Gravitomagnetic effects in the propagation of electromagnetic waves in variable gravitational fields of arbitrary-moving and spinning bodies

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The propagation of light in the gravitational field of self-gravitating spinning bodies moving with arbitrary velocities is discussed. The gravitational field is assumed to be "weak" everywhere. The equations of motion of a light ray are solved in the first post-Minkowskian approximation which is linear with respect to the universal gravitational constant *G*. We do not restrict ourselves to the approximation of a gravitational lens so that the solution of light geodesics is applicable for arbitrary locations of the source of light and the observer. This formalism is applied for studying corrections to the Shapiro time delay in binary pulsars caused by the rotation of the pulsar and its companion. We also derive the correction to the light deflection angle caused by the rotation of gravitating bodies in the solar system (Sun, planets) or a gravitational lens. The gravitational shift of frequency due to the combined translational and rotational motions of light-ray-deflecting bodies is analyzed as well. We give a general derivation of the formula describing the relativistic rotation of the plane of polarization of electromagnetic waves (Skrotskii effect). This formula is valid for arbitrary translational and rotational motion of gravitating bodies and greatly extends the results of previous researchers. Finally, we discuss the Skrotskii effect for gravitational waves emitted by localized sources such as a binary system. The theoretical results of this paper can be applied for studying various relativistic effects in microarcsecond space astrometry and developing corresponding algorithms for data processing in space astrometric missions such as FAME, SIM, and GAIA.

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I. INTRODUCTION

The influence of gravitation on the propagation of electromagnetic rays has been treated by many authors since Einstein first calculated the relativistic deflection of light by a spherically symmetric mass [1]. Currently there is much interest in several space missions dedicated to measuring astrometric positions, parallaxes, and proper motions of stars and quasars with an accuracy approaching 1 μ (arc sec) [2]. Thus progress in observational techniques has made it necessary to take into account the fact that electromagnetic rays are deflected not only by the monopole gravitoelectric field of light-deflecting bodies but also by their spin-gravitomagnetic and quadrupolar gravitoelectric fields. Propagation of light in the stationary gravitational fields of rotating oblate bodies with their centers of mass at rest is well known (see, e.g., [3–6], and references therein). Quite recently significant progress has been achieved in solving the problem of propagation of light rays in the field of an isolated source (such as a binary star) that has a time-dependent quadrupole moment and emits quadrupolar gravitational waves [7]. However, the translational motion of the source of gravitational waves was not taken into account in [7]; in effect, the source was assumed to be at rest.

We would like to emphasize the point that most gravitational sources are not in general static and move with respect to the observer in a variety of ways. As revealed by approximate estimates [8], the precision of planned astrometric space missions [2], binary pulsar timing tests of general relativity [9], very long baseline interferometry [10,11], etc., necessitate the development of more general methods of integration of the equations of light propagation that could account for such translational motions. Indeed, nonstationary sources emit gravitational waves that weakly perturb the propagation of electromagnetic signals. The cumulative effect of such gravitational waves may be quite large and, in principle, could be detectable [12]. It is necessary to know how large this influence is and whether one can neglect it or not in current and/or planned astrometric observations and experimental tests of general relativity. When considering the propagation of light in the field of moving bodies it is also worth keeping in mind that gravitational interaction propagates with the speed of light in linearized general relativity [13,14]. This retarded interaction has important consequences and can play a crucial role in the theoretical prediction of secondary relativistic effects in time delay, light deflection, polarization of light, etc.

The first crucial step in solving the problem of propagation of electromagnetic rays in the retarded gravitational field of arbitrary-moving bodies has been taken recently in [15]. The new formalism allows one to obtain a detailed description of the light-ray trajectory for unrestricted locations of the source of light and observer and to make unambiguous predictions for possible relativistic effects. An important feature of the formalism is that it is based on the post-Minkowskian solution [16–19] of the linearized Einstein field equations. Thus the amplitude of the gravitational potentials is assumed to be small compared to unity, but there are no *a priori* restrictions on the velocities, accelerations, etc., of the light-deflecting bodies. In this way the retarded character of the gravitational field is taken into account in the linear approximation. This is in contrast with the post-Newtonian approximation scheme [20–23], which assumes that the velocities of light-deflecting bodies must be small with respect to the speed of light. Such a treatment destroys the causal character of a gravitational null cone and makes the gravitational interaction appear to propagate instantaneously at each step of the iteration procedure. As proved in [15], this is the reason why the post-Newtonian metric gives correct answers for the time delay and light deflection angle only for a very restricted number of physical situations. As a rule, more subtle (secondary) effects in the propagation of light rays in time-dependent gravitational fields are not genuinely covered in the post-Newtonian approach, whereas the post-Minkowskian light-propagation formalism gives unique and unambiguous answers.

In the present paper, we extend the light-propagation formalism developed in [15] in order to be able to include the relativistic effects related to the gravitomagnetic field produced by the translational velocity-dependent terms in the metric tensor as well as the spin-dependent terms due to light-deflecting bodies. We shall start from the consideration of the energy-momentum tensor of moving and spinning bodies (Sec. II). Then we solve the Einstein field equations in terms of the retarded *Liénard-Wiechert* potentials (Sec. III), describe the light-ray trajectory in the field of these potentials (Sec. IV), and then calculate the deflection angle and time delay in the propagation of electromagnetic rays through the system of arbitrary-moving and spinning particles (Sec. V). We pay particular attention to the calculation of gravitomagnetic effects in ray propagation in pulsar timing and astrometry (Sec. VI). Moreover, we discuss the relativistic effect of the rotation of the plane of polarization of electromagnetic rays (Skrotskii effect [24–26]) in the gravitomagnetic field of the above-mentioned system of massive bodies (Sec. VII). This effect may be important for a proper interpretation of the number of astrophysical phenomena that take place in the accretion process of x-ray binaries [27–29] and/or supermassive black holes that may exist in active galactic nuclei. Finally, we derive an exact expression for the rotation of the plane of polarization of light caused by the quadrupolar gravitational waves emitted by localized sources (Sec. VIII). The treatment of the Skrotskii effect given in the final section may be important for understanding the effects of cosmological gravitational waves on the anisotropy of cosmic microwave background (CMB) radiation at different scales [30–32]. Details related to the calculation of integrals along the light-ray trajectory are relegated to the Appendices.

II. ENERGY-MOMENTUM TENSOR OF SPINNING BODY

We consider an ensemble of N self-gravitating bodies possessing mass and spin; higher-order multipole moments are neglected for the sake of simplicity; for their influence on light propagation via the mathematical techniques of the present paper see [33]. Propagation of light in the gravitational field of arbitrary-moving pointlike masses has been studied in [15]. The energy-momentum tensor $T^{\alpha\beta}$ of a spinning body is given by

$$T^{\alpha\beta}(t,\mathbf{x}) = T_M^{\alpha\beta}(t,\mathbf{x}) + T_S^{\alpha\beta}(t,\mathbf{x}), \tag{1}$$

where $T_M^{\alpha\beta}$ and $T_S^{\alpha\beta}$ are pieces of the tensor generated by, respectively, the mass and spin of the body, and t and $\mathbf{x}$ are the coordinate time and spatial coordinates of the underlying inertial coordinate system. In the case of several spinning bodies the total tensor of energy-momentum is a linear sum of tensors of the form (1) corresponding to each body. Therefore, in the linear approximation under consideration in this paper, the net gravitational field is simply a linear superposition of the fields due to individual bodies.

In Eq. (1), $T_M^{\alpha\beta}$ and $T_S^{\alpha\beta}$ are defined in terms of the Dirac δ function [34–36] as follows [37–39]:

$$T_M^{\alpha\beta}(t,\mathbf{x}) = \int_{-\infty}^{+\infty} p^{(\alpha}u^{\beta)}(-g)^{-1/2}\delta(t-z^0(\eta))\,\delta(\mathbf{x}-\mathbf{z}(\eta))\,d\eta, \tag{2}$$

$$T_S^{\alpha\beta}(t,\mathbf{x}) = -\nabla_\gamma \int_{-\infty}^{+\infty} S^{\gamma(\alpha}u^{\beta)}(-g)^{-1/2}\delta(t-z^0(\eta)) \times \delta(\mathbf{x}-\mathbf{z}(\eta))\,d\eta, \tag{3}$$

where η is the proper time along the world-line of the body's center of mass, $\mathbf{z}(\eta)$ are spatial coordinates of the body's center of mass at proper time η, $u^\alpha(\eta) = u^0(1,v^i)$ is the four-velocity of the body, $u^0(\eta) = (1-v^2)^{-1/2}$, $(v^i) \equiv \mathbf{v}(\eta)$ is the three-velocity of the body in space, $p^\alpha(\eta)$ is the body's linear momentum [in the approximation neglecting rotation of bodies $p^\alpha(\eta) = mu^\alpha$, where m is the invariant mass of the body], $S^{\alpha\beta}(\eta)$ is an antisymmetric tensor representing the body's spin angular momentum attached to the body's center of mass, ∇_γ denotes covariant differentiation with respect to the metric tensor $g_{\alpha\beta}$, and $g = \det(g_{\alpha\beta})$ is the determinant of the metric tensor.

The definition of $S^{\alpha\beta}$ is arbitrary up to the choice of a spin supplementary condition that is chosen as follows:

$$S^{\alpha\beta}u_\beta = 0; \tag{4}$$

this constraint is consistent with neglecting the internal structure of the bodies involved, so that we deal in effect with pointlike spinning particles [40,41]. We introduce a spin vector S^α which is related to the spin tensor by

$$S^{\alpha\beta} = \eta^{\alpha\beta\gamma\delta}u_\gamma S_\delta, \tag{5}$$

where $\eta^{\alpha\beta\gamma\delta}$ is the Levi-Civita̍ tensor related to the completely antisymmetric Minkowskian tensor $\epsilon_{\alpha\beta\gamma\delta}$ [42] as follows:

$$\eta^{\alpha\beta\gamma\delta} = -(-g)^{-1/2}\epsilon_{\alpha\beta\gamma\delta}, \qquad \eta_{\alpha\beta\gamma\delta} = (-g)^{1/2}\epsilon_{\alpha\beta\gamma\delta}, \tag{6}$$

where $\epsilon_{0123} = +1$. The spin vector is orthogonal to u^α by definition so that the identity

$$S^\alpha u_\alpha \equiv 0 \tag{7}$$

is always valid; this fixes the one remaining degree of freedom S^0. Hence, the four-vector S^α has only three independent spatial components.

The dynamical law $\nabla_\beta T^{\alpha\beta} = 0$, applied to Eqs. (1)–(3), leads to the equations of motion of a spinning particle in a gravitational field. Indeed, using a theorem of Schwartz that a distribution with a simple point as support is a linear combination of Dirac's delta function and a finite number of its derivatives, one can develop the theory of the motion of pointlike test bodies with multipole moments in general relativity [43]. In this way, it can be shown in particular that, for the "pole-dipole" particle under consideration in this paper, one is led uniquely [43] to the Mathisson-Papapetrou equations with the Pirani supplementary condition (4).

In what follows, we focus only on effects that are produced by the spin and are linear with respect to the spin and Newton's gravitational constant (the first post-Minkowskian approximation) in the underlying asymptotically inertial global coordinate system. The effects produced by the usual point-particle piece of the energy-momentum tensor have already been studied in [15]. Let us in what follows denote the components of the spin vector in the frame comoving with the body as $\mathcal{J}^\alpha=(0,\boldsymbol{\mathcal{J}})$. In this frame the temporal component of the spin vector vanishes as a consequence of Eq. (7) and, after making a Lorentz transformation from the comoving frame to the underlying "inertial" frame, we have in the post-Minkowskian approximation

$$S^0=\gamma(\mathbf{v}\cdot\boldsymbol{\mathcal{J}}),\quad S^i=\mathcal{J}^i+\frac{\gamma-1}{v^2}(\mathbf{v}\cdot\boldsymbol{\mathcal{J}})v^i, \tag{8}$$

where $\gamma\equiv(1-v^2)^{-1/2}$ and $\mathbf{v}=(v^i)$ is the velocity of the body with respect to the frame at rest.

III. GRAVITATIONAL FIELD EQUATIONS AND METRIC TENSOR

The metric tensor in the linear approximation can be written as

$$g_{\alpha\beta}(t,\mathbf{x})=\eta_{\alpha\beta}+h_{\alpha\beta}(t,\mathbf{x}), \tag{9}$$

where $\eta_{\alpha\beta}=\mathrm{diag}(-1,+1,+1,+1)$ is the Minkowski metric of flat space-time and the metric perturbation $h_{\alpha\beta}(t,\mathbf{x})$ is a function of time and spatial coordinates [36]. We split the metric perturbation into two pieces $h_M^{\alpha\beta}$ and $h_S^{\alpha\beta}$ that are linearly independent in the first post-Minkowskian approximation; that is,

$$h^{\alpha\beta}=h_M^{\alpha\beta}+h_S^{\alpha\beta}. \tag{10}$$

Thus, the solution for each piece can be found from the Einstein field equations with the corresponding energy-momentum tensor.

The point-particle piece $h_M^{\alpha\beta}$ of the metric tensor has already been discussed in [15] and is given by

$$h_M^{\alpha\beta}=4m\sqrt{1-v^2(s)}\frac{u^\alpha(s)u^\beta(s)+\frac{1}{2}\eta^{\alpha\beta}}{r(s)-\mathbf{v}(s)\cdot\mathbf{r}(s)}, \tag{11}$$

where $r(s)=|\mathbf{r}(s)|$, $\mathbf{r}(s)=\mathbf{x}-\mathbf{z}(s)$, and both the coordinates $\mathbf{z}$ and velocity $\mathbf{v}$ of the body are assumed to be time dependent and calculated at the retarded moment of time s defined by the light-cone equation

$$s+|\mathbf{x}-\mathbf{z}(s)|=t, \tag{12}$$

which has a vertex at the space-time point $(t,\mathbf{x})$ and describes the propagation of the gravitational field [44] on the unperturbed Minkowski space-time. The solution of this equation gives the retarded time s as a function of coordinate time t and spatial coordinates $\mathbf{x}$, that is, $s=s(t,\mathbf{x})$.

As for $h_S^{\alpha\beta}$, it can be found by solving the field equations which are given in the first post-Minkowskian approximation and in the harmonic gauge [45] as follows:

$$\Box h_S^{\alpha\beta}(t,\mathbf{x})=-16\pi\left[T_S^{\alpha\beta}(t,\mathbf{x})-\frac{1}{2}\eta^{\alpha\beta}T^{\lambda}_{S\lambda}(t,\mathbf{x})\right]. \tag{13}$$

Taking into account the orthogonality condition (4), the field equations (13) assume the form

$$\Box h_S^{\alpha\beta}(t,\mathbf{x})=16\pi\partial_\gamma\int_{-\infty}^{+\infty}d\eta[S^{\gamma(\alpha}(\eta)u^{\beta)}(\eta)\times\delta(t-z^0(\eta))\delta(\mathbf{x}-\mathbf{z}(\eta))], \tag{14}$$

where we have replaced the covariant derivative ∇_γ with a simple partial derivative $\partial_\gamma=\partial/\partial x^\gamma$ and $u^\alpha=\gamma(1,v^i)$, where $\gamma=(1-v^2)^{-1/2}$ is the Lorentz factor. The solution of these equations is given by the *Liénard-Wiechert* spin-dependent potentials

$$h_S^{\alpha\beta}(t,\mathbf{x})=-4\partial_\gamma\left[\frac{S^{\gamma(\alpha}(s)u^{\beta)}(s)}{r(s)-\mathbf{v}(s)\cdot\mathbf{r}(s)}\right]. \tag{15}$$

If we restricted ourselves to the linear approximation of general relativity, the sources under consideration here would have to be treated as a collection of free noninteracting spinning test particles each moving with arbitrary constant speed with its spin axis pointing in an arbitrary fixed direction. This simply follows from the Mathisson-Papapetrou equations in the underlying inertial coordinate system. Thus, in such a case carrying out the differentiation in Eq. (15), we arrive at

$$h_S^{\alpha\beta}(t,\mathbf{x})=4(1-v^2)\frac{r_\gamma S^{\gamma(\alpha}u^{\beta)}}{[r(s)-\mathbf{v}\cdot\mathbf{r}(s)]^3}, \tag{16}$$

where $\mathbf{v}$ and $S^{\alpha\beta}$ are treated as constants and we define $r^\alpha=(r,\mathbf{r})$. However, we may consider Eq. (9) as expressing the first two terms in a post-Minkowskian perturbation series; that is, at some "initial" time we turn on the gravitational interaction between the particles and keep track of terms in powers of the gravitational constant only, without making a Taylor expansion with respect to the ratio of the magnitudes of the characteristic velocities of bodies to the speed of light. The development of such a perturbative scheme involves many specific difficulties as discussed in [16–19]. In this approach one may relax the restrictions on the body's velocity and spin and think of $\mathbf{v}$ and $S^{\alpha\beta}$ in Eq. (16) as arbitrary functions of time. In this paper as well as [15], we limit our considerations to the first-order equations (10), (11), and (16); however, in the *application* of these results to the problem of ray propagation, we let $\mathbf{v}$ and $S^{\alpha\beta}$ in the *final* results based on Eqs. (11) and (16) be time dependent as required in the specific astrophysical situation under consideration. Therefore, the consistency of the final physical results with the requirements of our approximation scheme must be checked in every instance.

