sqrt{\det g_{AB}}$ on ∂V, in the following way:

$$\pi^{00}\delta\left(\frac{\pi^{03}}{\pi^{00}}\right) + \pi^{33}\delta\left(\frac{\pi^{30}}{\pi^{33}}\right) = \frac{1}{8\pi}\lambda\,\delta\alpha. \quad (82)$$

For the proof of this formula see Appendix D 2. Hence, we have

$$\int_V (\pi^{\mu\nu}\delta A^{0}_{\mu\nu})^{\cdot} + \int_{\partial V} \pi^{\mu\nu}\delta A^{\perp}_{\mu\nu}$$
$$= -\frac{1}{16\pi}\int_V (g_{kl}\,\delta P^{kl})^{\cdot} - \frac{1}{16\pi}\int_{\partial V} g_{ab}\,\delta\mathcal{Q}^{ab}$$
$$+ \frac{1}{8\pi}\int_{\partial V} (\lambda\,\delta\alpha)^{\cdot} \quad (83)$$

Now we perform the Legendre transformation both in the volume:

$$(g_{kl}\delta P^{kl})^{\cdot} = (\dot{g}_{kl}\delta P^{kl} - \dot{P}^{kl}\delta g_{kl}) + \delta(g_{kl}\dot{P}^{kl})$$

and on the boundary, $(\lambda\,\delta\alpha)^{\cdot} = (\dot{\lambda}\,\delta\alpha - \dot{\alpha}\,\delta\lambda) + \delta(\lambda\,\dot{\alpha})$.

In Appendix D 3 we prove the following formula:

$$-\frac{1}{16\pi}\int_V (g_{kl}\dot{P}^{kl}) + \frac{1}{8\pi}\int_{\partial V}\lambda\,\dot{\alpha}$$
$$= \frac{1}{8\pi}\int_V \sqrt{|g|}R^0{}_0 + \frac{1}{16\pi}\int_{\partial V}(\mathcal{Q}^{AB}g_{AB} - \mathcal{Q}^{00}g_{00}), \quad (84)$$

Then, we have

$$\frac{1}{16\pi}\int_V L_{\text{grav}} - \frac{1}{8\pi}\int_V \sqrt{|g|}R^0{}_0 = \frac{1}{8\pi}\int_V \sqrt{|g|}\left(\tfrac{1}{2}R - R^0{}_0\right)$$
$$= -\frac{1}{8\pi}\int_V \mathcal{G}^0{}_0. \quad (85)$$

Splitting the component $\mathcal{G}^0{}_0$ of the Einstein tensor into regular and singular parts, we obtain

$$\frac{1}{8\pi}\int_V \mathcal{G}^0{}_0 = \frac{1}{8\pi}\int_V \operatorname{reg}(\mathcal{G}^0{}_0) + \frac{1}{8\pi}\int_V \operatorname{sing}(\mathcal{G}^0{}_0). \quad (86)$$

The regular part of Einstein tensor density $\operatorname{reg}(\mathcal{G}^{\mu\nu})$ vanishes due to field equations. The singular part

$$\operatorname{sing}(\mathcal{G}^0{}_0) = \boldsymbol{\delta}(x^3)\mathbf{G}^0{}_0, \quad (87)$$

satisfies the matter Hamiltonian $\tau^0{}_0$ [see formula (77)] and becomes annihilated due to Einstein equations:

$$\frac{1}{8\pi}\int_{V\cap S}(\mathbf{G}^0{}_0 - 8\pi\tau^0{}_0) = 0. \quad (88)$$

Finally, we obtain the following generating formula [9]:

$$0 = \frac{1}{16\pi}\int_V (\dot{P}^{kl}\delta g_{kl} - \dot{g}_{kl}\delta P^{kl}) + \frac{1}{16\pi}\int_{\partial V}(\dot{\lambda}\,\delta\alpha - \dot{\alpha}\,\delta\lambda)$$
$$+ \int_{V\cap S}(\dot{p}^0{}_K\delta z^K - \dot{z}^K\delta p^0{}_K) - \frac{1}{16\pi}\int_{\partial V} g_{ab}\,\delta\mathcal{Q}^{ab}$$
$$+ \frac{1}{16\pi}\delta\int_{\partial V}(\mathcal{Q}^{AB}g_{AB} - \mathcal{Q}^{00}g_{00}). \quad (89)$$

Using results of Ref. [9] it may be easily shown that pushing the boundary ∂V to infinity and handling in a proper way the above three surface integrals over ∂V, one obtains in the asymptotically flat case the standard Hamiltonian formula for both gravitational and matter degrees of freedom, with the ADM mass (given by the resulting surface integral at infinity) playing role of the total Hamiltonian. More precisely, denoting the matter momenta by

$$\pi_K := p^0{}_K\boldsymbol{\delta}(x^3), \quad (90)$$

the final formula for $\partial V \to \infty$ reads

$$-\delta\mathcal{H} = \frac{1}{16\pi}\int_V (\dot{P}^{kl}\delta g_{kl} - \dot{g}_{kl}\delta P^{kl}) + \int_V (\dot{\pi}_K\delta z^K - \dot{z}^K\delta\pi_K), \quad (91)$$

where $\mathcal{H}$ is the total Hamiltonian, equal to the ADM mass at spatial infinity.[7]

VIII. CONSTRAINTS

Consider Cauchy data $(P^{kl}, g_{kl}, \pi_K, z^K)$ on a three-dimensional spacelike surface V_t and denote by $\tilde{g}^{kl}$ the three-dimensional metric inverse to g_{kl}. Moreover, we use the following notation: $\gamma := \sqrt{\det g_{kl}}$, $\overset{(3)}{R}$ is the three-dimensional scalar curvature of g_{kl}, $P := P^{kl} g_{kl}$ and $|$ is the three-dimensional covariant derivative with respect to g_{kl}.

We are going to prove that these data must fulfill constraints implied by Gauss-Codazzi equations for the components $\mathcal{G}^0{}_\mu$ of the Einstein tensor density. Standard decomposition of $\mathcal{G}^0{}_\mu$ into the spatial (tangent to V_t) part and the timelike (normal to V_t) part gives us, respectively,

$$\mathcal{G}^0{}_l = -P_l{}^k{}_{|k}, \tag{92}$$

and

$$2\mathcal{G}^0{}_\mu n^\mu = -\gamma \overset{(3)}{R} + \left(P^{kl}P_{kl} - \tfrac{1}{2}P^2\right)\frac{1}{\gamma}. \tag{93}$$

Here by n we have denoted the future orthonormal vector to Cauchy surface V_t:

$$n^\mu = -\frac{g^{0\mu}}{\sqrt{-g^{00}}}.$$

Vacuum Einstein equations outside and inside of S imply vanishing of the regular part of $\mathcal{G}^0{}_\mu$. Hence, the regular part of the vector constraint reads:

$$\mathrm{reg}(P_l{}^k{}_{|k}) = 0,$$

whereas the regular part of the scalar constraint reduces to

$$\mathrm{reg}\left(\gamma \overset{(3)}{R} - \left(P^{kl}P_{kl} - \tfrac{1}{2}P^2\right)\frac{1}{\gamma}\right) = 0.$$

The singular part of constraints, with support on the intersection sphere $S_t = V_t \cap S$, can be derived as follows.

