Thick brane worlds and their stability

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Three types of thick branes, i.e., Poincaré, de Sitter, and anti–de Sitter branes, are considered. They are realized as the nonsingular solutions of the Einstein equations with nontrivial dilaton and potentials. The scalar perturbations of these systems are also investigated. We find that the effective potentials of the master equations of the scalar perturbations are positive definite and consequently these systems are stable under small perturbations.

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I. INTRODUCTION

As the most promising candidate for a unified theory of everything, superstring theory has been investigated for a long time. The recent discovery of D-branes stimulated a rather old idea, the ''brane world'' [1,2]. In particular, the Randall-Sundrum (RS) model has been advocated as a simple model [3,4].

RS considered a 3-brane (the four-dimensional Minkowski spacetime) embedded in five-dimensional anti–de Sitter spacetime (AdS_5), and they found that there exists a massless graviton (0-mode) and massive gravitons (Kaluza-Klein modes). The massless graviton reproduces the Newtonian gravity on the 3-brane, and Kaluza-Klein modes, which are the effect of the existence of the higher dimension, give a correction to the Newtonian gravity [5,6]. RS showed that Newtonian gravity can be reproduced in a sufficiently low-energy limit, and the idea that our universe is a 3-brane embedded in higher-dimensional spacetime came to be investigated eagerly. Furthermore, the cosmological consequences of the RS model have been investigated and there has been no contradiction with observations until now [7–23]. The inflationary scenario also seems to be compatible with brane models [24–30].

Note that the RS model is motivated by unified theory such as the superstring theory. Such theories are necessary to represent the high-energy era of the history of our universe, and in superstring theory there seems to exist a minimum scale of length, thus we cannot consider an exact 0-width brane. That is, we cannot neglect the thickness of the brane at the string scale.

For this reason, ''thick brane world'' scenarios have been investigated [31–40]. In this paper, we pay special attention to the construction of thick branes. We consider three types of maximally symmetric branes, that is, Poincaré, de Sitter, and anti–de Sitter branes. As these branes are highly symmetric, they always exist as solutions of the Einstein equations if we introduce nontrivial dilatons and suitable scalar potentials, and these solutions are nonsingular in the whole spacetime. The thick de Sitter and anti–de Sitter branes, which are nonsingular, are found for the first time in this paper.

We also analyze the stability of these systems. We write down the master equations of the scalar perturbations and find the effective potential in the Poincaré, de Sitter, and anti–de Sitter brane case, respectively. As a result, due to the positivity of the effective potentials, we find that all of the systems are stable under the scalar perturbations.[1]

Finally, we refer to the relation between the thickness and the noncommutativity. As mentioned above, the thickness is necessary due to the existence of the minimum length, and the noncommutativity also arises from this length. Therefore, we can naively expect that the thickness has something to do with the noncommutativity.

The organization of this paper is as follows. In Sec. II, we review the setup of the thin brane models and construct three types of maximally symmetric branes. In order to see the effect of the thickness, we consider the behavior of gravitons in the thick brane backgrounds. In Sec. III, we analyze the scalar perturbations of the thick brane systems. We derive the effective potentials of the scalar perturbations and examine their behavior. Section IV is devoted to discussions and conclusions. The relation between the thickness and the noncommutativity is also discussed there.

II. THICK BRANE MODELS

A. Thin brane models

At first, we review the setup of the thin brane models. The simplest action of the thin brane model is

$$S=\int d^5x\sqrt{-g_5}\left(\tfrac{1}{2}R-\Lambda_5\right)-\sigma\int d^4x\sqrt{-g}, \qquad (2.1)$$

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[1] After completion of our work, we found the researches of Giovannini [38,39], which are related to our work. In [38,39], the thick Poincaré brane is investigated.

where we used the unit $8\pi G_5=1$ (G_5 is the five-dimensional gravitational constant), g_5 is the five-dimensional metric, and R denotes the five-dimensional Ricci scalar. If we consider the AdS_5, the five-dimensional cosmological constant Λ_5 is related to the AdS radius l as

$$\Lambda_5=-\frac{6}{l^2}. \tag{2.2}$$

The second term of the action is the action of the brane and σ denotes the tension of the brane. Now we consider the following type of metric:

$$ds^2=dy^2+e^{2\alpha(y)}\gamma_{\mu\nu}dx^\mu dx^\nu, \tag{2.3}$$

where y denotes the coordinate of the direction of the bulk, $\gamma_{\mu\nu}$ is the metric on the brane, and μ, ν denote the indices of the four-dimension of the brane. $e^{2\alpha(y)}$ is a so-called "warp factor." In this setup, we can get three types of branes whose geometry is maximally symmetric, that is, M^4, dS_4, or AdS_4. Solving the Einstein equations, we find

$$\alpha(y)=\begin{cases} y_0-|y| & \text{Poincaré brane,}\\ \log[\sinh(y_0-|y|)] & \text{de Sitter brane,}\\ \log[\cosh(y_0-|y|)] & \text{anti–de Sitter brane,}\end{cases} \tag{2.4}$$

where y_0 is a constant. In the case of the Poincaré brane, y_0 can be set to 0 without the loss of generality due to the Poincaré invariance. Therefore, we can say that y_0 does not have any physical meaning. In the case of the de Sitter brane, y_0 determines the range of the bulk. In fact, the warp factor becomes 0 at $y=y_0$, so the range of y is $-y_0\leqslant y\leqslant y_0$ and $y=y_0$ is the horizon of the AdS_5. On the contrary, in the case of the anti–de Sitter brane, y_0 denotes only the turning point of the warp factor because the metric does not become 0 at $y=y_0$. So there is no horizon in AdS_5 with the anti–de Sitter slicing. We consider the extension of the thin brane systems to the thick brane ones in the following subsections.