The metric tensor given by Eqs. (10), (11), and (16) can be used to solve the problem of propagation of light rays in the gravitational field of arbitrary-moving and spinning bodies.

IV. PROPAGATION LAWS FOR ELECTROMAGNETIC RADIATION

The general formalism describing the behavior of electromagnetic radiation in an arbitrary gravitational field is well known [46]. A high-frequency electromagnetic wave is defined as an approximate solution of the Maxwell equations of the form

$$F_{\alpha\beta}=\mathrm{Re}\{A_{\alpha\beta}\exp(i\varphi)\}, \tag{17}$$

where $A_{\alpha\beta}$ is a slowly varying function of position and φ is a rapidly varying phase of the electromagnetic wave. From Eq. (17) and Maxwell's equations one derives the following results (for more details see [47,48] and Appendix A). The electromagnetic wave vector $l_\alpha=\partial\varphi/\partial x^\alpha$ is real and null, that is, $g^{\alpha\beta}l_\alpha l_\beta=0$. The curves $x^\alpha=x^\alpha(\lambda)$ that have $l^\alpha=dx^\alpha/d\lambda$ as a tangent vector are null geodesics orthogonal to the surfaces of constant electromagnetic phase φ. The null vector l^α is parallel transported along itself according to the null geodesic equation

$$l^\beta\nabla_\beta l^\alpha=0. \tag{18}$$

The equation of the parallel transport (18) can be expressed as

$$\frac{dl^\alpha}{d\lambda}+\Gamma^\alpha_{\beta\gamma}l^\beta l^\gamma=0, \tag{19}$$

where λ is an affine parameter. The electromagnetic field tensor $F_{\alpha\beta}$ is a null field satisfying $F_{\alpha\beta}l^\beta=0$ whose propagation law in an arbitrary empty space-time is

$$D_\lambda F_{\alpha\beta}+\theta F_{\alpha\beta}=0, \quad D_\lambda\equiv\frac{D}{D\lambda}=\frac{dx^\alpha}{d\lambda}\nabla_\alpha, \tag{20}$$

where $\theta=\frac{1}{2}\nabla_\alpha l^\alpha$ is the expansion of the null congruence l_α.

Let us now construct a null tetrad $(l^\alpha,n^\alpha,m^\alpha,\overline{m}^\alpha)$, where the overbar indicates complex conjugation, and $n_\alpha l^\alpha=-1$ and $m_\alpha\overline{m}^\alpha=+1$ are the only nonvanishing products among the tetrad vectors. Then the electromagnetic tensor $F_{\alpha\beta}=\mathrm{Re}(\mathcal{F}_{\alpha\beta})$ can be written as (see, e.g., [49,50])

$$\mathcal{F}_{\alpha\beta}=\Phi l_{[\alpha}m_{\beta]}+\Psi l_{[\alpha}\overline{m}_{\beta]}, \tag{21}$$

where Φ and Ψ are complex scalar functions. In the rest frame of an observer with four-velocity u^α the components of the electric and magnetic field vectors are defined as $E^\alpha=F^{\alpha\beta}u_\beta$ and $H^\alpha=(-1/2)\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}u_\beta$, respectively.

In what follows it is useful to introduce a local orthonormal reference frame based on a restricted set of observers that all see the electromagnetic wave traveling in the $+z$ direction; i.e., the observers use a tetrad frame $e^\alpha_{(\beta)}$ such that

$$e^\alpha_{(0)}=u^\alpha, \quad e^\alpha_{(3)}=(-l_\alpha u^\alpha)^{-1}[l^\alpha+(l_\beta u^\beta)u^\alpha], \tag{22}$$

and $e^\alpha_{(1)},e^\alpha_{(2)}$ are two unit spacelike vectors orthogonal to each other as well as to both $e^\alpha_{(0)}$ and $e^\alpha_{(3)}$ [51,52]. The vectors $e^\alpha_{(1)}$ and $e^\alpha_{(2)}$ play a significant role in the discussion of polarized radiation. In fact, the connection between the null tetrad and the frame $e^\alpha_{(\beta)}$ is given by $l^\alpha=-(l_\gamma u^\gamma)(e^\alpha_{(0)}+e^\alpha_{(3)})$, $n^\alpha=-\frac{1}{2}(l_\gamma u^\gamma)(e^\alpha_{(0)}-e^\alpha_{(3)})$, $m^\alpha=2^{-1/2}(e^\alpha_{(1)}+ie^\alpha_{(2)})$, and $\overline{m}^\alpha=2^{-1/2}(e^\alpha_{(1)}-ie^\alpha_{(2)})$. The vectors $e^\alpha_{(1)}$ and $e^\alpha_{(2)}$ are defined up to an arbitrary rotation in space; their more specific definitions will be given in Sec. VII.

Vectors of the null tetrad $(l^\alpha,n^\alpha,m^\alpha,\overline{m}^\alpha)$ and those of $e^\alpha_{(\beta)}$ are parallel transported along the null geodesics. Thus, from the definition (21) and Eq. (20) it follows that the amplitude of the electromagnetic wave propagates according to the law

$$\frac{d\Phi}{d\lambda}+\theta\Phi=0, \quad \frac{d\Psi}{d\lambda}+\theta\Psi=0. \tag{23}$$

If $\mathcal{A}$ is the area of the cross section of a congruence of light rays, then

$$\frac{d\mathcal{A}}{d\lambda}=2\theta\mathcal{A}. \tag{24}$$

Thus, $\mathcal{A}|\Phi|^2$ and $\mathcal{A}|\Psi|^2$ remain constant along the congruence of light rays l^α.

The null tetrad frame $(l^\alpha,n^\alpha,m^\alpha,\overline{m}^\alpha)$ and the associated orthonormal tetrad frame $e^\alpha_{(\beta)}$ are not unique. In the JWKB (or geometric optics) approximation, the null ray follows a geodesic with tangent vector $l^\alpha=dx^\alpha/d\lambda$, where λ is an affine parameter. This parameter is defined up to a linear transformation $\lambda'=\lambda/A+\mathrm{const}$, where A is a nonzero constant, so that $l'^\alpha=dx^\alpha/d\lambda'=Al^\alpha$. The null tetrad is then $(l'^\alpha,n'^\alpha,m^\alpha,\overline{m}^\alpha)$ with $n'^\alpha=A^{-1}n^\alpha$; this affine transformation leaves the associated tetrad $e^\alpha_{(\beta)}$ unchanged. Moreover, we note that at each event the tetrad frame $e^\alpha_{(\beta)}$ is defined up to a Lorentz transformation. We are interested, however, in a subgroup of the Lorentz group that leaves l^α invariant, i.e., $\Lambda^\alpha_\beta l^\beta=l^\alpha$. This subgroup, which is the *little group* of the null vector l^α, is isomorphic to the Euclidean group in the plane. This consists of translations plus a rotation. The translation part is a two-dimensional Abelian subgroup given by

$$n'^\alpha=n^\alpha+Bm^\alpha+\overline{Bm^\alpha}+|B|^2l^\alpha, \tag{25}$$

$$m'^\alpha=m^\alpha+\overline{B}l^\alpha. \tag{26}$$

Let us note that this transformation leaves the electromagnetic tensor $F_{\alpha\beta}$ given in Eq. (21) invariant; hence, this is the *gauge subgroup* of the *little group* of l^α.

The rotation part of the subgroup under discussion is simply given by $n'^\alpha=n^\alpha$ and $m'^\alpha=Cm^\alpha$, where C

$=\exp(-i\Theta)$. This corresponds to a simple rotation by a constant angle Θ in the $(e^{\alpha}_{(1)},e^{\alpha}_{(2)})$ plane, i.e.,

$$e'^{\alpha}_{(1)}=\cos\Theta e^{\alpha}_{(1)}+\sin\Theta e^{\alpha}_{(2)}, \tag{27}$$

$$e'^{\alpha}_{(2)}=-\sin\Theta e^{\alpha}_{(1)}+\cos\Theta e^{\alpha}_{(2)}. \tag{28}$$

Finally, we note that a *null rotation* is an element of the four-parameter group of transformations given by

$$l'^{\alpha}=Al^{\alpha}, \tag{29}$$

$$n'^{\alpha}=A^{-1}n^{\alpha}+Bm^{\alpha}+\overline{Bm}^{\alpha}+|B|^2Al^{\alpha}, \tag{30}$$

$$m'^{\alpha}=C(m^{\alpha}+\bar{B}Al^{\alpha}). \tag{31}$$

A null rotation is the most general transformation of the local null tetrad frame that leaves the spatial *direction* of the null vector l^{α} invariant.

V. EQUATIONS OF LIGHT GEODESICS AND THEIR SOLUTIONS

We consider the motion of a light particle (''photon'') in the background gravitational field described by the metric (9). No back action of the photon on the gravitational field is assumed. Hence, we are allowed to use the equations of light geodesics (19) with $l^{\alpha}=dx^{\alpha}/d\lambda$ directly applying the metric tensor in question. Let the motion of the photon be defined by fixing the mixed initial-boundary conditions

$$\mathbf{x}(t_0)=\mathbf{x}_0, \quad \frac{d\mathbf{x}}{dt}(-\infty)=\mathbf{k}, \tag{32}$$

where $|\mathbf{k}|^2=1$ and, henceforth, the spatial components of vectors are denoted by bold letters. These conditions define the coordinates $\mathbf{x}_0$ of the photon at the moment of emission of light, t_0, and its velocity at the infinite past and infinite distance from the origin of the spatial coordinates (that is, at the so-called past null infinity denoted by $\mathcal{J}_-$ [47]).

In the underlying inertial frame of the background flat space-time the unperturbed trajectory of the light ray is a straight line

$$x^i(t)=x^i_N(t)=x^i_0+k^i(t-t_0),$$

where t_0, x^i_0, and $(k^i)=\mathbf{k}$ have been defined in Eq. (32). It is convenient to introduce a new independent parameter τ along the photon's trajectory according to the rule [7]

$$\tau=\mathbf{k}\cdot\mathbf{x}_N(t)=t-t_0+\mathbf{k}\cdot\mathbf{x}_0. \tag{33}$$

The time t_0 of the light signal's emission corresponds to $\tau=\tau_0$, where $\tau_0=\mathbf{k}\cdot\mathbf{x}_0$, and $\tau=0$ corresponds to the coordinate time $t=t^*$, where

$$t^*=t_0-\mathbf{k}\cdot\mathbf{x}_0. \tag{34}$$

This is the time of the closest approach of the unperturbed trajectory of the photon to the origin of an asymptotically flat harmonic coordinate system. We emphasize that the numerical value of the moment t^* is constant for a chosen trajectory of light ray and depends only on the space-time coordinates of the point of emission of the photon and the point of its observation. Thus, we find the relationship

$$\tau\equiv t-t^*, \tag{35}$$

which reveals that the differential identity $dt=d\tau$ is valid and, for this reason, the integration along the light ray's path with respect to time t can always be replaced by integration with respect to τ with a corresponding shift in the limits of integration.

Making use of the parameter τ, the equation of the unperturbed trajectory of the light ray can be represented as

$$x^i(\tau)=x^i_N(\tau)=k^i\tau+\xi^i. \tag{36}$$

The constant vector $(\xi^i)=\boldsymbol{\xi}=\mathbf{k}\times(\mathbf{x}_0\times\mathbf{k})$ is called the impact parameter of the unperturbed trajectory of the light ray with respect to the origin of the coordinates; $d=|\boldsymbol{\xi}|$ is the length of the impact parameter. We note that the vector $\boldsymbol{\xi}$ is transverse to the vector $\mathbf{k}$ and directed from the origin of the coordinate system toward the point of the closest approach of the unperturbed path of the light ray to the origin as depicted in Fig. 1.

The equations of light geodesics can be expressed in the first post-Minkowskian approximation as follows (for more details, see [7]):

$$\ddot{x}^i(\tau)=\frac{1}{2}k_\alpha k_\beta\hat{\partial}_i h^{\alpha\beta}(\tau,\boldsymbol{\xi})-\hat{\partial}_\tau\Big[k_\alpha h^{\alpha i}(\tau,\boldsymbol{\xi})+\frac{1}{2}k^i h^{00}(\tau,\boldsymbol{\xi}) - \frac{1}{2}k^i k_p k_q h^{pq}(\tau,\boldsymbol{\xi})\Big], \tag{37}$$

where an overdot denotes differentiation with respect to time, $\hat{\partial}_\tau\equiv\partial/\partial\tau$, $\hat{\partial}_i\equiv P_{ij}\partial/\partial\xi^j$, $k^\alpha=(1,k^i)$, $k_\alpha=(-1,k_i)$, $k^i=k_i$, $P_{ij}=\delta_{ij}-k_ik_j$ is the operator of projection onto the plane orthogonal to the vector $\mathbf{k}$, and $h^{\alpha\beta}(\tau,\boldsymbol{\xi})$ is simply $h^{\alpha\beta}(t,\mathbf{x})$ with $t=\tau+t^*$ and $\mathbf{x}=\mathbf{k}\tau+\boldsymbol{\xi}$. For a given null ray, all quantities on the right side of Eq. (37) depend on the running parameter τ and the parameter $\boldsymbol{\xi}$ which is assumed to be constant. Hence, Eq. (37) should be considered as an ordinary second-order differential equation in the variable τ.