The singular part of the three-dimensional derivatives of the ADM momentum P_{kl} consists of derivatives in the direction of x^3:

$$\mathrm{sing}(P_l{}^k{}_{|k}) = \mathrm{sing}(\partial_3 P_l{}^3) = \boldsymbol{\delta}(x^3)[P_l{}^3],$$

so the full vector constraint has the form

$$P_l{}^k{}_{|k} = [P_l{}^3]\boldsymbol{\delta}(x^3). \tag{94}$$

Components of the ADM momentum P^{kl} are regular; hence the singular part of the term $(P^{kl}P_{kl} - \frac{1}{2}P^2)$ vanishes. The singular part of the three-dimensional scalar curvature consists of derivatives in the direction of x^3 of the (three-dimensional) connection coefficients:

$$\begin{aligned}\mathrm{sing}(\overset{(3)}{R}) &= \mathrm{sing}(\partial_3(\Gamma^3_{kl}\tilde{g}^{kl} - \Gamma^m_{ml}\tilde{g}^{3l})) \\ &= \boldsymbol{\delta}(x^3)[\Gamma^3_{kl}\tilde{g}^{kl} - \Gamma^m_{ml}\tilde{g}^{3l}],\end{aligned} \tag{95}$$

and expression in the square brackets may be reduced to the following term:

$$\begin{aligned}\gamma[\Gamma^3_{kl}\tilde{g}^{kl} - \Gamma^m_{ml}\tilde{g}^{3l}] &= -2\sqrt{\tilde{g}^{33}}[\partial_3(\gamma\sqrt{\tilde{g}^{33}})] \\ &= -2\sqrt{\tilde{g}^{33}}\left[\partial_k\left(\frac{\gamma\tilde{g}^{3k}}{\sqrt{\tilde{g}^{33}}}\right)\right],\end{aligned} \tag{96}$$

because derivatives tangent to S are continuous. But the expression in square brackets is equal to the external curvature scalar k for the two-dimensional surface $S_t \subset V_t$:

$$\gamma k = -\partial_k\left(\frac{\gamma\tilde{g}^{3k}}{\sqrt{\tilde{g}^{33}}}\right). \tag{97}$$

So we get

$$\mathrm{sing}(\gamma \overset{(3)}{R}) = 2\gamma\sqrt{\tilde{g}^{33}}[k]\boldsymbol{\delta}(x^3) = 2[\lambda k]\boldsymbol{\delta}(x^3)$$

and finally

$$\gamma \overset{(3)}{R} - \left(P^{kl}P_{kl} - \tfrac{1}{2}P^2\right)\frac{1}{\gamma} = 2[\lambda k]\boldsymbol{\delta}(x^3). \tag{98}$$

Equations (94) and (98) give a generalization (in the sense of distributions) of the usual vacuum constraints (vector and scalar, respectively).

Now, we will show how the distributional matter located on S_t determines the four surface quantities $[P^3{}_k]$ and $[\lambda k]$, entering into the singular part of the constraints. The tangent (to S) part of $\mathcal{G}^0{}_\mu$ splits into the two-dimensional part tangent to S_t and the transversal part (along null rays).

The tangent to S_t part of Einstein equations gives the following,

$$\mathcal{G}^0{}_A = 8\pi\boldsymbol{\delta}(x^3)\tau^0{}_A, \tag{99}$$

which, due to Eqs. (92) and (94), implies the following two constraints:

$$[P^3{}_B] = -8\pi\tau^0{}_B. \tag{100}$$

The remaining null tangent part of Einstein equations reads

$$\mathcal{G}^0{}_\mu X^\mu = 8\pi\boldsymbol{\delta}(x^3)\tau^0{}_\mu X^\mu = 0, \tag{101}$$

because $\tau^0{}_\mu X^\mu = 0$. In Appendix E we show that this equation reduces to the following constraint:

[7] Formula (91) is analogous to the formula $-dH(q,p) = \dot{p}\,dq - \dot{q}\,dp$ in mechanics. In a nonconstrained case this formula is equivalent to the definition of the Hamiltonian vector field $(\dot{p},\dot{q})$ via Hamilton equations $\dot{p} = -\partial H/\partial q$, and $\dot{q} = \partial H/\partial p$. We stress, however, that the formula is much more general and is valid also for constrained systems, when the field is not unique, but given only "up to a gauge." For detailed analysis of this structure see Ref. [9].

$$\left[\frac{P^{33}}{\sqrt{\tilde{g}^{33}}}+\lambda k\right]=0. \tag{102}$$

We remind the reader that the singular part of $\mathcal{G}^0{}_3$ cannot be defined in any intrinsic way. Consequently, we have only three constraints for the singular parts (102) and (100). The fourth constraint (in a nondegenerate case) has been replaced here by the degeneracy condition $\det g_{ab}$ for the metric on S. Equations (100) and (102) together with (94) and (98) are the initial value constraints.

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APPENDIX A: STRUCTURE OF THE SINGULAR EINSTEIN TENSOR

We rewrite the Ricci tensor,

$$R_{\mu\nu}=\partial_\lambda\Gamma^\lambda_{\mu\nu}-\partial_{(\mu}\Gamma^\lambda_{\nu)\lambda}+\Gamma^\lambda_{\sigma\lambda}\Gamma^\sigma_{\mu\nu}-\Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\lambda}, \tag{A1}$$

in terms of the following combinations of Christoffel symbols [cf. Eq. (61) in Sec. V]:

$$A^\lambda_{\mu\nu}:=\Gamma^\lambda_{\mu\nu}-\delta^\lambda_{(\mu}\Gamma^\kappa_{\nu)\kappa}. \tag{A2}$$

We have

$$R_{\mu\nu}=\partial_\lambda A^\lambda_{\mu\nu}-A^\lambda_{\mu\sigma}A^\sigma_{\nu\lambda}+\tfrac{1}{3}A^\lambda_{\mu\lambda}A^\sigma_{\nu\sigma}. \tag{A3}$$

Terms quadratic in A's may have only steplike discontinuities. The derivatives along S are thus bounded and belong to the regular part of the Ricci tensor. The singular part of the Ricci tensor is obtained from the transversal derivatives only. In our adapted coordinate system, where x^3 is constant on S, we obtain

$$\mathrm{sing}(R_{\mu\nu})=\partial_3 A^3_{\mu\nu}=\boldsymbol{\delta}(x^3)[A^3_{\mu\nu}], \tag{A4}$$

where by $\boldsymbol{\delta}$ we denote the Dirac δ distribution and by square brackets we denote the jump of the value of the corresponding expression between the two sides of S. Consequently, the singular part of Einstein tensor density reads:

$$\mathrm{sing}(\mathcal{G}^\mu{}_\nu):=\sqrt{|g|}\,\mathrm{sing}(R^\mu{}_\nu-\tfrac{1}{2}R)=\boldsymbol{\delta}(x^3)\mathbf{G}^\mu{}_\nu, \tag{A5}$$

where

$$\mathbf{G}^\mu{}_\nu:=\sqrt{|g|}(\delta^\beta_\nu g^{\mu\alpha}-\tfrac{1}{2}\delta^\mu_\nu g^{\alpha\beta})[A^3_{\alpha\beta}]=[\tilde{Q}^\mu{}_\nu]. \tag{A6}$$