B. Construction of the thick brane models

In order to realize thick brane models, we consider the following action:

$$S=\int d^5x\sqrt{-g_5}\left[\tfrac{1}{2}R-\tfrac{1}{2}(\partial\varphi)^2-V(\varphi)\right]. \tag{2.5}$$

Here φ is the five-dimensional scalar field which depends only on the coordinate of the bulk and $V(\varphi)$ is its potential. We use the metric

$$ds^2=a^2(z)(dz^2+\gamma_{\mu\nu}dx^\mu dx^\nu), \tag{2.6}$$

where a conformal-like coordinate z is defined through the following equations:

$$z\equiv\int\frac{dy}{a},\quad a(z)=e^{\alpha(y(z))}. \tag{2.7}$$

Note that $\gamma_{\mu\nu}$ denotes the metric of the maximally symmetric four-dimensional spacetimes, so we can write the four-dimensional Ricci tensor and Ricci scalar as follows:

$$R^{(4)}_{\mu\nu}=3K\gamma_{\mu\nu}, \tag{2.8}$$

$$R^{(4)}=12K, \tag{2.9}$$

where K takes 0, 1, or -1 and these values correspond Poincaré, de Sitter, and anti–de Sitter branes, respectively.[2]

Now we can write down the Einstein equations and the equation of motion of the scalar field (matter),

$$(z,z):\quad 6\mathcal{H}^2-6K=\tfrac{1}{2}(\varphi')^2-a^2V(\varphi), \tag{2.10}$$

$$(\mu,\nu):\quad 3\mathcal{H}'+3\mathcal{H}^2-3K=-\tfrac{1}{2}(\varphi')^2-a^2V(\varphi), \tag{2.11}$$

$$\text{matter:}\quad \varphi''+3\mathcal{H}\varphi'=a^2\frac{\partial V}{\partial\varphi}. \tag{2.12}$$

Here a prime denotes the derivative with respect to z and we define $\mathcal{H}$ as follows:

$$\mathcal{H}\equiv\frac{a'}{a}. \tag{2.13}$$

From Eqs. (2.10) and (2.11), we get the following equations:

$$(\varphi')^2=-3\mathcal{H}'+3\mathcal{H}^2-3K, \tag{2.14}$$

$$V(\varphi)=-\frac{1}{2a^2}(3\mathcal{H}'+9\mathcal{H}^2-9K). \tag{2.15}$$

From Eqs. (2.14) and (2.15), we can see that one can construct a thick brane model starting from a given warp factor $a(z)$, as long as $a(z)$ satisfies the condition

$$(\varphi')^2\geqslant 0. \tag{2.16}$$

Here the equation of motion of the scalar field (2.12) is automatically satisfied due to the Bianchi identity. Equation (2.15) simply determines the functional form of the potential.

C. Thick Poincaré brane model

In this paper, we use the following warp factor in order to make a thick Poincaré brane:[3]

$$a^2(z)=e^{2\alpha(y(z))}=\left(\frac{1}{e^{-2n(y_0+y)}+e^{-2n(y_0-y)}}\right)^{1/n}. \tag{2.17}$$

As shown in Fig. 1, there is no jump of the value of the extrinsic curvature at $y=y_0$. When we take the limit n

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[2]From now on, we also set the AdS radius $l=1$ and we normalize the curvature radius of the four-dimensional spacetime to unity.

[3]There may be many ways to introduce the "thickness." Here we examine one of the possibilities. The parameter of the thickness which is related to the noncommutative geometry is discussed in Sec. IV.

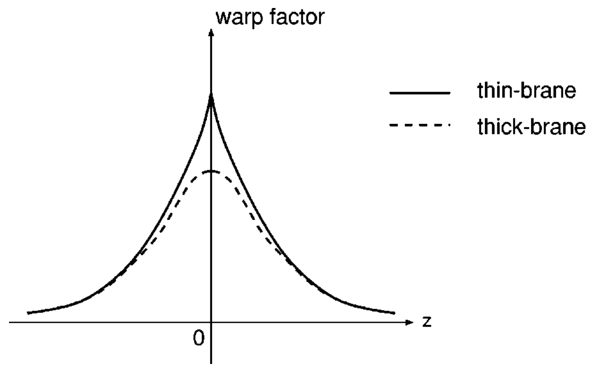

FIG. 1. The warp factors of the thin and the thick brane are shown. The solid line denotes the warp factor of the thin brane and the dashed line denotes that of the thick brane.

$\to\infty$, this warp factor approaches $e^{2(y_0-|y|)}$, which is the warp factor in the RS model. So the parameter n controls the ''thickness'' of the brane.