Perturbations of the trajectory of the photon are found by straightforward integration of the equations of light geodesics (37). Performing the calculations we find

$$\dot{x}^i(\tau)=k^i+\dot{\Xi}^i(\tau), \tag{38}$$

$$x^i(\tau)=x^i_N(\tau)+\Xi^i(\tau)-\Xi^i(\tau_0), \tag{39}$$

where $\tau=t-t^*$ and $\tau_0=t_0-t^*$ correspond, respectively, to the moments of observation and emission of the photon. The functions $\dot{\Xi}^i(\tau)$ and $\Xi^i(\tau)$ are given as follows:

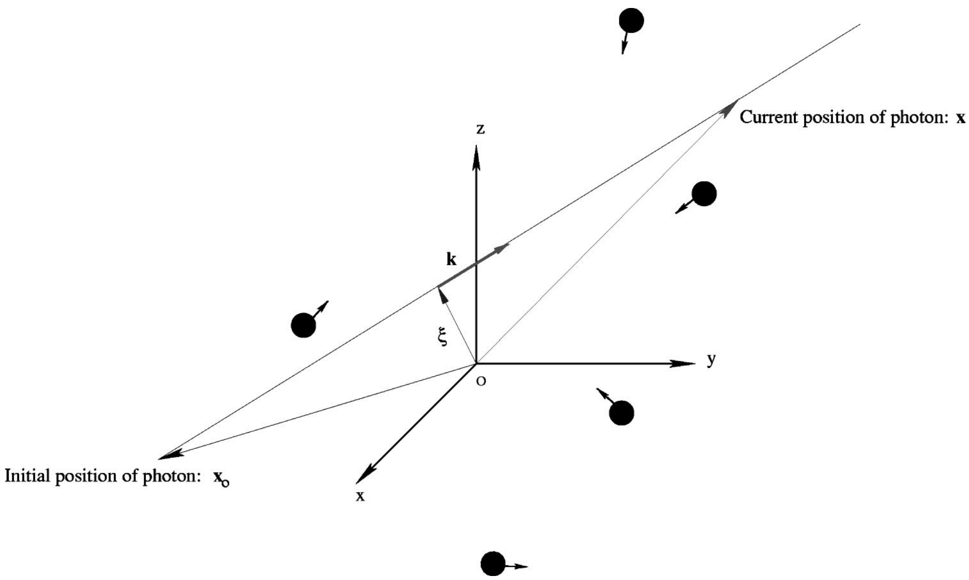

FIG. 1. Astronomical coordinate system used for the calculations. The origin of the coordinate system is at an arbitrary point in space. The unperturbed trajectory of a light ray is defined by the unit vector **k** directed from the source of light towards the observer. The impact parameter of the light ray is defined by the vector $\boldsymbol{\xi}$ which is orthogonal to **k**. Gravitating bodies with spin move along arbitrary world lines.

$$\dot{\Xi}^i(\tau)=\frac{1}{2}k_\alpha k_\beta \hat{\partial}_i B^{\alpha\beta}(\tau)-k_\alpha h^{\alpha i}(\tau)-\frac{1}{2}k^i h^{00}(\tau)+\frac{1}{2}k^i k_p k_q h^{pq}(\tau), \tag{40}$$

$$\Xi^i(\tau)=\frac{1}{2}k_\alpha k_\beta \hat{\partial}_i D^{\alpha\beta}(\tau)-k_\alpha B^{\alpha i}(\tau)-\frac{1}{2}k^i B^{00}(\tau)+\frac{1}{2}k^i k_p k_q B^{pq}(\tau), \tag{41}$$

where it is implicitly assumed that $h^{\alpha\beta}(-\infty)=0$, and the integrals $B^{\alpha\beta}(\tau)$ and $D^{\alpha\beta}(\tau)$ are given by

$$B^{\alpha\beta}(\tau)=B_M^{\alpha\beta}(\tau)+B_S^{\alpha\beta}(\tau),\quad B_M^{\alpha\beta}(\tau)=\int_{-\infty}^{\tau} h_M^{\alpha\beta}(\sigma,\boldsymbol{\xi})d\sigma,$$

$$B_S^{\alpha\beta}(\tau)=\int_{-\infty}^{\tau} h_S^{\alpha\beta}(\sigma,\boldsymbol{\xi})d\sigma, \tag{42}$$

$$D^{\alpha\beta}(\tau)=D_M^{\alpha\beta}(\tau)+D_S^{\alpha\beta}(\tau),$$

$$D_M^{\alpha\beta}(\tau)=\int_{-\infty}^{\tau} B_M^{\alpha\beta}(\sigma,\boldsymbol{\xi})d\sigma,$$

$$D_S^{\alpha\beta}(\tau)=\int_{-\infty}^{\tau} B_S^{\alpha\beta}(\sigma,\boldsymbol{\xi})d\sigma. \tag{43}$$

The integrals (42) and (43) and their derivatives are calculated in Appendix C by extending a method developed in [7] and [15]. It is important to emphasize, however, that in the case where the body moves along a straight line with constant velocity and spin one can obtain the same results by direct computation without employing the methods used in Appendix C.

Equation (39) can be used for the formulation of the boundary-value problem for the equation of light geodesics where the initial position $\mathbf{x}_0=\mathbf{x}(t_0)$ and final position $\mathbf{x}=\mathbf{x}(t)$ of the photon are prescribed for finding the solution of the light trajectory. This is in contrast to our original boundary-value problem, where the initial position $\mathbf{x}_0$ of the photon and the direction of light propagation **k** given at the past null infinity were specified. All that we need for the solution of the new boundary-value problem is the relationship between the unit vector **k** and the unit vector

$$\mathbf{K}=-\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|}, \tag{44}$$

which defines the geometric (coordinate) direction of the light propagation from the observer to the source of light as if the space-time were flat. The formulas (39) and (41) yield

$$k^i=-K^i-\beta^i(\tau,\boldsymbol{\xi})+\beta^i(\tau_0,\boldsymbol{\xi}), \tag{45}$$

where the relativistic corrections to the vector K^i are given by

$$\beta^i(\tau,\boldsymbol{\xi})=\frac{\frac{1}{2}k_\alpha k_\beta \hat{\partial}_i D^{\alpha\beta}(\tau)-k_\alpha P_{ij}B^{\alpha j}(\tau)}{|\mathbf{x}(\tau)-\mathbf{x}_0|}. \tag{46}$$

We emphasize that the vectors $(\beta^i)\equiv\boldsymbol{\beta}(\tau,\boldsymbol{\xi})$ and $(\beta_0^i)\equiv\boldsymbol{\beta}(\tau_0,\boldsymbol{\xi})$ are orthogonal to **k** and are evaluated at the points of observation and emission of the photon, respectively. The relationships obtained in this section are used for the discussion of observable relativistic effects in the following sections.

VI. GRAVITOMAGNETIC EFFECTS IN PULSAR TIMING, ASTROMETRY, AND DOPPLER TRACKING

A. Shapiro time delay in binary pulsars

We shall give in this paragraph the relativistic time delay formula for the case of the propagation of light through the *nonstationary* gravitational field of an arbitrary-moving and rotating body. The total time of propagation of an electromagnetic signal from the point $\mathbf{x}_0$ to the point $\mathbf{x}$ is derived from Eqs. (39) and (41). First, we use Eq. (39) to express the difference $\mathbf{x}-\mathbf{x}_0$ via the other terms of this equation. Then, we find the total coordinate time of propagation of light, $t-t_0$, from

$$t-t_0=|\mathbf{x}-\mathbf{x}_0|+\Delta_M(t,t_0)+\Delta_S(t,t_0), \tag{47}$$

where $|\mathbf{x}-\mathbf{x}_0|$ is the usual Euclidean distance between the points of emission, $\mathbf{x}_0$, and observation, $\mathbf{x}$, of the photon, $\Delta_M(t,t_0)$ is the Shapiro time delay produced by the gravitoelectric field of a pointlike massive body, and $\Delta_S(t,t_0)$ is the Shapiro time delay produced by the gravitomagnetic field of the spinning source. The term $\Delta_M(t,t_0)$ is discussed in detail in [53]. The new term $\Delta_S(t,t_0)$ is given by [cf. Eq. (C16) of Appendix C]

$$\Delta_S(t,t_0)=B_S(\tau)-B_S(\tau_0), \tag{48}$$

$$B_S(\tau)\equiv\frac{1}{2}k_\alpha k_\beta B_S^{\alpha\beta}(\tau)=2\frac{1-\mathbf{k}\cdot\mathbf{v}}{\sqrt{1-v^2}}\frac{k_\alpha r_\beta S^{\alpha\beta}}{(r-\mathbf{v}\cdot\mathbf{r})(r-\mathbf{k}\cdot\mathbf{r})}, \tag{49}$$

$$B_S(\tau_0)\equiv\frac{1}{2}k_\alpha k_\beta B_S^{\alpha\beta}(\tau_0)$$
$$=2\frac{1-\mathbf{k}\cdot\mathbf{v}_0}{\sqrt{1-v_0^2}}\frac{k_\alpha r_{0\beta}S_0^{\alpha\beta}}{(r_0-\mathbf{v}_0\cdot\mathbf{r}_0)(r_0-\mathbf{k}\cdot\mathbf{r}_0)}, \tag{50}$$

where the times $\tau=t-t^*$ and $\tau_0=t_0-t^*$ are related to the retarded times s and s_0 via Eq. (12), $S^{\alpha\beta}=S^{\alpha\beta}(s)$, $S_0^{\alpha\beta}=S^{\alpha\beta}(s_0)$, $\mathbf{r}=\mathbf{x}-\mathbf{z}$, $\mathbf{r}_0=\mathbf{x}_0-\mathbf{z}_0$, $r_0=|\mathbf{r}_0|$, $\mathbf{x}=\mathbf{x}(t)$, $\mathbf{x}_0=\mathbf{x}(t_0)$, $\mathbf{z}=\mathbf{z}(s)$, $\mathbf{z}_0=\mathbf{z}(s_0)$, $\mathbf{v}=\mathbf{v}(s)$, and $\mathbf{v}_0=\mathbf{v}(s_0)$. It is worthwhile to note that in the approximation where one can neglect relativistic terms in the relationship (45) between the vectors $\mathbf{k}$ and $-\mathbf{K}$, the expression for $\mathbf{r}$ is given by

$$\mathbf{r}=D\mathbf{k}+\mathbf{x}_0-\mathbf{z}(s), \tag{51}$$

where $D=|\mathbf{x}-\mathbf{x}_0|$. In the case of a binary pulsar, the distance D between the pulsar and the solar system is much larger than that between the point of emission of the radio pulse, $\mathbf{x}_0$, and the pulsar or its companion, $\mathbf{z}(s)$. Hence,

$$\mathbf{r}-\mathbf{k}r=\mathbf{k}\times[\mathbf{k}\times(\mathbf{x}_0-\mathbf{z})]-\frac{[\mathbf{k}\times(\mathbf{x}_0-\mathbf{z})]^2}{2D}\mathbf{k}, \tag{52}$$

where terms of higher order in the ratio $|\mathbf{x}_0-\mathbf{z}|/D$ are neglected.

The result (48) can be used, for example, to find the spin-dependent relativistic correction Δ_S to the timing formula of binary pulsars. The binary pulsar consists of two bodies—the pulsar itself (index "p") and its companion (index "c"); in what follows we let $\mathbf{r}_p=\mathbf{x}-\mathbf{z}_p(s)$, $\mathbf{r}_c=\mathbf{x}-\mathbf{z}_c(s)$, $\mathbf{r}_{0p}=\mathbf{x}_0-\mathbf{z}_p(s_0)$, and $\mathbf{r}_{0c}=\mathbf{x}_0-\mathbf{z}_c(s_0)$, where $\mathbf{z}_p$ and $\mathbf{z}_c$ are coordinates of the pulsar and its companion, respectively. According to [15], the difference between the instants of time s and s_0, s and t_0, and s_0 and t_0 for binary pulsars is of the order of the time required for light to cross the system, that is, of the order of a few seconds. Thus, we can expand all quantities depending on time in the neighborhood of the instant t_0 and make use of the approximations $\mathbf{r}_p=\mathbf{x}-\mathbf{z}_p(t_0)\equiv\boldsymbol{\rho}_p$, $\mathbf{r}_c=\mathbf{x}-\mathbf{z}_c(t_0)\equiv\boldsymbol{\rho}_c$, $\mathbf{r}_{0p}=\mathbf{x}_0-\mathbf{z}_p(t_0)\equiv\boldsymbol{\rho}_{0p}$, $\mathbf{r}_{0c}=\mathbf{x}_0-\mathbf{z}_c(t_0)\equiv\boldsymbol{\rho}_{0c}$, $\mathcal{J}_p=\mathcal{J}_{0p}$, and $\mathcal{J}_c=\mathcal{J}_{0c}$. The next approximation used in the calculations is

$$\mathbf{x}_0=\mathbf{z}_p(t_0)+\mathbf{k}X, \tag{53}$$

where X is the distance from the pulsar's center of mass to the point of emission of radio pulses [54]. One can also see that $\boldsymbol{\rho}_{0c}=\mathbf{R}+\mathbf{k}X$, where $\mathbf{R}=\mathbf{z}_p(t_0)-\mathbf{z}_c(t_0)$, i.e., the radius vector of the pulsar with respect to the companion. Taking these approximations into account, the following equalities hold:

$$\boldsymbol{\rho}_c-\mathbf{k}\rho_c=\mathbf{k}\times(\mathbf{R}\times\mathbf{k})-\frac{(\mathbf{k}\times\mathbf{R})^2}{2D}\mathbf{k}, \tag{54}$$

$$\boldsymbol{\rho}_p-\mathbf{k}\rho_p=0. \tag{55}$$

Moreover, it follows from Eq. (54) with $\rho_c\approx D$ that

$$\rho_c-\mathbf{k}\cdot\boldsymbol{\rho}_c=\frac{\rho_{0c}^2-(\mathbf{k}\cdot\boldsymbol{\rho}_{0c})^2}{2\rho_c}=\frac{R^2-(\mathbf{k}\cdot\mathbf{R})^2}{2\rho_c}. \tag{56}$$

The primary contribution to the function $\Delta_S(t,t_0)$ is obtained after expansion of the expressions for $B(\tau)$ and $B(\tau_0)$ in powers of v/c and picking up all velocity-independent terms. Taking account of Eqs. (54)–(56) and (D3) this procedure gives the time delay correction Δ_S to the standard timing formula as follows:

$$\Delta_S=\frac{2\mathcal{J}_p\cdot(\mathbf{k}\times\mathbf{r}_p)}{r_p(r_p-\mathbf{k}\cdot\mathbf{r}_p)}+\frac{2\mathcal{J}_c\cdot(\mathbf{k}\times\mathbf{r}_c)}{r_c(r_c-\mathbf{k}\cdot\mathbf{r}_c)}-\frac{2\mathcal{J}_{0p}\cdot(\mathbf{k}\times\mathbf{r}_{0p})}{r_{0p}(r_{0p}-\mathbf{k}\cdot\mathbf{r}_{0p})}-\frac{2\mathcal{J}_{0c}\cdot(\mathbf{k}\times\mathbf{r}_{0c})}{r_{0c}(r_{0c}-\mathbf{k}\cdot\mathbf{r}_{0c})}. \tag{57}$$

By means of Eqs. (53) and (55) we find that the first and third terms in Eq. (57) drop out. Neglecting terms of order X/R we can see that $\boldsymbol{\rho}_{0c}=\mathbf{R}$, which brings Δ_S into the form

$$\Delta_S=2\mathcal{J}_{0c}\cdot(\mathbf{k}\times\mathbf{R})\left[\frac{1}{\rho_c(\rho_c-\mathbf{k}\cdot\boldsymbol{\rho}_c)}-\frac{1}{R(R-\mathbf{k}\cdot\mathbf{R})}\right]. \tag{58}$$

Making use of Eq. (56) transforms expression (58) to the simpler form

$$\Delta_S=\frac{2\mathcal{J}_{0c}\cdot(\mathbf{k}\times\mathbf{R})}{R(R+\mathbf{k}\cdot\mathbf{R})}=-\frac{2\mathcal{J}_{0c}\cdot(\mathbf{K}\times\mathbf{R})}{R(R-\mathbf{K}\cdot\mathbf{R})}, \tag{59}$$

where the unit vector $\mathbf{K}$ is defined in Eq. (44).

Formula (59) coincides exactly with that obtained on the basis of the post-Newtonian expansion of the metric tensor and subsequent integration of the light-ray propagation in the static gravitational field of the pulsar companion [55]. In principle, the additional time delay caused by the spin of the companion might be used for testing whether the companion is a black hole or not [56]. However, as shown in [55], the time delay due to the spin is not separable from the delay caused by the bending of light rays in the gravitational field of the companion [57]. For this reason, the delay caused by the spin is not a directly measurable quantity and cannot be effectively used for testing the presence of the black hole companion of the pulsar [55].