We shall prove that the contravariant version of this quantity:

$$\mathrm{sing}(\mathcal{G})^{\mu\nu}=[\tilde{Q}^{\mu\nu}]\boldsymbol{\delta}(x^3),$$

is coordinate dependent and therefore does not define any geometric object. For this purpose we are going to relate the coordinate-dependent quantity $\tilde{Q}^{\mu\nu}$ with the external curvature $Q^a{}_b$ of S. We use the form of the metric introduced in Ref. [10]:

$$g_{\mu\nu}=\begin{bmatrix} n^A n_A & n_A & sM+m^A n_A \\ n_A & g_{AB} & m_A \\ sM+m^A n_A & m_A & (M/N)^2+m^A m_A\end{bmatrix}, \tag{A7}$$

and

$$g^{\mu\nu}=\begin{bmatrix} -(1/N)^2 & n^A/N^2-sm^A/M & s/M \\ n^A/N^2-sm^A/M & \tilde{\tilde{g}}^{AB}-n^A n^B/N^2+s(n^A m^B+m^A n^B)/M & -sn^A/M \\ s/M & -sn^A/M & 0\end{bmatrix}, \tag{A8}$$

where $M>0$, $s:=\mathrm{sgn}\,g^{03}=\pm1$, g_{AB} is the induced two-metric on surfaces $\{x^0=\mathrm{const}, x^3=\mathrm{const}\}$, and $\tilde{\tilde{g}}^{AB}$ is its inverse (contravariant) metric. Both $\tilde{\tilde{g}}^{AB}$ and g_{AB} are used to raise and lower indices A, $B=1$, 2 of the two-vectors n^A and m^A.

Formula (A7) implies: $\sqrt{|\det g_{\mu\nu}|}=\lambda M$. Moreover, the object Λ^a defined by formula (3), takes the form $\Lambda^a=\lambda X^a$, where λ is given by formula (2) and $X:=\partial_0-n^A\partial_A$. This means that we have chosen the following degeneracy field: $X^\mu=(1,-n^A,0)$.

For calculational purposes it is useful to rewrite the two-dimensional inverse metric $\tilde{\tilde{g}}^{AB}$ in three-dimensional notation, setting $\tilde{\tilde{g}}^{0a}:=0$. This object satisfies the obvious identity:

$$\tilde{\tilde{g}}^{ac}g_{cb}=\delta^a{}_b-X^a\delta^0{}_b.$$

Hence, the contravariant metric (A8) may be rewritten as follows:

$$g^{ab}=\tilde{\tilde{g}}^{ab}-\frac{1}{N^2}X^aX^b-\frac{s}{M}(m^aX^b+m^bX^a), \tag{A9}$$

where $m^a:=\tilde{\tilde{g}}^{aB}m_B$, so that $m^0:=0$, and

$$g^{3\mu}=\frac{s}{M}X^\mu.$$

It may be easily checked (see, e.g., Ref. [10], p. 406) that covariant derivatives of the field X *along* S are equal to:

$$\nabla_a X=-w_aX-l_{ab}\tilde{\tilde{g}}^{bc}\partial_c, \tag{A10}$$

where

$$w_a := -X^\mu \Gamma^0_{\mu a} \tag{A11}$$

and

$$l_{ab} := -g(\partial_b, \nabla_a X) = g(\nabla_a \partial_b, X) = X_\mu \Gamma^\mu_{ab}. \tag{A12}$$

Since X is orthogonal to S, we have $X_a = 0$. Because of Eq. (A7), the only nonvanishing component of X_μ is equal to $X_3 = sM$. Hence, we have $l_{ab} = sM\Gamma^3_{ab} = sMA^3_{ab}$ and, consequently,

$$\sqrt{|g|} A^3_{ab} = s\lambda l_{ab}. \tag{A13}$$

Because of the identity

$$X^a l_{ab} = X^a X^c \Gamma_{cab} = \frac{1}{2} X^c X^a g_{ca,b} \equiv 0, \tag{A14}$$

we also have $l_{ab}X^b = 0$ [10]. Now we are going to use the metricity condition for the connection Γ:

$$0 \equiv \nabla_a \pi^{3a} = \partial_a \pi^{3a} + \pi^{3\mu}\Gamma^a_{\mu a} + \pi^{\mu a}\Gamma^3_{\mu a} - \pi^{3a}\Gamma^\mu_{a\mu}$$
$$= \partial_a \pi^{3a} + \pi^{ab}\Gamma^3_{ab} = \partial_a \pi^{3a} + \pi^{ab} A^3_{ab}. \tag{A15}$$

Consequently,

$$\partial_c \Lambda^c = \partial_c (s\sqrt{|g|} g^{3c}) = s\pi^{3c}{}_{,c}$$
$$= -s\pi^{ab} A^3_{ab} = -\lambda g^{ab} l_{ab} = -\lambda \tilde{\tilde{g}}^{ab} l_{ab} = -\lambda l, \tag{A16}$$

where $l = \tilde{\tilde{g}}^{ab} l_{ab}$. Now, we want to calculate the component $A^3_{3a} = \Gamma^3_{3a} - (1/2)\Gamma^\mu_{\mu a}$. Because

$$\Gamma^\mu_{\mu a} = \partial_a \ln\sqrt{|g|} = \partial_a \ln(\lambda M),$$

it is sufficient to calculate Γ^3_{3a} according to the following formula:

$$\Gamma^3_{3a} = g^{3c}\Gamma_{c3a} = \frac{s}{M} X^c (g_{3c,a} - \Gamma_{3ca})$$
$$= \frac{s}{M} X^c g_{3c,a} - X^c g^{0\mu}\Gamma_{\mu ca} + X^c g^{0b}\Gamma_{bca}$$
$$= w_a + \frac{s}{M} X^c g_{3c,a} + \frac{s}{M} X^b m^c \Gamma_{bca} - \frac{s}{M} X^c m^b g_{bc,a}$$
$$= w_a + \frac{s}{M} m^c l_{ca} + \frac{s}{M}\{(X^c g_{3c})_{,a} - X^c_{,a}(g_{3c} - m^b g_{bc})\}$$
$$= w_a + \frac{s}{M} m^c l_{ca} + \frac{1}{M} M_{,a}. \tag{A17}$$

Finally, we obtain the following identity:

$$A^3_{3a} = w_a + \chi_a + \frac{s}{M} m^b l_{ba}, \tag{A18}$$

where $\chi_a := \frac{1}{2}\partial_a \ln(M/\lambda)$.

