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Localization of metric fluctuations on scalar branes

Massimo Giovannini*
Institute of Theoretical Physics, University of Lausanne, CH-1015 Lausanne, Switzerland
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The localization of metric fluctuations on scalar brane configurations breaking spontaneously five-dimensional Poincaré invariance is discussed. Assuming that the four-dimensional Planck mass is finite and that the geometry is regular, it is demonstrated that the vector and scalar fluctuations of the metric are not localized on the brane.

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If internal dimensions are not compact [1–4] all the fields describing the fundamental forces of our four-dimensional world should be localized on a higher dimensional topological defect [1,2]. Among the various interactions an important role is played by gravitational forces [5,6]: the localization of the metric fluctuations can lead to measurable deviations of Newton's law at short distances [7]. In [6] it has been shown that the zero mode related to the tensor fluctuations of the geometry is localized, provided the four-dimensional Planck mass is finite. For a recent review see [8].

If the four-dimensional world is Poincaré invariant, the higher-dimensional geometry will have not only tensor modes but also scalar and vector fluctuations. The localization of the various modes of the geometry will be the subject of the present analysis. The invariance of the fluctuations under infinitesimal coordinate transformations, i.e. gauge invariance [9], is the tool which will be used in order to address this problem. Gauge invariance guarantees that the equations for the fluctuations of the geometry do not change when moving from one coordinate system to the other. Since the scalar and vector modes of the geometry do depend on the specific coordinate system, the gauge-invariant fluctuations corresponding to these modes should be constructed and analyzed. The form of these equations allows a model-independent discussion of the localization properties of the geometry without resorting to any specific coordinate system.

The following five-dimensional action[1]

$$S=\int d^5x\sqrt{|G|}\left[-\frac{R}{2\kappa}+\frac{1}{2}G^{AB}\partial_A\varphi\partial_B\varphi-V(\varphi)\right] \quad (1)$$

can be used in order to describe the breaking of five-dimensional Poincaré symmetry. Consider a potential which is invariant under the $\varphi\to-\varphi$ symmetry. Then, non-singular domain-wall solutions can be obtained, for various potentials, in a metric

$$ds^2=\overline{G}_{AB}dx^A dx^B=a^2(w)[dt^2-d\vec{x}^2-dw^2]. \quad (2)$$

For instance, solutions of the type

$$a(w)=\frac{1}{\sqrt{b^2w^2+1}}, \quad \varphi=\varphi(w)=\sqrt{6}\arctan bw, \quad (3)$$

can be found for various classes of symmetry breaking potentials [10–12]. Solutions like Eqs. (3) represent a smooth version of the Randall-Sundrum scenario [5,6]. The assumptions of the present analysis will now be listed.

(i) The five-dimensional geometry is regular (in a technical sense) for any value of the bulk coordinate w. This implies that singularities in the curvature invariants are absent.

(ii) Five-dimensional Poincaré invariance is broken through a smooth five-dimensional domain-wall solution generated by a potential $V(\varphi)$ which is invariant under $\varphi\to-\varphi$. The warp factor $a(w)$ will then be assumed symmetric for $w\to-w$.

(iii) Four-dimensional Planck mass is finite because the following integral converges:

$$M_P^2\simeq M^3\int_{-\infty}^{\infty}dw\,a^3(w). \quad (4)$$

(iv) Five-dimensional gravity is described according to Eq. (1) and, consequently, the equations of motion for the warped background generated by the smooth wall are, in natural gravitational units,

$$\varphi'^2=6(\mathcal{H}^2-\mathcal{H}'), \quad Va^2=-3(3\mathcal{H}^2+\mathcal{H}'), \quad (5)$$

where the prime denotes derivation with respect to w and $\mathcal{H}=a'/a$. Using only the assumptions (i)–(iv) it will be shown that the gauge-invariant fluctuations corresponding to scalar and vector modes of the geometry are not localized on the wall, while the tensor modes of the geometry will be shown to be localized.