B. Deflection of light in gravitational lenses and by the solar system

Let us assume that the observer is at rest at an event with space-time coordinates $(t,\mathbf{x})$. The observed direction $\mathbf{s}$ to the source of light has been derived in [7] and is given by

$$\mathbf{s}=\mathbf{K}+\boldsymbol{\alpha}+\boldsymbol{\beta}-\boldsymbol{\beta}_0+\boldsymbol{\gamma}, \tag{60}$$

where the unit vector $\mathbf{K}$ is given in Eq. (44), the relativistic corrections $\boldsymbol{\beta}$ and $\boldsymbol{\beta}_0$ are defined in Eq. (46), $\boldsymbol{\alpha}$ describes the overall effect of deflection of the light-ray trajectory in the plane of the sky, and $\boldsymbol{\gamma}$ is related to the distortion of the local coordinate system of the observer with respect to the underlying global coordinate system used for the calculation of the propagation of light rays.

More precisely, the quantity $\boldsymbol{\alpha}$ can be expressed as

$$\alpha^i=\alpha^i_M+\alpha^i_S, \quad \alpha^i_M=-\hat{\partial}_i B_M(\tau)+k_\alpha P^i_j h^{\alpha j}_M(\tau),$$
$$\alpha^i_S=-\hat{\partial}_i B_S(\tau)+k_\alpha P^i_j h^{\alpha j}_S(\tau), \tag{61}$$

the quantities $\boldsymbol{\beta}=\boldsymbol{\beta}_M+\boldsymbol{\beta}_S$ and $\boldsymbol{\beta}_0=\boldsymbol{\beta}_{0M}+\boldsymbol{\beta}_{0S}$ are defined by Eq. (46), and

$$\gamma^i=\gamma^i_M+\gamma^i_S, \quad \gamma^i_M=-\frac{1}{2}k_n P^i_j h^{nj}_M(\tau),$$
$$\gamma^i_S=-\frac{1}{2}k_n P^i_j h^{nj}_S(\tau). \tag{62}$$

In what follows, we neglect all terms depending on the acceleration of the light-ray-deflecting body and the time derivative of its spin. Light-ray deflections represented by α^i_M, β^i_M, and γ^i_M caused by the mass-monopole part of the stress-energy tensor (1) have been calculated in [15] and will not be given here, as our primary interest in the present paper is the description of the spin-dependent gravitomagnetic effects. Using equation (49) for the function $B_S(\tau)$ and formula (D6), we obtain

$$\hat{\partial}_i B_S(\tau)=-2\frac{1-\mathbf{k}\cdot\mathbf{v}}{\sqrt{1-v^2}}\left[\frac{(1-v^2)k_\alpha r_\beta S^{\alpha\beta}P_{ij}r^j}{(r-\mathbf{v}\cdot\mathbf{r})^3(r-\mathbf{k}\cdot\mathbf{r})}+\frac{(1-\mathbf{k}\cdot\mathbf{v})k_\alpha r_\beta S^{\alpha\beta}P_{ij}r^j}{(r-\mathbf{v}\cdot\mathbf{r})^2(r-\mathbf{k}\cdot\mathbf{r})^2}-\frac{k_\alpha r_\beta S^{\alpha\beta}P_{ij}v^j}{(r-\mathbf{v}\cdot\mathbf{r})^2(r-\mathbf{k}\cdot\mathbf{r})}-\frac{P_{ij}k_\alpha S^{\alpha j}}{(r-\mathbf{v}\cdot\mathbf{r})(r-\mathbf{k}\cdot\mathbf{r})}\right]. \tag{63}$$

In addition, making use of formula (C11) yields

$$\frac{1}{2}k_\alpha k_\beta \hat{\partial}_i D^{\alpha\beta}_S(\tau)=-2\frac{1-\mathbf{k}\cdot\mathbf{v}}{\sqrt{1-v^2}}\frac{P_{ij}r^j}{r-\mathbf{v}\cdot\mathbf{r}}\frac{k_\alpha r_\beta S^{\alpha\beta}}{(r-\mathbf{k}\cdot\mathbf{r})^2}+\frac{2}{\sqrt{1-v^2}}\frac{k_\alpha P_{ij}S^{\alpha j}}{r-\mathbf{k}\cdot\mathbf{r}}, \tag{64}$$

$$\frac{1}{2}k_\alpha k_\beta \hat{\partial}_i D^{\alpha\beta}_S(\tau_0)=-2\frac{1-\mathbf{k}\cdot\mathbf{v}_0}{\sqrt{1-v_0^2}}\frac{P_{ij}r_0^j}{r_0-\mathbf{v}_0\cdot\mathbf{r}_0}\frac{k_\alpha r_{0\beta}S_0^{\alpha\beta}}{(r_0-\mathbf{k}\cdot\mathbf{r}_0)^2}+\frac{2}{\sqrt{1-v_0^2}}\frac{k_\alpha P_{ij}S_0^{\alpha j}}{r_0-\mathbf{k}\cdot\mathbf{r}_0}. \tag{65}$$

The light deflection vector, i.e., the vector connecting the undeflected image to the deflected image of the source of light, in the plane of the sky is defined in [15] and for the purely spin-induced part can be calculated from the third relation in Eq. (61). In that relation the first term is given in Eq. (63) and the second one can be calculated from Eq. (15), which results in

$$P_{ij}k_\alpha h^{\alpha j}_S=-2\sqrt{1-v^2}\left[(1-\mathbf{k}\cdot\mathbf{v})\frac{P_{ij}r_\alpha S^{\alpha j}}{(r-\mathbf{v}\cdot\mathbf{r})^3}-P_{ij}v^j\frac{k_\alpha r_\beta S^{\alpha\beta}}{(r-\mathbf{v}\cdot\mathbf{r})^3}\right]. \tag{66}$$

The most interesting physical application of the formalism given in the present section is the gravitomagnetic deflection of light in gravitational lenses and by bodies in the solar system. In both cases the impact parameter d of the light ray is considered to be extremely small in comparison with the distances from the body to the observer and the source of light. The gravitational lens approximation allows us to use the following relationships (for more details see Sec. VII B in [15]):

$$\mathbf{r}-\mathbf{k}r=\boldsymbol{\zeta}-\frac{d^2}{2r}\mathbf{k}, \quad \mathbf{r}_0+\mathbf{k}r_0=\boldsymbol{\zeta}_0, \tag{67}$$

where $\zeta^i=P^i_j(x^j-z^j(s))$ and $\zeta^i_0=P^i_j(x^j_0-z^j(s_0))$ are the impact parameters of the light ray with respect to the light-deflecting body evaluated at the moments of observation and emission of light. Assuming that $|\boldsymbol{\zeta}|\equiv d\ll\min[r,r_0]$ and $|\boldsymbol{\zeta}_0|\equiv d_0\ll\min[r,r_0]$, where $r=|\mathbf{x}-\mathbf{z}(s)|$ and $r_0=|\mathbf{x}_0-\mathbf{z}(s_0)|$ are the distances from the body to the observer and to the source of light, respectively, one can derive the following approximations:

$$r-\mathbf{k}\cdot\mathbf{r}=\frac{d^2}{2r}, \quad r_0-\mathbf{k}\cdot\mathbf{r}_0=2r_0, \tag{68}$$

along with

$$r-\mathbf{v}\cdot\mathbf{r}=r(1-\mathbf{k}\cdot\mathbf{v}), \quad r_0-\mathbf{v}_0\cdot\mathbf{r}_0=r_0(1+\mathbf{k}\cdot\mathbf{v}_0). \tag{69}$$

Hence, the relativistic spin-induced deflection of light α_S^i in Eq. (61) can be calculated from

$$\alpha_S^i = \frac{4}{\sqrt{1-v^2}}\left[\frac{2k_\alpha r_\beta S^{\alpha\beta}\zeta^i}{d^4} - \frac{k_\alpha P_{ij}S^{\alpha j}}{d^2}\right], \tag{70}$$

where we have neglected all residual terms of $O(d/r)$. After substituting the expressions given in Eqs. (D2) and (D3) in Eq. (70), one obtains the analytic representation of the relativistic deflection of light valid for the body having arbitrary high speed $\mathbf{v}$ and spin $\mathcal{J}$:

$$\alpha_S^i = \frac{8\gamma^2}{d^4}\left[\mathcal{J}\cdot(\mathbf{k}\times\boldsymbol{\zeta}) + \mathcal{J}\cdot(\boldsymbol{\zeta}\times\mathbf{v}) + \frac{1-\gamma}{\gamma v^2}(\mathbf{v}\cdot\mathcal{J})(\mathbf{k}\times\boldsymbol{\zeta})\cdot\mathbf{v}\right]\zeta^i - \frac{4\gamma^2}{d^2}\left[P_{ij}(\mathbf{v}\times\mathcal{J})^j - (\mathbf{k}\times\mathcal{J})^i - \frac{1-\gamma}{\gamma v^2}(\mathbf{v}\cdot\mathcal{J})(\mathbf{k}\times\mathbf{v})^i\right]. \tag{71}$$

In the case of slow motion, the Taylor expansion of Eq. (71) with respect to the parameter v/c yields

$$\alpha_S^i = \frac{8\mathcal{J}\cdot(\mathbf{k}\times\boldsymbol{\zeta})\zeta^i}{d^4} + \frac{4(\mathbf{k}\times\mathcal{J})^i}{d^2}, \tag{72}$$

which exactly coincides with the previously known formula for a stationary rotating gravitational lens [see, e.g., the leading terms of Eq. (6.28) in [58]] derived using a different mathematical technique. One can also recast Eq. (72) in a simpler gradient form:

$$\alpha_S^i = 4\frac{\partial\psi_S}{\partial\zeta^i}, \quad \psi_S = 4(\mathbf{k}\times\mathcal{J})^j\partial\ln d/\partial\zeta^j, \tag{73}$$

where ψ_S is a gravitomagnetic component of the gravitational lens potential [cf. the second term of Eq. (153) in [15]].

C. Gravitational shift of frequency and Doppler tracking

The special relativistic treatment of the Doppler frequency shift in an inertial system of Cartesian coordinates is well known. It is based on two facts: proper time runs differently for identical clocks moving with different speeds and electromagnetic waves propagate along straight lines in such an inertial system in flat space-time (see [59] and [60] for more details). The general relativistic formulation of the frequency shift in curved space-time is more involved.

Two definitions of the Doppler shift are used [61]—in terms of energy (A) and in terms of frequency (B):

$$(\mathrm{A})\ \ \frac{\nu}{\nu_0} = \frac{u^\alpha l_\alpha}{u_0^\alpha l_{0\alpha}}, \quad (\mathrm{B})\ \ \frac{\nu}{\nu_0} = \frac{dT_0}{dT}, \tag{74}$$

where ν_0 and ν are the emitted and observed electromagnetic frequencies of light; here $(T_0,\, u_0^\alpha)$ and $(T,\, u^\alpha)$ are, respectively, the proper time and four-velocity of the source of light and observer, and $l_{0\alpha}$ and l_α are null four-vectors of the light ray at the points of emission and observation, respectively. Despite the apparent difference in the two definitions, they are identical as equalities $u^\alpha = dx^\alpha/dT$ and $l_\alpha = \partial\varphi/\partial x^\alpha$ hold. In order to connect various physical quantities at the points of emission and observation of light one has to integrate the equations of light propagation.

The integration of null geodesic equation in the case of space-times possessing symmetries has been known for a long time and extensively used in astronomical practice (see, e.g., [62] and references therein). However, interesting astronomical phenomena in the propagation of light rays in curved space-time are also caused by small time-dependent perturbations of the background geometry. Usually, the first post-Newtonian approximation in the relativistic N-body problem with fixed or uniformly moving bodies has been applied in order to consider the effects of the N-body system on electromagnetic signals [58]. Unfortunately, this approximation works properly if, and only if, the time of propagation of light is much shorter than the characteristic Keplerian time of the N-body problem. An adequate treatment of the effects in the propagation of light must account for the retardation in the propagation of the gravitational field from the light-deflecting body to the point of interaction of the field with the electromagnetic signal.

A theory of light propagation and Doppler shift that takes account of such retardation effects has been constructed in the first post-Minkowskian approximation by Kopeikin and Schäfer [15] and Kopeikin [63]. For the sake of brevity, we do not reproduce the formalism here and restrict ourselves to consideration of the case of the gravitational lens only. Some details of this approximation have been given in the previous section. Making use of either of the definitions (74), we obtain for the gravitational shift of frequency

$$\left(\frac{\delta\nu}{\nu_0}\right)_{gr} = \left(-\mathbf{v} + \frac{r_0}{D}\boldsymbol{v} + \frac{r}{D}\boldsymbol{v}_0\right)\cdot(\boldsymbol{\alpha}_M + \boldsymbol{\alpha}_S), \tag{75}$$

where $D = |\mathbf{x}-\mathbf{x}_0|$ is the distance between the source of light and the observer, $r = |\mathbf{x}-\mathbf{z}(s)|$ is the distance between the lens and observer, $r_0 = |\mathbf{x}_0 - \mathbf{z}(s_0)|$ is the distance between the lens and the source of light, $\boldsymbol{v} = d\mathbf{x}(t)/dt$ is the velocity of the observer, $\boldsymbol{v}_0 = d\mathbf{x}_0(t_0)/dt_0$ is the velocity of the source of light, and $\boldsymbol{\alpha}_M$ and $\boldsymbol{\alpha}_S$ represent the vectors of the deflection of light by the gravitational lens given in Eq. (61). Formula (75) can be applied to the processing of Doppler tracking data from spacecrafts in deep space. In the case of superior conjunction of such a spacecraft with the Sun or a planet, the result shown in Eq. (75) should be doubled since the light passes the light-deflecting body twice—the first time on its way from the emitter to the spacecraft and the second time on its way back to the receiver. The Doppler tracking formula after subtracting the special relativistic corrections (for details see [15] and [63]) assumes the following form:

$$\left(\frac{\delta\nu}{\nu_0}\right)_{gr} = \left(\frac{8GM}{c^3d^2} + \frac{16G(\mathbf{k}\times\mathcal{J})\cdot\boldsymbol{\zeta}}{c^4d^4}\right)\left(\mathbf{v} - \frac{r_0}{D}\boldsymbol{v} - \frac{r}{D}\boldsymbol{v}_0\right)\cdot\boldsymbol{\zeta} - \frac{8G}{c^4d^2}\left(\mathbf{v} - \frac{r_0}{D}\boldsymbol{v} - \frac{r}{D}\boldsymbol{v}_0\right)\cdot(\mathbf{k}\times\mathcal{J}). \tag{76}$$

Only terms proportional to the mass M of the deflector were known previously (see, e.g., [15] and [64]). With the mathematical techniques of the present paper general relativistic corrections due to the intrinsic rotation of the gravitational lens can now be calculated. For the light ray grazing the limb of the light-ray-deflecting body, the gravitomagnetic Doppler shift due to the body's rotation is smaller than the effect produced by the body's mass by terms of order $\omega_{\rm rot}L/c$, where $\omega_{\rm rot}$ is the body's angular frequency and L is its characteristic radius. In the case of the Sun, the effect reaches a magnitude of about 0.8×10^{-14}, which is measurable in practice taking into account the current stability and accuracy of atomic time and frequency standards ($\sim 10^{-16}$, cf. [65] and [66]). For Jupiter, the corresponding estimate of the gravitomagnetic Doppler shift is 0.7×10^{-15}, which is also, in principle, measurable.