To express $\tilde{Q}$ in terms of l_{ab} and w_a, we observe that

$$s\tilde{Q}^a{}_b = \lambda(g^{ac} l_{cb} - \tfrac{1}{2}\delta^a{}_b l) + \Lambda^a \Lambda^3_{3b} - \delta^a{}_b \Lambda^c A^3_{3c}, \tag{A19}$$

$$s\tilde{Q}^3{}_3 = -\frac{1}{2}\lambda l, \tag{A20}$$

$$s\tilde{Q}^3{}_a = 0. \tag{A21}$$

The missing component $\tilde{Q}^a{}_3$ is much more complicated,

$$\tilde{Q}^a{}_3 = \sqrt{|g|} g^{a\beta} A^3_{\beta 3} = \lambda M (g^{3a} A^3_{33} + g^{ab} A^3_{b3})$$
$$= s\Lambda^a A^3_{33} + \lambda M \left\{ \tilde{\tilde{g}}^{ab} + \frac{1}{N^2} X^a X^b - \frac{s}{M}(m^a X^b + m^b X^a) \right\} A^3_{b3}, \tag{A22}$$

and depends upon A^3_{33},

$$A^3_{33} = \Gamma^3_{33} - \Gamma^\mu_{3\mu} = -\Gamma^a_{3a} = \tfrac{1}{2}(g^{ab} g_{ab,3} + g^{a3} g_{33,a})$$
$$= -\partial_3 \ln\lambda + \frac{s}{M} m^a X^b g_{ab,3} - \tfrac{1}{2} g^{a3} g_{33,a}, \tag{A23}$$

where we have used the identity

$$\tfrac{1}{2}\tilde{\tilde{g}}^{ab} g_{ab,3} = \partial_3 \ln\lambda.$$

We are ready to prove the following.

Lemma A.2. The object $\tilde{Q}^a{}_b$ is related to $Q^a{}_b$ as follows:

$$s\tilde{Q}^a{}_b = sQ^a{}_b - \tfrac{1}{2}\lambda l \delta^a{}_b + \Lambda^a \chi_b - \delta^a{}_b \Lambda^c \chi_c, \tag{A24}$$

where $\chi_c := (1/2)\partial_c \ln(M/\lambda)$.

Proof. Using Eqs. (A19), (A18), and (A9) we obtain:

$$s\tilde{Q}^a{}_b = \lambda(\tilde{\tilde{g}}^{ac} l_{cb} - \tfrac{1}{2}\delta^a{}_b l) + \Lambda^a w_b - \delta^a{}_b \Lambda^c w_c + \Lambda^a \chi_b - \delta^a{}_b \Lambda^c \chi_c. \tag{A25}$$

From definition (19) and property (A10), one can check that

$$sQ^a{}_b = \lambda\delta^a{}_b \nabla_c X^c - \lambda \nabla_b X^a - \delta^a{}_b \partial_c \Lambda^c$$
$$= -\lambda\delta^a{}_b (w_c X^c + l) + \lambda(w_b X^a + \tilde{\tilde{g}}^{ac} l_{cb}) + \delta^a{}_b \lambda l$$
$$= \lambda \tilde{\tilde{g}}^{ac} l_{cb} + \Lambda^a w_b - \delta^a{}_b \Lambda^c w_c, \tag{A26}$$

and thus we obtain Eq. (A24). ■

Remark. Formula (A26), together with $l_{ab}X^b = 0 = g_{ab}X^b$, gives us the orthogonality condition $Q^a{}_b X^b = 0$ and symmetry of the tensor $Q_{ab} := g_{ac} Q^c{}_b$.

Now, we would like to examine the properties of $\mathbf{G}^{\mu\nu} = [\tilde{Q}^{\mu\nu}]$. From continuity of the metric across S we obtain

$$[l_{ab}] = sM[A^3_{ab}] = sM[\Gamma^3_{ab}] = X^c[\Gamma_{cab}] = 0. \tag{A27}$$

On the other hand, the jump of $A^3_{3\mu}$ is in general nonvanishing. From Eq. (A18) we have

$$[A^3_{3a}]=[w_a]. \tag{A28}$$

Formulas (A19)–(A21) and (A27) imply

$$s[\tilde{Q}^a{}_b]=\Lambda^a[A^3_{3b}]-\delta^a{}_b\Lambda^c[A^3_{3c}] \\ =\Lambda^a[w_b]-\delta^a{}_b\Lambda^c[w_c]=s[Q^a{}_b], \tag{A29}$$

$$[\tilde{Q}^3{}_\mu]=0. \tag{A30}$$

Moreover, we have

$$[\tilde{Q}^a{}_3]=s\Lambda^a\left([A^3_{33}]+s\frac{M}{N^2}X^b[w_b]-m^b[w_b]\right) \\ +(\lambda M\tilde{\tilde{g}}^{ab}-sm^a\Lambda^b)[w_b]. \tag{A31}$$

On the other hand the jump of A^3_{33} may be obtained from Eq. (A23):

$$[A^3_{33}]=-[\partial_3\ln\lambda]+2m^b[w_b], \tag{A32}$$

where we have used

$$[w_a]=-X^bg^{03}[\Gamma_{3ba}]=\frac{s}{2M}X^b[g_{ab,3}]. \tag{A33}$$

However,

$$X^a[w_a]=\frac{s}{2M}[X^aX^bg_{ab,3}]=0. \tag{A34}$$

Hence

$$[\tilde{Q}^a{}_3]=s\Lambda^a\{-[\partial_3\ln\lambda]+m^b[w_b]\}+M\lambda\tilde{\tilde{g}}^{ab}[w_b]. \tag{A35}$$

Using these results we calculate components of $[\tilde{Q}^{\mu\nu}]=\mathbf{G}^{\mu\nu}$. From Eq. (A30) we can easily check the property (26)

$$\mathbf{G}^{33}=[\tilde{Q}^{33}]=g^{33}[\tilde{Q}^3{}_3]+g^{3b}[\tilde{Q}^3{}_b]=0,$$

$$\mathbf{G}^{3a}=[\tilde{Q}^{3a}]=g^{33}[\tilde{Q}^a{}_3]+g^{3b}[\tilde{Q}^a{}_b]=-\frac{s}{M}[X^bQ^a{}_b]=0,$$

where we used the property $[\tilde{Q}^a{}_b]=[Q^a{}_b]$, which is crucial for showing that the object $\mathbf{G}^a{}_b$ is a well-defined geometric object on S. On the contrary, the object $\mathbf{G}^{ab}$ is not a geometric object because depends on a choice of coordinates. This can be seen when we calculate the component $\mathbf{G}^{00}$:

$$\mathbf{G}^{00}=[\tilde{Q}^{00}]=g^{03}[\tilde{Q}^0{}_3]+g^{0b}[\tilde{Q}^0{}_b] \\ =\frac{\lambda}{M}(-[\partial_3\ln\lambda]+m^b[w_b])-s\left(\frac{1}{N^2}X^b+\frac{s}{M}m^b\right)\lambda[w_b] \\ =-\frac{1}{M}[\partial_3\lambda]. \tag{A36}$$