In this case, we can write down $\varphi(y)$ and $V(\varphi)$ explicitly. In fact, substituting the warp factor (2.17) into Eqs. (2.14) and (2.15), we get

$$\varphi(y)=\pm\sqrt{\frac{6}{n}}\arctan(e^{2ny}), \tag{2.18}$$

$$V(\varphi)=-6+3(n+2)\sin^2\left(\frac{\sqrt{6n}}{3}\varphi\right). \tag{2.19}$$

Now, in order to see the effect of the ''thickness,'' let us consider the gravitational perturbation in this system. Here the graviton $h_{\mu\nu}$ is the tensor perturbation of the metric, so it can be written as

$$ds^2=a^2(z)[dz^2+(\gamma_{\mu\nu}+h_{\mu\nu})dx^\mu dx^\nu]. \tag{2.20}$$

Here $h_{\mu\nu}$ satisfies the transverse-traceless condition

$$h^\mu_\mu=h^{|\nu}_{\mu\nu}=0, \tag{2.21}$$

where the vertical bar denotes the covariant derivative with respect to $\gamma_{\mu\nu}$. In this system, the equation for $h_{\mu\nu}$ becomes

$$h''_{\mu\nu}+3\mathcal{H}'_{\mu\nu}+h^{|\lambda}_{\mu\nu|\lambda}-2Kh_{\mu\nu}=0. \tag{2.22}$$

Here we can define the four-dimensional momentum p as follows:

$$h^{|\lambda}_{\mu\nu|\lambda}-2Kh_{\mu\nu}=p^2h_{\mu\nu}. \tag{2.23}$$

Furthermore, we can decompose $h_{\mu\nu}$ using the polarization tensor $\varepsilon_{\mu\nu}$, which depends only on the four-dimensional coordinates x^ρ as follows:

$$h_{\mu\nu}(z,x^\rho)=\varepsilon_{\mu\nu}(x^\rho)X(z), \tag{2.24}$$

where $\varepsilon_{\mu\nu}$ satisfies the transverse-traceless condition

$$\varepsilon^\mu_\mu=\varepsilon^{|\nu}_{\mu\nu}=0. \tag{2.25}$$

Now, regardless of the value of K, we can rewrite Eq. (2.22) as

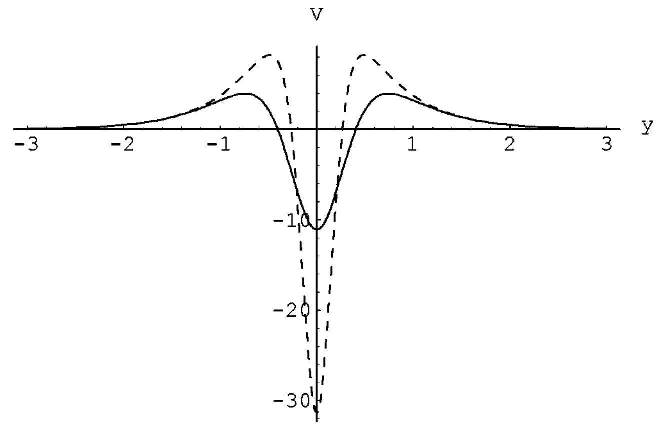

FIG. 2. The effective potential of the graviton in the thick Poincaré brane system. Here we set $y_0=1$. The solid line denotes the $n=1$ case and the dashed line denotes the $n=2$ case, respectively.

$$X''+3\mathcal{H}X'=-p^2X. \tag{2.26}$$

To see the behavior of $h_{\mu\nu}$, we transform this equation into the Schrödinger-type equation and write down the effective potential.

Now we introduce a new function $\chi(z)$, which satisfies the following equation:

$$X(z)=a(z)^{-3/2}\chi(z). \tag{2.27}$$

Substituting Eq. (2.27) into Eq. (2.26), we find that the equation for $\chi(z)$ becomes

$$-\chi''(z)+V(z)\cdot\chi(z)=-p^2\chi(z), \tag{2.28}$$

$$V(z)=\tfrac{3}{2}\mathcal{H}'+\tfrac{9}{4}\mathcal{H}^2, \tag{2.29}$$

where $V(z)$ is the effective potential of the graviton. Furthermore, substituting the warp factor (2.17) into Eq. (2.29), we get

$$V(z)=\frac{3e^{2y_0}[5e^{4ny}-(16n+10)+5e^{-4ny}]}{4(e^{2ny}+e^{-2ny})^{2+(1/n)}}. \tag{2.30}$$

We show the effective potentials in Fig. 2. The effective potentials show that the gravitons can localize around the brane. In fact, there is a 0-mode solution,

$$\chi_0(z)\sim a^{3/2}(z). \tag{2.31}$$

Here we neglect the normalization constant.

The width of the brane becomes wider as n approaches 0. On the other hand, when n approaches ∞, the effective potential approaches the ''volcano'' potential, which can be seen in the RS model, and the 0-mode solution $\chi_0(z)$ coincides with the massless graviton in the RS model. So we can say that if we take a sufficiently large n and the low-energy limit, the Newtonian gravity can be reproduced in this model.[4]

[4]In [34], it is shown that the Newtonian gravity is reproduced.

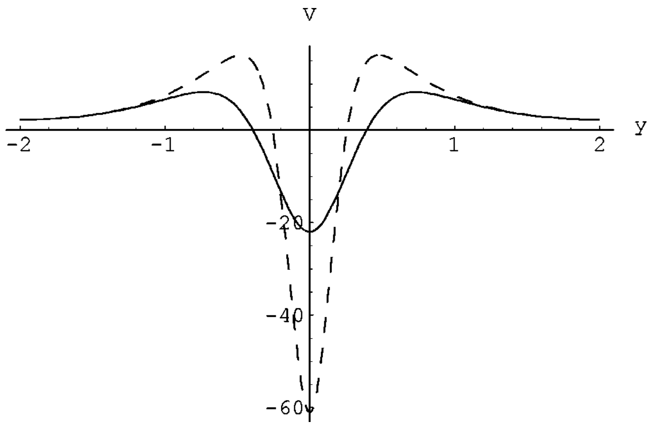

FIG. 3. The effective potential of the graviton in the thick de Sitter brane system. The solid line denotes the $n=1$ case and the dashed line denotes the $n=2$ case, respectively. Both approach $\frac{9}{4}H^2$ as $y\to y_0$. Here we set $y_0=2$.

