Gravitational waves from inspiraling compact binaries: Energy flux to third post-Newtonian order

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The multipolar-post-Minkowskian approach to gravitational radiation is applied to the problem of the generation of waves by the compact binary inspiral. We investigate specifically the third post-Newtonian (3PN) approximation in the total energy flux. The new results are the computation of the mass quadrupole moment of the binary to the 3PN order, and the current quadrupole and mass octupole to the 2PN order. Wave tails and tails of tails in the far zone are included up to the 3.5PN order. The recently derived 3PN equations of binary motion are used to compute the time derivatives of the moments. We find perfect agreement to the 3.5PN order with perturbation calculations of black holes in the test-mass limit for one body. Technical inputs in our computation include a model of point particles for describing the compact objects, and the Hadamard self-field regularization. Because of a physical incompleteness of the Hadamard regularization at the 3PN order, the energy flux depends on one unknown physical parameter, which is a combination of a parameter λ in the equations of motion, and a new parameter θ coming from the quadrupole moment.

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I. INTRODUCTION

Inspiraling compact binaries are systems of two neutron stars and/or black holes undergoing an adiabatic orbital decay by gravitational radiation emission. These systems constitute an important target for the gravitational-wave detectors such as the Laser Interferometric Gravitational Wave Observatory (LIGO) and VIRGO. The currently favored theory for describing the binary inspiral is the post-Newtonian approximation. Since inspiraling compact binaries are very relativistic the Newtonian description (corresponding to the quadrupole approximation) is grossly inadequate for constructing the theoretical templates to be used in the signal analysis of detectors. In fact, from several measurement-accuracy analyses [1–9] it follows that the third post-Newtonian (3PN) approximation, corresponding to the order $1/c^6$ when the speed of light $c \to +\infty$, constitutes a necessary achievement in this field. Note that the 3PN approximation is needed to compute the time evolution of the binary's orbital phase, that depends via an energy balance equation on the total gravitational-wave energy flux. The energy flux is therefore a crucial quantity to predict.

Following the earliest computations at the 1PN level [10,11] (at a time where post-Newtonian corrections were of purely academic interest), the energy flux generated by compact binaries was determined to the 2PN order [12–16], by means of a formalism based on multipolar and post-Minkowskian approximations [17–21], and independently using a direct integration of the relaxed Einstein equations [14,22,23] (see also Refs. [24,25]). Since then the calculations have been extended to include the nonlinear effects of tails at higher post-Newtonian orders. The tails at the 2.5PN and 3.5PN orders were computed in Refs. [26,27] (this extended the computation of tails at the dominant 1.5PN order [28–30]), and the contribution of tails generated by the tails themselves (so-called "tails of tails") at the 3PN order was obtained in Ref. [27]. However, unlike the 1.5PN, 2.5PN and 3.5PN orders that are entirely composed of tail terms, the 3PN approximation involves also, besides the tails of tails, many non-tail contributions coming from the relativistic corrections in the multipole moments of the binary.

The present paper is devoted to the computation of the multipole moments, chiefly the quadrupole moment at the 3PN order, in the case where the binary's orbit is circular (the relevant case for most inspiraling binaries). We reduce some general expressions for the multipole moments of a slowly-moving extended system [21] to the case of a point-particle binary at the 3PN order. The self-field of point-particles is systematically regularized by means of Hadamard's concept of "partie finie" [31–33]. The time-derivatives of the 3PN quadrupole moment are computed with the help of the equations of binary motion at the 3PN order in harmonic coordinates (the coordinate system chosen for this computation). The 3PN equations of motion have been derived recently by two groups working independently with different methods: Arnowitt-Deser-Misner- (ADM-) Hamiltonian formulation of general relativity [34–38], and direct post-Newtonian iteration of the field equations in harmonic coordinates [40–44]. There is complete physical equivalence between the results given by the two approaches [38,44]. We shall find that our end result for the energy flux at the 3.5PN order is in perfect agreement, in the test-body limit for one body, with the result of black-hole perturbation theory, which is currently known up to the higher 5.5PN approximation [45–47] (see Ref. [48] for a review). In a separate work [49] we report the computation of the 3.5PN-

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accurate orbital phase which constitutes the crucial component of the theoretical template of inspiraling binaries.

One conclusion of the investigation of the equations of motion of compact binaries is that from the 3PN order the model of point-particles (described by Dirac distributions) might become physically incomplete, in the sense that the equations involve one undetermined coefficient, $\omega_{\rm static}$ in the ADM-Hamiltonian formalism [34–38] (see, however, [39]) and λ in the harmonic-coordinate approach [40–44]. Technically this is due to some subtle features of the self-field regularization in the manner of Hadamard. In the present paper, we shall be led to introduce a second undetermined coefficient, called θ, coming from our computation of the 3PN quadrupole moment. However, we shall find that the total energy flux contains only one unknown parameter, which is a certain linear combination of θ and λ entering the 3PN coefficient. All other terms in the flux up to the 3.5PN order are completely specified.

The plan of this paper is as follows. Sections II–IV are devoted to the basic expressions of the moments we shall apply. Section V presents the needed information concerning our point-particle model, and Secs. VI–IX deal with the computation of all the different types of terms in the required multipole moments. Section X explains our introduction of the θ-ambiguity. Finally we present our results for the moments and energy flux in Secs. XI and XII. The intermediate values for all the terms composing the moments in the case of circular orbits are relegated to the Appendix.

II. EXPRESSIONS OF THE MULTIPOLE MOMENTS

In this section we give a short summary of the expressions of multipole moments in the post-Newtonian approximation. The moments describe some general isolated sources that are weakly self-gravitating and slowly-moving, i.e., whose internal velocities are much smaller than the speed of light: $v \ll c$. In this paper we order all expressions according to the formal order in $1/c$, and we pose $\mathcal{O}(n) \equiv \mathcal{O}(1/c^n)$. In addition, the moments are *a priori* valid only in the case where the source is continuous (for instance a hydrodynamical fluid); however, we shall apply these moments to the case of point-particles by supplementing the above expressions with a certain regularization ansatz based on Hadamard's concept of ''partie finie'' [31–33]. We adopt a system of harmonic coordinates, which means

$$\partial_\nu h^{\mu\nu} = 0, \tag{2.1a}$$

$$h^{\mu\nu} = |g|^{1/2} g^{\mu\nu} - \eta^{\mu\nu}, \tag{2.1b}$$

where $g^{\mu\nu}$ and g denote respectively the inverse and the determinant of the covariant metric $g_{\mu\nu}$, and where $\eta^{\mu\nu}$ denotes the Minkowski metric with signature $+2$. The Einstein field equations, relaxed by the harmonic-coordinate condition, take the form of d'Alembertian equations for all the components of the field variable,

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu}, \tag{2.2a}$$

$$\tau^{\mu\nu} \equiv |g| T^{\mu\nu} + \frac{c^4}{16\pi G} \Lambda^{\mu\nu}, \tag{2.2b}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and where we have introduced the effective stress-energy (pseudo-)tensor $\tau^{\mu\nu}$ of the matter and gravitational fields in harmonic coordinates. The matter stress-energy is described by $T^{\mu\nu}$ and the gravitational stress-energy by the nonlinear interaction term $\Lambda^{\mu\nu}$. The latter is given in terms of the metric by the exact expression

$$\begin{aligned}\Lambda^{\mu\nu} = & - h^{\rho\sigma}\partial^2_{\rho\sigma} h^{\mu\nu} + \partial_\rho h^{\mu\sigma}\partial_\sigma h^{\nu\rho} + \frac{1}{2} g^{\mu\nu} g_{\rho\sigma}\partial_\lambda h^{\rho\tau}\partial_\tau h^{\sigma\lambda} \\ & - g^{\mu\rho} g_{\sigma\tau}\partial_\lambda h^{\nu\tau}\partial_\rho h^{\sigma\lambda} - g^{\nu\rho} g_{\sigma\tau}\partial_\lambda h^{\mu\tau}\partial_\rho h^{\sigma\lambda} \\ & + g_{\rho\sigma} g^{\lambda\tau}\partial_\lambda h^{\mu\rho}\partial_\tau h^{\nu\sigma} + \frac{1}{8}(2 g^{\mu\rho} g^{\nu\sigma} - g^{\mu\nu} g^{\rho\sigma}) \\ & \times (2 g_{\lambda\tau} g_{\epsilon\pi} - g_{\tau\epsilon} g_{\lambda\pi})\partial_\rho h^{\lambda\pi}\partial_\sigma h^{\tau\epsilon}.\end{aligned} \tag{2.3}$$

Both the matter and gravitational contributions in $\tau^{\mu\nu}$ depend on the field h, with the gravitational term $\Lambda^{\mu\nu}$ being at least quadratic in h and its space-time derivatives.

The multipole moments of slowly-moving sources are in the form of some functionals of the (formal) post-Newtonian expansion of the pseudo-tensor $\tau^{\mu\nu}$; we denote the formal post-Newtonian expansion with an overbar, so $\overline{\tau}^{\mu\nu} = \mathrm{PN}(\tau^{\mu\nu})$. It is convenient to introduce the auxiliary notation

$$\Sigma = \frac{\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2}; \quad \Sigma_i = \frac{\overline{\tau}^{0i}}{c}; \quad \Sigma_{ij} = \overline{\tau}^{ij}. \tag{2.4}$$

From a general study [20,21] of the matching between the exterior gravitational field of the source and the inner post-Newtonian field, we obtain some ''natural'' definitions for the lth order mass-type (I_L) and current-type (J_L) multipole moments of the source. The physics of the isolated source, as seen in its exterior, is extracted from these multipole moments when they are connected, in a consistent way, to the observables of the radiative field at (Minkowskian) future null infinity, given in this formalism by the so-called radiative multipole moments. The connection between I_L and J_L and the mass-type (U_L) and current-type (V_L) radiative moments at infinity involves up to say the 3.5PN order many tail effects and even a particular ''tail-of-tail'' effect arising specifically at 3PN. All these effects are known [27] and therefore will not be investigated here but simply added at the end of our computation in Sec. XII. Here we focus our attention on the reduction to point-particle binaries of the general *source* multipole moments (in symmetric-tracefree form), whose complete expressions are given by

$$I_L(t)=\mathop{\mathrm{FP}}_{B=0}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\int_{-1}^{1}dz\Bigg\{\delta_l(z)\hat{x}_L\Sigma - \frac{4(2l+1)}{c^2(l+1)(2l+3)}\delta_{l+1}(z)\hat{x}_{iL}\dot{\Sigma}_i + \frac{2(2l+1)}{c^4(l+1)(l+2)(2l+5)}\delta_{l+2}(z)\hat{x}_{ijL}\ddot{\Sigma}_{ij}\Bigg\} \times(\mathbf{x},t+z|\mathbf{x}|/c), \tag{2.5a}$$

$$J_L(t)=\mathop{\mathrm{FP}}_{B=0}\varepsilon_{ab<i_l}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\int_{-1}^{1}dz\Bigg\{\delta_l(z)\hat{x}_{L-1>a}\Sigma_b - \frac{2l+1}{c^2(l+2)(2l+3)}\delta_{l+1}(z)\hat{x}_{L-1>ac}\dot{\Sigma}_{bc}\Bigg\} \times(\mathbf{x},t+z|\mathbf{x}|/c). \tag{2.5b}$$

Our notation is as follows. $L=i_1 i_2 \dots i_l$ is a multi-index composed of l indices; a product of l spatial vectors $x^i \equiv x_i$ is denoted $x_L = x_{i_1}x_{i_2}\dots x_{i_l}$; the symmetric-tracefree (STF) part of that product is denoted using a hat: $\hat{x}_L=\mathrm{STF}(x_L)$, for instance $\hat{x}_{ij}=x_ix_j-\frac{1}{3}\delta_{ij}$, $\hat{x}_{ijk}=x_ix_jx_k-\frac{1}{5}(x_i\delta_{jk}+x_j\delta_{ki}+x_k\delta_{ij})$; the STF projection is also denoted using angular brackets surrounding the indices, e.g., $\hat{x}_{ij}\equiv x_{\langle ij\rangle}$, $x_{\langle i}v_{j\rangle}=\frac{1}{2}(x_iv_j+x_jv_i)-\frac{1}{3}\delta_{ij}x_kv_k$; ε_{ijk} denotes the usual Levi-Civita symbol ($\varepsilon_{000}=+1$); the dots refer to the time differentiation. The matter densities Σ, Σ_i, and Σ_{ij} in Eqs. (2.5) are evaluated at the position $\mathbf{x}$ and at time $t+z|\mathbf{x}|/c$. The function $\delta_l(z)$ is given by

$$\delta_l(z)=\frac{(2l+1)!!}{2^{l+1}l!}(1-z^2)^l;\qquad \int_{-1}^{1}dz\,\delta_l(z)=1. \tag{2.6}$$

This function tends to the Dirac distribution when $l\to+\infty$. Each of the terms composing I_L and J_L is to be understood in the sense of post-Newtonian expansion, and computed using the (infinite) post-Newtonian series

$$\int_{-1}^{1}dz\,\delta_l(z)S(\mathbf{x},t+z|\mathbf{x}|/c) = \sum_{j=0}^{\infty}\frac{(2l+1)!!}{2^j j!(2l+2j+1)!!}|\mathbf{x}|^{2j}\left(\frac{\partial}{c\,\partial t}\right)^{2j}S(\mathbf{x},t). \tag{2.7}$$

Finally the symbol $\mathrm{FP}_{B=0}$ in front of the integrals in Eqs. (2.5) refers to a specific finite part operation defined by analytic continuation (see Ref. [21] for the details). Such a finite part is crucial because the integrals have a non-compact support due to the gravitational contribution in the pseudo-tensor, and would be otherwise divergent at infinity (when $|\mathbf{x}|\to+\infty$). The integral involves the regularization factor $|\tilde{\mathbf{x}}|^B=|\mathbf{x}/r_0|^B$, where B is a complex number and r_0 denotes an arbitrary length scale. It is defined by complex analytic continuation for any $B\in\mathbb{C}$ except at isolated poles in $\mathbb{Z}$, including in general the value of interest $B=0$. We expand the integral as a Laurent expansion when $B\to 0$ and pick up the finite part (in short $\mathrm{FP}_{B=0}$), or coefficient of the zeroth power of B in that expansion. This finite part is in fact equivalent to the Hadamard "partie finie" [31].

Thus, the moments depend *a priori* on the constant r_0 introduced in this analytic continuation process. This constant can be thought of as due to the "regularization" of the field at infinity; the moments will depend explicitly on r_0 when the integral develops a polar part at $B=0$ due to the behavior of the integrand when $|\mathbf{x}|\to+\infty$. As we shall see, the source moments start depending explicitly on r_0 from the 3PN order. However, we know that the metric is actually independent of r_0 [more precisely, r_0 cancels out between the two terms of the multipole expansion given by Eq. (3.11) in Ref. [21]]. Indeed, as a good check of the calculation, we shall see that because of non-linear tail effects in the wave zone the constant r_0 is canceled out, so the physical energy flux does not depend on it.