It may be easily checked [10] that the quantity above transforms in a homogeneous way with respect to coordinate transformation on S. This proves that the components $\mathbf{G}^{ab}$ do not define any tensor density on S. An independent argument for this statement may be produced as follows. Begin with a coordinate system in which we have $X=\partial_0$ (i.e., $n^A=0$) and perform the following coordinate transformation:

$$\tilde{x}^0=x^0+b_Ax^A,\quad \tilde{x}^A=x^A,\quad \tilde{x}^3=x^3, \tag{A37}$$

where b_A are constant. According to Eq. (A8) we have

$$\frac{s}{\tilde{M}}=g(d\tilde{x}^0,d\tilde{x}^3)=g(dx^0,dx^3)+b_Ag(dx^A,dx^3) \\ =\frac{s}{M}(1-b_An^A)=\frac{s}{M}, \tag{A38}$$

whence we get $\tilde{M}=M$. Moreover, the new tetrad $(\tilde{X},\tilde{\partial}_{\tilde{B}},\tilde{\partial}_{\tilde{3}})$ may be calculated as follows:

$$\tilde{X}=X, \tag{A39}$$

$$\tilde{\partial}_{\tilde{B}}=\frac{\partial x^0}{\partial\tilde{x}^{\tilde{B}}}\partial_0+\frac{\partial x^A}{\partial\tilde{x}^{\tilde{B}}}\partial_A=\delta^A_{\tilde{B}}\partial_A-b_{\tilde{B}}X, \tag{A40}$$

$$\tilde{\partial}_3=\partial_3. \tag{A41}$$

This implies $\tilde{\lambda}=\lambda$, and, consequently,

$$\tilde{\mathbf{G}}^{\tilde{0}\tilde{0}}=-\frac{1}{\tilde{M}}[\partial_{\tilde{3}}\tilde{\lambda}]=-\frac{1}{M}[\partial_3\lambda]=\mathbf{G}^{00}. \tag{A42}$$

On the other hand, we have $d\tilde{x}^0=dx^0+b_Adx^A$ and $\det(\partial x^a/\partial\tilde{x}^{\tilde{b}})=1$. Hence,

$$\tilde{\mathbf{G}}^{\tilde{0}\tilde{0}}-\mathbf{g}^{00}=\mathbf{G}(d\tilde{x}^0,d\tilde{x}^0)-\mathbf{G}(dx^0,dx^0) \\ =2b_A\mathbf{G}^{0A}+\mathbf{G}^{AB}b_Ab_B,$$

which does not need to vanish in a generic case.

APPENDIX B: GAUSS-CODAZZI EQUATIONS

We begin with the Lie derivative of a connection Γ with respect to a vector field W [17]:

$$\mathcal{L}_W\Gamma^\lambda_{\mu\nu}=\nabla_\mu\nabla_\nu W^\lambda-W^\sigma R^\lambda{}_{\nu\mu\sigma}. \tag{B1}$$

For the coordinate field $W=\partial_a$ (i.e., $W^\mu=\delta^\mu_a$), Lie derivative reduces to the partial derivative: $\mathcal{L}_W\Gamma^\lambda_{\mu\nu}=\partial_a\Gamma^\lambda_{\mu\nu}$. Hence, taking appropriate traces of Eq. (B1) and denoting $\pi^{\mu\nu} := \sqrt{|g|}g^{\mu\nu}$, we obtain

$$\begin{aligned}\pi^{\mu\nu}\partial_a A^\alpha_{\mu\nu} &= (\delta^\alpha_\lambda\pi^{\mu\nu}-\delta^\mu_\lambda\pi^{\alpha\nu})\partial_\alpha\Gamma^\lambda_{\mu\nu}\\ &= (\delta^\alpha_\lambda\pi^{\mu\nu}-\delta^\mu_\lambda\pi^{\alpha\nu})(\nabla_\mu\nabla_\nu W^\lambda - W^\sigma R^\lambda{}_{\nu\mu\sigma})\\ &= \sqrt{|g|}\{\nabla_\mu(\nabla^\mu W^\alpha-\nabla^\alpha W^\mu)+2R^\alpha_\sigma W^\sigma\}\\ &= \partial_\mu\{\sqrt{|g|}(\nabla^\mu W^\alpha-\nabla^\alpha W^\mu)\}+2\sqrt{|g|}R^\alpha{}_\sigma W^\sigma.\end{aligned}$$

We apply this formula for $\alpha=3$. In this way we have

$$\begin{aligned}\pi^{\mu\nu}\partial_a A^3_{\mu\nu} &= \partial_\mu\{\sqrt{|g|}(\nabla^\mu W^3-\nabla^3 W^\mu)\}+2\mathcal{R}^3{}_a\\ &= \partial_b\{\sqrt{|g|}(\nabla^b W^3-\nabla^3 W^b)\}+2\mathcal{R}^3{}_a,\end{aligned} \tag{B2}$$

where $\mathcal{R}^3{}_a := \sqrt{|g|}R^3{}_a$. But

$$\nabla_\mu W^\nu = \Gamma^\nu_{a\mu}.$$

Hence

$$\begin{aligned}\nabla^b W^3-\nabla^3 W^b &= \tfrac{1}{2}(g^{b\lambda}g^{3\mu}-g^{3\lambda}g^{b\mu})(g_{\mu\lambda,a}+g_{\mu a,\lambda}-g_{\lambda a,\mu})\\ &= g^{b\lambda}g^{3\mu}(g_{\mu a,\lambda}-g_{\lambda a,\mu})\\ &= 2g^{b\lambda}\Gamma^3_{\lambda a}-g^{b\lambda}g^{3\mu}g_{\mu\lambda,a}\\ &= 2g^{b\lambda}A^3_{\lambda a}+g^{b3}\Gamma^\mu_{a\mu}+g^{b3}{}_{,a},\end{aligned}$$

and, consequently,

$$\sqrt{|g|}(\nabla^b W^3-\nabla^3 W^b) = 2\pi^{b\lambda}A^3_{\lambda a}+\pi^{3b}{}_{,a}, \tag{B3}$$

Inserting this into Eq. (B2) we obtain

$$\mathcal{R}^3{}_a+\partial_b\{\pi^{b\lambda}A^3_{\lambda a}-\tfrac{1}{2}\delta^b_a(\pi^{\mu\nu}A^3_{\mu\nu}-\pi^{3c}{}_{,c})\} = -\frac{1}{2}\pi^{\mu\nu}{}_{,a}A^3_{\mu\nu}. \tag{B4}$$

However,

$$\begin{aligned}-\pi^{\mu\nu}{}_{,a}A^3_{\mu\nu} &= -(g^{\mu\nu}\partial_a\sqrt{|g|}+\sqrt{|g|}g^{\mu\alpha}g^{\nu\beta}g_{\alpha\beta,a})A^3_{\mu\nu}\\ &= (-\tfrac{1}{2}g^{\alpha\beta}\pi^{\mu\nu}+g^{\alpha\mu}\pi^{\beta\nu})A^3_{\mu\nu}g_{\alpha\beta,a}\\ &= \tilde{Q}^{\alpha\beta}g_{\alpha\beta,a},\end{aligned} \tag{B5}$$

where we used definition (62), namely,

$$\begin{aligned}\tilde{Q}^\mu{}_\nu &:= \sqrt{|g|}(g^{\mu\alpha}A^3_{\alpha\nu}-\tfrac{1}{2}\delta^\mu{}_\nu g^{\alpha\beta}A^3_{\alpha\beta})\\ &= \pi^{\mu\alpha}A^3_{\alpha\nu}-\tfrac{1}{2}\delta^\mu{}_\nu\pi^{\alpha\beta}A^3_{\alpha\beta}.\end{aligned} \tag{B6}$$

Hence, we obtain the following identity:

$$\mathcal{G}^3{}_a+\partial_b\{\tilde{Q}^b{}_a+\tfrac{1}{2}\delta^b{}_a\pi^{3c}{}_{,c}\}-\tfrac{1}{2}\tilde{Q}^{\alpha\beta}g_{\alpha\beta,a}\equiv 0. \tag{B7}$$

To calculate the last term of Eq. (B7), we use the following.