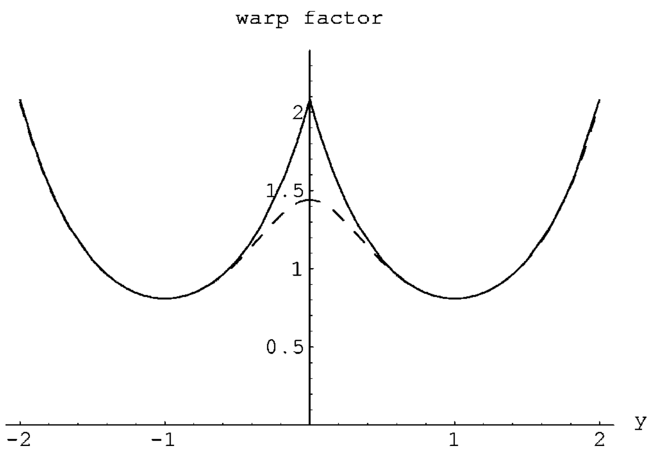

FIG. 4. The squares of the warp factors of a thin and thick AdS brane. The solid line denotes the warp factor of the thin AdS brane and the dashed line denotes that of the thick AdS brane. Here we set $y_0=1$.

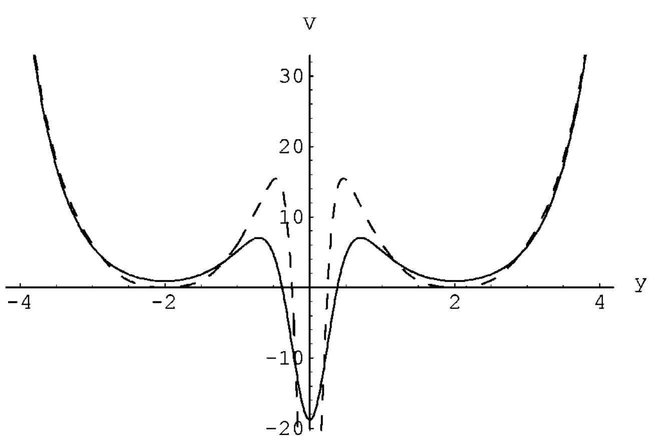

FIG. 5. The effective potential of the graviton in the thick AdS brane system. The solid line denotes the $n=1$ case and the dashed line denotes the $n=2$ case, respectively. Here we set $\epsilon=0.4$ and $y_0=2$.

D. Thick de Sitter brane model

Similarly, we set the warp factor of a thick de Sitter brane as follows:

$$a^2(z)=\left(\frac{1}{\sinh^{-2n}(y_0+y)+\sinh^{-2n}(y_0-y)}\right)^{1/n}. \tag{2.32}$$

In this case, we cannot calculate $\varphi(y)$ and $V(\varphi)$ analytically for an arbitrary n, so we have to calculate them numerically.

Let us consider the behavior of the graviton in the thick de Sitter brane system as well as in the previous subsection. Substituting the warp factor (2.32) into Eq. (2.30), we get the effective potential of the graviton propagating in the thick de Sitter brane system. The result is shown in Fig. 3.

The shape of the effective potential in this case is very similar to that in the case of the Poincaré brane. There is a hole around the location of the brane and we can find a bound state of the graviton which localizes around the brane.

Here, we have to note that the asymptotic value of the effective potential in this case is different from that in the case of the Poincaré brane. The value of the effective potential of the graviton in the case of the thick de Sitter brane approaches $\frac{9}{4}H^2$ as $y\to\pm\infty$ (where H is the Hubble constant). This phenomenon can be seen in the analysis of the thin brane, too [24,27].

E. Thick anti–de Sitter brane model

Finally, we construct the warp factor of the thick anti–de Sitter brane as follows:

$$a^2(z)=\left[\left(1+\frac{1}{\sinh^{-2n}(y_0+y)+\sinh^{-2n}(y_0-y)}\right)^{1/2n}-\epsilon\right]^2, \tag{2.33}$$

where ϵ is a constant which satisfies $0<\epsilon<1$. ϵ is necessary for the spacetime to become an exact AdS_5 only at infinity (i.e., $y=\pm\infty$). Without ϵ, we find that the spacetime also becomes an exact AdS_5 at $y=\pm y_0$. This will cause technical trouble in the analysis of the scalar perturbation discussed in Sec. III, and we have to calculate the fluctuations numerically in this case as well. We show the warp factor of the thick anti–de Sitter brane in Fig. 4 and the effective potential of the graviton propagating in this system in Fig. 5. Both of them diverge as $y\to\infty$, which is different from the previous two cases.

III. STABILITY ANALYSIS

A. Perturbation of scalar-gravity coupled systems

In the previous section, we constructed three types of thick brane systems. Next, we have to investigate their stability. To do so, let us examine the scalar perturbation of these systems.