In the literature there exist different approaches to metric fluctuations in brane cosmology but the present approach is different. In [13–17] fluctuations have been decomposed according to three-dimensional rotations (as previously done in a Kaluza-Klein context [18]): decoupled fluctuations have not been obtained and scalar sources have not been considered. In the present case, fluctuations are decomposed using 4D Poincaré transformations and the full system is disentangled by exploiting gauge invariance. Sometimes scalar fluctuations (but not vector) are analyzed in the context of

*Electronic address: massimo.giovannini@ipt.unil.ch

[1]Latin (uppercase) indices run over the five-dimensional space-time whereas Greek indices run over the four-dimensional space-time. Notice that natural gravitational units $2\kappa=16\pi G_5=16\pi/M_5^3=$ will be often used.

the radion problem but within specific gauges (often adapted to the particular form of the metric) and in two (or even three [19]) brane systems (see also [20] and references therein). In the present case we deal only with one brane but the analysis, as stressed, holds for general backgrounds and independently on the coordinate system. This discussion differs from the one of [21] where a specific gauge has been selected and the thin brane case was only investigated.

The entangled equations for the fluctuations can be written in general terms as

$$\delta R_{AB}=\frac{1}{2}\partial_A\varphi\partial_B\chi+\frac{1}{2}\partial_A\chi\partial_B\varphi-\frac{1}{3}\frac{\partial V}{\partial\varphi}\chi\overline{G}_{AB}-\frac{V}{3}\delta G_{AB}, \tag{6}$$

$$\delta G^{AB}(\partial_A\partial_B\varphi-\overline{\Gamma}^C_{AB}\partial_C\varphi)+\overline{G}^{AB}(\partial_A\partial_B\chi-\overline{\Gamma}^C_{AB}\partial_C\chi-\delta\Gamma^C_{AB}\partial\varphi)+\frac{\partial^2 V}{\partial\varphi^2}\chi=0, \tag{7}$$

where the metric and the scalar field have been separated into their background and perturbation parts:

$$G_{AB}(x^\mu,w)=\overline{G}_{AB}(w)+\delta G_{AB}(x^\mu,w),$$

$$\varphi(x^\mu,w)=\varphi(w)+\chi(x^\mu,w). \tag{8}$$

In Eqs. (6) and (7), $\delta\Gamma^C_{AB}$ and δR_{AB} are, respectively, the fluctuations of the Christoffel connections and of the Ricci tensors, whereas $\overline{\Gamma}^C_{AB}$ are the values of the connections computed using the background metric (2). In Eq. (6) and (7), the fluctuations of the brane source are coupled to the scalar fluctuations of the geometry whose modes are classified using Poincaré invariance in four dimensions

$$\delta G_{AB}=a^2(w)\begin{pmatrix} 2h_{\mu\nu}+(\partial_\mu f_\nu+\partial_\nu f_\mu)+2\eta_{\mu\nu}\psi+2\partial_\mu\partial_\nu E & D_\mu+\partial_\mu C \\ D_\mu+\partial_\mu C & 2\xi \end{pmatrix}. \tag{9}$$

On top of $h_{\mu\nu}$ which is divergenceless and traceless [i.e. $\partial_\mu h^\mu_\nu=0$, $h^\mu_\mu=0$] there are four scalars (i.e. E, ψ, ξ and C) and two divergenceless vectors (D_μ and f_μ). For infinitesimal coordinate transformations $x^A\rightarrow\tilde{x}^A=x^A+\epsilon^A$ the tensors are invariant whereas the vectors and the scalars transform non-trivially. The four-dimensional part of the infinitesimal shift $\epsilon_A=a^2(w)(\epsilon_\mu,-\epsilon_w)$ can be decomposed as $\epsilon_\mu=\partial_\mu\epsilon+\zeta_\mu$, where ζ_μ is a divergenceless vector and ϵ is a scalar.

The transformations for the scalars involve two gauge functions ϵ and ϵ_w. The transformations for the vectors involve ζ_μ. Two scalars and one divergenceless vector can be gauged away by fixing the scalar and the vector gauge functions. In a different perspective [9], since there are two scalar gauge functions and four scalar fluctuations of the metric (9), two gauge-invariant (scalar) variables can be defined. In the present case the gauge-invariant scalar variables can be usefully chosen to be

$$\Psi=\psi-\mathcal{H}(E'-C),$$

$$\Xi=\xi-\frac{1}{a}[a(C-E')]'. \tag{10}$$

By shifting infinitesimally the coordinate system from x^A to $\tilde{x}^A$ the metric fluctuations change as

$$\delta G_{AB}(x^\mu,w)\rightarrow\delta\tilde{G}_{AB}=\delta G_{AB}-\nabla_A\epsilon_B-\nabla_B\epsilon_A, \tag{11}$$

where the covariant derivatives are computed using the backround metric of Eq. (2). In spite of this, $\tilde{\Psi}=\Psi$, $\tilde{\Xi}=\Xi$ and $\tilde{h}_{\mu\nu}=h_{\mu\nu}$. The scalar field fluctuation of Eq. (8) is not gauge-invariant and the gauge-invariant variable associated with it is