VII. SKROTSKII EFFECT FOR ARBITRARY-MOVING POLE-DIPOLE MASSIVE BODIES

A. Relativistic description of polarized radiation

The polarization properties of electromagnetic radiation are defined in terms of the electric field measured by an observer. Let us start with a general electromagnetic radiation field $F_{\alpha\beta}$ and define the complex field $\mathcal{F}_{\alpha\beta}$ such that $F_{\alpha\beta}=\mathrm{Re}(\mathcal{F}_{\alpha\beta})$ and $E_\alpha=\mathrm{Re}(\mathcal{E}_\alpha)$, where $\mathcal{E}_\alpha=\mathcal{F}_{\alpha\beta}u^\beta$ is the complex electric field. In the rest frame of an observer with four-velocity u^α, the intensity and polarization properties of the radiation are describable in terms of the tensor

$$J_{\alpha\beta}=\langle \mathcal{E}_\alpha \overline{\mathcal{E}_\beta}\rangle, \tag{77}$$

where the angular brackets represent an ensemble average and $J_{\alpha\beta}u^\beta=0$. The electromagnetic Stokes parameters are defined with respect to two of the four vectors of the tetrad $e^\alpha_{(\beta)}$, introduced in Eq. (22), as follows (cf. [67] and [68]):

$$S_0=J_{\alpha\beta}[e^\alpha_{(1)}e^\beta_{(1)}+e^\alpha_{(2)}e^\beta_{(2)}], \tag{78}$$

$$S_1=J_{\alpha\beta}[e^\alpha_{(1)}e^\beta_{(1)}-e^\alpha_{(2)}e^\beta_{(2)}], \tag{79}$$

$$S_2=J_{\alpha\beta}[e^\alpha_{(1)}e^\beta_{(2)}+e^\alpha_{(2)}e^\beta_{(1)}], \tag{80}$$

$$S_3=iJ_{\alpha\beta}[e^\alpha_{(1)}e^\beta_{(2)}-e^\alpha_{(2)}e^\beta_{(1)}]. \tag{81}$$

Using Eq. (77), the Stokes parameters can be expressed in the standard way [67] in a linear polarization basis as

$$S_0=\langle|\mathcal{E}_{(1)}|^2+|\mathcal{E}_{(2)}|^2\rangle, \tag{82}$$

$$S_1=\langle|\mathcal{E}_{(1)}|^2-|\mathcal{E}_{(2)}|^2\rangle, \tag{83}$$

$$S_2=\langle\mathcal{E}_{(1)}\overline{\mathcal{E}}_{(2)}+\overline{\mathcal{E}}_{(1)}\mathcal{E}_{(2)}\rangle, \tag{84}$$

$$S_3=i\langle\mathcal{E}_{(1)}\overline{\mathcal{E}}_{(2)}-\overline{\mathcal{E}}_{(1)}\mathcal{E}_{(2)}\rangle, \tag{85}$$

where $\mathcal{E}_{(n)}=\mathcal{E}_\alpha e^\alpha_{(n)}$ for $n=1,2$. Under the gauge subgroup of the little group of l^α, the Stokes parameters remain invariant. However, for a constant rotation of angle Θ in the $(e^\alpha_{(1)},e^\alpha_{(2)})$ plane, $S_0'=S_0$, $S_1'=S_1\cos 2\Theta+S_2\sin 2\Theta$, $S_2'=-S_1\sin 2\Theta+S_2\cos 2\Theta$, and $S_3'=S_3$. This is what would be expected for a spin-1 field. That is, under a duality rotation of $\Theta=\pi/2$, one linear polarization state turns into the other.

For the null field (21) under consideration in this paper,

$$\mathcal{E}_\alpha=\frac{1}{2}\omega(\Phi m_\alpha+\Psi\overline{m}_\alpha), \tag{86}$$

where $\omega=-l^\alpha u_\alpha$ is the *constant* frequency of the ray measured by the observer with four-velocity u^α. Changing from the circular polarization basis (86) with amplitudes $\omega\Phi/2$ and $\omega\Psi/2$ to the linear polarization basis $\mathcal{E}^\alpha=\mathcal{E}_{(1)}e^\alpha_{(1)}+\mathcal{E}_{(2)}e^\alpha_{(2)}$, we find that $\mathcal{E}_{(1)}=\omega(\Phi+\Psi)/\sqrt{8}$ and $\mathcal{E}_{(2)}=i\omega(\Phi-\Psi)/\sqrt{8}$. The variation of the Stokes parameters along the ray are essentially given by Eq. (23), since the frequency ω is simply a constant parameter along the ray given by $\omega=dt/d\lambda=d\tau/d\lambda$.

The polarization vector $\mathbf{P}$ and the degree of polarization $P=|\mathbf{P}|$ can be defined in terms of the Stokes parameters $(S_0,\mathbf{S})$ by $\mathbf{P}=\mathbf{S}/S_0$ and $P=|\mathbf{S}|/S_0$, respectively. Any partially polarized wave may be thought of as an incoherent superposition of a completely polarized wave with Stokes parameters $(PS_0,\mathbf{S})$ and a completely unpolarized wave with Stokes parameters $(S_0-PS_0,0)$, so that $(S_0,\mathbf{S})=(PS_0,\mathbf{S})+(S_0-PS_0,0)$. For completely polarized waves, $\mathbf{P}$ describes the surface of the unit sphere introduced by Poincaré. The center of the Poincaré sphere corresponds to unpolarized radiation and the interior to partially polarized radiation. Orthogonally polarized waves represent any two conjugate points on the Poincaré sphere; in particular, $P_1=\pm 1$ and $P_3=\pm 1$ represent orthogonally polarized waves corresponding to the linear and circular polarization bases, respectively.

Any stationary or time-dependent axisymmetric gravitational field in general causes a relativistic effect of the rotation of the polarization plane of electromagnetic waves. This effect was first discussed by Skrotskii [25] and later by many other researchers (see, e.g., [69] and [26] and references therein). We generalize the results of previous authors to the case of spinning bodies that can move arbitrarily fast.

B. Reference tetrad field

The rotation of the polarization plane is not conceivable without an unambiguous definition of a local reference frame (tetrad) constructed along the null geodesic. The null tetrad frame based on four null vectors $(l^\alpha,n^\alpha,m^\alpha,\overline{m}^\alpha)$ introduced in Sec. IV is a particular choice. As discussed in Sec. IV, this null tetrad is intimately connected with the local frame $(e^\alpha_{(0)},e^\alpha_{(1)},e^\alpha_{(2)},e^\alpha_{(3)})$, where the vectors $e^\alpha_{(0)}$ and $e^\alpha_{(3)}$ are defined in Eq. (22) and the spacelike vectors $e^\alpha_{(1)}$ and $e^\alpha_{(2)}$ are directly related to the polarization of the electromagnetic wave. Each vector of the tetrad $(e^\alpha_{(0)},e^\alpha_{(1)},e^\alpha_{(2)},e^\alpha_{(3)})$ depends upon time and is parallel transported along the null geodesic defined by its tangent vector l^α. To characterize the variation of $e^\alpha_{(\beta)}$ along the ray, it is necessary to have access to a fiducial field of tetrad frames for reference purposes. To this end, let us choose the reference frame based on the set of static observers in the background space-time. Then, at the past null infinity, where according to our assumption the space-time is asympotically flat, one has

$$e^{\alpha}_{(0)}(-\infty)=(1,0,0,0), \quad e^{\alpha}_{(1)}(-\infty)=(0,a^1,a^2,a^3),$$
$$e^{\alpha}_{(2)}(-\infty)=(0,b^1,b^2,b^3), \quad e^{\alpha}_{(3)}(-\infty)=(0,k^1,k^2,k^3). \tag{87}$$

Here the spatial vectors $\mathbf{a}=(a^1,a^2,a^3)$, $\mathbf{b}=(b^1,b^2,b^3)$, and $\mathbf{k}=(k^1,k^2,k^3)$ are orthonormal in the Euclidean sense and the vector $\mathbf{k}$ defines the spatial direction of propagation of the light ray [see Eq. (32)]. At the same time, the four vectors (87) also form a basis $\boldsymbol{\epsilon}_{(\beta)}=\epsilon^{\alpha}_{(\beta)}\partial/\partial x^{\alpha}$ of the global harmonic coordinate system at each point of space-time, where $\epsilon^{\alpha}_{(\beta)}=e^{\alpha}_{(\beta)}(-\infty)$. However, it is worth noting that this coordinate basis is not orthonormal at an arbitrary point in space-time. Nevertheless, it is possible to construct an orthonormal basis $\omega^{\alpha}_{(\beta)}$ at each point by making use of a linear transformation $\omega^{\alpha}_{(\beta)}=\Lambda^{\alpha}_{\ \gamma}\epsilon^{\gamma}_{(\beta)}$ such that the transformation matrix is given in the linear approximation by

$$\Lambda^0_0=1+\frac{1}{2}h_{00}, \quad \Lambda^0_i=h_{0i},$$
$$\Lambda^i_0=0, \quad \Lambda^i_j=\delta^i_j-\frac{1}{2}h_{ij}. \tag{88}$$

A local orthonormal basis $\omega^{\alpha}_{(\beta)}$ is then defined by

$$\omega^{\alpha}_{(0)}=\left(1+\frac{1}{2}h_{00},0,0,0\right), \quad \omega^{\alpha}_{(1)}=\left(h_{0j}a^j,a^i-\frac{1}{2}h_{ij}a^j\right),$$
$$\omega^{\alpha}_{(2)}=\left(h_{0j}b^j,b^i-\frac{1}{2}h_{ij}b^j\right), \quad \omega^{\alpha}_{(3)}=\left(h_{0j}k^j,k^i-\frac{1}{2}h_{ij}k^j\right). \tag{89}$$

By definition, the tetrad frame $e^{\alpha}_{(\beta)}$ is parallel transported along the ray. The propagation equations for these vectors are thus obtained by applying the operator D_λ of the parallel transport [see Eq. (20)]. Hence,

$$\frac{de^{\alpha}_{(\mu)}}{d\lambda}+\Gamma^{\alpha}_{\beta\gamma}l^{\beta}e^{\gamma}_{(\mu)}=0, \tag{90}$$

where λ is an affine parameter along the light ray. Using the definition of the Christoffel symbols (B1) and changing over to the variable τ with $dx^{\alpha}/d\tau=k^{\alpha}+O(h)$, one can recast Eq. (90) in the form

$$\frac{d}{d\tau}\left(e^{\alpha}_{(\mu)}+\frac{1}{2}h^{\alpha}_{\beta}e^{\beta}_{(\mu)}\right)=\frac{1}{2}\eta^{\alpha\nu}(\partial_{\nu}h_{\gamma\beta}-\partial_{\gamma}h_{\nu\beta})k^{\beta}e^{\gamma}_{(\mu)}. \tag{91}$$

Equation (91) is the main equation for the discussion of the Skrotskii effect.

C. Skrotskii effect

We have chosen the null tetrad frame along the ray such that at $t=-\infty$ the tetrad has the property that $e^0_{(i)}(-\infty)=0$. It follows that in general $e^0_{(i)}=O(h)$. The propagation equation for the spatial components $e^i_{(\mu)}$ is therefore given by

$$\frac{d}{d\tau}\left(e^i_{(\mu)}+\frac{1}{2}h_{ij}e^j_{(\mu)}\right)=\frac{1}{2}(\partial_i h_{j\beta}-\partial_j h_{i\beta})k^{\beta}e^j_{(\mu)}. \tag{92}$$

Furthermore, we are interested only in solving Eqs. (91) for the vectors $e^{\alpha}_{(1)}$ and $e^{\alpha}_{(2)}$ that are used in the description of the polarization of light; hence

$$\frac{d}{d\tau}\left(e^i_{(n)}+\frac{1}{2}h_{ij}e^j_{(n)}\right)+\varepsilon_{ijl}e^j_{(n)}\Omega^l=0 \qquad (n=1,2), \tag{93}$$

where we have defined the quantity Ω^i as

$$\Omega^i=-\varepsilon_{ijl}\partial_j\left(\frac{1}{2}h_{l\alpha}k^{\alpha}\right). \tag{94}$$

Therefore, $e^i_{(n)}$ can be obtained from the integration of Eq. (93). Moreover, $l_{\alpha}e^{\alpha}_{(n)}=0$ implies that

$$e^0_{(n)}=k_ie^i_{(n)}+h_{0i}e^i_{(n)}+h_{ij}k^ie^j_{(n)}+\delta_{ij}\dot{\Xi}^ie^j_{(n)}. \tag{95}$$

It is worth noting that, as a consequence of definition $l^{\alpha}=\omega(1,\dot{x}^i)$ and Eq. (38), one has $\mathbf{l}=\omega(\mathbf{k}+\dot{\boldsymbol{\Xi}})$. Moreover, from the condition $l_{\alpha}l^{\alpha}=0$, it follows that $\mathbf{k}\cdot\dot{\boldsymbol{\Xi}}=-(1/2)h_{\alpha\beta}k^{\alpha}k^{\beta}$.

Let us decompose Ω^i into components that are parallel and perpendicular to the unit vector k^i, i.e.,

$$\Omega^i=(\mathbf{k}\cdot\boldsymbol{\Omega})k^i+P^i_j\Omega^j. \tag{96}$$

Then, Eq. (93) can be expressed as

$$\frac{d}{d\tau}\left(e^i_{(n)}+\frac{1}{2}h_{ij}e^j_{(n)}\right)+(\mathbf{k}\cdot\boldsymbol{\Omega})\varepsilon_{ijl}e^j_{(n)}k^l + \varepsilon_{ijl}e^j_{(n)}P^l_q\Omega^q=0 \quad (n=1,2). \tag{97}$$

Integrating this equation from $-\infty$ to τ taking into account the initial conditions (87) and equalities $\varepsilon_{ijl}a^jk^l=-b^i$ and $\varepsilon_{ijl}b^jk^l=a^i$, we obtain

$$e^i_{(1)}=a^i-\frac{1}{2}h_{ij}a^j+\left(\int_{-\infty}^{\tau}\mathbf{k}\cdot\boldsymbol{\Omega}d\sigma\right)b^i-\varepsilon_{ijl}a^jP^l_q\int_{-\infty}^{\tau}\Omega^q d\sigma, \tag{98}$$

$$e^i_{(2)}=b^i-\frac{1}{2}h_{ij}b^j-\left(\int_{-\infty}^{\tau}\mathbf{k}\cdot\boldsymbol{\Omega}d\sigma\right)a^i-\varepsilon_{ijl}b^jP^l_q\int_{-\infty}^{\tau}\Omega^q d\sigma. \tag{99}$$

To interpret these results properly, let us note that a rotation in the $(\omega^i_{(1)},\omega^i_{(2)})$ plane by an angle ϕ at time τ leads to

$$\omega^i_{(1)}=\left(a^i-\frac{1}{2}h_{ij}a^j\right)\cos\phi+\left(b^i-\frac{1}{2}h_{ij}b^j\right)\sin\phi, \tag{100}$$

$$\omega^i_{(2)}=-\left(a^i-\frac{1}{2}h_{ij}a^j\right)\sin\phi+\left(b^i-\frac{1}{2}h_{ij}b^j\right)\cos\phi, \tag{101}$$

so that if the angle $\phi=O(h)$ then we have

$$\omega^i_{(1)}=a^i-\frac{1}{2}h_{ij}a^j+\phi b^i,\quad \omega^i_{(2)}=b^i-\frac{1}{2}h_{ij}b^j-\phi a^i. \tag{102}$$

Comparing these vectors with Eqs. (98) and (99), we recognize that as vectors $\boldsymbol{e}_{(1)}$ and $\boldsymbol{e}_{(2)}$ propagate along the light ray they are rotating with an angle

$$\phi(\tau)=\int_{-\infty}^{\tau}\mathbf{k}\cdot\boldsymbol{\Omega}d\sigma \tag{103}$$

about $\mathbf{k}$ in the local $(\boldsymbol{\omega}_{(1)},\boldsymbol{\omega}_{(2)})$ plane. Moreover, $\boldsymbol{e}_{(1)}$ and $\boldsymbol{e}_{(2)}$ rotate to $O(h)$ toward the direction of light propagation $\mathbf{k}$ [see the very last terms on the right-hand sides of Eqs. (98) and (99)].