Lemma B.3. The following equality holds:

$$\begin{aligned}s\tilde{Q}^{\alpha\beta}g_{\alpha\beta,a} &= \lambda(g^{be}g^{cd}l_{ed}-\tfrac{1}{2}lg^{bc})g_{bc,a}+(\Lambda^b g^{cd}+\Lambda^c g^{bd}\\ &\quad -\Lambda^d g^{cb})A^3_{3d}g_{bc,a}+2s\tilde{Q}^3{}_3\Big(\partial_a\ln M\\ &\quad +\frac{s}{M}m_B n^B_{,a}\Big).\end{aligned} \tag{B8}$$

Proof. From Eqs. (A20) and (A21) we obtain

$$\tilde{Q}^{33}=0$$

and

$$\tilde{Q}^{3b}=g^{3b}\tilde{Q}^3{}_3,$$

so

$$\tilde{Q}^{\alpha\beta}g_{\alpha\beta,a}=2\tilde{Q}^3{}_3 g^{3b}g_{3b,a}+\tilde{Q}^{bc}g_{bc,a}.$$

Moreover, from Eqs. (A7) and (A8) we have

$$g^{3b}g_{3b,a}=\partial_a\ln M+\frac{s}{M}m_B n^B_{,a}$$

and

$$\tilde{Q}^{ab}=(\delta^a{}_c g^{bd}+g^{ad}g^{3b}g_{3c})\tilde{Q}^c{}_d+\frac{X^aX^b}{M^2}\tilde{Q}_{33}.$$

Using Eq. (A19) and taking into account that $X^aX^bg_{ab,c}=0$ we get

$$\begin{aligned}s\tilde{Q}^{bc}g_{bc,a} &= \lambda(g^{be}g^{cd}l_{ed}-\tfrac{1}{2}lg^{bc})g_{bc,a}+\Lambda^b A^3_{3d}g^{cd}\\ &\quad +\Lambda^c A^3_{3d}g^{bd}-\Lambda^d A^3_{3d}g^{cb}g_{bc,a}\end{aligned} \tag{B9}$$

and finally

$$\begin{aligned}s\tilde{Q}^{\alpha\beta}g_{\alpha\beta,a} &= \lambda(g^{be}g^{cd}l_{ed}-\tfrac{1}{2}lg^{bc})g_{bc,a}+(\Lambda^b g^{cd}+\Lambda^c g^{bd}\\ &\quad -\Lambda^d g^{cb})A^3_{3d}g_{bc,a}+2s\tilde{Q}^3{}_3\Big(\partial_a\ln M\\ &\quad +\frac{s}{M}m_B n^B_{,a}\Big).\end{aligned} \tag{B10}$$

■

Now, the proof of Eq. (20) is roughly a straightforward calculation starting from Eq. (B7) and a consequent reexpression of all ingredients in terms of the connection objects l_{ab}, w_a and the metric objects M, m^A, N, X^a, g_{ab} describing the four-dimensional metric $g_{\mu\nu}$. It turns out that the terms containing M, N, and m^A drop out. Inserting Eqs. (A24) and (B8) into (B7) and using Eqs. (A18), (A20), and (A9), we obtain

$$s\mathcal{G}^3{}_a = -s\partial_b\{\tilde{Q}^b{}_a + \tfrac{1}{2}\delta^b{}_a\pi^{3c}{}_{,c}\} + s\tfrac{1}{2}\tilde{Q}^{\alpha\beta}g_{\alpha\beta,a}$$
$$= \partial_b\{-sQ^b{}_a + \delta^b{}_a\lambda l - \Lambda^a\chi_b + \delta^a{}_b\Lambda^c\chi_c\}$$
$$-\tfrac{1}{2}\lambda l\left(\partial_a \ln M + \frac{s}{M}m_B n^B_{,a}\right)$$
$$+\tfrac{1}{2}g_{bc,a}(\Lambda^b g^{cd} + \Lambda^c g^{bd} - \Lambda^d g^{cb})\left(w_d + \chi_d\right.$$
$$\left. + \frac{s}{M}m^B l_{Bd}\right) + \tfrac{1}{2}\lambda g_{bc,a}(g^{be}g^{cd}l_{ed} - \tfrac{1}{2}lg^{bc})$$
$$= -s\partial_b Q^b{}_a + \tfrac{1}{2}sQ^{bc}g_{bc,a} + \lambda\partial_a l, \tag{B11}$$

where we have used the formula

$$sQ^{ab} = \lambda\tilde{\tilde{g}}^{ac}\tilde{\tilde{g}}^{bd}l_{cd} + (\Lambda^a\tilde{\tilde{g}}^{bc} + \Lambda^b\tilde{\tilde{g}}^{ac} - \tilde{\tilde{g}}^{ab}\Lambda^c)w_c .$$

Formula (B11) is equivalent to Eq. (20) if we use Eq. (A16), and keep in mind the "gauge" condition $X(x^0) = 1$, used thoroughly in this proof.

APPENDIX C: PROOF OF LEMMA V.1 AND EXAMPLES OF INVARIANT LAGRANGIANS

Since the matter Lagrangian (33) is an invariant scalar density, its value may be calculated in any coordinate system. For purposes of the proof let us restrict ourselves to local coordinate systems (x^a) on S, which are compatible with the degeneracy of the metric, i.e., such that $X \coloneqq \partial_0$ is null-like.

Suppose that (x^a) and (y^a) are two such local systems in a neighborhood of a point $x \in S$. Suppose, moreover, that both vectors ∂_0 coincide. It is easy to see that these conditions imply the following form of the transformation between the two systems:

$$y^A = y^A(x^B), \tag{C1}$$

$$y^0 = x^0 + \psi(x^A). \tag{C2}$$

A three-dimensional Jacobian of such a transformation is equal to the two-dimensional one: $\det(\partial y^A/\partial x^B)$. Observe that the two-dimensional part g_{AB} of the metric g_{ab} transforms according to the same two-dimensional matrix and whence its determinant λ gets multiplied by the same two-dimensional Jacobian when transformed from (x^a) to (y^a). So does the volume v_X. This means that the function

$$f \coloneqq \frac{L}{v_X} \tag{C3}$$

does not change value during such a transformation. *A priori*, we could have:

$$f = f(z^K; z^K{}_0; z^K{}_A; g_{ab}), \tag{C4}$$

but we are going to prove that, in fact, it cannot depend upon derivatives $z^K{}_A$. For this purpose consider new coordinates:

$$y^A = x^A, \tag{C5}$$

$$y^0 = x^0 - \epsilon_1 x^1 - \epsilon_2 x^2. \tag{C6}$$

This implies that

$$\frac{\partial}{\partial y^A} = \frac{\partial}{\partial x^A} + \epsilon_A\frac{\partial}{\partial x^0}.$$

Passing from (x^a) to (y^a), the value of $z^K{}_A$ will be thus replaced by $z^K{}_A + \epsilon_A z^K{}_0$, whereas the remaining variables of the function (C4) (and also its value) will remain unchanged. This implies the following identity:

$$f(z^K; z^K{}_0; z^K{}_A; g_{ab}) = f(z^K; z^K{}_0; z^K{}_A + \epsilon_A z^K{}_0; g_{ab}), \tag{C7}$$

which must be valid for any configuration of the field z^K. Such a function cannot depend upon $z^K{}_A$. But in our coordinate system we have $z^K{}_0 = z^K{}_a X^a = \mathcal{L}_X z^K$. Thus, we have proved that

$$f = f(z^K; \mathcal{L}_X z^K; g_{ab}). \tag{C8}$$

Relaxing condition (C2) and admitting arbitrary time coordinates y^0, we easily see that the dependence of Eq. (C8) upon its second variable must annihilate the (homogeneous of degree minus one) dependence of the density v_X upon the field X in formula (34). This proves that f must be homogeneous of degree one in $\mathcal{L}_X z^K$.