To the 1PN order the expressions (2.5a) and (2.5b) are equivalent to some alternative forms obtained earlier in Refs. [17] and [18], respectively. The multipole moments in the form (2.5) were derived in [20] up to the 2PN order, and shown subsequently in [21] to be in fact valid up to any post-Newtonian order (formally). On the other hand, both Eqs. (2.5a) and (2.5b) reduce to the expressions obtained in Ref. [50] in the limit of linearized gravity, where we can replace $\tau^{\mu\nu}$ by the compact-support matter tensor $T^{\mu\nu}$ (hence there is no need in this limit to consider a finite part). Note that the source multipole moments I_L and J_L parametrize, by definition, the linearized approximation to the vacuum metric outside the source [21], but take into account all the nonlinearities due to the inner (near-zone) field of the source. The nonlinearities in the exterior field can be obtained by some specific post-Minkowskian algorithm (see Ref. [21] for proof and details). The inclusion of these nonlinearities permits one to relate the source moments I_L and J_L to the radiative ones U_L and V_L. Some other source moments W_L, X_L, Y_L and Z_L should also be taken into account (see Ref. [21] for discussion), but these parametrize a (linearized) gauge transformation and do not contribute to the radiation field up to a high post-Newtonian order. We shall check that these moments do not affect the present calculation.

III. DEFINITIONS OF POTENTIALS

Our first task is to work out the expressions (2.5) to the 3PN order in the case of I_L and 2PN order in the case of J_L. In this paper we shall use some convenient retarded potentials, and then, from these, the corresponding "instantaneous" potentials. For insertion into the pseudo-tensor $\overline{\tau}^{\mu\nu}$ (and, most importantly, its gravitational part $\overline{\Lambda}^{\mu\nu}$) we need the components of the metric $\overline{h}^{\mu\nu}$ developed to post-Newtonian order $\mathcal{O}(8, 7, 8)$. By this we mean $\mathcal{O}(8)\equiv\mathcal{O}(1/c^8)$ in the 00 and ij components of the metric, and $\mathcal{O}(7)\equiv\mathcal{O}(1/c^7)$ in the $0i$ components. With this precision the metric reads

$$\overline{h}^{00}=-\frac{4}{c^2}V-\frac{2}{c^4}(\hat{W}+4V^2)-\frac{8}{c^6}\left(\hat{Z}+2\hat{X}+V\hat{W}+\frac{4}{3}V^3\right)+\mathcal{O}(8), \tag{3.1a}$$

$$\overline{h}^{0i}=-\frac{4}{c^3}V_i-\frac{8}{c^5}(\hat{R}_i+VV_i)+\mathcal{O}(7), \tag{3.1b}$$

$$\overline{h}^{ij}=-\frac{4}{c^4}\left(\hat{W}_{ij}-\frac{1}{2}\delta_{ij}\hat{W}\right)-\frac{16}{c^6}\left(\hat{Z}_{ij}-\frac{1}{2}\delta_{ij}\hat{Z}\right)+\mathcal{O}(8). \tag{3.1c}$$

The potentials are generated by the components of the matter tensor $T^{\mu\nu}$ or, rather, using a notation similar to Eq. (2.4), by

$$\sigma=\frac{T^{00}+T^{ii}}{c^2},\quad \sigma_i=\frac{T^{0i}}{c},\quad \sigma_{ij}=T^{ij}. \tag{3.2}$$

The potential V is a retarded version of the Newtonian potential and is defined by the retarded integral $\Box_R^{-1}$ acting on the source σ,

$$V(\mathbf{x},t)=\Box_R^{-1}\{-4\pi G\sigma\}\equiv G\int\frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\sigma(\mathbf{y},t-|\mathbf{x}-\mathbf{y}|/c). \tag{3.3}$$

To the 1PN order we have the potentials V_i and $\hat{W}_{ij}$ (together with the spatial trace $\hat{W}=\hat{W}_{ii}$), which are generated by the current and stress σ_i and σ_{ij}, respectively,

$$V_i=\Box_R^{-1}\{-4\pi G\sigma_i\}, \tag{3.4a}$$

$$\hat{W}_{ij}=\Box_R^{-1}\{-4\pi G(\sigma_{ij}-\delta_{ij}\sigma_{kk})-\partial_i V\partial_j V\}. \tag{3.4b}$$

To the 2PN order there are the potentials $\hat{R}_i$, $\hat{Z}_{ij}$, $\hat{X}$ (and also $\hat{Z}=\hat{Z}_{ii}$) whose expressions read

$$\hat{X}=\Box_R^{-1}\left\{-4\pi G\sigma_{ii}V+\hat{W}_{ij}\partial^2_{ij}V+2V_i\partial_t\partial_i V+V\partial_t^2 V+\frac{3}{2}(\partial_t V)^2-2\partial_i V_j\partial_j V_i\right\}, \tag{3.5a}$$

$$\hat{R}_i=\Box_R^{-1}\left\{-4\pi G(\sigma_i V-\sigma V_i)-2\partial_k V\partial_i V_k-\frac{3}{2}\partial_t V\partial_i V\right\}, \tag{3.5b}$$

$$\hat{Z}_{ij}=\Box_R^{-1}\left\{-4\pi G(\sigma_{ij}-\delta_{ij}\sigma_{kk})V-2\partial_{(i}V\partial_t V_{j)}+\partial_i V_k\partial_j V_k+\partial_k V_i\partial_k V_j-2\partial_{(i}V_k\partial_k V_{j)}-\delta_{ij}\partial_k V_m(\partial_k V_m-\partial_m V_k)-\frac{3}{4}\delta_{ij}(\partial_t V)^2\right\}. \tag{3.5c}$$

Next we expand the retardations and define some associated instantaneous potentials. The highest-order expansion is needed for the V potential, up to $\mathcal{O}(5)$, while $\mathcal{O}(3)$ is sufficient for V_i and $\hat{W}_{ij}$. We write these expansions in the form

$$V=U+\frac{1}{2c^2}\partial_t^2\chi-\frac{2G}{3c^3}\frac{d^3Q}{dt^3}+\frac{1}{24c^4}\partial_t^4 P+\mathcal{O}(5), \tag{3.6a}$$

$$V_i=U_i+\frac{1}{2c^2}\partial_t^2\chi_i+\mathcal{O}(3), \tag{3.6b}$$

$$\hat{W}_{ij}=U_{ij}-\frac{G}{2c}\frac{d^3Q_{ij}}{dt^3}+\frac{1}{2c^2}\partial_t^2\chi_{ij}-\frac{1}{c^2}K_{ij}+\mathcal{O}(3), \tag{3.6c}$$

where the instantaneous potentials are given by the Poisson-type integrals

$$U=\Delta^{-1}\{-4\pi G\sigma\}\equiv G\int\frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\sigma(\mathbf{y},t), \tag{3.7a}$$

$$U_i=\Delta^{-1}\{-4\pi G\sigma_i\}, \tag{3.7b}$$

$$U_{ij}=\Delta^{-1}\{-4\pi G(\sigma_{ij}-\delta_{ij}\sigma_{kk})-\partial_i U\partial_j U\}, \tag{3.7c}$$

$$\chi=2\Delta^{-1}U=G\int d^3\mathbf{y}|\mathbf{x}-\mathbf{y}|\sigma(\mathbf{y},t), \tag{3.7d}$$

$$\chi_i=2\Delta^{-1}U_i, \tag{3.7e}$$

$$\chi_{ij}=2\Delta^{-1}U_{ij}, \tag{3.7f}$$

$$P=24\Delta^{-2}U=G\int d^3\mathbf{y}|\mathbf{x}-\mathbf{y}|^3\sigma(\mathbf{y},t), \tag{3.7g}$$

$$K_{ij}=\Delta^{-1}\{\partial_{(i}U\partial_{j)}\partial_t^2\chi\}. \tag{3.7h}$$

In addition, the Newtonian precision $\mathcal{O}(1)$ is required for the other potentials $\hat{X}$, $\hat{R}_i$ and $\hat{Z}_{ij}$. For simplicity in the notation, we shall keep the same names for the Newtonian approximations to these potentials, henceforth redefined as

$$\hat{X}=\Delta^{-1}\left\{-4\pi G\sigma_{ii}U+U_{ij}\partial^2_{ij}U+2U_i\partial_t\partial_i U+U\partial_t^2 U+\frac{3}{2}(\partial_t U)^2-2\partial_i U_j\partial_j U_i\right\}, \tag{3.8a}$$

$$\hat{R}_i=\Delta^{-1}\left\{-4\pi G(\sigma_i U-\sigma U_i)-2\partial_k U\partial_i U_k-\frac{3}{2}\partial_t U\partial_i U\right\}, \tag{3.8b}$$

$$\hat{Z}_{ij}=\Delta^{-1}\left\{-4\pi G(\sigma_{ij}-\delta_{ij}\sigma_{kk})U-2\partial_{(i}U\partial_t U_{j)}+\partial_i U_k\partial_j U_k+\partial_k U_i\partial_k U_j-2\partial_{(i}U_k\partial_k U_{j)}-\delta_{ij}\partial_k U_m(\partial_k U_m-\partial_m U_k)-\frac{3}{4}\delta_{ij}(\partial_t U)^2\right\}. \tag{3.8c}$$

Finally the "odd" terms in Eqs. (3.6) (having an odd power of $1/c$ in factor) are simple functions of time parametrized by

$$Q_{ij}(t)=\int d^3\mathbf{x}(x_i x_j-\mathbf{x}^2\delta_{ij})\sigma(\mathbf{x},t), \tag{3.9a}$$

$$Q(t)=\int d^3\mathbf{x}\mathbf{x}^2\sigma(\mathbf{x},t). \tag{3.9b}$$

(Beware that $Q\neq Q_{ii}$.)

IV. NOMENCLATURE OF TERMS

The post-Newtonian metric (3.1) is inserted into the pseudo-tensor (2.2b), in which notably the term $\Lambda^{\mu\nu}$, given by Eq. (2.3), is developed up to quartic order h^4. Making use of the formula (2.7) we obtain the source moments $I_L(t)$ and $J_L(t)$ as some functionals of all the retarded potentials, and, then, of all the "instantaneous" potentials defined by Eqs. (3.6)–(3.9). We transform some of the terms by integration by parts, being careful to take into account the presence of the analytic continuation factor $|\tilde{\mathbf{x}}|^B$. The surface terms are always zero by analytic continuation (starting from the case where the real part of B is a large negative number). Notice that we use the Leibniz rule, which is surely valid in the case of potentials corresponding to smooth (fluid) sources. However, when we shall insert for the potentials some singular expressions corresponding to point-like particles, and shall replace the derivatives by some appropriate distributional derivatives, the Leibniz rule will no longer be satisfied in general. This will be a source of some indeterminacy discussed in Sec. X.

We find that the moments are quite complicated, so it is useful to devise a good nomenclature of terms. First, we distinguish in I_L and J_L the contributions which are due to the source densities Σ, Σ_i, and Σ_{ij} [see Eq. (2.5)], and we refer to them as scalar (S), vector (V) and tensor (T) respectively. Furthermore, we split each of these contributions according to the value of the summation index j in Eq. (2.7): for instance the S-type term denoted SI is defined by the set of terms in I_L coming from the "scalar" Σ in which we have used the formula (2.7) with only the contribution of the index $j=0$ (there are no S-type terms in J_L); similarly we denote by SII, using Roman letters, the S terms corresponding to $j=1$ (these terms involve a factor $\mathbf{x}^2$ and a second time-derivative); and for instance VII denotes the set of terms in both I_L and J_L coming from the "vector" Σ_i and which have $j=2$. With this notation the mass moment to the 3PN order can be written as

$$I_L = \mathrm{SI}+\mathrm{SII}+\mathrm{SIII}+\mathrm{SIV}+\mathrm{VI}+\mathrm{VII}+\mathrm{VIII}+\mathrm{TI}+\mathrm{TII}+\mathcal{O}(7). \tag{4.1}$$

For simplicity's sake we omit writing the multi-index L on each of these separate pieces (there can be no confusion from the context). Second, the numerous terms are numbered according to their order of appearance in the following formulas. For instance the piece SI which is part of the mass moment (4.1) will be composed of the terms SI(1), SI(2), etc; similarly VII is made of terms VII(1) and so on. The numbering of terms is indicated in round brackets at the right of each term in Eq. (4.2) (it should not be confused with, e.g., a differentiation or a power). The explicit expressions of all the separate pieces forming I_L are as follows:

$$\begin{aligned}
\mathrm{SI} = \mathop{\mathrm{FP}}_{B=0}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_L\bigg\{ & \sigma^{(1)} - \frac{1}{2\pi Gc^2}\Delta(U^2)^{(2)} + \frac{4U}{c^4}\sigma^{aa(3)} - \frac{2}{\pi Gc^4}U_i\partial_t\partial_i U^{(4)} - \frac{1}{\pi Gc^4}(\partial^2_{ij}U)U_{ij}{}^{(5)} - \frac{1}{2\pi Gc^4}(\partial_t U)^{2(6)} \\
& + \frac{2}{\pi Gc^4}\partial_i U_j\partial_j U_i{}^{(7)} - \frac{2}{3\pi Gc^4}\Delta(U^3)^{(8)} - \frac{1}{2\pi Gc^4}\Delta(U\partial_t^2\chi)^{(9)} - \frac{1}{2\pi Gc^4}\Delta(UU_{aa})^{(10)} - \frac{2}{3}\frac{G}{c^5}\sigma\frac{d^3Q}{dt^3}{}^{(11)} \\
& + \frac{1}{2\pi c^5}(\partial^2_{ij}U)\frac{d^3Q_{ij}}{dt^3}{}^{(12)} + \frac{16}{c^6}\sigma U_a U_a{}^{(13)} + \frac{8}{c^6}\sigma^{aa}U^{2(14)} + \frac{2}{c^6}\sigma^{aa}\partial_t^2\chi^{(15)} + \frac{4}{c^6}U_{jk}\sigma_{jk}{}^{(16)} - \frac{1}{\pi Gc^6}U_i\partial_i\partial_t^3\chi^{(17)} \\
& - \frac{1}{\pi Gc^6}(\partial_t^2\chi_i)(\partial_t\partial_i U)^{(18)} - \frac{1}{2\pi Gc^6}(\partial^2_{ij}U)(\partial_t^2\chi_{ij})^{(19)} + \frac{1}{\pi Gc^6}(\partial^2_{ij}U)K^{ij(20)} - \frac{1}{2\pi Gc^6}(\partial^2_{ij}\partial_t^2\chi)U_{ij}{}^{(21)} \\
& + \frac{1}{2\pi Gc^6}U_{aa}\partial_t^2 U^{(22)} + \frac{1}{2\pi Gc^6}U\partial_t^2 U_{aa}{}^{(23)} - \frac{1}{2\pi Gc^6}\partial_t U\partial_t^3\chi^{(24)} + \frac{2}{\pi Gc^6}\partial_i U_j\partial_t^2\partial_j\chi_i{}^{(25)} - \frac{2}{\pi Gc^6}U(\partial_t U)^{2(26)} \\
& - \frac{6}{\pi Gc^6}U_a\partial_t U\partial_a U^{(27)} - \frac{12}{\pi Gc^6}UU_a\partial_t\partial_a U^{(28)} + \frac{4}{\pi Gc^6}U\partial_a U\partial_t U_a{}^{(29)} - \frac{8}{\pi Gc^6}U_a(\partial_b U_a)\partial_b U^{(30)} + \frac{2}{\pi Gc^6}(\partial_t U_a)^{2(31)} \\
& - \frac{1}{\pi Gc^6}(\partial_t U_{aa})\partial_b U_b{}^{(32)} + \frac{4}{\pi Gc^6}\partial_i U_j\partial_t U_{ij}{}^{(33)} + \frac{8}{\pi Gc^6}UU_i\partial_t\partial_i U^{(34)} - \frac{4}{\pi Gc^6}\hat{Z}_{ij}\partial^2_{ij}U^{(35)} - \frac{4}{\pi Gc^6}U\partial_i U\partial_t U_i{}^{(36)} \\
& - \frac{4}{\pi Gc^6}(\partial_t\partial_i U)\hat{R}_i{}^{(37)} + \frac{8}{\pi Gc^6}\partial_i U_j\partial_j\hat{R}_i{}^{(38)} - \frac{2}{3\pi Gc^6}\Delta(U^4)^{(39)} - \frac{1}{\pi Gc^6}\Delta(U^2U_{bb})^{(40)} - \frac{1}{\pi Gc^6}\Delta(U^2\partial_t^2\chi)^{(41)} \\
& - \frac{1}{8\pi Gc^6}\Delta(\partial_t^2\chi\partial_t^2\chi)^{(42)} - \frac{1}{24\pi Gc^6}\Delta(U\partial_t^4 P)^{(43)} - \frac{1}{4\pi Gc^6}\Delta(U_{aa}\partial_t^2\chi)^{(44)} - \frac{1}{4\pi Gc^6}\Delta(\partial_t^2\chi_{aa}U)^{(45)} \\
& + \frac{1}{2\pi Gc^6}\Delta(K^{aa}U)^{(46)} - \frac{1}{4\pi Gc^6}\Delta(U_{aa}U_{bb})^{(47)} + \frac{1}{2\pi Gc^6}\Delta(U_{jk}U_{jk})^{(48)} - \frac{4}{\pi Gc^6}\Delta(U\hat{X})^{(49)} - \frac{2}{\pi Gc^6}\Delta(U\hat{Z}_{aa})^{(50)}\bigg\},
\end{aligned} \tag{4.2a}$$

$$\mathrm{SII}=\frac{1}{2c^2(2l+3)}\mathop{\mathrm{FP}}_{B=0}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\partial_t^2\Big\{|\mathbf{x}|^2\hat{x}_L\Big[\sigma^{(1)}+\frac{4U}{c^4}\sigma^{aa(2)}-\frac{2}{\pi Gc^4}U_i\partial_t\partial_iU^{(3)}-\frac{1}{\pi Gc^4}(\partial^2_{ij}U)U_{ij}{}^{(4)}-\frac{1}{2\pi Gc^4}(\partial_tU)^{2(5)}$$
$$+\frac{2}{\pi Gc^4}\partial_iU_j\partial_jU_i{}^{(6)}\Big]-\frac{2l+3}{\pi Gc^2}\hat{x}_LU^{2(7)}-\frac{1}{2\pi Gc^2}\partial_i[\partial_i(U^2)|\mathbf{x}|^2\hat{x}_L-U^2\partial_i(|\mathbf{x}|^2\hat{x}_L)]^{(8)}-\frac{2l+3}{\pi Gc^4}\hat{x}_LU\partial_t^2\chi^{(9)}$$
$$-\frac{1}{2\pi Gc^4}\partial_i[\partial_i(U\partial_t^2\chi)|\mathbf{x}|^2\hat{x}_L-U\partial_t^2\chi\partial_i(|\mathbf{x}|^2\hat{x}_L)]^{(10)}-\frac{2l+3}{\pi Gc^4}\hat{x}_LUU_{aa}{}^{(11)}-\frac{1}{2\pi Gc^4}\partial_i[\partial_i(UU_{aa})|\mathbf{x}|^2\hat{x}_L$$
$$-UU_{aa}\partial_i(|\mathbf{x}|^2\hat{x}_L)]^{(12)}-\frac{4(2l+3)}{3\pi Gc^4}\hat{x}_LU^{3(13)}-\frac{2}{3\pi Gc^4}\partial_i[\partial_i(U^3)|\mathbf{x}|^2\hat{x}_L-U^3\partial_i(|\mathbf{x}|^2\hat{x}_L)]^{(14)}\Big\}, \tag{4.2b}$$

$$\mathrm{SIII}=\frac{1}{8c^4(2l+3)(2l+5)}\mathop{\mathrm{FP}}_{B=0}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\partial_t^4\Big\{|\mathbf{x}|^4\hat{x}_L\sigma^{(1)}-\frac{2(2l+5)}{\pi Gc^2}|\mathbf{x}|^2\hat{x}_LU^{2(2)}$$
$$-\frac{1}{2\pi Gc^2}\partial_i[\partial_i(U^2)|\mathbf{x}|^4\hat{x}_L-U^2\partial_i(|\mathbf{x}|^4\hat{x}_L)]^{(3)}\Big\}, \tag{4.2c}$$

$$\mathrm{SIV}=\frac{1}{48c^6(2l+3)(2l+5)(2l+7)}\mathop{\mathrm{FP}}_{B=0}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B|\mathbf{x}|^6\hat{x}_L\partial_t^6\sigma^{(1)}, \tag{4.2d}$$

$$\mathrm{VI}=\frac{-4(2l+1)}{c^2(l+1)(2l+3)}\mathop{\mathrm{FP}}_{B=0}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_{aL}\partial_t\Big\{\sigma_a{}^{(1)}+\frac{2}{c^2}\sigma_aU^{(2)}-\frac{2}{c^2}\sigma U_a{}^{(3)}+\frac{1}{\pi Gc^2}\partial_kU\partial_aU_k{}^{(4)}+\frac{3}{4\pi Gc^2}\partial_tU\partial_aU^{(5)}$$
$$-\frac{1}{2\pi Gc^2}\Delta(UU_a)^{(6)}+\frac{\sigma_a}{c^4}\partial_t^2\chi^{(7)}+\frac{2}{c^4}\sigma_aU^{2(8)}-\frac{1}{c^4}\sigma\partial_t^2\chi_a{}^{(9)}-\frac{4}{c^4}\sigma\hat{R}_a{}^{(10)}+\frac{2}{c^4}U_a\sigma_{ss}{}^{(11)}+\frac{2}{c^4}U_{ak}\sigma_k{}^{(12)}$$
$$+\frac{2}{c^4}U_k\sigma_{ak}{}^{(13)}+\frac{1}{2\pi Gc^4}\partial_kU\partial_a\partial_t^2\chi_k{}^{(14)}+\frac{1}{2\pi Gc^4}(\partial_k\partial_t^2\chi)(\partial_aU_k)^{(15)}+\frac{1}{2\pi Gc^4}U_a\partial_t^2U^{(16)}$$
$$+\frac{3}{8\pi Gc^4}\partial_tU(\partial_a\partial_t^2\chi)^{(17)}+\frac{3}{8\pi Gc^4}(\partial_t^3\chi)\partial_aU^{(18)}-\frac{1}{2\pi Gc^4}U\partial_t^2U_a{}^{(19)}+\frac{1}{\pi Gc^4}\partial_tU\partial_tU_a{}^{(20)}$$
$$-\frac{2}{\pi Gc^4}U_k(\partial_k\partial_tU_a)^{(21)}+\frac{3}{2\pi Gc^4}U\partial_tU\partial_aU^{(22)}-\frac{1}{\pi Gc^4}U_a\partial_kU\partial_kU^{(23)}+\frac{3}{2\pi Gc^4}U_k\partial_aU\partial_kU^{(24)}$$
$$+\frac{2}{\pi Gc^4}\partial_kU\partial_a\hat{R}_k{}^{(25)}-\frac{1}{\pi Gc^4}U_{kl}\partial^2_{kl}U_a{}^{(26)}+\frac{1}{\pi Gc^4}\partial_tU_{ak}\partial_kU^{(27)}-\frac{1}{\pi Gc^4}\partial_kU_l\partial_aU_{kl}{}^{(28)}+\frac{1}{\pi Gc^4}\partial_kU_{al}\partial_lU_k{}^{(29)}$$
$$-\frac{1}{2\pi Gc^4}\Delta(U^2U_a)^{(30)}-\frac{1}{4\pi Gc^4}\Delta(U\partial_t^2\chi_a)^{(31)}-\frac{1}{4\pi Gc^4}\Delta(\partial_t^2\chi U_a)^{(32)}-\frac{1}{\pi Gc^4}\Delta(U\hat{R}_a)^{(33)}$$
$$-\frac{1}{2\pi Gc^4}\Delta(U_{kk}U_a)^{(34)}+\frac{1}{2\pi Gc^4}\Delta(U_{ak}U_k)^{(35)}\Big\}, \tag{4.2e}$$

$$\mathrm{VII}=\frac{-2(2l+1)}{c^4(l+1)(2l+3)(2l+5)}\mathop{\mathrm{FP}}_{B=0}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\partial_t^3\Big\{|\mathbf{x}|^2\hat{x}_{aL}\Big[\sigma_a{}^{(1)}+\frac{2}{c^2}\sigma_aU^{(2)}-\frac{2}{c^2}\sigma U_a{}^{(3)}+\frac{1}{\pi Gc^2}\partial_kU\partial_aU_k{}^{(4)}$$
$$+\frac{3}{4\pi Gc^2}\partial_tU\partial_aU^{(5)}\Big]-\frac{2l+5}{\pi Gc^2}\hat{x}_{aL}UU_a{}^{(6)}-\frac{1}{2\pi Gc^2}\partial_i[\partial_i(UU_a)|\mathbf{x}|^2\hat{x}_{aL}-UU_a\partial_i(|\mathbf{x}|^2\hat{x}_{al})]^{(7)}\Big\}, \tag{4.2f}$$

$$\mathrm{VIII}=\frac{-(2l+1)}{2c^6(l+1)(2l+3)(2l+5)(2l+7)}\mathop{\mathrm{FP}}_{B=0}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_{aL}|\mathbf{x}|^4\partial_t^5\sigma^{a(1)}, \tag{4.2g}$$

$$\mathrm{TI}=\frac{2(2l+1)}{c^4(l+1)(l+2)(2l+5)}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_{abL}\partial_t^2\bigg\{\sigma_{ab}{}^{(1)}+\frac{1}{4\pi G}\partial_aU\partial_bU^{(2)}+\frac{4}{c^2}\sigma_{ab}U^{(3)}-\frac{4}{c^2}\sigma_{(a}U_{b)}{}^{(4)}$$
$$+\frac{1}{4\pi Gc^2}\partial_{(a}U\partial_{b)}\partial_t^2\chi^{(5)}+\frac{2}{\pi Gc^2}\partial_{(a}U\partial_tU_{b)}{}^{(6)}-\frac{1}{\pi Gc^2}\partial_aU_k\partial_bU_k{}^{(7)}+\frac{2}{\pi Gc^2}\partial_{(a}U_k\partial_kU_{b)}{}^{(8)}-\frac{1}{2\pi Gc^2}\Delta(U_aU_b)^{(9)}\bigg\},\tag{4.2h}$$

$$\mathrm{TII}=\frac{2l+1}{c^6(l+1)(l+2)(2l+5)(2l+7)}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_{abL}|\mathbf{x}|^2\partial_t^4\bigg\{\sigma_{ab}{}^{(1)}+\frac{1}{4\pi G}\partial_aU\partial_bU^{(2)}\bigg\}.\tag{4.2i}$$

In the case of the 2PN current moment we write similarly

$$J_L=\mathrm{VI}+\mathrm{VII}+\mathrm{VIII}+\mathrm{TI}+\mathrm{TII}+\mathcal{O}(5).\tag{4.3}$$

The expressions of these separate pieces have the same structure as the corresponding V and T terms in the 3PN mass moment I_L. The differences lie only in the over-all coefficient, in the number of time-derivatives, and in the presence of a Levi-Civita symbol. We have

$$\mathrm{VI}=-\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\varepsilon_{ab<i_l}\hat{x}_{L-1>b}\ \{\text{same as in the curly brackets of Eq. (4.2e)}\},\tag{4.4a}$$

$$\mathrm{VII}=-\frac{1}{2c^2(2l+3)}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\varepsilon_{ab<i_l}\hat{x}_{L-1>b}\partial_t^2\ \{\text{same as in Eq. (4.2f)}\},\tag{4.4b}$$

$$\mathrm{VIII}=-\frac{1}{8c^4(2l+3)(2l+5)}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B|\mathbf{x}|^4\varepsilon_{ab<i_l}\hat{x}_{L-1>b}\partial_t^4\ \{\text{same as in Eq. (4.2g)}\},\tag{4.4c}$$

$$\mathrm{TI}=\frac{2l+1}{c^2(l+2)(2l+3)}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\varepsilon_{ac<i_l}\hat{x}_{L-1>bc}\partial_t\ \{\text{same as in Eq. (4.2h)}\},\tag{4.4d}$$

$$\mathrm{TII}=\frac{2l+1}{2c^4(l+2)(2l+3)(2l+5)}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B|\mathbf{x}|^2\varepsilon_{ac<i_l}\hat{x}_{L-1>bc}\partial_t^3\ \{\text{same as in Eq. (4.2i)}\}.\tag{4.4e}$$

We explained that we denote the terms in the previous formulas by SI(1), SI(2), ..., SI(50), SII(1), ..., TII(2).[1] Our convention is that this notation means that the terms involve their complete coefficient in front; for instance,

$$\mathrm{SI}(5)=-\frac{1}{\pi Gc^4}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_LU_{ij}\partial_{ij}^2U,\tag{4.5a}$$

$$\mathrm{SII}(14)=-\frac{1}{3\pi Gc^6(2l+3)}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\partial_t^2\partial_i[\partial_i(U^3)|\mathbf{x}|^2\hat{x}_L-U^3\partial_i(|\mathbf{x}|^2\hat{x}_L)],\tag{4.5b}$$

$$\mathrm{TI}(7)=\frac{2(2l+1)}{\pi Gc^6(l+1)(l+2)(2l+5)}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_{abL}\partial_t^2\{\partial_aU_k\partial_bU_k\}.\tag{4.5c}$$

The notation means also that the terms include all the post-Newtonian corrections relevant to obtain the 3PN order in the energy flux. Consistently with that order we shall have to compute the mass quadrupole moment I_{ij} to the 3PN order, the mass octupole I_{ijk} and current quadrupole J_{ij} to the 2PN order only. Look for instance at the term SI(5) given by Eq. (4.5a): this is a 2PN term since it carries a factor $1/c^4$. Thus, in the mass quadrupole I_{ij} we need to compute SI(5) with 1PN relative precision, while in the mass octupole I_{ijk} the Newtonian precision is sufficient [the term SI(5) does not exist in the current moments]. Note also that a term such as SII(14) given by Eq. (4.5b) includes in fact two terms (which come from an operation by parts). Furthermore, since the different pieces (of types V and T) composing the current moments have exactly the same structure as in the mass moments, we employ the same notation for these terms in both

[1]Though SI(29) and SI(36) cancel out among themselves, they are computed and included in our presentation.