We are mostly interested in finding the rotational angle ϕ in the plane perpendicular to $\mathbf{k}$. It is worth noting that the Euclidean dot product $\mathbf{k}\cdot\boldsymbol{\Omega}$ can be expressed in terms of partial differentiation with respect to the impact parameter ξ^i only. This can be done by making use of Eq. (C4) and noting that $\varepsilon_{ijp}k^jk^p\equiv 0$, so that

$$\mathbf{k}\cdot\boldsymbol{\Omega}=\frac{1}{2}k^{\alpha}k^{i}\varepsilon_{i\hat{p}\hat{q}}\hat{\partial}_q h_{\alpha\hat{p}}, \tag{104}$$

where the caret over spatial indices denotes the projection onto the plane orthogonal to the propagation of the light ray, for instance, $A^{\hat{i}}\equiv P^i_jA^j$. Hence, the transport equation for the angle ϕ assumes the form

$$\frac{d\phi}{d\tau}=\mathbf{k}\cdot\boldsymbol{\Omega}\frac{1}{2}k^{\alpha}k^{i}\varepsilon_{i\hat{p}\hat{q}}\hat{\partial}_q h_{\alpha\hat{p}}, \tag{105}$$

which is useful for integration. Formula (105) constitutes a significant generalization of a result that was first discussed by Skrotskii ([25] and [70]) and bears his name.

For a stationary "pole-dipole" source of the gravitational field at rest at the origin of the coordinates, Eqs. (10), (11), and (16) for the metric perturbations imply that

$$h_{00}=\frac{2m}{r},\quad h_{0i}=-\frac{2(\boldsymbol{\mathcal{J}}\times\mathbf{r})^i}{r^3},\quad h_{ij}=\frac{2m}{r}\delta_{ij}. \tag{106}$$

It follows from Eq. (94) that in this case

$$\boldsymbol{\Omega}=\mathbf{B}_g-\frac{m(\mathbf{k}\times\mathbf{r})}{r^3}, \tag{107}$$

where $\mathbf{B}_g$ is the dipolar gravitomagnetic field associated with the source

$$\mathbf{B}_g=\boldsymbol{\nabla}\times\left(\frac{\boldsymbol{\mathcal{J}}\times\mathbf{r}}{r^3}\right)=\frac{\mathcal{J}}{r^3}[3(\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\mathcal{J}}})\hat{\mathbf{r}}-\hat{\boldsymbol{\mathcal{J}}}], \tag{108}$$

where $\hat{\mathbf{r}}=\mathbf{r}/r$ and $\hat{\boldsymbol{\mathcal{J}}}=\boldsymbol{\mathcal{J}}/\mathcal{J}$ are unit vectors, and $\mathcal{J}$ is the magnitude of the angular momentum of the source. Thus, $\mathbf{k}\cdot\boldsymbol{\Omega}=\mathbf{k}\cdot\mathbf{B}_g$, and, in this way, we recover Skrotskii's original result $d\phi/d\tau=\mathbf{k}\cdot\mathbf{B}_g$, which is the natural gravitational analogue of the Faraday effect in electrodynamics. The Skrotskii effect has a simple physical interpretation in terms of the gravitational Larmor theorem [71]. We note that in the particular case under consideration here, the gravitomagnetic field can be expressed as $\mathbf{B}_g=-\boldsymbol{\nabla}(\boldsymbol{\mathcal{J}}\cdot\mathbf{r}/r^3)$ for $\mathbf{r}\neq 0$. Thus for radiation propagating in the space-time exterior to the source, $\phi=-\boldsymbol{\mathcal{J}}\cdot\mathbf{r}/r^3+\text{const}$. In particular, it follows from this result that the net angle of rotation of the plane of polarization from $-\infty$ to $+\infty$ is zero. This conclusion is confirmed later in Eq. (114) as well.

In the general case, where the bodies generating the gravitational field are both moving and rotating, the integration of Eq. (105) is also straightforward and is accomplished with the help of a mathematical technique described in Appendix B. The result is

$$\phi(\tau)=\phi_0+\delta\phi(\tau)-\delta\phi(\tau_0),$$

$$\delta\phi(\tau)=\frac{1}{2}k^{\alpha}k^{i}\varepsilon_{i\hat{p}\hat{q}}\hat{\partial}_q B_{\alpha\hat{p}}(\tau,\boldsymbol{\xi}), \tag{109}$$

where ϕ_0 is a constant angle defining the orientation of the polarization plane of the electromagnetic wave under discussion at τ_0, $\delta\phi(\tau)$ and $\delta\phi(\tau_0)$ are relativistic rotations of the polarization plane with respect to its orientation at infinity at the moments of observation, τ, and emission of light, τ_0, respectively, and the function $B_{\alpha\beta}$ is defined in Eq. (42).

We note that according to the definition (42) the tensor function $B_{\alpha\beta}$ consists of two parts, the first of which, $B_M^{\alpha\beta}$, relates to the action of the mass-monopole field of the particles and the second one, $B_S^{\alpha\beta}$, describes the action of spin dipole fields on the rotation of the polarization plane. Therefore, the angle of rotation can be represented as an algebraic sum of two components

$$\delta\phi=\delta\phi_M+\delta\phi_S, \tag{110}$$

where the monopole part can be calculated using Eq. (C10),

$$\delta\phi_M=\frac{1}{2}k_{\alpha}k_i\varepsilon^{i\hat{p}\hat{q}}\hat{\partial}_q B_M^{\alpha\hat{p}}=2m\frac{1-\mathbf{k}\cdot\mathbf{v}}{\sqrt{1-v^2}}\frac{\mathbf{k}\cdot(\mathbf{v}\times\boldsymbol{\xi})}{(r-\mathbf{k}\cdot\mathbf{r})(r-\mathbf{v}\cdot\mathbf{r})}. \tag{111}$$

The spin part is given by

$$\delta\phi_S=\frac{1}{2}k_{\alpha}k_i\varepsilon^{i\hat{p}\hat{q}}\hat{\partial}_q B_S^{\alpha\hat{p}}, \tag{112}$$

and has a rather complicated form if written explicitly by making use of the partial derivative of the function $B_S^{\alpha\beta}$ given in Eq. (C9). In the gravitational lens approximation when light goes from $-\infty$ to $+\infty$, Eqs. (111) and (112) simplify and assume the form

$$\delta\phi_M(+\infty) = -\frac{4Gm}{c^3 d^2}\mathbf{k}\cdot(\boldsymbol{\xi}\times\mathbf{v}), \tag{113}$$

$$\delta\phi_S(+\infty) = \frac{4G}{c^3 d^2}\left[(\mathbf{k}\cdot\boldsymbol{\mathcal{J}}) + \frac{(\mathbf{k}\times\boldsymbol{\xi})\cdot(\boldsymbol{\xi}\times\boldsymbol{\mathcal{J}})}{d^2}\right] \equiv 0. \tag{114}$$

Equations (113) and (114) make it evident that in the case of gravitational lensing the integrated effect discussed by Skrotskii [25] is due only to the translational motion of the lens. There is no contribution to the effect caused by spin of the body and proportional to $1/d^2$, where d is the impact parameter of the light ray with respect to the body. This rather remarkable fact was first noted by Kobzarev and Selivanov [72] who criticized the final conclusions of the papers by Skrotskii [25] and others (see [73] and [74]). We would like to stress, however, that all the results of the paper by Skrotskii [25] are correct, since the final formulas of his paper relate to the situation where light is emitted from or near the surface of the rotating body. Such a case obviously is not reduced to that of a gravitational lens. Thus, the angle of rotation of the polarization plane does not equal zero as correctly shown by Skrotskii.

The absence of the Skrotskii effect for electromagnetic radiation propagating from $-\infty$ to $+\infty$ in the case of a stationary rotating body reveals that the coupling of the polarization vector of the photon with the spin of the rotating body cannot be amplified by the presence of a gravitational lens.

VIII. SKROTSKII EFFECT BY QUADRUPOLAR GRAVITATIONAL WAVES FROM LOCALIZED SOURCES

A. Quadrupolar gravitational wave formalism

We shall consider in this section the Skrotskii effect associated with the emission of gravitational waves from a localized astronomical system like a binary star, supernova explosion, etc. For simplicity we shall restrict our consideration to the quadrupole approximation only. The direct way to tackle the problem would be to use the Taylor expansion of the formula for the Skrotskii effect given in the previous section. However, it is instructive to make use of an approach outlined in [7] that is based on the multipole expansion of the radiative gravitational field of a localized astronomical source [75].

To this end, let us consider the propagation of a light ray taking place always outside a source (with its center of mass at rest) that emits gravitational waves. The metric perturbation $h_{\mu\nu}$ can be split into a canonical part $h_{\mu\nu}^{can.}$, which contains symmetric trace-free (STF) tensors only (for more details on STF tensors see [76]), and a gauge part, i.e., $h_{\mu\nu} = h_{\mu\nu}^{can.} + \partial_\mu w_\nu + \partial_\nu w_\mu$. In the case of mass-monopole, spin-dipole, and mass-quadrupole source moments and in the harmonic gauge the canonical part of the metric perturbation is given by

$$h_{00}^{can.} = \frac{2\mathcal{M}}{r} + \partial_{pq}\left[\frac{\mathcal{I}_{pq}(t-r)}{r}\right], \tag{115}$$

$$h_{0i}^{can.} = -\frac{2\varepsilon_{ipq}\mathcal{J}_p x_q}{r^3} + 2\partial_j\left[\frac{\dot{\mathcal{I}}_{ij}(t-r)}{r}\right], \tag{116}$$

$$h_{ij}^{can.} = \delta_{ij}h_{00}^{can.} + \frac{2}{r}\ddot{\mathcal{I}}_{ij}(t-r). \tag{117}$$

Here the mass $\mathcal{M}$ and spin $\mathcal{J}^i$ of the source are constants, whereas its quadrupole moment $\mathcal{I}_{ij}$ is a function of the retarded time $t-r$. Dependence of the mass and spin on time would be caused by the process of emission of energy and angular momentum which are then carried away from the source by gravitational waves. If necessary, the time dependence of mass and spin can be treated in the framework of the same calculational scheme as applied in the present paper. Moreover, it is assumed in Eqs. (115)–(117) that the center of mass of the source of gravitational waves is at the origin of the coordinate system and does not move. Hence, the radial coordinate r is the distance from the center of mass to the field point in space.

The explicit expressions for the gauge functions w^μ relating $h_{\mu\nu}^{can.}$ to $h_{\mu\nu}$ are chosen in the form [7]

$$w^0 = \frac{1}{2}\nabla_k\nabla_l\left[\frac{{}^{(-1)}\mathcal{I}_{kl}(t-r)}{r}\right], \tag{118}$$

$$w^i = \frac{1}{2}\nabla_i\nabla_k\nabla_l\left[\frac{{}^{(-2)}\mathcal{I}_{kl}(t-r)}{r}\right] - 2\nabla_k\left[\frac{\mathcal{I}_{ki}(t-r)}{r}\right], \tag{119}$$

where we have introduced the definitions of time integrals of the quadrupole moment

$${}^{(-1)}\mathcal{I}_{ij}(t) \equiv \int_{-\infty}^{t} dv\,\mathcal{I}_{ij}(v),$$

$${}^{(-2)}\mathcal{I}_{ij}(t) \equiv \int_{-\infty}^{t} dv\,{}^{(-1)}\mathcal{I}_{ij}(v). \tag{120}$$

The gauge functions (118) and (119) make space-time coordinates satisfy both harmonic and Arnowitt-Deser-Misner (ADM) gauge conditions simultaneously [7]. The harmonic-ADM coordinates are especially useful for integrating equations of light propagation and for description of the motions of a free falling source of light and observer. It turns out that the source of light and the observer do not "experience" the influence of gravitational waves in ADM coordinates and move only under the action of the stationary part of the gravitational field created by the mass and spin of the source of the gravitational waves (for more details concerning the construction of the harmonic ADM coordinates see [7]).

The rotation of the polarization plane is described by Eq. (105), where the rotation frequency $\mathbf{\Omega}$ is defined in Eq. (94). Using expressions (115)–(117) together with the gauge freedom, the angular velocity is given by

$$\mathbf{k}\cdot\mathbf{\Omega}=\hat{\partial}_\tau\left(-\frac{\mathcal{J}_i x^i}{r^3}\right)+k^i\varepsilon_{i\hat{p}\hat{q}}\hat{\partial}_{qj}\left[\frac{\dot{\mathcal{I}}_{jp}(t-r)}{r}\right]+k^i\varepsilon_{i\hat{p}\hat{q}}\hat{\partial}_{q\tau}\left[\frac{k^j\dot{\mathcal{I}}_{jp}(t-r)}{r}\right]-\hat{\partial}_\tau\left[\frac{1}{2}\mathbf{k}\cdot(\nabla\times\mathbf{w})\right], \tag{121}$$

where $\mathbf{w}\equiv(w^i)$ is given by Eq. (119), $r=\sqrt{\tau^2+d^2}$, and $d=|\boldsymbol{\xi}|$ is the impact parameter of the light ray.

Integration of Eq. (105) with respect to time results in

$$\delta\phi(\tau)=-\frac{\mathcal{J}_i x^i}{r^3}-\frac{1}{2}\mathbf{k}\cdot(\nabla\times\mathbf{B}), \tag{122}$$

where $\mathbf{B}\equiv(B^i)$ with

$$B^i=\frac{2\xi^j\dot{\mathcal{I}}_{ij}(t-r)}{ry}+\frac{2k^j\dot{\mathcal{I}}_{ij}(t-r)}{r}-2\nabla_k\left[\frac{\mathcal{I}_{ki}(t-r)}{r}\right] \tag{123}$$

and $y=\tau-r$. Making use of formula (119) for w^i and taking partial derivatives brings Eq. (122) to the following more explicit form:

$$\delta\phi(\tau)=-\frac{\mathcal{J}_i x^i}{r^3}+\frac{(\mathbf{k}\times\boldsymbol{\xi})^i\xi^j}{ry}\left[-\frac{\dot{\mathcal{I}}_{ij}}{ry}+\frac{\dot{\mathcal{I}}_{ij}}{r^2}+\frac{\ddot{\mathcal{I}}_{ij}}{r}\right]+\frac{(\mathbf{k}\times\boldsymbol{\xi})^i x^j}{r^3}\left[\ddot{\mathcal{I}}_{ij}+\frac{3\dot{\mathcal{I}}_{ij}}{r}+\frac{3\mathcal{I}_{ij}}{r^2}\right]+\frac{(\mathbf{k}\times\boldsymbol{\xi})^i k^j}{r^2}\left[\ddot{\mathcal{I}}_{ij}+\frac{\dot{\mathcal{I}}_{ij}}{r}\right], \tag{124}$$

where the quadrupole moments $\mathcal{I}_{ij}$ are all evaluated at the retarded time $s=t-r$. It is straightforward to obtain the two limiting instances of this formula related to the cases of a gravitational lens and a plane gravitational wave.

B. Gravitational lens approximation

In the event of gravitational lensing one has [7]

$$y\equiv\tau-r=\sqrt{r^2-d^2}-r=-\frac{d^2}{2r}+\cdots, \tag{125}$$

so that

$$ry=-\frac{d^2}{2},\qquad t-r=t^*-\frac{d^2}{2r}+\cdots, \tag{126}$$

where t^* is the instant of closest approach of the light ray to the center of mass of the gravitational lens. After making use of the approximations shown above, Eq. (124) simplifies and assumes the form

$$\delta\phi(t^*)=-\frac{4\dot{\mathcal{I}}_{ij}(t^*)(\mathbf{k}\times\boldsymbol{\xi})^i\xi^j}{d^4}. \tag{127}$$

One can see that the first nonvanishing contribution to the Skrotskii effect comes from the time derivative of a mass-quadrupole moment and does not depend on the spin of the source of gravitational waves. It agrees with the conclusions of Sec. VII. Formula (127) also shows that if the source of gravitational waves is periodic, such as a binary star system, the polarization plane of the electromagnetic wave will experience periodic changes of its orientation with a characteristic frequency that is twice that of the source [72].