We use the following perturbed metric:

$$ds^2=(g_{MN}+\delta g_{MN})dx^M dx^N$$
$$=a^2(z)\{(1+2\phi)dz^2-2B_{|\mu}dz\,dx^\mu$$
$$+[(1+2\psi)\gamma_{\mu\nu}-E_{|\mu\nu}]dx^\mu dx^\nu\}. \tag{3.1}$$

In this paper, we use the longitudinal gauge $(B=E=0)$ and we get the following metric:

$$ds^2=a^2(z)[(1+2\phi)dz^2+(1+2\psi)\gamma_{\mu\nu}dx^\mu dx^\nu]. \tag{3.2}$$

Now we get equations for the scalar perturbation:

$$(z,z):\quad 3\gamma^{\rho\lambda}\psi_{|\rho\lambda}+12\mathcal{H}\psi'-12\mathcal{H}^2\phi+12K\psi$$
$$=\varphi_0'\delta\varphi'-\phi(\varphi_0')^2-a^2\frac{\partial V}{\partial\varphi_0}\delta\varphi, \tag{3.3}$$

$$(z,\mu):\quad -3\psi'_{|\mu}+3\mathcal{H}\phi_{|\mu}=\varphi_0'\delta\varphi_{|\mu}, \tag{3.4}$$

$$(\mu,\nu):\quad (3\psi''-6\mathcal{H}'\phi-3\mathcal{H}\phi'+9\mathcal{H}\psi'-6\mathcal{H}^2\phi$$
$$+\gamma^{\rho\lambda}\phi_{|\rho\lambda}+2\gamma^{\rho\lambda}\psi_{|\rho\lambda}+6K\psi)\delta^\mu_\nu-\gamma^{\mu\rho}\phi_{|\rho\nu}$$
$$-2\gamma^{\mu\rho}\psi_{|\rho\nu}$$
$$=\left(-\varphi_0'\delta\varphi'+\phi(\varphi_0')^2-a^2\frac{\partial V}{\partial\varphi_0}\delta\varphi\right)\delta^\mu_\nu, \tag{3.5}$$

$$\text{matter:}\quad \delta\varphi''+3\mathcal{H}\delta\varphi'+(4\psi'-\phi'-6\mathcal{H}\phi)\varphi_0'$$
$$-2\phi\varphi''+\gamma^{\rho\lambda}\delta\varphi_{|\rho\lambda}=a^2\frac{\partial^2 V}{\partial\varphi_0^2}\delta\varphi, \tag{3.6}$$

where the vertical bar denotes the covariant derivative with respect to $\gamma_{\mu\nu}$ just as in Sec. II, and we split φ into its background φ_0 and its perturbations $\delta\varphi$. Now let us derive the master equation of this system. At first, from Eq. (3.4), we get

$$\delta\varphi=\frac{1}{\varphi_0'}(-3\psi'+3\mathcal{H}\phi), \tag{3.7}$$

and from the off-diagonal part of Eq. (3.5), we get

$$\phi+2\psi=0. \tag{3.8}$$

Substituting Eqs. (3.6), (3.7), and (3.8) into Eqs. (3.3) and (3.5), we find the master equation of the system,

$$\psi''+\gamma^{\rho\lambda}\psi_{|\rho\lambda}+\left(3\mathcal{H}-2\frac{\varphi_0''}{\varphi_0'}\right)\psi'+\left(4\mathcal{H}'-4\mathcal{H}\frac{\varphi_0''}{\varphi_0'}+6K\right)\psi$$
$$=0. \tag{3.9}$$

We transform this equation into a form of the Schrödinger equation in order to examine the stability of this system. To do so, we define a new function $F(z,x^\mu)$ as

$$\psi(z,x^\mu)=\frac{\varphi_0'(z)}{a(z)^{3/2}}F(z,x^\mu). \tag{3.10}$$

We get the Schrödinger-type equation for the scalar perturbation,

$$-F''(z,x^\mu)+V_e(z)\cdot F(z,x^\mu)=\gamma^{\rho\lambda}F(z,x^\mu)_{|\rho\lambda}, \tag{3.11}$$

where V_e is the effective potential for the perturbation and its concrete form becomes

$$V_e=-\tfrac{5}{2}\mathcal{H}'+\tfrac{9}{4}\mathcal{H}^2+\mathcal{H}\frac{\varphi_0''}{\varphi_0'}-\frac{\varphi_0'''}{\varphi_0'}+2\left(\frac{\varphi_0''}{\varphi_0'}\right)^2-6K. \tag{3.12}$$

We analyze the effective potentials of three types of branes separately.

B. Thick Poincaré brane case

In the case of the thick Poincaré brane, we can expand $F(z,x^\mu)$ as follows:

$$F(z,x^\mu)=\int\frac{d^4p}{(\sqrt{2\pi})^4}f_p(z)e^{ip_\mu x^\mu}, \tag{3.13}$$

and we find that the equation for $f_p(z)$ becomes

$$-f_p''(z)+V_e(z)\cdot f_p(z)=m^2 f_p(z), \tag{3.14}$$

where m is the four-dimensional mass which satisfies $m^2=-p^2$. In the four-dimensional flat spacetime, we have the relation $p^2=-\omega^2+k^2$. Here, ω is the eigenvalue of the time direction and k is the norm of the three-dimensional momentum. If there is only time dependence in this system (i.e., $k=0$), $\omega=m$. So if there is a solution which has an imaginary m, we can say that this system is unstable.

Next we examine the effective potential V_e. In this system, V_e becomes

$$V_e=\frac{e^{2y_0}[(16n^2+16n+3)e^{4ny}+32n^2+80n-6+(16n^2+16n+3)e^{-4ny}]}{4(e^{2ny}+e^{-2ny})^{2+(1/n)}}. \tag{3.15}$$

Clearly, V_e is positive definite and it approaches 0 as $y\to\pm\infty$.[5] The solution which has an imaginary m cannot exist because it will necessarily diverge either at $y=\infty$ or at $y=-\infty$. So we can conclude that the thick Poincaré brane is stable under the scalar perturbation. It is interesting that there is no bound state on the brane, which is different from the tensor perturbation. The thickness parameter n determines the height of the effective potential. The effective potential becomes higher as n becomes larger. In the thin brane limit (i.e., $n\to\infty$), the height of the effective potential goes to infinity, and the thin brane becomes a singular object which has no width (see Fig. 6).