I_L and J_L. For instance TI(7) denotes both the term in the mass moment as given by Eq. (4.5c) and the corresponding term in the current moment (with a little experience there can be no confusion). Finally, in some cases we split the term into subterms according to the nature of a potential therein, either "compact" or "non-compact" potential. The compact (respectively non-compact) part of a potential is that part which is generated by a source with compact (non-compact) support. For instance the term SI(5), which contains the potential U_{ij} given by Eq. (3.7c), is naturally split into the two contributions

$$\mathrm{SI}(5)=\mathrm{SI}(5\mathrm{C})+\mathrm{SI}(5\mathrm{NC}), \tag{4.6}$$

where U_{ij} is replaced by its compact (C) or non-compact (NC) parts given by

$$U_{ij}=U_{ij}^{(\mathrm{C})}+U_{ij}^{(\mathrm{NC})}, \tag{4.7a}$$

$$U_{ij}^{(\mathrm{C})}=\Delta^{-1}\{-4\pi G(\sigma_{ij}-\delta_{ij}\sigma_{kk})\}, \tag{4.7b}$$

$$U_{ij}^{(\mathrm{NC})}=\Delta^{-1}\{-\partial_i U\partial_j U\}. \tag{4.7c}$$

We shall split similarly all the terms containing the potentials U_{ij}, χ_{ij}, $\hat{R}_i$, $\hat{Z}_{ij}$ and $\hat{X}$. This splitting into C and NC parts is fairly obvious from the expressions of the potentials: for instance,

$$\hat{R}_i^{(\mathrm{NC})}=\Delta^{-1}\left\{-2\partial_k U\partial_i U_k-\frac{3}{2}\partial_t U\partial_i U\right\}. \tag{4.8}$$

When computing the terms in the moments (4.1)–(4.4) we shall separate them into various categories, according to the way their computation is performed. This entails introducing some new terminology for the various classes. For instance we shall consider the compact-support terms like SI(1), or so-called Y-terms made of the quadratic product of two U-type potentials [examples are VI(4) and also SI(5C)], or so-called non-compact terms like SI(5NC) or SII(4NC). These categories of terms are defined when we tackle their computation. The resulting nomenclature is complicated but turned out to be useful during the explicit computation and the many associated checks, since it delineates clearly the different problems posed by the different categories of terms.

V. APPLICATION TO POINT-PARTICLES

Our aim is to compute the multipole moments for a system of two point-like particles. One is not allowed *a priori* to use the expressions (2.5) as they have been obtained in Ref. [21] under the assumption of a continuous (smooth) source. Applying them to a system of point-particles, we find that the integrals are divergent at the location of the particles, i.e., when $\mathbf{x}\to\mathbf{y}_1(t)$ or $\mathbf{y}_2(t)$, where $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ denote the two trajectories. Therefore we must supplement the computation by a prescription for how to remove the infinite part of these integrals. In this paper, we systematically employ the Hadamard regularization [31,32] (see Ref. [33] for an entry to the mathematical literature). The usefulness of this regularization for problems involving point-particles in general relativity has been shown by numerous works (see, e.g., [51]). Recently the properties of the Hadamard regularization have been re-visited and a new set of generalized functions (distributional forms) associated with this regularization was introduced [41,42].

The functions $F(\mathbf{x})$ we need to deal with are smooth on $\mathbb{R}^3$ excised of the two points $\mathbf{y}_1$ and $\mathbf{y}_2$, and admit when $r_1=|\mathbf{x}-\mathbf{y}_1|\to 0$ (and similarly when $r_2=|\mathbf{x}-\mathbf{y}_2|\to 0$) a singular expansion of the type

$$\forall n\in\mathbb{N},\quad F(\mathbf{x})=\sum_{a_0\leqslant a\leqslant n} r_1^a\,{}_1f_a(\mathbf{n}_1)+o(r_1^n), \tag{5.1}$$

where the coefficients ${}_1f_a$ of the various powers of r_1 in the expansion depend on the unit direction $\mathbf{n}_1=(\mathbf{x}-\mathbf{y}_1)/r_1$. The powers a of r_1 are real, range in discrete steps [i.e., a belongs to some countable set $(a_i)_{i\in\mathbb{N}}$] and are bounded from below $(a_0\leqslant a)$. The functions like F are said to belong to the class of functions $\mathcal{F}$ (see Ref. [41] for precise definitions). If F and G belong to $\mathcal{F}$ so do the ordinary (pointwise) product FG and the ordinary gradient $\partial_i F$. The Hadamard "partie finie" of F at the location of particle 1 is defined as

$$(F)_1=\int\frac{d\Omega_1}{4\pi}\,{}_1f_0(\mathbf{n}_1), \tag{5.2}$$

where $d\Omega_1=d\Omega(\mathbf{n}_1)$ is the solid angle element centered on $\mathbf{y}_1$ and of direction $\mathbf{n}_1$. On the other hand, the Hadamard partie finie (Pf) of the integral $\int d^3\mathbf{x}F$, divergent because of the two singular points $\mathbf{y}_1$ and $\mathbf{y}_2$, is defined by

$$\mathrm{Pf}_{u_1,u_2}\int d^3\mathbf{x}F=\lim_{u\to 0}\left\{\int_{\substack{r_1>u\\ r_2>u}} d^3\mathbf{x}F+4\pi\sum_{a+3<0}\frac{u^{a+3}}{a+3}\left(\frac{F}{r_1^a}\right)_1 + 4\pi\ln\left(\frac{u}{u_1}\right)(r_1^3F)_1+1\leftrightarrow 2\right\}. \tag{5.3}$$

The first term represents the integral on $\mathbb{R}^3$ excluding two spherical volumes of radius u surrounding the singularities. The other terms are such that they cancel out the divergent part of the latter integral when $u\to 0$ (the symbol $1\leftrightarrow 2$ means the terms obtained by exchanging the labels 1 and 2). Notice the presence of a logarithmic term, which depends on an arbitrary constant u_1, and similarly u_2 for the other singularity. In this paper we shall keep the constants u_1 and u_2 all the way through our calculation. We assume nothing about these constants, for instance they are different *a priori* from similar constants s_1 and s_2 introduced in the equations of motion (Sec. II in [43]). We shall see that the multipole moments do depend on u_1 and u_2 (as well as on r_0) at the 3PN order.

The strategy we adopt in this paper is to insert into the source multipole moments (2.5) the following expression of the matter stress-energy tensor $T^{\mu\nu}$ for two point-masses:

$$T^{\mu\nu}_{\text{point-particle}}=m_1 v_1^\mu v_1^\nu\left(\frac{dt}{d\tau}\right)_1\left(\frac{1}{\sqrt{-g}}\right)_1\delta(\mathbf{x}-\mathbf{y}_1)+1\leftrightarrow 2, \tag{5.4a}$$

$$\left(\frac{dt}{d\tau}\right)_1=\frac{1}{\sqrt{-(g_{\rho\sigma})_1 v_1^\rho v_1^\sigma/c^2}}, \tag{5.4b}$$

where m_1 is the (Schwarzschild) mass, $\mathbf{y}_1(t)$ the trajectory, and $\mathbf{v}_1(t)=d\mathbf{y}_1/dt$ the velocity of body 1 [with $v_1^\mu=(c,\mathbf{v}_1)$]. This stress-energy tensor constitutes a "naive" model to describe the particles, since the factors of the Dirac distribution have been evaluated at the point 1 by means of the regularization defined by Eq. (5.2). However, because of the so-called non-distributivity of the Hadamard partie finie, other tensors are possible as well. In particular, we discuss in Sec. X the effect of choosing another stress-energy tensor, which is particularly natural within the context of the Hadamard regularization, and that we proposed in Ref. [42]. After $T^{\mu\nu}_{\text{point-particle}}$ is substituted inside them, the moments are comprised of many divergent integrals and we define each of these integrals by means of the Hadamard partie finie (5.3). Therefore our ansatz for applying the general "fluid" formalism to the ill-defined case of point-particles is

$$(I_L)_{\text{point-particle}}=\text{Pf}\{I_L[T^{\mu\nu}_{\text{point-particle}}]\}, \tag{5.5a}$$

$$(J_L)_{\text{point-particle}}=\text{Pf}\{J_L[T^{\mu\nu}_{\text{point-particle}}]\}, \tag{5.5b}$$

where the functionals I_L and J_L are exactly the ones given by Eq. (2.5) or Eqs. (4.1)–(4.4) (including in particular the finite part $\text{FP}_{B=0}$ at infinity). In what follows we shall carefully apply this prescription, but in order to reduce clutter we generally omit writing the partie-finie symbol Pf.

The relative position and velocity of two bodies in harmonic coordinates are denoted by

$$x^i=y_1^i-y_2^i, \quad \text{and} \quad v^i=\frac{dx^i}{dt}=v_1^i-v_2^i. \tag{5.6}$$

To the 2PN order (only needed in this paper) the relation between the absolute trajectories in a center-of-mass frame and the relative ones reads, in the case of a circular orbit (see, e.g., Ref. [13]), as

$$y_1^i=\frac{m_2+3\nu\gamma^2\delta m}{m}x^i+\mathcal{O}(5), \tag{5.7a}$$

$$y_1^i=\frac{-m_1+3\nu\gamma^2\delta m}{m}x^i+\mathcal{O}(5). \tag{5.7b}$$

Here m_1 and m_2 are the two masses, with $m=m_1+m_2$, $\nu=m_1m_2/m^2$ (such that $0<\nu\leqslant 1/4$) and $\delta m=m_1-m_2$. Furthermore,

$$\gamma=\frac{Gm}{rc^2}, \tag{5.8}$$

represents a small post-Newtonian parameter of order $\mathcal{O}(2)$, with $r=|\mathbf{x}|$, often also denoted r_{12}, the distance between the two masses in harmonic coordinates.

When computing the multipole moments we get many terms involving accelerations and derivatives of accelerations. These are reduced to the consistent post-Newtonian order by means of the binary's equations of motion. To control the moments at the 3PN order we need the equations of motion at the 2PN order. For circular orbits these equations are (see, e.g., [13])

$$\frac{d\mathbf{v}}{dt}=-\omega^2\mathbf{x}+\mathcal{O}(5), \tag{5.9a}$$

$$\omega^2=\frac{Gm}{r^3}\left\{1+[-3+\nu]\gamma+\left[6+\frac{41}{4}\nu+\nu^2\right]\gamma^2+\mathcal{O}(\gamma^3)\right\}. \tag{5.9b}$$

The content of these equations lies in the relation (5.9b) between the orbital frequency ω and the coordinate separation r in harmonic coordinates. However, note that the precision given by Eqs. (5.9) is insufficient to obtain the (second and higher) time-derivatives of the moments at the 3PN order. Evidently for this we need the more accurate 3PN equations of motion. These will be given in Sec. XII when we compute the total energy flux [see Eq. (12.3) below]. In addition, we shall also need for some intermediate computations the equations of motion for general (not necessarily circular) orbits but at the 1PN order. These are given by

$$\frac{d\mathbf{v}_1}{dt}=-\frac{Gm_2}{r^2}\mathbf{n}+\frac{Gm_2}{c^2r^2}\left\{\mathbf{n}\left[-v_1^2-2v_2^2+4(v_1v_2)+\frac{3}{2}(nv_2)^2+5\frac{Gm_1}{r}+4\frac{Gm_2}{r}\right]+\mathbf{v}[4(nv_1)-3(nv_2)]\right\}+\mathcal{O}(4) \tag{5.10}$$

(and *idem* for $1\leftrightarrow 2$). The notation (nv_1) for instance means the usual scalar product between the vectors $\mathbf{n}=\mathbf{x}/r$ (sometimes denoted also $\mathbf{n}_{12}$) and $\mathbf{v}_1$. With these preliminary inputs in place, we are in a position to tackle the computation of each of the terms composing the multipole moments (4.1)–(4.4).

VI. COMPACT TERMS

In this category we consider all the terms in Eqs. (4.1)–(4.4) whose integrand involves explicitly the matter densities σ, σ_i, or σ_{ij} as a factor, and thus which extend only over the spatially compact support of the source. For these terms the finite part operation $\text{FP}_{B=0}$ (which deals with the bound at infinity of the integral) can be dropped out. With the present notation the compact terms are (i) compact term at Newtonian order SI(1); (ii) compact terms at 1PN order: SII(1), VI(1); (iii) compacts at 2PN:SI(3), SIII(1), VI(2), VI(3), VII(1), TI(1); (iv) compacts at 3PN: SI(13), SI(14), SI(15), SI(16C), SII(2), SIV(1), VI(7), VI(8), VI(9), VI(10C), VI(11), VI(12C), VI(13), VII(2), VII(3), VIII(1), TI(3), TI(4), TII(1).

As explained earlier, it is convenient, when the potential is composed of both compact and non-compact parts, to separate out these pieces. Thus we shall also have the compact terms involving the non-compact part of a potential, namely

SI(16NC), VI(10NC), VI(12NC).

Evidently we have to compute the "Newtonian" term SI(1) with the maximal 3PN precision, while for instance a term which appears at 3PN needs only the Newtonian precision. We devote this section to the computation of the Newtonian term SI(1), and to one example of a compact term with non-compact potential: SI(16NC); the computation of the other compact terms is similar, or does not present any difficulty, so we only list the final results in the Appendix.