As an example of an invariant Lagrangian consider a theory of a lightlike "elastic media" described by material variables z^A, $A = 1, 2$, considered as coordinates in a two-dimensional material space Z, equipped with a Riemannian "material metric" γ_{AB}. Moreover, take a scalar field ξ. Then for numbers α and $\beta > 0$, satisfying identity $2\alpha + \beta = 1$, and for any function ψ of one variable, the following Lagrangian density,

$$L = \lambda\psi(\xi)\left(X^a\frac{\partial z^K}{\partial x^a}X^b\frac{\partial z^L}{\partial x^b}\gamma_{KL}(z^A)\right)^\alpha\left(X^c\frac{\partial \xi}{\partial x^c}\right)^\beta, \tag{C9}$$

fulfills properties listed in Lemma V.1 and therefore is invariant. If ψ is constant, a possible physical interpretation of the variable ξ as a "thermodynamical potential" may be found in [14].

APPENDIX D: REDUCTION OF THE GENERATING FORMULA

1. Proof of formulas (78) and (79)

We reduce the generating formula with respect to constraints implied by identities $\nabla_k\pi^{0k} = 0$ and $\nabla_k\pi^{00} = 0$. In fact, expressing the left-hand sides in terms of $\pi^{\mu\nu}$ and $A^0_{\mu\nu}$ we immediately get the following constraints:

$$A^0_{00} = \frac{1}{\pi^{00}}(\partial_k\pi^{0k} + A^0_{kl}\pi^{kl}), \tag{D1}$$

$$A^0_{0k} = -\frac{1}{2\pi^{00}}(\partial_k\pi^{00} + 2A^0_{kl}\pi^{0l}). \tag{D2}$$

It is easy to see that they imply the following formula:

$$\pi^{\mu\nu}\delta A^0_{\mu\nu} = \pi^{kl}A^0_{kl} + 2\pi^{0k}\delta A^0_{0k} + \pi^{00}\delta A^0_{00} = -\frac{1}{16\pi}g_{kl}\delta P^{kl} + \partial_k\left(\pi^{00}\delta\left(\frac{\pi^{0k}}{\pi^{00}}\right)\right), \tag{D3}$$

where we have denoted

$$P^{kl} := \sqrt{\det g_{mm}}(K\tilde{g}^{kl} - K^{kl}), \quad K_{kl} := -\frac{1}{\sqrt{|g^{00}|}}\Gamma^0_{kl} = -\frac{1}{\sqrt{|g^{00}|}}A^0_{kl}, \tag{D4}$$

and $\tilde{g}^{kl}$ is the three-dimensional inverse with respect to the induced metric g_{kl} on V.

Let us exchange now the role of x^3 and x^0. Identities (D1) and (D2) become constraints on the boundary of the world-tube $\partial\mathcal{U}$:

$$A^3_{33} = \frac{1}{\pi^{33}}(\partial_a\pi^{3a} + A^3_{ab}\pi^{ab}), \tag{D5}$$

$$A^3_{3a} = -\frac{1}{2\pi^{33}}(\partial_a\pi^{33} + 2A^3_{ab}\pi^{3b}). \tag{D6}$$

They imply

$$\pi^{\mu\nu}\delta A^3_{\mu\nu} = \pi^{ab}\delta A^3_{ab} + 2\pi^{3a}\delta A^3_{3a} + \pi^{33}\delta A^3_{33} = -\frac{1}{16\pi}g_{ab}\delta Q^{kl} + \partial_a\left(\pi^{33}\delta\left(\frac{\pi^{3a}}{\pi^{33}}\right)\right), \tag{D7}$$

where we have denoted

$$\mathcal{Q}^{ab} = \sqrt{|\det g_{cd}|}(L\tilde{g}^{ab} - L^{ab}), \quad L_{ab} = -\frac{1}{\sqrt{g^{33}}}\Gamma^3_{ab}, \tag{D8}$$

and $\tilde{g}^{ab}$ is the three-dimensional inverse with respect to the induced metric g_{ab} on the world-tube.

2. Proof of formula (82)

Write the right-hand side as follows:

$$\pi^{00}\delta\left(\frac{\pi^{03}}{\pi^{00}}\right) + \pi^{33}\delta\left(\frac{\pi^{30}}{\pi^{33}}\right) = 2\sqrt{|\pi^{00}\pi^{33}|}\,\delta\frac{\pi^{30}}{\sqrt{|\pi^{00}\pi^{33}|}} \tag{D9}$$

and

$$2\sqrt{|\pi^{00}\pi^{33}|} = \frac{2}{16\pi}\sqrt{|g|}\sqrt{|g^{00}g^{33}|} = \frac{1}{8\pi}\frac{\sqrt{\det g_{AB}}}{\sqrt{1+q^2}}. \tag{D10}$$

This automatically implies

$$\pi^{00}\delta\left(\frac{\pi^{03}}{\pi^{00}}\right) + \pi^{33}\delta\left(\frac{\pi^{30}}{\pi^{33}}\right) = \frac{\lambda}{8\pi}\frac{\delta q}{\sqrt{1+q^2}} = \frac{\lambda}{8\pi}\delta\alpha. \tag{D11}$$

3. Proof of formula (84)

To prove Eq. (84), consider first the following identity

$$\int_V g_{kl}\dot{P}^{kl} = -\int_V D + \int_V \partial_k\left(\pi^{00}\partial_0\left(\frac{\pi^{0k}}{\pi^{00}}\right)\right), \tag{D12}$$

where we denote:

$$D := -\frac{1}{16\pi}g_{kl}\dot{P}^{kl} + \partial_k\left(\pi^{00}\partial_0\left(\frac{\pi^{0k}}{\pi^{00}}\right)\right) = \pi^{\mu\nu 0}_{\lambda}\partial_0\Gamma^\lambda_{\mu\nu} = \pi^{\mu\nu 0}_{\lambda}\mathcal{L}_X\Gamma^\lambda_{\mu\nu},$$

with $X = \partial/\partial x^0$, i.e., $X^\mu = \delta^\mu_0$ and $\mathcal{L}_X$ being the Lie derivative with respect to the field X:

$$\mathcal{L}_X\Gamma^\lambda_{\mu\nu} = \nabla_\mu\nabla_\nu X^\lambda - X^\sigma R^\lambda{}_{\nu\mu\sigma}$$