Equation (127) can be derived from Eq. (113) as well. Indeed, let us assume that the gravitational lens is comprised of N point particles forming a self-gravitating body. For N point particles, Eq. (113) implies

$$\delta\phi_M(+\infty)=-4\sum_{a=1}^{N}\frac{Gm_a}{c^3|\boldsymbol{\xi}-\boldsymbol{\xi}_a|^2}\mathbf{k}\cdot[(\boldsymbol{\xi}-\boldsymbol{\xi}_a)\times\mathbf{v}_a], \tag{128}$$

where the point particles are enumerated by the index a running from 1 to N, m_a is the mass of the ath particle, $\mathbf{x}_a$ and $\mathbf{v}_a$ are the coordinates and velocity of the ath particle, $\xi^i_a=P^i_j x^j_a$, and ξ^i is the impact parameter of the light ray with respect to the origin of the coordinate system, which we assume coincides with the center of mass of the lens, $\mathcal{I}^i$, defined by the equation

$$\mathcal{I}^i(t^*)=\sum_{a=1}^{N}m_a x^i_a(t^*)=0. \tag{129}$$

As a consequence of Eq. (129), we conclude that all the time derivatives of $\mathcal{I}^i$ vanish identically. We also define the spin of the lens and its tensor of inertia as follows

$$\mathcal{J}^i=\sum_{a=1}^{N}m_a(\mathbf{x}_a\times\mathbf{v}_a)^i,\qquad \mathcal{I}^{ij}(t^*)=\sum_{a=1}^{N}m_a x^i_a(t^*)x^j_a(t^*). \tag{130}$$

The spin of the lens is conserved and does not depend on time, while the tensor of inertia is, in general, a function of time.

Let us expand the right-hand side of Eq. (128) in a power series with respect to the parameter $|\boldsymbol{\xi}_a|/|\boldsymbol{\xi}|$ which is presumed to be small. We find that

$$\delta\phi_M(+\infty) = -\frac{4G}{c^3 d^2}\left[\mathbf{k}\cdot\left(\boldsymbol{\xi}\times\sum_{a=1}^{N} m_a \mathbf{v}_a\right) - \mathbf{k}\cdot\sum_{a=1}^{N} m_a(\boldsymbol{\xi}_a\times\mathbf{v}_a)\right] - \frac{8(\mathbf{k}\times\boldsymbol{\xi})^i}{d^4} \times\sum_{a=1}^{N} m_a v_a^i(\boldsymbol{\xi}_a\cdot\boldsymbol{\xi}) + O\left(v_a\frac{\xi_a}{d}\right)^2, \quad (131)$$

where $d=|\boldsymbol{\xi}|$. Taking into account definitions of the center of mass, spin, tensor of inertia of the lens, as well as the identity

$$\sum_{a=1}^{N} 2m_a[(\mathbf{k}\times\boldsymbol{\xi})\cdot\mathbf{v}_a](\boldsymbol{\xi}_a\cdot\boldsymbol{\xi}) = (\mathbf{k}\times\boldsymbol{\xi})^i[\dot{\mathcal{I}}_{ij}\xi^j - (\boldsymbol{\xi}\times\boldsymbol{\mathcal{J}})^i] \quad (132)$$

together with formula (113), we arrive at the result shown in (127). Comparing expression (127) with the corresponding result derived by Kobzarev and Selivanov [[72], formulas (11) and (12)], we conclude that the result given by these authors [72] is misleading.

C. Plane gravitational wave approximation

In the limit of a plane gravitational wave, we assume that the distance D between the observer and source of light is much smaller than r and r_0, their respective distances from the deflector. The following exact equalities hold:

$$d = r\sin\vartheta, \quad y = \tau - r = r(\cos\vartheta - 1), \quad (133)$$

where ϑ is the angle between the directions from the observer to the source of light and the source of gravitational waves. Furthermore, we note that the vector $\boldsymbol{\xi}$, corresponding to impact parameter d, can be represented as

$$\xi^i = r(N^i - k^i\cos\vartheta), \quad (134)$$

where $N^i = x^i/r$ and $|\mathbf{N}|=1$. Making asymptotic expansions up to leading terms of order $1/r$ and $1/r_0$, and neglecting all residual terms of order $1/r^2$ and $1/r_0^2$, would lead to the following result:

$$\delta\phi(\tau) = \frac{(\mathbf{k}\times\mathbf{N})^i(N^j\cos\vartheta - k^j)\ddot{\mathcal{I}}_{ij}(t-r)}{r(\cos\vartheta-1)} = \frac{(\mathbf{k}\times\mathbf{N})^i k^j \ddot{\mathcal{I}}^{TT}_{ij}(t-r)}{r(1-\cos\vartheta)}, \quad (135)$$

where the definition of the "transverse-traceless" tensor [47] with respect to the direction $\mathbf{N}$ is taken into account,

$$\mathcal{I}^{TT}_{ij} = \mathcal{I}_{ij} + \frac{1}{2}(\delta_{ij} + N_i N_j)N_p N_q \mathcal{I}_{pq} - (\delta_{ip}N_j N_q + \delta_{jp}N_i N_q)\mathcal{I}_{pq}, \quad (136)$$

and the projection is onto the plane orthogonal to the unit vector $\mathbf{N}$. In terms of the transverse-traceless metric perturbation $h^{TT}_{ij} = 2\ddot{\mathcal{I}}^{TT}_{ij}(t-r)/r$, the angle of rotation of the polarization plane of the electromagnetic wave emitted at past null infinity is given by

$$\delta\phi(\tau) = \frac{1}{2}\frac{(\mathbf{k}\times\mathbf{N})^i k^j}{1-\cos\vartheta} h^{TT}_{ij}(t-r). \quad (137)$$

In the case of an electromagnetic wave emitted at instant t_0 and at the distance r_0 from the source of the gravitational waves and received at instant t at the distance r, the overall angle of rotation of the polarization plane is defined as the difference

$$\delta\phi(\tau) - \delta\phi(\tau_0) = \frac{1}{2}\left[\frac{(\mathbf{k}\times\mathbf{N})^i k^j}{1-\cos\vartheta} h^{TT}_{ij}(t-r) - \frac{(\mathbf{k}\times\mathbf{N}_0)^i k^j}{1-\cos\vartheta_0} h^{TT}_{ij}(t_0-r_0)\right], \quad (138)$$

where $N_0^i = x_0^i/r_0$ and ϑ_0 is the angle between the vectors $\mathbf{k}$ and $\mathbf{N}_0$.

It is worthwhile to compare the results obtained in the present section to the standard approach used for calculating effects caused by plane gravitational waves. Ordinarily in such an approach a plane gravitational wave is assumed to be monochromatic with the property that at infinity the space-time remains asymptotically flat. Actually, this means that one deals with a localized packet of waves with boundaries asymptotically approaching plus and minus null infinity uniformly. The metric tensor of such a plane gravitational wave in the transverse-traceless gauge can be expressed as [47]

$$h_{00} = h_{0i} = 0, \quad h^{TT}_{ij} = \mathrm{Re}\{\hat{a}_{ij}\exp(i\hat{p}_\alpha x^\alpha)\}, \quad (139)$$

where the spatial tensor $\hat{a}_{ij}$ is symmetric and traceless ($\hat{a}_{kk} = 0$), $\hat{p}_\alpha = \omega_{gr}(-1,\hat{\mathbf{p}})$ is the propagation four-vector of the gravitational wave with constant frequency ω_{gr} and constant unit spatial propagation vector $\hat{\mathbf{p}}$, and $\hat{a}_{ij}\hat{p}^j = 0$ [77]. One can easily see that such a localized plane gravitational wave does not interact at all with an electromagnetic ray propagating from minus to plus null infinity.

Indeed, the integration of the light ray equation (37) leads to integrals involving the exponential function entering Eq. (139) along the unperturbed light ray path of the form

$$\lim_{T\to\infty}\int_{-T}^{+T} e^{i\hat{p}_\alpha k^\alpha \tau} d\tau = 2\pi\delta(\hat{p}_\alpha k^\alpha) = 2\pi\omega_{gr}^{-1}\delta(\mathbf{k}\cdot\hat{\mathbf{p}} - 1), \quad (140)$$

where $\delta(x)$ is the Dirac delta function. Now, if the direction of the propagation of electromagnetic ray does not coincide with that of the gravitational wave ($\mathbf{k}\neq\hat{\mathbf{p}}$) the argument of the delta function in Eq. (140) is not zero, i.e., $\mathbf{k}\cdot\hat{\mathbf{p}} < 1$, and the result of integration along the light-ray trajectory vanishes. On the other hand, if $\mathbf{k} = \hat{\mathbf{p}}$ there is again no effect since the right side of Eq. (37) vanishes due to the transversality of h^{TT}_{ij}, revealing that in the case under consideration $h^{TT}_{ij}k^j = h^{TT}_{ij}\hat{p}^j = 0$. This result has been proved by Damour and Esposito-Farèse [78] who studied the deflection of light and the integrated time delay caused by the time-dependent gravitational field generated by a localized material source lying close to the line of sight [79,80].

The absence of the interaction of plane monochromatic gravitational waves with electromagnetic rays propagating from minus to plus null infinity makes it evident that all relativistic effects taking place in the gravitational lens approximation have little to do with gravitational waves—i.e., only the near-zone gravitational field of the lens contributes to the overall effects. The other conclusion is that the plane gravitational wave disturbs the propagation of electromagnetic signals if and only if the signals go to finite distances [81,82]. However, another limitation has to be kept in mind, namely, as follows for instance from Eq. (138), the standard plane gravitational wave approximation (139) is applicable only for gravitational waves with dominant wavelengths $\lambda_{gr} \ll \min(r,r_0)$. This remark is especially important for having a self-consistent analysis of such intricate problems as the detection of the solar g modes by interferometric gravitational wave detectors [83], the theoretical prediction of low-frequency pulsar timing noise (see, e.g., [84–87], and references therein), and the anisotropy of cosmic microwave background radiation [88] caused by primordial gravitational waves.

APPENDIX A: GEOMETRIC OPTICS APPROXIMATION FROM THE MAXWELL EQUATIONS

Here we give a brief derivation of the equations of the geometric optics approximation, discussed in Sec. IV, from the Maxwell equations for electromagnetic waves propagating in empty space-time. The source-free Maxwell equations are given by [67]

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0, \tag{A1}$$

$$\nabla_\beta F^{\alpha\beta} = 0, \tag{A2}$$

where ∇_α denotes covariant differentiation [89]. Taking a covariant derivative from Eq. (A1) and using Eq. (A2), we obtain the covariant wave equation for the electromagnetic field tensor

$$\Box_g F_{\alpha\beta} + R_{\alpha\beta\gamma\delta} F^{\gamma\delta} - R_{\alpha\gamma} F^{\gamma}_{\beta} + R_{\beta\gamma} R^{\gamma}_{\alpha} = 0, \tag{A3}$$

where $\Box_g \equiv \nabla^\alpha \nabla_\alpha$, $R_{\alpha\beta\gamma\delta}$ is the Riemann curvature tensor, and $R_{\alpha\beta} = R^{\gamma}_{\alpha\gamma\beta}$ is the Ricci tensor.

Let us now assume that the electromagnetic tensor $F_{\alpha\beta}$ has the form shown in Eq. (17). We introduce a dimensionless perturbation parameter ε and assume an expansion of the field of the form

$$F_{\alpha\beta} = \mathrm{Re}\left[(a_{\alpha\beta} + \varepsilon b_{\alpha\beta} + \varepsilon^2 c_{\alpha\beta} + \cdots) \exp\left(\frac{i\varphi}{\varepsilon} \right) \right]; \tag{A4}$$

see Ref. [48] for a critical examination of this procedure and its underlying physical assumptions. Substituting the expansion (A4) into Eq. (A1), taking into account the definition $l_\alpha = \partial\varphi/\partial x^\alpha$, and arranging terms with similar powers of ε leads to the chain of equations

$$l_\alpha a_{\beta\gamma} + l_\beta a_{\gamma\alpha} + l_\gamma a_{\alpha\beta} = 0, \tag{A5}$$

$$i(\nabla_\alpha a_{\beta\gamma} + \nabla_\beta a_{\gamma\alpha} + \nabla_\gamma a_{\alpha\beta}) = l_\alpha b_{\beta\gamma} + l_\beta b_{\gamma\alpha} + l_\gamma b_{\alpha\beta}, \tag{A6}$$

etc., where we have assumed that the effects of curvature are negligibly small. Similarly, Eq. (A2) gives a chain of equations

$$l_\beta a^{\alpha\beta} = 0, \tag{A7}$$

$$\nabla_\beta a^{\alpha\beta} + i l_\beta b^{\alpha\beta} = 0, \tag{A8}$$

and so on.

Equation (A7) implies that the electromagnetic field tensor is orthogonal in the four-dimensional sense to vector l_α. Contracting Eq. (A5) with l_α and taking into account Eq. (A7), we find that l_α is null, $l_\alpha l^\alpha = 0$. Taking the covariant derivative of this equality and using the fact that $\nabla_{[\beta} l_{\alpha]} = 0$ since $l_\alpha = \nabla_\alpha \varphi$, one can show that the vector l_α obeys the null geodesic Eq. (18). Finally, equation (A3) can be used to show that

$$\nabla_\gamma (a_{\alpha\beta} l^\gamma) + l^\gamma \nabla_\gamma a_{\alpha\beta} = 0, \tag{A9}$$

which immediately leads to Eq. (20) of the propagation law for the electromagnetic field tensor. In this way, one can prove the validity of the equations of the geometric optics approximation displayed in Sec. IV.

APPENDIX B: APPROXIMATE EXPRESSIONS FOR THE CHRISTOFFEL SYMBOLS

Making use of the general definition of the Christoffel symbols [47]

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} (\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma}), \quad \partial_\alpha \equiv \partial/\partial x^\alpha, \tag{B1}$$

and applying the expansion of the metric tensor (9) results in the approximate post-Minkowskian expressions

$$\Gamma^{0}_{00} = -\frac{1}{2} \partial_t h_{00}(t,\mathbf{x}), \quad \partial_t \equiv \frac{\partial}{\partial t}, \tag{B2}$$

$$\Gamma^{0}_{0i} = -\frac{1}{2} \partial_i h_{00}(t,\mathbf{x}), \quad \partial_i \equiv \frac{\partial}{\partial x^i}, \tag{B3}$$

$$\Gamma^{0}_{ij} = -\frac{1}{2} [\partial_i h_{0j}(t,\mathbf{x}) + \partial_j h_{0i}(t,\mathbf{x}) - \partial_t h_{ij}(t,\mathbf{x})], \tag{B4}$$

$$\Gamma^{i}_{00} = \partial_t h_{0i}(t,\mathbf{x}) - \frac{1}{2} \partial_i h_{00}(t,\mathbf{x}), \tag{B5}$$

$$\Gamma^i_{0j}=\frac{1}{2}[\partial_j h_{0i}(t,\mathbf{x})-\partial_i h_{0j}(t,\mathbf{x})+\partial_t h_{ij}(t,\mathbf{x})], \tag{B6}$$

$$\Gamma^i_{jp}=\frac{1}{2}[\partial_j h_{ip}(t,\mathbf{x})+\partial_p h_{ij}(t,\mathbf{x})-\partial_i h_{jp}(t,\mathbf{x})]. \tag{B7}$$

These expressions are used for the calculation of the right-hand side of the null geodesic equation (19).