From the stress-energy tensor (5.4) we find that the matter source densities (3.2) are given by

$$\sigma(\mathbf{x},t)=\tilde{\mu}_1\delta[\mathbf{x}-\mathbf{y}_1(t)]+1\leftrightarrow 2, \tag{6.1a}$$

$$\sigma_i(\mathbf{x},t)=\mu_1 v_1^i\delta[\mathbf{x}-\mathbf{y}_1(t)]+1\leftrightarrow 2, \tag{6.1b}$$

$$\sigma_{ij}(\mathbf{x},t)=\mu_1 v_1^i v_1^j\delta[\mathbf{x}-\mathbf{y}_1(t)]+1\leftrightarrow 2, \tag{6.1c}$$

where we have introduced some "effective" masses μ_1 and $\tilde{\mu}_1$ defined by

$$\mu_1(t)=m_1\left(\frac{dt}{d\tau}\right)_1\left(\frac{1}{\sqrt{-g}}\right)_1, \tag{6.2a}$$

$$\tilde{\mu}_1(t)=\mu_1(t)\left[1+\frac{\mathbf{v}_1^2}{c^2}\right]. \tag{6.2b}$$

These effective masses are some mere functions of time t through the dependence over the particle trajectories and velocities (the accelerations are order-reduced). Notice that, had we used the stress-energy tensor proposed in Sec. V of [42] (see also the discussion in Sec. X below), we would have found that μ_1 and $\tilde{\mu}_1$ depend both on time and space, as they contain the factor $1/\sqrt{-g}$ that is given at any field point $\mathbf{x}$. Using the metric (3.1), expressed in terms of the retarded potentials (3.3)–(3.5), we find the expressions of the two required factors entering the effective masses (6.2) up to the 3PN order: namely,

$$\left(\frac{1}{\sqrt{-g}}\right)_1=\Bigg(1-\frac{2}{c^2}V+\frac{1}{c^4}[-2\hat{W}+2V^2]+\frac{1}{c^6}\left[-8\hat{Z}-8\hat{X}+4V\hat{W}-8V_iV_i-\frac{4}{3}V^3\right]\Bigg)_1+\mathcal{O}(8), \tag{6.3a}$$

$$\left(\frac{dt}{d\tau}\right)_1=\Bigg(1+\frac{1}{c^2}\left[V+\frac{1}{2}v_1^2\right]+\frac{1}{c^4}\left[\frac{1}{2}V^2+\frac{5}{2}Vv_1^2-4V_iv_1^i+\frac{3}{8}v_1^4\right]+\frac{1}{c^6}\Big[4\hat{X}+4V_iV_i-8\hat{R}_iv_1^i+2\hat{W}_{ij}v_1^iv_1^j-12VV_iv_1^i-6V_iv_1^iv_1^2+\frac{1}{6}V^3+\frac{25}{4}V^2v_1^2+\frac{27}{8}Vv_1^4+\frac{5}{16}v_1^6\Big]\Bigg)_1+\mathcal{O}(8), \tag{6.3b}$$

where the subscript 1 means that all the potentials are to be evaluated following the regularization (5.2). In these expressions there are no problems associated with the non-distributivity of the Hadamard partie-finie; that is, we can assume $(FG)_1=(F)_1(G)_1$ for this computation (see, however, Sec. X). Most of the regularized values of the needed potentials at 1 (for general orbits) have been computed in Ref. [51] (see the Appendix B there). Here we simply report the appropriate formulas [where $r_{12}=|\mathbf{y}_1-\mathbf{y}_2|$, $\mathbf{n}_{12}=(\mathbf{y}_1-\mathbf{y}_2)/r_{12}$]:

$$(V)_1=\frac{Gm_2}{r_{12}}\Bigg\{1+\frac{1}{c^2}\left[\frac{-3}{2}\frac{Gm_1}{r_{12}}+2v_2^2-\frac{1}{2}(n_{12}v_2)^2\right]+\frac{4}{3}\frac{Gm_1}{r_{12}c^3}(n_{12}v_{12})+\frac{Gm_1}{r_{12}c^4}\Big[\frac{11}{2}\frac{Gm_1}{r_{12}}+\frac{5}{4}\frac{Gm_2}{r_{12}}+\frac{15}{8}v_1^2-\frac{7}{4}(v_1v_2)-\frac{25}{8}v_2^2+\frac{1}{8}(n_{12}v_1)^2-\frac{25}{4}(n_{12}v_1)(n_{12}v_2)+\frac{33}{8}(n_{12}v_2)^2\Big]+\frac{1}{c^4}\left[2v_2^4-\frac{3}{2}(n_{12}v_2)^2v_2^2+\frac{3}{8}(n_{12}v_2)^4\right]\Bigg\}+\mathcal{O}(5), \tag{6.4a}$$

$$(V_i)_1=\frac{Gm_2}{r_{12}}\Bigg\{v_2^i+\frac{v_2^i}{c^2}\left[-2\frac{Gm_1}{r_{12}}+v_2^2-\frac{1}{2}(n_{12}v_2)^2\right]+\frac{1}{2}\frac{Gm_1}{r_{12}c^2}v_1^i+\frac{Gm_1}{r_{12}c^2}n_{12}^i\left[-\frac{3}{2}(n_{12}v_1)+\frac{1}{2}(n_{12}v_2)\right]\Bigg\}+\mathcal{O}(3), \tag{6.4b}$$

$$(\hat{W}_{ij})_1=\frac{Gm_2}{r_{12}}\Bigg\{v_2^{ij}-\delta^{ij}v_2^2+\frac{Gm_1}{r_{12}}[-2n_{12}^{ij}+\delta^{ij}]+\frac{Gm_2}{4r_{12}}[n_{12}^{ij}-\delta^{ij}]\Bigg\}+\mathcal{O}(1), \tag{6.4c}$$

$$(\hat{R}_i)_1=\frac{G^2m_1m_2}{r_{12}^2}\left[-\frac{3}{4}v_1^i+\frac{5}{4}v_2^i-\frac{1}{2}(n_{12}v_1)n_{12}^i-\frac{1}{2}(n_{12}v_2)n_{12}^i\right]+\frac{G^2m_2^2}{r_{12}^2}\left[-\frac{1}{8}v_2^i+\frac{1}{8}(n_{12}v_2)n_{12}^i\right]+\mathcal{O}(1). \tag{6.4d}$$

$$(\hat{X})_1=\frac{G^2m_1m_2}{r_{12}^2}\left[-\frac{3}{2}\frac{Gm_1}{r_{12}}+\frac{1}{4}v_1^2-2(v_1v_2)+\frac{9}{4}v_2^2\right.$$
$$\left.-\frac{11}{4}(n_{12}v_1)^2+\frac{9}{2}(n_{12}v_1)(n_{12}v_2)-\frac{11}{4}(n_{12}v_2)^2\right]$$
$$+\frac{G^2m_2^2}{r_{12}^2}\left[\frac{1}{12}\frac{Gm_2}{r_{12}}-\frac{1}{8}v_2^2+\frac{1}{8}(n_{12}v_2)^2\right]+\mathcal{O}(1). \tag{6.4e}$$

Notice that during the computation of the potential V at the 2PN order we used the 1PN equations of motion for general orbits: these are given by Eq. (5.10). In addition to the above, we need the trace $\hat{W}=\hat{W}_{ii}$ at 1PN order. [To the order considered in Eq. (6.4c) we have $U_{ij}=\hat{W}_{ij}$.] By a computation similar to those of Ref. [51] we get

$$(\hat{W})_1=\frac{Gm_2}{r_{12}}\left[\frac{Gm_1}{r_{12}}-\frac{1}{2}\frac{Gm_2}{r_{12}}-2v_2^2\right]-\frac{2G^2m_1m_2}{r_{12}^2c}(n_{12}v_{12})$$
$$+\frac{G^2m_1m_2}{r_{12}^2c^2}\left[-3\frac{Gm_1}{r_{12}}+\frac{1}{2}\frac{Gm_2}{r_{12}}+\frac{3}{2}v_1^2+\frac{13}{2}v_2^2\right.$$
$$\left.+(n_{12}v_1)^2+2(n_{12}v_1)(n_{12}v_2)-2(n_{12}v_2)^2\right]$$
$$+\frac{G^2m_2^2}{r_{12}^2c^2}\left[-\frac{9}{4}v_2^2+\frac{3}{4}(n_{12}v_2)^2\right]$$
$$+\frac{Gm_2}{r_{12}c^2}[-2v_2^4+(n_{12}v_2)^2v_2^2]+\mathcal{O}(3). \tag{6.5}$$

Inserting these expressions into Eq. (6.3) we obtain the 3PN $\tilde{\mu}_1$ and then straightforwardly compute SI(1). In the quadrupole case $l=2$ it is given by

$$\mathrm{SI}(1)=\int d^3\mathbf{x}\tilde{\mu}_1\hat{x}^{ij}\delta_1+1\leftrightarrow 2$$
$$=\tilde{\mu}_1y_1^{\langle i}y_1^{j\rangle}+1\leftrightarrow 2. \tag{6.6}$$

The final result for circular orbits [using the relations (5.7) and (5.8)] reads then

$$\mathrm{SI}(1)=m\nu\left[1+\frac{\gamma}{2}(1-5\nu)-\frac{\gamma^2}{8}(13-61\nu+25\nu^2)\right.$$
$$\left.+\frac{\gamma^3}{16}(149-573\nu+354\nu^2-29\nu^3)\right]\hat{x}_{ij}. \tag{6.7}$$

The sensitivity of this result to the choice of stress-energy tensor for point-particles (in accordance with the "non-distributivity" of the particle finie) is discussed in Sec. X.

Other interesting terms in this category are

$$\mathrm{SI(16NC)}=\frac{4}{c^6}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_L\sigma_{ab}U_{ab}^{(\mathrm{NC})}, \tag{6.8}$$

and the similar VI(10NC) and VI(12NC). Applying our computation rules we get

$$\mathrm{SI(16NC)}=\frac{4m_1}{c^6}v_1^{ab}[(y_1+r_1n_1)^{\langle i}(y_1+r_1n_1)^{j\rangle}U_{ab}^{(\mathrm{NC})}]_1$$
$$+1\leftrightarrow 2, \tag{6.9}$$

where we have written $x^i=y_1^i+r_1n_1^i$ valid in the vicinity of the point 1. The result follows from applying the regularization (5.2), with the help of the Newtonian approximation of the NC potential. The interesting point is that the regularized factor in Eq. (6.9) is different from $y_1^{\langle i}y_1^{j\rangle}(U_{ab}^{(\mathrm{NC})})_1$ as a consequence of the non-distributivity. See Sec. X.

VII. QUADRATIC TERMS

In this category we consider all the terms whose support is spatially non-compact (hence the finite part operation $\mathrm{FP}_{B=0}$ plays a crucial role), and which are made of the integral of a product of two derivatives of compact-support potentials. Furthermore we subdivide the quadratic terms into subcategories *Y*-, *S*-, and *T*-terms named after the functions Y_L, S_L and T_L defined below, and we classify all these terms according to their dominant post-Newtonian order. The exhaustive list follows. (i) *Y*-terms at 2PN: SI(4), SI(6), SI(7), SII(7), VI(4), VI(5), TI(2); (ii) *Y*-terms at 3PN: SI(31), SI(35C), SI(37C), SI(38C), VI(16), VI(20), VII(6), VI(19), VI(21), VI(25C) VI(26C), VI(27C), VI(29C), TI(6), TI(7), TI(8); (iii) *S*-terms at 3PN: SII(3), SII(4C), SII(5), SII(6), SIII(2), VII(4), VII(5), TII(2); (iv) *T*-terms at 3PN: SI(17), SI(19C), SI(21C), SI(24), SI(25), SII(9), VI(14), VI(15), VI(17), VI(18), TI(5).

The *Y*- and *S*-terms involve the product of two compact-support potentials U, U_i or $U_{ij}^{(\mathrm{C})}$, while the *T*-terms involve a product of one of the latter potentials (of type U) and a potential of the type χ, χ_i or $\chi_{ij}^{(\mathrm{C})}$ [see Eq. (3.7)]. Compared to *Y*-terms, the *S*-terms contain in addition a factor $|\mathbf{x}|^2$ inside their integrand. In the two-body case these compact-support U-type potentials read

$$U=\frac{G\tilde{\mu}_1}{r_1}+1\leftrightarrow 2, \tag{7.1a}$$

$$U_i=\frac{G\mu_1}{r_1}v_1^i+1\leftrightarrow 2, \tag{7.1b}$$

$$U_{ij}^{(\mathrm{C})}=\frac{G\mu_1}{r_1}(v_1^{ij}-\delta^{ij}v_1^2)+1\leftrightarrow 2. \tag{7.1c}$$

The potentials of type χ are obtained by replacing $1/r_1$ by r_1 in these expressions. Then from the structure $\sim 1/r_1+1/r_2$ or $\sim r_1+r_2$ it is not difficult to express all the *Y*-, *S*-, and *T*-terms with the help of three and only three types of elementary integrals Y_L, S_L, and T_L, respectively (where $L=i_1i_2\ldots i_l$ denotes the multipolar index). Two examples in the quadrupole case ij are

$$\mathrm{SI}(4) = -\frac{4G}{c^4}\mu_1[\tilde{\mu}_2 v_1^a v_2^b \,{}_2\partial_{ab}Y_{ij} + \dot{\tilde{\mu}}_2 v_1^a \,{}_2\partial_a Y_{ij}] + 1 \leftrightarrow 2, \tag{7.2a}$$

$$\mathrm{SII}(4\mathrm{C}) = \frac{G}{7c^6}m_1 m_2 \frac{d^2}{dt^2}[(v_2^{ab} - \delta^{ab}v_2^2)\,{}_1\partial_{ab}S_{ij}] + 1 \leftrightarrow 2. \tag{7.2b}$$

[We denote, e.g., ${}_2\partial_a \equiv \partial/\partial y_2^a$.] Since SI(4) is a 2PN term it needs the relative 1PN precision (for simplicity we do not write the post-Newtonian remainders). The elementary integrals are defined by

$$Y_L(\mathbf{y}_1,\mathbf{y}_2) = -\frac{1}{2\pi}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B \frac{\hat{x}_L}{r_1 r_2}, \tag{7.3a}$$

$$S_L(\mathbf{y}_1,\mathbf{y}_2) = -\frac{1}{2\pi}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B |\mathbf{x}|^2 \frac{\hat{x}_L}{r_1 r_2}, \tag{7.3b}$$

$$T_L(\mathbf{y}_1,\mathbf{y}_2) = -\frac{1}{2\pi}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B \hat{x}_L \frac{r_1}{r_2}. \tag{7.3c}$$

In these definitions, the finite part at infinity is absolutely crucial (it comes directly from the formalism [20,21]). However, it is easily seen that the integrals are convergent near the two bodies so the Hadamard partie finie is not needed. The integral Y_L agrees with the definition used in [20,13] and is equivalent with the alternative form proposed in Ref. [18].