(due to Bianchi identities the right-hand side is automatically symmetric with respect to lower indices). Hence

$$\begin{aligned} D &:= (\delta^0_\lambda\pi^{\mu\nu} - \delta^\mu_\lambda\pi^{0\nu})(\nabla_\mu\nabla_\nu X^\lambda - X^\sigma R^\lambda{}_{\nu\mu\sigma}) \\ &= \frac{\sqrt{|g|}}{16\pi}\{\nabla_\mu(\nabla^\mu X^0 - \nabla^0 X^\mu) + 2R^0{}_\sigma X^\sigma\} \\ &= \frac{1}{16\pi}\{\partial_k(\sqrt{|g|}(\nabla^k X^0 - \nabla^0 X^k)) + 2\sqrt{|g|}R^0{}_0\}. \end{aligned} \tag{D13}$$

The covariant derivative ∇_μ has been replaced in the last equation by the partial derivative ∂_μ, because they both coincide when acting on antisymmetric, covariant bivector densities. We use also identity

$$\nabla^\mu X^\nu = g^{\mu\lambda}X^\sigma\Gamma^\nu_{\sigma\lambda} = g^{\mu\lambda}\Gamma^\nu_{0\lambda}. \tag{D14}$$

which finally implies:

$$\int_V D = \frac{1}{16\pi}\int_V \partial_\nu(\sqrt{|g|}(g^{\nu\mu}\Gamma^0_{0\mu} - g^{0\mu}\Gamma^\nu_{0\mu})) + \frac{1}{8\pi}\int_V \sqrt{|g|}R^0{}_0. \tag{D15}$$

D is regular, because singular expressions contained in its definition cancel out, as implied by Eq. (D12). Hence, we treat D as a regular expression, and there is no need to integrate it in a distributional sense. Hence we have

$$\int_V g_{kl}\dot{P}^{kl} = -\frac{1}{8\pi}\int_V \sqrt{|g|}R^0{}_0 - \frac{1}{16\pi}\int_{\partial V}\sqrt{|g|}\left(g^{3\mu}\Gamma^0_{0\mu}g^{0\mu}\Gamma^3_{0\mu} - \pi^{00}\partial_0\left(\frac{\pi^{03}}{\pi^{00}}\right)\right). \tag{D16}$$

From the definition of α we also have:

$$\lambda\dot{\alpha} = 8\pi\left(\pi^{00}\partial_0\left(\frac{\pi^{03}}{\pi^{00}}\right) + \pi^{33}\partial_0\left(\frac{\pi^{30}}{\pi^{33}}\right)\right). \tag{D17}$$

Using the formula above we may write

$$-\frac{1}{16\pi}\int_V (g_{kl}\dot{P}^{kl}) + \frac{1}{8\pi}\int_{\partial V}\lambda\dot{\alpha} = \frac{1}{8\pi}\int_V \sqrt{|g|}R^0{}_0 + \frac{1}{16\pi}\int_{\partial V}\sqrt{|g|}\left(g^{3\mu}\Gamma^0_{0\mu} - g^{0\mu}\Gamma^3_{0\mu} + g^{33}\partial_0\left(\frac{\pi^{30}}{\pi^{33}}\right)\right). \tag{D18}$$

The left-hand side of the equation above is regular, but on the right-hand side singular terms such as $(1/8\pi)\int_V\sqrt{|g|}R^0{}_0$ and $(1/16\pi)\int_{\partial V}\sqrt{|g|}(g^{3\mu}\Gamma^0_{0\mu} - g^{0\mu}\Gamma^3_{0\mu})$ arise. The latter quantity, although it is a boundary term, originates from the volume term $(1/16\pi)\int_V\partial_\nu[\sqrt{|g|}(g^{\nu\mu}\Gamma^0_{0\mu} - g^{0\mu}\Gamma^\nu_{0\mu})]$ via the Stokes theorem. From derivatives in the x^3 direction there are singular terms that cancel out the singular part of $R^0{}_0$, giving a regular expression as a final result.

We may rewrite expressions in Eq. (D18) in terms of the quantity $\mathcal{Q}^{ab}$ [defined by Eq. (D8)]:

$$\frac{1}{16\pi}\int_{\partial V}\sqrt{|g|}\left(g^{3\mu}\Gamma^0_{\mu 0} - g^{0\mu}\Gamma^3_{\mu 0} + g^{33}\partial_0\left(\frac{\pi^{30}}{\pi^{33}}\right)\right) = \frac{1}{16\pi}\int_{\partial V}(\mathcal{Q}^{AB}g_{AB} - \mathcal{Q}^{00}g_{00}), \tag{D19}$$

what completes the proof of formula (84).

APPENDIX E: PROOF OF THE CONSTRAINT (102)

Using the decomposition (A7) and (A8) of the metric, one can express the vector n orthonormal to V_t as follows:

$$n = \frac{1}{N}\left(\partial_0 - n^A\partial_A + s\frac{N^2}{M}m^A\partial_A - s\frac{N^2}{M}\partial_3\right).$$

Choosing $X = \partial_0 - n^A\partial_A$, we have

$$\frac{1}{N}X = s\frac{N}{M}(\partial_3 - m^A\partial_A) + n. \tag{E1}$$

Consequently, we can rewrite the left-hand side of Eq. (101) as follows:

$$\frac{1}{N}\mathcal{G}^0{}_\mu X^\mu = s\frac{N}{M}\mathcal{G}^0{}_3 - s\frac{N}{M}m^A\mathcal{G}^0{}_A + \mathcal{G}^0{}_\mu n^\mu. \tag{E2}$$

Expressing $\mathcal{G}^0{}_\mu$ in terms of the canonical ADM momentum P_{kl} [Eqs. (92) and (93)], Eq. (101) takes the form

$$0 = \frac{1}{N}\mathcal{G}^0{}_\mu X^\mu = s\frac{N}{M}(P_3{}^k{}_{|k} - m^A P_A{}^k{}_{|k}) + \frac{1}{2}\left(\gamma\overset{(3)}{R} - (P^{kl}P_{kl} - \tfrac{1}{2}P^2)\frac{1}{\gamma}\right). \tag{E3}$$

Equations (94) and (98) give us the following result:

$$s\frac{N}{M}([P_3{}^3] - m^A[P_A{}^3]) + [\lambda k] = 0. \tag{E4}$$

Due to Eq. (A7), one can express the three-dimensional inverse metric $\tilde{g}^{kl}$ as follows:

$$\tilde{g}^{kl} = \left(\frac{N}{M}\right)^2\begin{bmatrix}\left[\left(\frac{M}{N}\right)^2 + m^A m_A\right]\tilde{\tilde{g}}^{AB} & -m^A \\ -m^A & 1\end{bmatrix}. \tag{E5}$$

The above form of $\tilde{g}^{kl}$ can be used to rewrite the canonical momentum part of Eq. (E4):

$$s\frac{N}{M}([P_3{}^3] - m^A[P_A{}^3]) = s\frac{M}{N}[P^{33}] = \frac{\gamma}{\lambda}[P^{33}] = \left[\frac{P^{33}}{\sqrt{\tilde{g}^{33}}}\right], \tag{E6}$$

and finally we obtain the constraint (102).

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