C. Thick de Sitter brane case

In the case of the thick de Sitter brane, we expand $F(z,x^\mu)$ as follows:

$$F(z,x^\mu)=\int d^3\mathbf{k}\,dm\ \ f_m(z)g_{km}(t)e^{i\mathbf{kx}}. \tag{3.16}$$

Here we introduce the four-dimensional mass m through the equation for $g_{km}(t)$ as follows:

$$\ddot g_{km}(t)+3H\dot g_{km}(t)+(\mathbf{k}^2e^{-2t}+m^2)g_{km}(t)=0, \tag{3.17}$$

where a dot denotes the derivative with respect to t, and H is the Hubble constant. Then solving Eq. (3.17), we find that g_{km} becomes

$$g_{km}(\eta)=\frac{\sqrt{\pi}}{2}H(-\eta)^{3/2}e^{-\pi\beta/2}H^{(1)}_{i\beta}(-k\eta), \tag{3.18}$$

where $H^{(1)}$ is the Hankel function of the first kind.[6] Here, η is conformal time defined as

$$\eta\equiv -e^{-t} \tag{3.19}$$

and β is defined as follows:

$$\beta\equiv\sqrt{m^2-\tfrac{9}{4}}. \tag{3.20}$$

In order to investigate the time evolution of the scalar perturbation, we examine $g_{km}(\eta)$ for various m. At first, we expand $g_{km}(\eta)$ near $\eta\sim 0(t\to\infty)$ as

$$\begin{aligned}g_{km}(\eta)=\frac{\sqrt{\pi}}{2}e^{-\pi\beta/2}\Bigg[&\frac{k^{i\beta}}{2^{i\beta}\Gamma(1+i\beta)}\left(1+i\,\frac{\cos(i\beta\pi)}{\sin(i\beta\pi)}\right)\\&\times(-\eta)^{(3/2)+i\beta}-\frac{ik^{-i\beta}}{2^{-i\beta}\sin(i\beta\pi)\Gamma(1-i\beta)}\\&\times(-\eta)^{(3/2)-i\beta}\Bigg].\end{aligned} \tag{3.21}$$

For $m^2\geqslant\frac{9}{4}$, β becomes a real number, so g_{km} oscillates near $\eta\sim 0$ and we can say that g_{km} is stable. Next, for $0<m^2\leqslant\frac{9}{4}$, β becomes an imaginary number. Now we define a new variable ζ as

$$\beta=\sqrt{m^2-\tfrac{9}{4}}\equiv i\zeta. \tag{3.22}$$

The first term in the square brackets of Eq. (3.21) behaves as $(-\eta)^{(3/2)-\zeta}$ and the second term behaves as $(-\eta)^{(3/2)+\zeta}$. So both terms converge to 0 as η approaches 0 because ζ is $0\leqslant\zeta<\frac{3}{2}$. From the above discussion, we can conclude that the solution whose m^2 satisfies $0<m^2\leqslant\frac{9}{4}$ is also stable.

Finally, for $m^2\leqslant 0$, ζ becomes larger than $\frac{3}{2}$, so the first term of Eq. (3.21) does not converge to 0 when η goes to 0. This means that if there is a solution with $m^2\leqslant 0$, this system must be unstable.

Then we examine the effective potential for the thick de Sitter brane. We calculate it numerically, and the result is shown in Fig. 7. In this case, $y=\pm y_0$ is the horizon of AdS_5, so the spacetime is defined in the region $-y_0\leqslant y\leqslant y_0$. Figure 7 shows that the effective potential of the scalar perturbation is positive for $-y_0\leqslant y\leqslant y_0$ and we can conclude that there is no solution with $m^2\leqslant 0$ because it cannot be normalizable. So unstable solutions do not exist, and as a

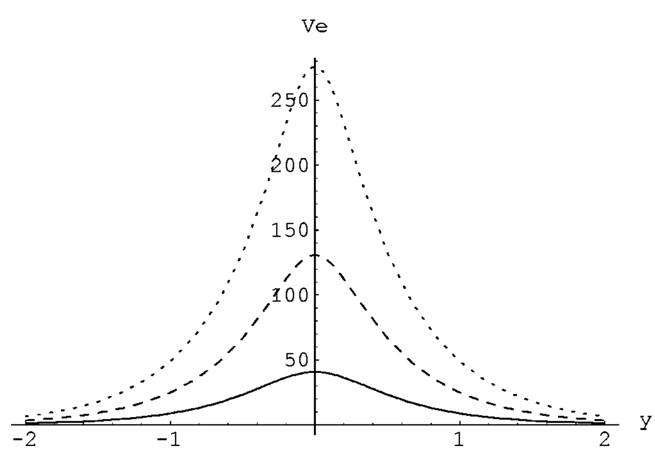

FIG. 6. The effective potential of the scalar perturbation in the case of the thick Poincaré brane. The solid line, dashed line, and dotted line denote the $n=1$, $n=2$, and $n=3$ case, respectively.

[5] Note that we can discuss using y instead of z because z is a monotonic function of y from Eq. (2.7).

[6] We set $H=1$ from now on.