We present several derivations of the closed-form expressions of these integrals for arbitrary l. This permits us to introduce some techniques which are necessary when we compute some more complicated integrals in Secs. VIII and IX. The first method consists of writing the multipolarity factor $\hat{x}_L$ in the form

$$\hat{x}_L = \sum_{p=0}^{l}\binom{l}{p} r_1^{\langle P}y_1^{L-P\rangle}, \tag{7.4}$$

where $\binom{l}{p}$ denotes the binomial coefficient (and $\langle\rangle$ refers to the STF projection). Inserting this into the integral Y_L, it is easy to obtain the equivalent expression

$$Y_L = -\frac{1}{2\pi}\sum_{p=0}^{l}\binom{l}{p}\frac{(-)^p}{(2p-1)!!}y_1^{\langle L-P}\,{}_1\partial_{P\rangle} \times\left\{\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B \frac{r_1^{2p-1}}{r_2}\right\}. \tag{7.5}$$

Next we compute the integral inside the curly brackets of Eq. (7.5). Let us show that the polar part of this integral when $B\to 0$ is zero. We replace the integrand by its expansion when $|\mathbf{x}|\to\infty$ (any pole at $B=0$ necessarily comes from the behavior of the integral at infinity), we integrate over the angles and look for radial integrals of the type $\int^{+\infty} d|\mathbf{x}||\mathbf{x}|^{B-1}$ which are the only ones to produce a pole. However these radial integrals do not exist since after the angular integration the powers of $|\mathbf{x}|$ are only of the type $B+2k$ where k is an integer. So the integral in Eq. (7.5) can be computed by analytic continuation down to the value $B=0$. We obtain ($\forall p\in\mathbb{N}$)

$$\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B \frac{r_1^{2p-1}}{r_2} = -\frac{2\pi r_{12}^{2p+1}}{(p+1)(2p+1)}, \tag{7.6}$$

which is a particular case of the Riesz formula [52], valid for any a, $b\in\mathbb{C}$ except at some isolated poles:

$$\int d^3\mathbf{x}\, r_1^a r_2^b = \pi^{3/2}\frac{\Gamma\left(\frac{a+3}{2}\right)\Gamma\left(\frac{b+3}{2}\right)\Gamma\left(-\frac{a+b+3}{2}\right)}{\Gamma\left(-\frac{a}{2}\right)\Gamma\left(-\frac{b}{2}\right)\Gamma\left(\frac{a+b+6}{2}\right)} \times r_{12}^{a+b+3} \tag{7.7}$$

(Γ denotes the Eulerian function). A closely related reasoning to prove Eq. (7.6) is to replace the regularization factor $|\tilde{\mathbf{x}}|^B$ by its expansion when $B\to 0$, i.e.,

$$|\tilde{\mathbf{x}}|^B = \tilde{r}_1^B\left\{1 + \frac{B}{2}\ln\left[1 + 2\frac{(n_1 y_1)}{r_1} + \frac{y_1^2}{r_1^2}\right] + \mathcal{O}(B^2)\right\}. \tag{7.8}$$

Since the integral does not develop any pole when $B\to 0$, the term of order B cannot contribute, nor any of the higher-order terms $\mathcal{O}(B^2)$. This means that we can replace the regularization factor $|\tilde{\mathbf{x}}|^B$ by $\tilde{r}_1^B$ (where $\tilde{r}_1 = r_1/r_0$). From the Riesz formula, with $a = B+2p-1$ and $b=-1$, and computation of the limit $B\to 0$ we get the same result.

Thus, plugging Eq. (7.6) into Eq. (7.5) we find the explicit expression of Y_L as

$$Y_L(\mathbf{y}_1,\mathbf{y}_2) = r_{12}\sum_{p=0}^{l}\binom{l}{p}\frac{(-)^p}{p+1}y_1^{\langle L-P}y_{12}^{P\rangle}, \tag{7.9}$$

where $y_{12}^i = y_1^i - y_2^i$ and $r_{12} = |\mathbf{y}_{12}|$. In terms of y_1^i and y_2^i the expression is simpler:

$$Y_L = \frac{r_{12}}{l+1}\sum_{q=0}^{l} y_1^{\langle L-Q}y_2^{Q\rangle}. \tag{7.10}$$

Using exactly the same method we find for the S_L-integral,

$$\begin{aligned} S_L &= r_{12}\sum_{p=0}^{l}\binom{l}{p}(-)^p y^{\langle L-P}y_{12}^{P\rangle}\left[\frac{\mathbf{y}_{12}^2}{(p+2)(p+3)}\left(p+1-\frac{2l}{3}\right) - \frac{2\mathbf{y}_1\cdot\mathbf{y}_{12}}{p+2} + \frac{\mathbf{y}_1^2}{p+1}\right] \\ &= \frac{r_{12}}{(l+1)(l+2)}\sum_{q=0}^{l} y_1^{\langle L-Q}y_2^{Q\rangle}\left[(l+1-q)\mathbf{y}_1^2 - \frac{2}{3}(q+1) \times(l+1-q)\mathbf{y}_{12}^2 + (q+1)\mathbf{y}_2^2\right], \end{aligned} \tag{7.11}$$

and, for the T_L-integral,

$$T_L = \frac{r_{12}^3}{3} \sum_{p=0}^{l} \binom{l}{p} \frac{(-)^p}{p+2} y_1^{\langle L-P} y_{12}^{P\rangle}$$
$$= \frac{r_{12}^3}{3(l+1)(l+2)} \sum_{q=0}^{l} (q+1) y_1^{\langle L-Q} y_2^{Q\rangle}. \quad (7.12)$$

Notice that S_L can be deduced from T_L and Y_L using the formula

$$S_L = (1 - 2y_1^i\,{}_1\partial_i) T_L + \mathbf{y}_1^2 Y_L. \quad (7.13)$$

The integrals Y_L, S_L and T_L vanish in the limit $\mathbf{y}_1 \to \mathbf{y}_2$. As is clear from the defining expressions (7.3) there is no problem with the latter limit, in the sense that it does not introduce any singularity at the point 1. This justifies *a posteriori* our neglect of all the "self" contributions (proportional to m_1^2 and m_2^2) in the quadratic terms; see the examples given by Eqs. (7.2). However, when we compute the cubic and non-compact terms in Secs. VIII and IX we shall find some important non-zero self contributions.

Another method for the computation of the integrals (7.3) is based on the set of functions defined by

$$g = \ln(r_1 + r_2 + r_{12}), \quad (7.14a)$$

$$f = \frac{1}{6}(r_1^2 + r_2^2 - r_{12}^2)\left(g - \frac{1}{3}\right) + \frac{1}{6}(r_{12}r_1 + r_{12}r_2 - r_1 r_2), \quad (7.14b)$$

$$f^{12} = \frac{1}{6}(r_1^2 + r_{12}^2 - r_2^2)\left(g - \frac{1}{3}\right) + \frac{1}{6}(r_1 r_2 + r_{12}r_2 - r_1 r_{12}), \quad (7.14c)$$

$$f^{21} = \frac{1}{6}(r_2^2 + r_{12}^2 - r_1^2)\left(g - \frac{1}{3}\right) + \frac{1}{6}(r_1 r_2 + r_{12}r_1 - r_2 r_{12}), \quad (7.14d)$$

which satisfy, in the sense of distribution theory,

$$\Delta g = \frac{1}{r_1 r_2}, \quad \Delta_1 g = \frac{1}{r_1 r_{12}}, \quad \Delta_2 g = \frac{1}{r_2 r_{12}}, \quad (7.15a)$$

$$\Delta f = 2g, \quad \Delta_1 f = \frac{r_1}{r_{12}}, \quad \Delta_2 f = \frac{r_2}{r_{12}}, \quad (7.15b)$$

$$\Delta f^{12} = \frac{r_1}{r_2}, \quad \Delta_1 f^{12} = 2g, \quad \Delta_2 f^{12} = \frac{r_{12}}{r_2}, \quad (7.15c)$$

$$\Delta f^{21} = \frac{r_2}{r_1}, \quad \Delta_1 f^{21} = \frac{r_{12}}{r_1}, \quad \Delta_2 f^{21} = 2g, \quad (7.15d)$$

where the Laplacians $\Delta = \partial_i \partial_i$, $\Delta_1 = {}_1\partial_i \cdot {}_1\partial_i$, $\Delta_2 = {}_2\partial_i \cdot {}_2\partial_i$. Let us take the example of the integral Y_L. With the help of Eq. (7.14a) it can be rewritten as

$$Y_L = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L \Delta g. \quad (7.16)$$

We operate the Laplacian by parts, discard the B-dependent surface term which is zero by analytic continuation, and use the formula $\Delta(|\mathbf{x}|^B \hat{x}_L) = B(B+2l+1)|\mathbf{x}|^{B-2}\hat{x}_L$. Hence,

$$Y_L = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \left\{ B(B+2l+1) \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B |\mathbf{x}|^{-2} \hat{x}_L g \right\}. \quad (7.17)$$

Because there is an explicit factor B in front of the integral we need to look only at the polar part when $B \to 0$, which depends only on the behavior of the integrand at the upper bound $r \equiv |\mathbf{x}| \to +\infty$ (this r should not be confused with $r = r_{12}$ as we sometimes denote the orbital separation). Thus we are allowed to replace the function g in Eq. (7.17) by its expansion at infinity. It can be checked that the (simple) pole of the integral in Eq. (7.17) is produced exclusively by the term in the expansion of g of order r^{-l-1}. Let us consider the quadrupole case $l=2$. We have

$$g = \ln(2r) + \frac{1}{r}\{\cdots\} + \frac{1}{r^2}\{\cdots\}$$
$$+ \frac{1}{r^3}\left\{ \frac{r_{12}}{4}[(ny_1)^2 + (ny_1)(ny_2) + (ny_2)^2] + \cdots \right\}$$
$$+ \mathcal{O}\left(\frac{1}{r^4}\right), \quad (7.18)$$

where the dots indicate some terms which yield no contribution to the present computation, either because they do not belong to the relevant order r^{-3} or they will be zero after angular integration. Thus the formula (7.17) becomes in this case

$$Y_{ij} = \underset{B=0}{\mathrm{FP}} \left\{ -2B(B+5) \int^{+\infty} dr \tilde{r}^B r^{-1} \int \frac{d\Omega}{4\pi} \hat{n}_{ij} \frac{r_{12}}{4} \right.$$
$$\left. \times [(ny_1)^2 + (ny_1)(ny_2) + (ny_2)^2] \right\}. \quad (7.19)$$

The notation for the radial integral means that only the bound at infinity contributes to its value. The latter expression is easily transformed into

$$Y_{ij} = \frac{r_{12}}{3}[y_1^{\langle ij\rangle} + y_1^{\langle i} y_2^{j\rangle} + y_2^{\langle ij\rangle}], \quad (7.20)$$

in agreement with the more general result (7.10). The same method works for S_L as well, but one performs two successive integrations by parts using the functions g and f. Concerning T_L, one integration by parts is sufficient but using the function f^{12}.

With the latter expressions of the elementary integrals Y_L, S_L and T_L we obtain all the quadratic terms. The results in the case of circular orbits are displayed in Appendix A.

VIII. CUBIC TERMS

By cubic terms we refer to all the terms which are made of a product between three (derivatives of) compact-support potentials U and U_i [there are no such terms involving the tensor potential $U_{ij}^{(C)}$]. From Eq. (4.2) we can check that the only cubic terms appear at the 3PN order. These are

$$\mathrm{SI}(26),\mathrm{SI}(27),\mathrm{SI}(28),\mathrm{SI}(29),\mathrm{SI}(30),\mathrm{SI}(34),\mathrm{SI}(36),\mathrm{SII}(13),$$
$$\mathrm{VI}(22),\mathrm{VI}(23),\mathrm{VI}(24).$$

Let us proceed in a way similar to the computation of the quadratic terms, i.e., by expressing the terms as functionals of some elementary integrals that are computed separately. Since the cubic terms are 3PN, their computation can be done using the Newtonian potentials

$$U=\frac{Gm_1}{r_1}+\mathcal{O}(2)+1\leftrightarrow 2, \tag{8.1a}$$

$$U_i=\frac{Gm_1}{r_1}v_1^i+\mathcal{O}(2)+1\leftrightarrow 2. \tag{8.1b}$$

For simplicity we gather in one computation the sum of all the cubic terms in SI [and similarly in VI, there is only one cubic term in SII, which is SII(13)]. In the case of mass-type moments we get

$$\begin{aligned}\mathrm{SI}(26+27+28+29+30+34+36)=&\frac{G^2m_1^3}{c^6}\left\{-\frac{32}{15}v_1^iv_1^j\;{}_1\partial_{ij}Y_L{}^{(-3,0)}+\frac{88}{5}v_1^2Y_L{}^{(-5,0)}+\frac{512}{225}v_1^{ab}\;{}_1\partial_{ab}\hat{y}_1^L\right\}\\&+\frac{G^2m_1^2m_2}{c^6}\left\{\left(-\frac{9}{2}v_1^iv_2^j-\frac{5}{2}v_1^iv_1^j\right){}_1\partial_{ij}Y_L{}^{(-2,-1)}-8v_1^iv_2^j\;{}_2\partial_{ij}Y_L{}^{(-2,-1)}\right.\\&+[-2v_1^iv_2^j-6v_1^iv_1^j+8\delta^{ij}(v_1v_2)+8\delta^{ij}v_1^2]\;{}_1\partial_i\;{}_2\partial_jY_L{}^{(-2,-1)}\\&\left.+[15(v_1v_2)+3v_1^2]Y_L{}^{(-4,-1)}\right\}+1\leftrightarrow 2,\end{aligned} \tag{8.2a}$$

$$\mathrm{SII}(13)=\frac{4G^2}{3c^6}\frac{d^2}{dt^2}\{m_1^3Y_L{}^{(-3,0)}+3m_1^2m_2Y_L{}^{(-2,-1)}\}+1\leftrightarrow 2, \tag{8.2b}$$

$$\begin{aligned}\mathrm{VI}(22+23+24)=&\frac{8G^2(2l+1)}{c^6(l+1)(2l+3)}\frac{d}{dt}\left\{-m_1^3v_1^aY_{aL}{}^{(-5,0)}+m_1^2m_2\left[\frac{3}{4}(v_1^k-v_2^k)\;{}_1\partial_a\;{}_2\partial_k\right.\right.\\&\times Y_{aL}{}^{(-2,-1)}-v_1^a\;{}_1\partial_k\;{}_2\partial_kY_{aL}{}^{(-2,-1)}-\frac{3}{16}(v_1^k-v_2^k)\;{}_1\partial_{ak}Y_{aL}{}^{(-2,-1)}\\&\left.\left.-\left(\frac{3}{8}v_1^a+\frac{5}{8}v_2^a\right)Y_{aL}{}^{(-4,-1)}\right]\right\}+1\leftrightarrow 2.\end{aligned} \tag{8.2c}$$

In the case of the current-type moments there are only the VI-terms, which admit a formula analogous to Eq. (8.2c). As we see, we could express all the cubic terms by means of a single type of elementary integral,

$$Y_L{}^{(n,p)}(\mathbf{y}_1,\mathbf{y}_2)=-\frac{1}{2\pi}\mathop{\mathrm{FP}}_{B=0}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_Lr_1^nr_2^p, \tag{8.3}$$

of which some particular cases used in the previous section read $Y_L=Y_L{}^{(-1,-1)}$ and $T_L=Y_L{}^{(1,-1)}$. The integral (8.3) is well-defined in the vicinity of the points $\mathbf{y}_1$ and $\mathbf{y}_2$ only when $n>-3$ and $p>-3$. When this is not the case—for instance the integral $Y_L{}^{(-3,0)}$ appearing in Eq. (8.2)—one should add the Hadamard partie-finie operation Pf defined by Eq. (5.3) and depending *a priori* on two constants u_1 and u_2. According to our convention we generally do not write such parties finies, but they are always implicitly understood.