$$X=\chi-\varphi'(E'-C). \tag{12}$$

Since there is one vector gauge function, i.e. ζ_μ, one gauge-invariant variable can be constructed out of D_μ and f_μ:

$$V_\mu=D_\mu-f'_\mu. \tag{13}$$

The choice of gauge-invariant fluctuations proposed in this paper is relevant: in terms of the variables defined in Eqs. (10)–(12) and (13) the perturbed system of Eqs. (6) and (7) can be written (and decoupled) with a procedure which is gauge-invariant at every step. The equation for the tensors, as expected, decouples from the very beginning:

$$\mu''_{\mu\nu}-\partial_\alpha\partial^\alpha\mu_{\mu\nu}-\frac{(a^{3/2})''}{a^{3/2}}\mu_{\mu\nu}=0. \tag{14}$$

where $\mu_{\mu\nu}=a^{3/2}h_{\mu\nu}$ is the canonical normal mode of the of the action (1) perturbed to second order in the amplitude of tensor fluctuations.

The scalar variables (10) and (12) form a dynamical system defined by the diagonal components of Eq. (6),

$$\Psi''+7\mathcal{H}\Psi'+\mathcal{H}\Xi'+2(\mathcal{H}'+3\mathcal{H}^2)\Xi+\frac{1}{3}\frac{\partial V}{\partial\varphi}a^2X-\partial_\alpha\partial^\alpha\Psi=0, \tag{15}$$

$$-\partial_\alpha\partial^\alpha\Xi-4[\Psi''+\mathcal{H}\Psi']-4\mathcal{H}\Xi'-\varphi'X'-\frac{1}{3}\frac{\partial V}{\partial\varphi}a^2X+\frac{2}{3}Va^2\Xi=0, \tag{16}$$

supplemented by the gauge-invariant version of the perturbed scalar field equation (7)

$$\partial_\alpha\partial^\alpha X-X''-3\mathcal{H}X'+\frac{\partial^2V}{\partial\varphi^2}a^2X-\varphi'[4\Psi'+\Xi']-2\Xi(\varphi''+3\mathcal{H}\varphi')=0, \tag{17}$$

and subjected to the constraints

$$\partial_\mu\partial_\nu[\Xi-2\Psi]=0, \tag{18}$$

$$6\mathcal{H}\,\Xi+6\Psi'+X\,\varphi'=0. \tag{19}$$

coming from the off-diagonal components [i.e. (μ,w)] of Eq. (6). From Eq. (6) the evolution of the gauge-invariant vector variable (13) is

$$\partial_\alpha\partial^\alpha\mathcal{V}_\mu=0,\quad \mathcal{V}_\mu'+\frac{3}{2}\mathcal{H}\mathcal{V}_\mu=0, \tag{20}$$

where $\mathcal{V}_\mu=a^{3/2}V_\mu$ is the canonical normal mode of the action (1) perturbed to second order in the amplitude of vector fluctuations of the metric.

Using repeatedly the constraints of Eqs. (18) and (19), together with the background relations (5), the scalar system can be reduced to the following two equations:

$$\Phi''-\partial_\alpha\partial^\alpha\Phi-z\left(\frac{1}{z}\right)''\Phi=0, \tag{21}$$

$$\mathcal{G}''-\partial_\alpha\partial^\alpha\mathcal{G}-\frac{z''}{z}\mathcal{G}=0, \tag{22}$$

where

$$\Phi=\frac{a^{3/2}}{\varphi'}\Psi,\quad \mathcal{G}=a^{3/2}X-z\Psi. \tag{23}$$

The same equation satisfied by Ψ is also satisfied by Ξ by virtue of Eq. (18). In Eqs. (22) and (23) the background dependence appears only in terms of the ‘‘universal’’ function $z(w)$:

$$z(w)=\frac{a^{3/2}\varphi'}{\mathcal{H}}. \tag{24}$$

In deriving Eqs. (21), (22) and (23), (24) no specific background dependence has been assumed, but only the equations of motion, i.e. Eqs. (5), which come directly from Eq. (1) and hold for any choice of the potential generating the scalar brane configuration. The effective Schrödinger-like ‘‘potentials’’ appearing in Eqs. (21) and (22) are dual with respect to $z\to1/z$. Finally, the gauge-invariant function $\mathcal{G}$ is the normal mode of the action perturbed to second order in the amplitude of the scalar fluctuations (see, for comparison, also [18] which deals with the case of compact extra dimensions).