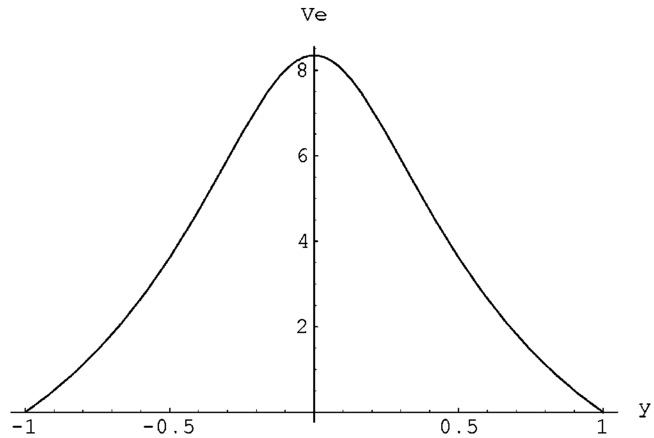

FIG. 7. The effective potential of the scalar perturbation in the case of the thick de Sitter brane. Here we set $n=1$, $y_0=1$.

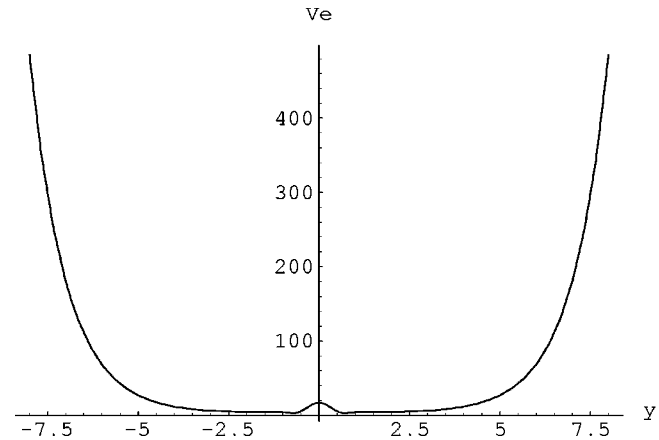

FIG. 8. The effective potential of the scalar perturbation in the case of the thick AdS brane. Here we set $n=1$, $y_0=1$, $\epsilon=0.4$.

result this system is stable under the scalar perturbation just as in the case of the thick Poincaré brane.

Notice that the shape of the effective potential is very different from that of the graviton or that of the free test scalar field. The effective potentials of the graviton or the free scalar field have holes in their potentials at the location of the branes. On the contrary, the effective potential of the scalar perturbation has no hole at the location of the brane.

Furthermore, the mass gap has disappeared in the effective potential of the scalar perturbation. There exist mass gaps in the effective potentials of the graviton or the free test scalar fields. That is, the effective potentials approach $\frac{9}{4}H^2$ (H is the Hubble constant), not 0, as $y\to\pm y_0$, as shown in Fig. 3. The disappearance of the mass gap is peculiar to the scalar perturbation for the thick de Sitter brane.

D. Thick anti–de Sitter brane case

Finally, let us consider the thick AdS brane case. In the case of the AdS brane, we decompose $F(z,x^\mu)$ in Eq. (3.11) as

$$F(z,x^\mu)=\int dm\, f_m(z)g_m(x^\mu), \tag{3.23}$$

and we get the following equation for $g_m(x^\mu)$:

$$\gamma^{\rho\lambda}g_m(x^\mu)_{|\rho\lambda}=m^2 g_m(x^\mu), \tag{3.24}$$

where m is the four-dimensional mass. It is known that Eq. (3.24) can be solved with suitable harmonic functions, and there is the Breitenlohner-Freedman bound which allows the tachyonic mass to some extent from the condition of the normalization [40–45]. From the Breitenlohner-Freedman bound, the mass m is bounded as

$$m^2\geqslant-\tfrac{9}{4}. \tag{3.25}$$

This means that even when there are solutions with $-\frac{9}{4}\leqslant m^2<0$, such solutions are stable in spite of the tachyonic mass.

Then we examine the effective potential in the case of the thick AdS brane. Substituting the warp factor (2.33) into Eq. (3.12) and setting $K=-1$, we can get V_e. We show the numerical result in Fig. 8.

Clearly, the effective potential is positive definite also in this case. As the effective potential is positive, there is no solution with $m^2<-\frac{9}{4}$, so we can conclude that the thick AdS brane is stable under the scalar perturbation as well as the thick Poincaré and the thick de Sitter brane.

Next we pay attention to the shape of the effective potential. The upheaval appears around $y=0$ just as in the case of the Poincaré and of the de Sitter brane. This upheaval demonstrates that the thick branes behave repulsively. But the effective potential in the AdS case diverges as $y\to\pm\infty$. This behavior is unique to the AdS brane system. This divergence is caused by the divergence of the warp factor at $y\to\pm\infty$.

Note that we treat the metric which coincides with an exact AdS_5 only at infinity. We may be able to consider the metric which coincides with an exact AdS_5 at finite y, but if we use such a metric, φ' becomes 0 at that point and the effective potential appears to diverge there. In fact, if we take $\epsilon=0$ in Eq. (2.33), we find that $\varphi'=0$ at $y=\pm y_0$ and the effective potential diverges there. But note that the warp factor itself does not diverge at $y=\pm y_0$. This shows that this divergence has a different origin from the divergence at infinity, and we can say that the divergence at $y=\pm y_0$ has no physical meaning. In fact, if we take a suitable variable and rewrite the effective potential with it, we will see no divergence at $y=\pm y_0$.

IV. CONCLUSIONS AND DISCUSSIONS

In this paper, we proposed three types of thick brane models and analyzed their stability. The three types of thick branes are the solutions of the Einstein equations with nontrivial dilatons and potentials. Furthermore, these solutions are nonsingular in the whole spacetime, even at the location of the brane.