The integral $Y_L{}^{(-2,-1)}$ is perfectly well-behaved near the two bodies (like Y_L, S_L, and T_L considered in Sec. VII), so it does not need the partie finie. We substitute in it a formula obtained from Eq. (7.4) by exchanging the labels 1 and 2, obtaining

$$\begin{aligned}Y_L{}^{(-2,-1)}=&-\frac{1}{2\pi}\sum_{p=0}^{l}\binom{l}{p}\frac{(-1)^p}{(2p-1)!!}y_2^{\langle L-P}\;{}_2\partial_{P\rangle}\\&\times\left\{\mathop{\mathrm{FP}}_{B=0}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\frac{r_2^{2p-1}}{r_1^2}\right\}.\end{aligned} \tag{8.4}$$

Next we replace the regularization factor $|\tilde{x}|^B$ by its expansion around $B=0$ already written in Eq. (7.8). Since the integral can develop simple poles at most, we can limit ourselves to the first order in B. Then the integral in the brackets of Eq. (8.4) reads

$$\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\frac{r_2^{2p-1}}{r_1^2} = \underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}\tilde{r}_1^B\frac{r_2^{2p-1}}{r_1^2}+\mathrm{FP}_{B=0}\left\{\frac{B}{2}\int d^3\mathbf{x}\tilde{r}_1^B\frac{r_2^{2p-1}}{r_1^2}\times\ln\left[1+2\frac{(n_1y_1)}{r_1}+\frac{y_1^2}{r_1^2}\right]\right\}. \tag{8.5}$$

The first term follows from the Riesz formula (7.7), and the second term depends only on the poles developed by the integral at infinity (because of the explicit factor B in front). Now, contrary to the case of the integral $Y_L\equiv Y_L{}^{(-1,-1)}$ investigated in Sec. VII, we find that this second term gives a net contribution to the integral, straightforwardly obtained from expanding the integrand when $r=|\mathbf{x}|\to+\infty$. The final values that we obtain in the quadrupole and octupole cases ($l=2$ and $l=3$) of interest are

$$Y_{ij}{}^{(-2,-1)}=y_1^{\langle ij\rangle}\left[\frac{16}{15}\ln\tilde{r}_{12}-\frac{188}{225}\right]+y_1^{\langle i}y_2^{j\rangle}\left[\frac{8}{15}\ln\tilde{r}_{12}-\frac{4}{225}\right]+y_2^{\langle ij\rangle}\left[\frac{2}{5}\ln\tilde{r}_{12}-\frac{2}{25}\right], \tag{8.6a}$$

$$Y_{ijk}{}^{(-2,-1)}=y_1^{\langle ijk\rangle}\left[\frac{32}{35}\ln\tilde{r}_{12}-\frac{2552}{3675}\right]+y_1^{\langle ij}y_2^{k\rangle}\left[\frac{16}{35}\ln\tilde{r}_{12}+\frac{124}{3675}\right]+y_1^{\langle i}y_2^{jk\rangle}\left[\frac{12}{35}\ln\tilde{r}_{12}+\frac{66}{1225}\right]+y_2^{\langle ijk\rangle}\left[\frac{2}{7}\ln\tilde{r}_{12}-\frac{2}{49}\right]. \tag{8.6b}$$

Note the occurrence of some logarithms of $\tilde{r}_{12}=r_{12}/r_0$. Applying on these values the point-1 Laplacian $\Delta_1={}_1\partial_{ii}$, and using $\Delta_1 r_1^{-2}=2r_1^{-4}$ (a statement valid in the sense of distributions), we obtain

$$Y_{ij}{}^{(-4,-1)}=\frac{1}{r_{12}^2}\left(\frac{8}{3}y_1^{\langle ij\rangle}-\frac{4}{3}y_1^{\langle i}y_2^{j\rangle}-\frac{1}{3}y_2^{\langle ij\rangle}\right), \tag{8.7a}$$

$$Y_{ijk}{}^{(-4,-1)}=\frac{1}{r_{12}^2}\left(\frac{16}{5}y_1^{\langle ijk\rangle}-\frac{8}{5}y_1^{\langle ij}y_2^{k\rangle}-\frac{2}{5}y_1^{\langle i}y_2^{jk\rangle}-\frac{1}{5}y_2^{\langle ijk\rangle}\right). \tag{8.7b}$$

Alternatively, the results (8.7) can also be obtained by the same technique as used previously for $Y_L{}^{(-2,-1)}$ (i.e., from the Riesz formula and search for the pole at infinity).

The computation of the integral $Y_L{}^{(-3,0)}$, defined by

$$Y_L{}^{(-3,0)}(\mathbf{y}_1)=-\frac{1}{2\pi}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\frac{\hat{x}_L}{r_1^3}, \tag{8.8}$$

is *a priori* more tricky because this integral necessitates the Hadamard partie finie for curing the divergence at the point $\mathbf{y}_1$. Actually, the same method as before, based on the Riesz formula, could be used because we know that the Hadamard partie finie can also be obtained as an analytic continuation (see, e.g., [41]). We prefer here to vary the techniques and to present some other derivations. We split the integration domain $\mathbb{R}^3$ into a ball centered on $\mathbf{y}_1$ with some fixed radius $\mathcal{R}_1$, and the complementary domain, i.e., $r_1>\mathcal{R}_1$. The partie finie applies only on the "inner" domain, surrounding the singularity 1, and the finite part $\mathrm{FP}_{B=0}$ applies only on the integral extending to infinity. Hence,

$$Y_L{}^{(-3,0)}=-\frac{1}{2\pi}\mathrm{Pf}_{u_1}\int_{r_1<\mathcal{R}_1}d^3\mathbf{x}\frac{\hat{x}_L}{r_1^3}-\frac{1}{2\pi}\underset{B=0}{\mathrm{FP}}\int_{r_1>\mathcal{R}_1}d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\frac{\hat{x}_L}{r_1^3}. \tag{8.9}$$

In the first term we recall that the partie finie depends on a constant u_1 [see the definition (5.3)]. For this term we readily find

$$-\frac{1}{2\pi}\mathrm{Pf}_{u_1}\int_{r_1<\mathcal{R}_1}d^3\mathbf{x}\frac{\hat{x}_L}{r_1^3}=-2\hat{y}_1^L\ln\left(\frac{\mathcal{R}_1}{u_1}\right). \tag{8.10}$$

On the other hand, one must replace into the second term the factor $|\tilde{\mathbf{x}}|^B$ by its B-expansion as given by Eq. (7.8). This yields two contributions: one is immediately computed using the properties of the analytic continuation, the other contains an explicit factor B and therefore relies on the existence of poles at infinity:

$$-\frac{1}{2\pi}\underset{B=0}{\mathrm{FP}}\int_{r_1>\mathcal{R}_1}d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\frac{\hat{x}_L}{r_1^3} =2\hat{y}_1^L\ln\left(\frac{\mathcal{R}_1}{r_0}\right)-\frac{1}{2\pi}\mathrm{FP}_{B=0}\left\{\frac{B}{2}\int_{\mathcal{R}_1}^{+\infty}d^3\mathbf{x}\tilde{r}_1^B\frac{\hat{x}_L}{r_1^3}\times\ln\left[1+2\frac{(n_1y_1)}{r_1}+\frac{y_1^2}{r_1^2}\right]\right\}. \tag{8.11}$$

As expected, the sum of the two contributions (8.10) and (8.11) is independent of the intermediate length scale $\mathcal{R}_1$. Indeed, the integral in the second term of Eq. (8.11) does not in fact depend on $\mathcal{R}_1$ as it depends only on the infinite bound. We obtain

$$Y_L{}^{(-3,0)} = 2\hat{y}_1^L \ln\left(\frac{u_1}{r_0}\right) - \frac{1}{4\pi}\mathrm{FP}_{B=0}\left\{B\int^{+\infty} d^3\mathbf{x}\tilde{r}_1^B \frac{\hat{x}_L}{r_1^3} \times \ln\left[1+2\frac{(n_1y_1)}{r_1}+\frac{y_1^2}{r_1^2}\right]\right\}. \tag{8.12}$$

The computation of the second term proceeds along the same line as for the reduction of Y_L in Eq. (7.17). We expand the log-term up for instance to the order $1/r_1^2$ necessary to get the quadrupole case $l=2$,

$$\ln\left[1+2\frac{(n_1y_1)}{r_1}+\frac{y_1^2}{r_1^2}\right] = 2\frac{(n_1y_1)}{r_1}+\frac{1}{r_1^2}[-2(n_1y_1)^2+y_1^2] + \mathcal{O}\left(\frac{1}{r_1^3}\right). \tag{8.13}$$

Therefore,

$$Y_{ij}{}^{(-3,0)} = 2\hat{y}_1^{ij}\ln\left(\frac{u_1}{r_0}\right) - \mathrm{FP}_{B=0}\left\{B\int^{+\infty} dr_1 \tilde{r}_1^B r_1^{-1} \times \int\frac{d\Omega_1}{4\pi}\hat{n}_1^{ij}[4n_1^{\langle i}y_1^{j\rangle}(n_1y_1)-2n_1^{\langle ij\rangle}(n_1y_1)^2]\right\}. \tag{8.14}$$

The integral follows immediately. This method yields the results (cases $l=2,3$)

$$Y_{ij}{}^{(-3,0)} = \left[2\ln\left(\frac{u_1}{r_0}\right)+\frac{16}{15}\right]y_1^{\langle ij\rangle}, \tag{8.15a}$$

$$Y_{ijk}{}^{(-3,0)} = \left[2\ln\left(\frac{u_1}{r_0}\right)+\frac{142}{105}\right]y_1^{\langle ijk\rangle}. \tag{8.15b}$$

The results depend on the Hadamard-regularization constant u_1.

We present another derivation of the integral $Y_L{}^{(-3,0)}$, based on the interesting formula of distribution theory (see, e.g., [33])

$$\Delta\left[\frac{1}{r_1}\ln\left(\frac{r_1}{u_1}\right)\right] = -\mathrm{Pf}_{u_1}\left(\frac{1}{r_1^3}\right)+4\pi\delta(\mathbf{x}-\mathbf{y}_1). \tag{8.16}$$

[Notice the sign of the distributional term, $+4\pi\delta_1$, opposite to the sign in the more famous formula $\Delta(1/r_1)=-4\pi\delta_1$.] With Eq. (8.16) one can re-express $Y_L{}^{(-3,0)}$ in the form

$$Y_L{}^{(-3,0)} = -2\hat{y}_1^L + \frac{1}{2\pi}\mathrm{FP}_{B=0}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_L\Delta\left[\frac{1}{r_1}\ln\left(\frac{r_1}{u_1}\right)\right]. \tag{8.17}$$

Here the first term comes from the delta-function in Eq. (8.16). Integrating the second term by parts, we get

$$Y_L{}^{(-3,0)} = -2\hat{y}_1^L + \frac{1}{2\pi}\mathrm{FP}_{B=0} \times\left\{B(B+2l+1)\int^{+\infty} d^3\mathbf{x}|\tilde{\mathbf{x}}|^B|\mathbf{x}|^{-2}\frac{\hat{x}_L}{r_1}\ln\left(\frac{r_1}{u_1}\right)\right\}. \tag{8.18}$$

Following the same principle as before, we compute the remaining integral by looking at the pole at infinity. The result is in agreement with the earlier derivation (as we checked in the case $l=2$). Let us also mention that still another method to compute $Y_L{}^{(-3,0)}$ consists of taking the limit $\mathbf{y}_2\to\mathbf{y}_1$ of the integral $Y_L{}^{(-2,-1)}$. The limit is singular since $Y_L{}^{(-2,-1)}$ diverges when the two particles merge together. In fact the limit must be taken in the sense of the Hadamard partie finie (5.2). Indeed, applying Eq. (5.5) in Ref. [41], we obtain the following limit relation between $Y_L{}^{(-3,0)}$ and $Y_L{}^{(-2,-1)}$:

$$Y_L{}^{(-3,0)}(\mathbf{y}_1) = \left(Y_L{}^{(-2,-1)}(\mathbf{y}_1,\mathbf{x}) - 2\left[\ln\left(\frac{r_1}{u_1}\right)-1\right]y_1^{\langle L\rangle}\right)_1. \tag{8.19}$$

Inserting for instance the result for $Y_{ij}{}^{(-2,-1)}$ obtained in Eq. (8.6a) we recover exactly the function $Y_{ij}{}^{(-3,0)}$ given by Eq. (8.15a).

Finally it is easy to see that the function $Y_L{}^{(-5,0)}$, also needed in the cubic terms (8.2), is identically zero. We apply the point-1 Laplacian Δ_1 onto the expression of $Y_L{}^{(-3,0)}$ using the known formula of distribution theory

$$\Delta_1\left(\frac{1}{r_1^3}\right) = \frac{6}{r_1^5} - \frac{10\pi}{3}\Delta_1\delta_1, \tag{8.20}$$

and readily obtain, for any l,

$$Y_L{}^{(-5,0)} = 0. \tag{8.21}$$

The results for the cubic terms in the case of circular orbits are reported in the Appendix.

IX. NON-COMPACT TERMS

The most difficult part of the present analysis is the computation of the so-called "non-compact" terms, which are cubically nonlinear terms (like the cubic terms) made of the product of a compact-support potential like U and a quadratic "non-compact" potential like $U_{ij}^{(\mathrm{NC})}$. The complete list of non-compact terms is

SI(5NC),SI(19NC),SI(20),SI(21NC),SI(33NC),SI(35NC), SI(37NC),SI(38NC),SII(4NC),VI(25NC),VI(26NC), VI(27NC),VI(28NC),VI(29NC).