The lowest mass eigenstate of Eq. (14), i.e. $\mu(w)=\mu_0a^{3/2}(w)$, is normalized if

$$|\mu_0|^2\int_{-\infty}^{\infty}a^3dw=2|\mu_0|^2\int_0^{\infty}a^3(w)dw=1, \tag{25}$$

where the $w\to-w$ symmetry of the background geometry [see (ii)] has been exploited. Using now the assumptions (i), (ii) and (iii) the tensor zero mode turns out to be normalizable [5,6].

Equation (20) indicates that the vector fluctuations are always massless and the corresponding zero mode, i.e. $\mathcal{V}(w)\sim\mathcal{V}_0a^{-3/2}$, is localized if

$$2|\mathcal{V}_0|^2\int_0^{\infty}\frac{dw}{a^3(w)}=1, \tag{26}$$

which cannot be true if assumptions (i), (ii) and (iii) hold.

From Eq. (21) the lowest mass eigenstate of the metric fluctuation, i.e. $\Phi(w)=\Phi_0z^{-1}(w)$, is normalized if

$$2|\Phi_0|^2\int_0^{\infty}\frac{dw}{z^2(w)}=1. \tag{27}$$

The integrand appearing in Eq. (27) is non-convergent at infinity if the geometry is regular in the same limit. In fact, according to assumption (i), the curvature invariants pertaining to the geometry (2) [i.e. R^2, $R_{MNAB}R^{MNAB}$ and $R_{MN}R^{MN}$] should be regular for any w and, in particular, at infinity. Since $a(w)$ must converge at infinity, $a(w)\sim w^{-\gamma}$ with[2] $1/3\leq\gamma\leq1$. Notice that $\gamma\geq1/3$ comes from the convergence (at infinity) of the integral of Eq. (4) and that $\gamma\leq1$ is implied by the regularity of the curvature invariants since, at infinity, $R_{MN}R^{MN}\sim R_{MNAB}R^{MNAB}\sim w^{4(\gamma-1)}$ should converge. Using Eq. (24) and Eqs. (5) the integrand of Eq. (27) can be written as

$$\frac{1}{z^2}=\frac{\mathcal{H}^2}{a^3\varphi'^2}=\frac{1}{6a^3}\left(\frac{\mathcal{H}^2}{\mathcal{H}^2-\mathcal{H}'}\right). \tag{28}$$

Assuming the regularity of the curvature invariants at infinity [i.e. $a(w)\sim w^{-\gamma}$ with $0<\gamma\leq1$] it can be demonstrated that $1/z^2$ diverges, at infinity, *at least*[3] as a^{-3}.

The second (linearly independent) solution to Eq. (21) which is given by $z^{-1}(w)\int^w z^2(x)\,dx$ has poles at infinity and for $w\to0$. The poles appearing for $w\to0$ will now be

[2]The power γ measures only the degree of convergence of a given integral.

[3]In fact, $\gamma=1$, $1/z^2$ diverges even more as it can be argued from Eq. (28) which has a further pole for $\gamma^2=\gamma$. The example given in Eqs. (3) corresponds to a behavior at infinity given by $\gamma=1$ and a direct calculation shows that $1/z^2$ diverges, in this case, as w^5.

discussed since they are needed in order to prove that the zero mode of Eq. (22), i.e. $\mathcal{G}(w)=\mathcal{G}_0 z(w)$, is not localized.

Provided the assumptions (i)–(iv) are satisfied, the integral

$$2|\mathcal{G}_0|^2\int_0^\infty z^2 dw \tag{29}$$

is divergent not because of the behavior at infinity but because of the behavior of the solution close to the core of the wall, i.e. $w\to 0$. According to assumptions (i) and (ii) $a(w)$ and φ should be regular for any w and close to the core of the wall $\varphi\to 0$ and $a(w)\to$const because of the $w\to -w$ symmetry. Hence the following regular expansions can be written for small w

$$a(w)\simeq a_0-a_1 w^{\beta}+\cdots,\quad \beta>0, \tag{30}$$

$$\varphi(w)\simeq\varphi_1 w^{\alpha}+\cdots,\quad \alpha>0, \tag{31}$$

for $w\to 0$. Inserting the expansion (30), (31) into Eqs. (5) the following relations can be obtained:

$$\beta=2\alpha,\quad \alpha^2\varphi_1^2=6\frac{a_1}{a_0}\beta(\beta-1). \tag{32}$$