At first, we considered gravitons in these thick brane systems. In all cases, the effective potentials of the gravitons show that there are bound states which localize around the brane. Such gravitons make the four-dimensional gravity on the brane if we take the thin brane and low-energy limit. Here, note that there are ambiguities with regard to estimating the corrections to the four-dimensional gravity due to the KK modes. This issue warrants further investigation.

Next we analyzed the scalar perturbations of the thick brane systems. We write the master equations of the scalar perturbations explicitly, and we get the effective potentials of the scalar perturbations in the three cases, respectively. As a result, we see that all of the thick branes are stable under the scalar perturbations. This is because the effective potentials of the scalar perturbations are positive definite. But the shape of the effective potentials of the scalar perturbations is very different from that of the gravitons. In all cases, there is no bound state because there is no hole around the location of the brane in the effective potentials of the scalar perturbations. Furthermore, there is an upheaval around the location of the brane. So we conclude that the three types of thick branes behave repulsively against the scalar perturbations.

The asymptotic behavior of the effective potential of the scalar perturbation in the case of the thick de Sitter brane is also different from that of the graviton or the free test scalar field propagating in the same background. As $y \to \pm y_0$, the effective potential of the graviton approaches $\frac{9}{4}H^2$, not 0. This mass gap can be seen in various analyses on de Sitter brane models [24]. But for the scalar perturbation in the thick de Sitter brane system, the mass gap disappeared. That is, the effective potential of the scalar perturbation approaches 0 as $y \to \pm y_0$. This phenomenon is peculiar to the scalar perturbation including the backreaction.

In addition, the shape of the effective potential in the case of the thick AdS brane is distinctive. It diverges as $y \to \pm\infty$, which is caused by the divergence of the warp factor at $y = \pm\infty$. In the case of the thick Poincaré brane or of the de Sitter brane, the warp factors do not diverge in the whole spacetimes.

Consequently, we have concluded that all of the three branes are stable under the scalar perturbations. But we should note that these analyses have been done classically. In the analogy of the two analyses on the Schwarzschild black holes (i.e., the analysis by Regge-Wheeler and the analysis by Hawking), the thick de Sitter brane might be unstable quantum mechanically. In fact, as the de Sitter spacetime has a temperature, the spacetime may radiate and get to the Poincaré spacetime [46]. So we may have to treat this system quantum mechanically. We leave this issue for future work.

Finally, we refer to the relation between the thickness and the noncommutativity. As we have mentioned, we can make smooth warp factors in various ways. For example, to construct the thick Poincaré brane, we can introduce another "thickness" parameter $\lambda\theta^2$,

$$a^2(z) = e^{2\alpha(y(z))} = \frac{1}{e^{2y} + \lambda\theta^2 e^{-2y}}. \tag{4.1}$$

If we set $\lambda\theta^2$ to 0, AdS_5 spacetime with the Poincaré slicing is recovered. Here we introduce the new variable ξ, which is defined as

$$e^{-2\xi} \equiv \sqrt{\lambda\theta^2} e^{-2y}. \tag{4.2}$$

Using ξ, we can rewrite the warp factor (4.1) into the following form:

$$a^2(\xi) = \frac{1}{\sqrt{\lambda\theta^2}} \cdot \frac{1}{e^{2\xi} + e^{-2\xi}}. \tag{4.3}$$

So Eq. (4.1) coincides with Eq. (2.17) with $n=1$ and $e^{2y_0} = 1/\sqrt{\lambda\theta^2}$. From the above equation, we can say that Eq. (4.1) is one of the warp factors of the thick Poincaré brane models and that $\lambda\theta^2$ is one of the parameters which determine the thickness.

On the other hand, $\lambda\theta^2$ seems to be related to the noncommutative geometry. From the discussion of the AdS/CFT correspondence, it is known that there is the classical solution of the supergravity which corresponds to $\mathcal{N}=4$ super-Yang-Mills theory in the noncommutative spacetime [47,48]. This classical solution is given by

$$ds^2 = dy^2 + \left(\frac{1}{e^{2y} + \lambda\theta^2 e^{-2y}} \right) \eta_{\mu\nu} dx^\mu dx^\nu + d\Omega_5^2, \tag{4.4}$$

where $d\Omega_5^2$ is the metric of S^5 and clearly Eq. (4.1) coincides with the warp factor of Eq. (4.4).

$\mathcal{N}=4$ super-Yang-Mills theory in the noncommutative spacetime is realized on the D3-branes with the nonzero expectation value of the B field. In this context, λ is the 't Hooft coupling of $\mathcal{N}=4$ super-Yang-Mills theory in the noncommutative spacetime and θ is the expectation value of the B field. So we can call $\lambda\theta^2$ the noncommutative parameter.

From the above discussion, we can interpret $\lambda\theta^2$ in two ways, that is, as the thickness parameter and as the noncommutative parameter. So we can expect that Eq. (4.1) contains some effects due to the noncommutativity.

There is another thing we should mention here. In this paper, we construct the thick de Sitter and the thick anti–de Sitter brane system. In the analogy of the case of the thick Poincaré brane, we cannot deny the possibility that the thick de Sitter and the thick anti–de Sitter brane systems also correspond to some field theories. If so, the corresponding theories may be the quantum field theories in the curved spacetime with the noncommutative coordinate. As we have not known of any theories in the noncommutative curved spacetime, this topic is very interesting.

In any case, we want to derive thick brane systems from the ten-dimensional supergravity or from the eleven-dimensional M theory. We expect that we might be able to interpret the noncommutativity in the context of these theories. We leave these themes for future work.

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