APPENDIX C: CALCULATION OF INTEGRALS ALONG THE LIGHT-RAY TRAJECTORY

In order to calculate the integrals (42) and (43), it is useful to change in the integrands the time argument σ to the new one ζ defined by the light-cone Eq. (12), which after substituting the unperturbed light trajectory (36) for $\mathbf{x}$ can be expressed as follows:

$$\sigma+t^*=\zeta+|\boldsymbol{\xi}+\mathbf{k}\sigma-\mathbf{z}(\zeta)|, \tag{C1}$$

where t^*, $\boldsymbol{\xi}$, and $\mathbf{k}$ are considered as parameters such that the orthogonality condition $\xi_i k^i=0$ holds. The differentiation of Eq. (C1) yields the partial derivatives of the retarded time with respect to the parameters

$$\frac{\partial\zeta}{\partial t^*}=\frac{r}{r-\mathbf{v}\cdot\mathbf{r}},\quad P_{ij}\frac{\partial\zeta}{\partial\xi^j}=-\frac{P_{ij}r^j}{r-\mathbf{v}\cdot\mathbf{r}},\quad \frac{\partial\zeta}{\partial k^i}=-\frac{\sigma r^i}{r-\mathbf{v}\cdot\mathbf{r}}, \tag{C2}$$

and the relationship between the time differentials along the world line of the photon

$$d\sigma=d\zeta\frac{r-\mathbf{v}\cdot\mathbf{r}}{r-\mathbf{k}\cdot\mathbf{r}}. \tag{C3}$$

It is worth noting that in the formula (C2) one can write $P_{ij}r^j=(\mathbf{k}\times(\mathbf{r}\times\mathbf{k}))^i\equiv r^i-k^i(\mathbf{k}\cdot\mathbf{r})$. Moreover, $\xi_i=P_{ij}\xi^j$ since $P^{ij}=\delta^{ij}-k^ik^j$ and $k_i\xi^i=0$, i.e., ξ^i has only two independent components.

Two principal differential identities applied to any smooth function $F(t,\mathbf{x})$ are used in calculations throughout the paper, namely, as proved in [7],

$$\left[\frac{\partial F(t,\mathbf{x})}{\partial x^i}+k_i\frac{\partial F(t,\mathbf{x})}{\partial t}\right]_{\mathbf{x}=\mathbf{k}(t-t_0)+\mathbf{x}_0}=P_{ij}\frac{\partial F(\tau+t^*,\mathbf{k}\tau+\boldsymbol{\xi})}{\partial\xi^j}+k_i\frac{\partial F(\tau+t^*,\mathbf{k}\tau+\boldsymbol{\xi})}{\partial\tau} \tag{C4}$$

and [15]

$$\left[\frac{\partial F(t,\mathbf{x})}{\partial t}\right]_{t=\sigma+t^*;\mathbf{x}=\mathbf{k}\sigma+\boldsymbol{\xi}}=\frac{\partial F(\sigma+t^*,\mathbf{k}\sigma+\boldsymbol{\xi})}{\partial t^*}. \tag{C5}$$

The relationship (C4) allows one to change the order of operations of partial differentiation and substitution for the unperturbed light-ray trajectory, while Eq. (C5) shows how to change differentiation from the time t to the parameter t^* making use of the reparametrization of the light-ray trajectory. The parameters ξ^i and t^* do not depend on time and, for this reason, derivatives with respect to them can be taken out of the integrals along the light-ray trajectory. Hence, applying Eqs. (C4) and (C5) to the integral (42) and taking account of Eq. (C3) we obtain

$$B_S^{\alpha\beta}(\tau,\boldsymbol{\xi})=4\frac{\partial}{\partial t^*}\int_{-\infty}^{s(\tau,t^*)}\frac{k_\gamma S^{\gamma(\alpha}u^{\beta)}}{r-\mathbf{k}\cdot\mathbf{r}}d\zeta-4\frac{\partial}{\partial\xi^i}\times\int_{-\infty}^{s}\frac{S^{i(\alpha}u^{\beta)}}{r-\mathbf{k}\cdot\mathbf{r}}d\zeta-4\frac{k_iS^{i(\alpha}u^{\beta)}}{r-\mathbf{v}\cdot\mathbf{r}}. \tag{C6}$$

where the expression

$$r-\mathbf{k}\cdot\mathbf{r}=t^*+\mathbf{k}\cdot\mathbf{z}(\zeta)-\zeta \tag{C7}$$

is a function of the retarded argument ζ and the (constant) parameter t^* only. Differentiation with respect to the parameters t^* and ξ^i in Eq. (C6) results in

$$B_S^{\alpha\beta}(\tau,\boldsymbol{\xi})=\frac{4r_\gamma S^{\gamma(\alpha}u^{\beta)}}{(r-\mathbf{v}\cdot\mathbf{r})(r-\mathbf{k}\cdot\mathbf{r})}-4\int_{-\infty}^{s}\frac{k_\gamma S^{\gamma(\alpha}u^{\beta)}}{(r-\mathbf{k}\cdot\mathbf{r})^2}d\zeta, \tag{C8}$$

where $r^\alpha=(r,r^i)$ and $r_\alpha=(-r,r^i)$. A useful formula is obtained after differentiating $B_S^{\alpha\beta}$ with respect to the impact parameter. Specifically, we have

$$\hat{\partial}_iB_S^{\alpha\beta}=\frac{4P_{ij}S^{j(\alpha}u^{\beta)}}{(r-\mathbf{v}\cdot\mathbf{r})(r-\mathbf{k}\cdot\mathbf{r})}+\frac{4r_\gamma S^{\gamma(\alpha}u^{\beta)}}{r-\mathbf{k}\cdot\mathbf{r}}\frac{P_{ij}v^j}{(r-\mathbf{v}\cdot\mathbf{r})^2}+4P_{ij}r^j\left[\frac{k_\gamma S^{\gamma(\alpha}u^{\beta)}}{(r-\mathbf{v}\cdot\mathbf{r})(r-\mathbf{k}\cdot\mathbf{r})^2}-\frac{(1-v^2)r_\gamma S^{\gamma(\alpha}u^{\beta)}}{(r-\mathbf{k}\cdot\mathbf{r})(r-\mathbf{v}\cdot\mathbf{r})^3}-\frac{(1-\mathbf{k}\cdot\mathbf{v})r_\gamma S^{\gamma(\alpha}u^{\beta)}}{(r-\mathbf{v}\cdot\mathbf{r})^2(r-\mathbf{k}\cdot\mathbf{r})^2}\right]. \tag{C9}$$

For comparison, in the case of pure monopole particles we have [15]

$$\hat{\partial}_iB_M^{\alpha\beta}(\tau)=-4m(1-v^2)^{1/2}\frac{u^\alpha u^\beta+\frac{1}{2}\eta^{\alpha\beta}}{r-\mathbf{v}\cdot\mathbf{r}}\frac{P_{ij}r^j}{r-\mathbf{k}\cdot\mathbf{r}}. \tag{C10}$$

As for the integral $D_S^{\alpha\beta}$, one can see from Eqs. (40) and (41) that in the calculation of the light-ray trajectory we do not need $D^{\alpha\beta}$ directly but only its partial derivative with respect to the parameter ξ^i. Thus, using the definition of $D_S^{\alpha\beta}$ and the expression for $B_S^{\alpha\beta}$ given in Eq. (C8), one can prove that

$$\hat{\partial}_iD_S^{\alpha\beta}(\tau)=-4\frac{P_{ij}r^j}{r-\mathbf{v}\cdot\mathbf{r}}\frac{r_\gamma S^{\gamma(\alpha}u^{\beta)}}{(r-\mathbf{k}\cdot\mathbf{r})^2}+4\int_{-\infty}^{s}\frac{P_{ij}S^{j(\alpha}u^{\beta)}}{(r-\mathbf{k}\cdot\mathbf{r})^2}\times d\zeta+8\int_{-\infty}^{s}\frac{P_{ij}r^jk_\gamma S^{\gamma(\alpha}u^{\beta)}}{(r-\mathbf{k}\cdot\mathbf{r})^3}d\zeta. \tag{C11}$$

The remaining integrals in Eqs. (C8) and (C11) have the following form:

$$\mathcal{I}(s)=\int_{-\infty}^{s}\frac{1-\mathbf{k}\cdot\mathbf{v}(\zeta)}{(r-\mathbf{k}\cdot\mathbf{r})^{n}}F(\zeta)d\zeta,\quad (n=2,3),\tag{C12}$$

where $F(\zeta)$ is a smooth function of the four-velocity and/or spin of the bodies at the retarded time. The integral (C12) can be calculated by making use of the new variable [cf. Eq. (125)]

$$y=\mathbf{k}\cdot\mathbf{r}-r=\zeta-t^{*}-\mathbf{k}\cdot\mathbf{z}(\zeta),\quad dy=[1-\mathbf{k}\cdot\mathbf{v}(\zeta)]d\zeta,\tag{C13}$$

so that the above integral (C12) can be expressed as

$$\mathcal{I}(s)=(-1)^{n}\int_{-\infty}^{y(s)}\frac{F(y)dy}{y^{n}}=\frac{(-1)^{n+1}}{n-1}\int_{-\infty}^{y(s)}F(y)\frac{d}{dy}\left(\frac{1}{y^{n-1}}\right)dy.\tag{C14}$$

Integration by parts results in

$$\mathcal{I}(s)=\frac{1}{n-1}\frac{F(y(s))}{(r-\mathbf{k}\cdot\mathbf{r})^{n-1}}-\frac{1}{n-1}\int_{-\infty}^{s}\frac{\dot{F}(\zeta)d\zeta}{(r-\mathbf{k}\cdot\mathbf{r})^{n-1}}.\tag{C15}$$

The last integral in Eq. (C15) can be neglected under ordinary circumstances for it includes the acceleration and/or a time derivative of the spin of the body. Omitting such terms in accordance with our general approximation scheme, we finally arrive at

$$B_{S}^{\alpha\beta}(\tau)=\frac{4r_{\gamma}S^{\gamma(\alpha}u^{\beta)}}{(r-\mathbf{v}\cdot\mathbf{r})(r-\mathbf{k}\cdot\mathbf{r})}-\frac{4}{1-\mathbf{k}\cdot\mathbf{v}}\frac{k_{\gamma}S^{\gamma(\alpha}u^{\beta)}}{(r-\mathbf{k}\cdot\mathbf{r})},\tag{C16}$$

$$\hat{\partial}_{i}D_{S}^{\alpha\beta}(\tau)=-\frac{4P_{ij}r^{j}}{r-\mathbf{v}\cdot\mathbf{r}}\frac{r_{\gamma}S^{\gamma(\alpha}u^{\beta)}}{(r-\mathbf{k}\cdot\mathbf{r})^{2}}+\frac{4P_{ij}r^{j}}{1-\mathbf{k}\cdot\mathbf{v}}\frac{k_{\gamma}S^{\gamma(\alpha}u^{\beta)}}{(r-\mathbf{k}\cdot\mathbf{r})^{2}}+\frac{4P_{ij}v^{j}}{(1-\mathbf{k}\cdot\mathbf{v})^{2}}\frac{k_{\gamma}S^{\gamma(\alpha}u^{\beta)}}{(r-\mathbf{k}\cdot\mathbf{r})}+\frac{P_{ij}S^{j(\alpha}u^{\beta)}}{(1-\mathbf{k}\cdot\mathbf{v})(r-\mathbf{k}\cdot\mathbf{r})}.\tag{C17}$$

APPENDIX D: AUXILIARY ALGEBRAIC AND DIFFERENTIAL RELATIONSHIPS

In this Appendix we give several algebraic relationships that can be used for the calculation of observable effects. Making use of Eq. (8) and definition (5) we have

$$S^{i0}=\gamma(\mathbf{v}\times\boldsymbol{\mathcal{J}})^{i},\tag{D1}$$

$$S^{ij}=\gamma\varepsilon_{ijk}\mathcal{J}^{k}+\frac{1-\gamma}{v^{2}}(\mathbf{v}\cdot\boldsymbol{\mathcal{J}})\varepsilon_{ijk}v^{k},\tag{D2}$$

$$k_{\alpha}r_{\beta}S^{\alpha\beta}=\gamma[\boldsymbol{\mathcal{J}}\cdot(\mathbf{k}\times\mathbf{r})+\boldsymbol{\mathcal{J}}\cdot((\mathbf{r}-\mathbf{k}r)\times\mathbf{v})]+\frac{1-\gamma}{v^{2}}(\mathbf{v}\cdot\boldsymbol{\mathcal{J}})(\mathbf{k}\times\mathbf{r})\cdot\mathbf{v},\tag{D3}$$

where $\gamma=(1-v^{2})^{-1/2}$ and $\varepsilon_{ijk}\equiv\epsilon_{0ijk}$ [42].

In addition, one has the following formulas of differentiation with respect to the impact parameter ξ^{i} (a dot over any quantity denotes differentiation with respect to time t):

$$\hat{\partial}_{i}r=\frac{P_{ij}r^{j}}{r-\mathbf{v}\cdot\mathbf{r}},\quad \hat{\partial}_{i}\equiv P_{ij}\frac{\partial}{\partial\xi^{j}},\tag{D4}$$

$$\hat{\partial}_{i}r^{j}=P_{ij}+\frac{v^{j}P_{ik}r^{k}}{r-\mathbf{v}\cdot\mathbf{r}},\tag{D5}$$

$$\hat{\partial}_{i}(r_{\beta}S^{\alpha\beta})=P_{ij}S^{\alpha j}-\frac{r_{\beta}\dot{S}^{\alpha\beta}P_{ij}r^{j}}{r-\mathbf{v}\cdot\mathbf{r}},\tag{D6}$$

$$\hat{\partial}_{i}\left(\frac{1}{r-\mathbf{k}\cdot\mathbf{r}}\right)=-\frac{(1-\mathbf{k}\cdot\mathbf{v})P_{ij}r^{j}}{(r-\mathbf{v}\cdot\mathbf{r})(r-\mathbf{k}\cdot\mathbf{r})^{2}},\tag{D7}$$

$$\hat{\partial}_{i}\left(\frac{1}{r-\mathbf{v}\cdot\mathbf{r}}\right)=-\frac{(1-v^{2}+\mathbf{a}\cdot\mathbf{r})P_{ij}r^{j}}{(r-\mathbf{v}\cdot\mathbf{r})^{3}}+\frac{P_{ij}v^{j}}{(r-\mathbf{v}\cdot\mathbf{r})^{2}},\tag{D8}$$

where $\mathbf{a}=\dot{\mathbf{v}}$ is the light-ray-deflecting body's acceleration and it has been assumed that $\mathbf{r}=\mathbf{x}-\mathbf{z}(s)=\mathbf{k}\tau+\boldsymbol{\xi}-\mathbf{z}(s)$.

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$$\int_{-\infty}^{+\infty} \delta(t-t')\,\delta(\mathbf{x}-\mathbf{x}')\,dt\,d^3\mathbf{x}=1.$$

Here $\delta(\mathbf{x})=\delta(x^1)\,\delta(x^2)\,\delta(x^3)$, where $\mathbf{x}=(x^1,x^2,x^3)$ are Cartesian coordinates in three-dimensional Euclidean space.
[35] Throughout this paper, we use units such that Newton's gravitational constant G and the speed of light c equal unity.
[36] Greek indices run from 0 to 3 and Latin indices from 1 to 3. Repeated indices assume the Einstein summation rule. Bold letters denote three-dimensional Euclidean vectors.
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$$h^{\beta}_{\alpha,\beta}-\frac{1}{2}h^{\beta}_{\beta,\alpha}=0,$$

where a comma denotes partial differentiation.
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