Inserting now Eqs. (30), (31) into Eq. (24) and exploiting the first of Eqs. (32) we have

$$\lim_{w\to 0} z^2(w)\simeq w^{2(\alpha-\beta)}=w^{-2\alpha},\quad \alpha>0 \tag{33}$$

Using Eqs. (30) into the curvature invariants we get that $R_{AB}R^{AB}\sim R_{MNAB}R^{MNAB}\sim w^{2\beta-4}$, which implies $\beta\geqslant 2$ in order to have regular invariants for $w\to 0$. Since, from Eq. (32), $\beta=2\alpha$, in Eq. (33) it must be $\alpha\geqslant 1$. As in the case of Eq. (21), Eq. (22) also has a second (linearly independent) solution for the lowest mass eigenvalue, namely $z(w)\int^w dx\, z^{-2}(x)$ which has poles at infinity. In fact, a direct check shows that, at infinity, this quantity goes as $w^{(3/2)\gamma+1}$ where, as usual, $1/3\leqslant\gamma\leqslant 1$ for the convergence of the Planck mass and of the curvature invariants at infinity.

The gauge-invariant techniques developed in the thick brane limit can be compared with what happens in the thin brane limit. Indeed it has been shown [10,11] that the action of Eq. (1) can be viewed as the thick version of a thin brane action of the type

$$S_{\rm thin}=\int d^5x\sqrt{|G|}[-R-\Lambda]-\frac{\lambda}{2}\int d^4\rho\sqrt{|\gamma|}\,\gamma^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N G_{MN} \tag{34}$$

where $\gamma_{\alpha\beta}$ is the induced metric, λ is the brane tension and Λ is the bulk cosmological constant. Then the equations of motion which have to be perturbed are

$$R_{AB}=-\frac{\Lambda}{3}G_{AB}+\frac{\lambda}{2}\int d^4\rho\frac{\sqrt{|\gamma|}}{\sqrt{|G|}}\gamma^{\alpha\beta}\partial_\alpha X^M\partial\beta X^N\times\left(G_{AM}G_{BN}-\frac{1}{3}G_{MN}G_{AB}\right)\delta(w), \tag{35}$$

where the delta function comes about since, in the variation, the brane location is fixed. From Eq. (35) the background equations are simply

$$6(\mathcal{H}^2-\mathcal{H}')=\lambda a\,\delta(w),$$
$$6[3\mathcal{H}^2+\mathcal{H}']+2\Lambda a^2+\lambda a\,\delta(w)=0, \tag{36}$$

which can be solved, in particular, by $a(w)=(b|w|+1)^{-1}$ with $b=\sqrt{-\Lambda/12}$. By consistently perturbing Eq. (35) and using Eqs. (36) in the obtained equations for the fluctuations it is found that, also in the thin brane case, the normalizability conditions for the tensor and for the gauge-invariant vectors (i.e. $\mathcal{V}_\mu$) are exactly the same as the ones obtained in the thick case. This conclusion comes from the tensor and vector projection of the perturbed (μ,ν) component of Eq. (35). Moreover, from the perturbed vector projection of the (μ,w) component of Eq. (35), it is found that the gauge-invariant vector is massless. Hence, if the four-dimensional Planck mass is finite, the gauge-invariant vector fluctuations are not normalizable. The only scalar fluctuations appearing in the thin brane limit are the ones associated with Ψ. In this case the scalar projection of the (μ,w) component of Eqs. (35) leads to the constraint $\Psi'+2\mathcal{H}\Psi=0$ from which $\Psi\sim a^{-2}$. Using this constraint into the other dynamical equations together with the background relations (36) we get that the equation for $\overline{\Psi}=a^{3/2}\Psi$ is

$$\overline{\Psi}''-\partial_\alpha\partial^\alpha\overline{\Psi}-\frac{(a^{-1/2})''}{a^{-1/2}}\overline{\Psi}=0. \tag{37}$$

Hence the integral which should converge is in this case $\int dw\,a(w)^{-1}$, which never converges if the 4D Planck mass is finite.

In conclusion, it has been demonstrated that under assumptions (i)–(iv) the scalar and vector fluctuations of the five-dimensional metric decouple from the wall. Heeding experimental tests [7], the present results suggest that under the assumptions (i)–(iv) no vector or scalar component of the Newtonian potential at short distances should be expected.

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