A. Expressions of the NC terms

As before, here again our strategy is to express the non-compact terms as functionals of certain elementary integrals, that are computed separately. We substitute inside the sources of non-compact terms the appropriate post-Newtonian potentials computed for two particles on a general orbit. The compact potentials U, U_i, and $U^{(\mathrm{C})}_{ij}$ (and similar expressions for the χ's) were already given by Eq. (7.1). Here we list all the non-compact potentials needed for this computation [see Eqs. (3.7) and (3.8) for definitions]. The potential $U^{(\mathrm{NC})}_{ij}$ is the only one which is needed at 1PN order; the other potentials are Newtonian:

$$U^{(\mathrm{NC})}_{ij}=-\frac{1}{8}\tilde{\mu}_1^2\left(\partial_{ij}\ln r_1+\frac{\delta_{ij}}{r_1^2}\right)-\tilde{\mu}_1\tilde{\mu}_2\,{}_ig_j+1\leftrightarrow 2, \tag{9.1a}$$

$$\chi^{(\mathrm{NC})}_{ij}=-\frac{1}{4}m_1^2\left\{\partial_{ij}\left[\frac{r_1^2}{6}\left(\ln r_1-\frac{5}{6}\right)\right]+\delta_{ij}\ln r_1\right\}-m_1m_2\,{}_if_j+1\leftrightarrow 2, \tag{9.1b}$$

$$\hat{R}^{(\mathrm{NC})}_{i}=-\frac{1}{16}m_1^2v_1^k\left(\partial_{ik}\ln r_1+\frac{\delta_{ik}}{r_1^2}\right)-2m_1m_2\left(v_1^k-\frac{3}{4}v_2^k\right){}_ig_k+1\leftrightarrow 2, \tag{9.1c}$$

$$\begin{aligned}\hat{Z}^{(\mathrm{NC})}_{ij}=m_1^2\bigg\{&a_1^{(i}\ {}_1\partial_{j)}\ln r_1+\frac{1}{8}v_1^2\ {}_1\partial_{ij}\ln r_1\\&+\frac{1}{32}\delta^{ij}v_1^{km}\ {}_1\partial_{km}\ln r_1+\frac{1}{2}\frac{v_1^{ij}}{r_1^2}-\frac{11}{32}\delta^{ij}\frac{v_1^2}{r_1^2}\bigg\}\\&+m_1m_2\bigg\{2a_1^{(i}g_{j)}+2v_1^{k(i}\ {}_kg_{j)}-2v_1^{(i}v_2^k\ {}_kg_{j)}\\&+(v_1v_2)_{(i}g_{j)}+v_1^{(i}v_2^{j)}\ {}_kg_k-\frac{3}{4}\delta_{ij}v_1^kv_2^m\ {}_kg_m\\&+\delta_{ij}v_1^mv_2^k\ {}_kg_m-\delta_{ij}(v_1v_2)\ {}_kg_k\bigg\}+1\leftrightarrow 2,\end{aligned} \tag{9.1d}$$

$$\begin{aligned}K_{ij}=m_1^2\bigg\{&\left[\frac{1}{48}a_1^k\ {}_1\partial_{ijk}+\frac{1}{96}v_1^{km}\ {}_1\partial_{ijkm}\right](r_1^2\ln r_1)\\&+\left[-\frac{1}{8}\delta^{ij}a_1^k\ {}_1\partial_k+\frac{1}{4}a_1^{(i}\ {}_1\partial_{j)}-\frac{1}{16}\delta^{ij}v_1^{km}\ {}_1\partial_{km}\right.\\&\left.+\frac{1}{16}v_1^2\ {}_1\partial_{ij}\right](\ln r_1)+\left[\frac{1}{16}\delta^{ij}v_1^2+\frac{1}{8}v_1^{ij}\right](r_1^{-2})\bigg\}\\&+m_1m_2[a_1^k\ {}_1\partial_k+v_1^{km}\ {}_1\partial_{km}]_{(i}f_{j)}{}^{12}+1\leftrightarrow 2.\end{aligned} \tag{9.1e}$$

Here, g, f, f^{12} and f^{21} are defined by Eq. (7.14), and we denote, e.g., ${}_ig_j={}_1\partial_i\,{}_2\partial_jg$ (see Ref. [51] for the expression of ${}_ig_j$); the acceleration is $a_1^i=dv_1^i/dt$; the parenthesis around indices denotes the symmetrization (and $G=1$).

Notice that we have chosen to express the non-compact potentials by means of g, f, f^{12}, and f^{21}. But these functions constitute merely some *particular* solutions of the Laplace equations (7.15) we have to solve, and the question arises of which solution is the correct one. The most general solution will be obtained by adding to the particular one a *homogeneous* term, solving a source-free Laplace-type equation. We have checked that the only possible homogeneous solutions, that are regular at the origin, are constants or linear functions of the position, and that these are always either canceled by some spatial or time derivatives, or disappear at the end of our computations. This justifies our use of the particular solutions (7.14). (Similarly, we found that the same happens in the computation of the 3PN equations of motion, where these particular solutions are sufficient [43].)

The potentials (9.1) contain a "self" part, proportional to m_1^2 or m_2^2 (before replacement of the accelerations), and an "interaction" part, proportional to m_1m_2. Similarly the sources of the non-compact terms will involve a self part, proportional to m_1^3 or m_2^3, and an interaction part, proportional to $m_1^2m_2$ or $m_1m_2^2$. At the 2PN level, all the self parts canceled out in the multipole moments [13]. At the 3PN level, we shall find that the self parts bring a contribution to the moments. [Actually, we shall argue in Sec. X that the self parts are unknown.] For treating the NC terms we used the standard distributional derivative [32,33]. Thus, we have, for instance,

$$\Delta\frac{1}{r_1}=-4\pi\delta_1, \tag{9.2a}$$

$$\partial_{ij}\left(\frac{1}{r_1}\right)=\frac{3n_1^{ij}-\delta^{ij}}{r_1^3}-\frac{4\pi}{3}\delta^{ij}\delta_1, \tag{9.2b}$$

$$\Delta\left(\frac{1}{r_1^3}\right)=\frac{6}{r_1^5}-\frac{10\pi}{3}\Delta\delta_1, \tag{9.2c}$$

$$\partial_{ij}\left(\frac{1}{r_1^3}\right)=\frac{15n_1^{ij}-3\delta^{ij}}{r_1^5}-\frac{2\pi}{5}\delta^{ij}\Delta\delta_1-\frac{32\pi}{15}\partial_{ij}\delta_1. \tag{9.2d}$$

However, the use of the standard Schwartz derivative can be justified only when the terms involved are multiplied by some smooth functions. In the case of the self parts of NC terms, this will not be true in general, so the Schwartz derivative gives some ill-defined contributions, composed of the product of a delta-function and a singular function. In Sec. X we consider a well-defined way to do the computation of the self terms, which is based on the distributional derivatives proposed in Ref. [41]. From the discussion in Sec. X we conclude that one must add to the present computation some undetermined terms taking into account the ambiguities in the choice of the regularization and distributional derivatives. All the expressions in Eq. (9.3) below are modulo these ill-defined contributions and we can safely proceed with the knowledge that our procedure is unambiguous and complete. We are securely protected from such ill-defined contributions at this stage since we shall add such terms with an arbitrary coefficient in Sec. X. We obtain the following expressions of the non-compact terms, as functionals of several new types of elementary integrals (we pose $D={}_1\partial_i\,{}_2\partial_i$ and $G=1$). In the case of the mass-type moments:

$$\mathrm{SI}(5\mathrm{NC})=\frac{\tilde{\mu}_1^3}{c^4}Y_L{}^{(-5,0)}+\frac{\tilde{\mu}_1^2\tilde{\mu}_2}{c^4}\left\{-\frac{1}{2}\hat{y}_2^L r_{12}^{-2}-\frac{1}{4}D^2N_L{}^{(0,-1)}-4\,{}_2\partial_s G_L^s\right\}+1\leftrightarrow 2, \tag{9.3a}$$

$$\begin{aligned}\mathrm{SI}(19\mathrm{NC})=&\frac{m_1^3}{c^6}\left[-\frac{1}{30}v_1^{ab}\,{}_1\partial_{ab}Y_L{}^{(-3,0)}+\frac{2}{5}v_1^2Y_L{}^{(-5,0)}+\frac{8}{225}v_1^{ab}\,{}_1\partial_{ab}\hat{y}_1^L\right]\\&+\frac{m_1^2m_2}{c^6}\left\{\left(-\frac{1}{2}\frac{(n_{12}a_1)}{r_{12}}+\frac{(n_{12}v_1)^2}{r_{12}^2}-\frac{1}{2}\frac{v_1^2}{r_{12}^2}\right)\hat{y}_2^L-\frac{1}{24}(a_1^a\,{}_1\partial_a+v_1^{ab}\,{}_1\partial_{ab})D^2\right.\\&\left.\times\left[N_L{}^{(2,-1)}-\frac{5}{6}Y_L{}^{(2,-1)}\right]-2\left(\frac{d}{dt}\right)^2{}_2\partial_iF_L^i-4v_1^j\frac{d}{dt}({}_2\partial_iF_L^{ij})-2a_1^j\,{}_2\partial_iF_L^{ij}-2v_1^{jk}\,{}_2\partial_iF_L^{ijk}\right\}+1\leftrightarrow 2,\end{aligned} \tag{9.3b}$$

$$\begin{aligned}\mathrm{SI}(20)=&\frac{m_1^3}{c^6}\left\{\left[\frac{1}{3}a_1^a\,{}_1\partial_a-\frac{1}{15}v_1^{ab}\,{}_1\partial_{ab}\right]Y_L{}^{(-3,0)}+\frac{4}{5}v_1^2Y_L{}^{(-5,0)}-\frac{2}{9}a_1^s\,{}_1\partial_s\hat{y}_1^L+\frac{16}{225}v_1^{ab}\,{}_1\partial_{ab}\hat{y}_1^L\right\}\\&+\frac{m_1^2m_2}{c^6}\left\{\left[\frac{1}{2}(n_{12}a_1)r_{12}^{-1}-\frac{1}{2}(n_{12}v_1)^2r_{12}^{-2}\right]\hat{y}_2^L+\left[-\frac{1}{24}a_1^a\,{}_1\partial_a-\frac{1}{48}v_1^{ab}\,{}_1\partial_{ab}\right]D^2N_L{}^{(2,-1)}\right.\\&+\left[-\frac{1}{2}a_1^a\,{}_2\partial_a-\frac{1}{8}v_1^2D\right]DN_L{}^{(0,-1)}-\frac{1}{4}v_1^{ab}\,{}_2\partial_{ab}Y_L{}^{(-2,-1)}-2[a_2^a\,{}_2\partial_a+v_2^{ab}\,{}_2\partial_{ab}]_2\partial_sF_L^{s21}-2[a_1^a\,{}_1\partial_a\\&\left.+v_a^{ab}\,{}_1\partial_{ab}]_2\partial_sF_L^{s12}-2[a_1^a+2v_1^{ab}\,{}_1\partial_b]_2\partial_sF_L^{sa12}-2v_1^{ab}\,{}_2\partial_sF_L^{sab12}\right\}+1\leftrightarrow 2,\end{aligned} \tag{9.3c}$$

$$\begin{aligned}\mathrm{SI}(21\mathrm{NC})=&\frac{m_1^3}{c^6}\left\{\left[-\frac{1}{6}a_1^s\,{}_1\partial_s-\frac{2}{15}v_1^{su}\,{}_1\partial_{su}\right]Y_L{}^{(-3,0)}+\frac{3}{5}v_1^2Y_L{}^{(-5,0)}+\frac{1}{9}a_1^s\,{}_1\partial_s\hat{y}_1^L+\frac{32}{225}v_1^{ab}\,{}_1\partial_{ab}\hat{y}_1^L\right\}\\&+\frac{m_1^2m_2}{c^6}\left\{-\frac{1}{8}(a_2^a\,{}_2\partial_a+v_2^{ab}\,{}_2\partial_{ab})D^2N_L{}^{(0,1)}-\frac{1}{4}(a_2^a\,{}_2\partial_a+v_2^{ab}\,{}_2\partial_{ab})Y_L{}^{(-2,-1)}+2a_1^a\,{}_2\partial_sH_L^{as}\right.\\&\left.-2v_1^{ab}\,{}_2\partial_sH_L^{abs}\right\}+1\leftrightarrow 2,\end{aligned} \tag{9.3d}$$

$$\begin{aligned}\mathrm{SI}(33\mathrm{NC})=&\frac{m_1^3}{c^6}\left[\frac{2}{15}v_1^{ab}\,{}_1\partial_{ab}Y_L{}^{(-3,0)}-\frac{8}{5}v_1^2Y_L{}^{(-5,0)}-\frac{32}{225}v_1^{ab}\,{}_1\partial_{ab}\hat{y}_1^L\right]+\frac{m_1^2m_2}{c^6}\{-v_1^av_2^b[{}_1\partial_{ab}DN_L{}^{(0,-1)}\\&+{}_1\partial_a\,{}_2\partial_bY_L{}^{(-2,-1)}]+8v_1^b\partial_t[{}_2\partial_bG_L+{}_2\partial_a({}_bG_L^a)]+8v_1^{bc}[{}_2\partial_b(G_L^c)+{}_2\partial_a({}_bG_L^{ac})]\}+1\leftrightarrow 2,\end{aligned} \tag{9.3e}$$

$$\begin{aligned}\mathrm{SI}(35\mathrm{NC})=&\frac{m_1^3}{c^6}\left[-\frac{16}{3}a_1^i\,{}_1\partial_iY_L{}^{(-3,0)}+\frac{4}{5}v_1^{ij}\partial_{1ij}Y_L{}^{(-3,0)}-\frac{28}{5}v_1^2Y_L{}^{(-5,0)}+\frac{32}{9}a_1^s\,{}_1\partial_s\hat{y}_1^L-\frac{64}{75}v_1^{ab}\,{}_1\partial_{ab}\hat{y}_1^L\right]+\frac{m_1^2m_2}{c^6}\\&\times\{[-5v_1^2-(n_{12}v_1)^2]\hat{y}_2^Lr_{12}^{-2}+[12(v_1v_2)+8(n_{12}v_1)(n_{12}v_2)]\hat{y}_1^Lr_{12}^{-2}+8a_1^i\,{}_2\partial_iDN_L{}^{(0,-1)}+v_1^2D^2\\&\times N_L{}^{(0,-1)}+4v_1^{ij}\,{}_2\partial_{ij}Y_L{}^{(-2,-1)}+16a_1^i\,{}_2\partial_jK_L^{ij}+16[a_2^i-v_2^i(v_1^k-v_2^k)_2\partial_k+(v_1v_2)_2\partial_i]G_L^i\\&+16v_1^i(v_1^k-v_2^k)_2\partial_j\,{}_kG_L^{ij}+16v_1^iv_2^jU_L^{ij}\}+1\leftrightarrow 2,\end{aligned} \tag{9.3f}$$

$$\mathrm{SI(37NC)}=\frac{m_1^3}{c^6}\left[\frac{1}{15}v_1^{ab}\ {}_1\partial_{ab}Y_L{}^{(-3,0)}-\frac{4}{5}v_1^2Y_L{}^{(-5,0)}-\frac{16}{225}v_1^{ab}\ {}_1\partial_{ab}\hat{y}_1^L\right]+\frac{m_1^2m_2}{c^6}\left\{\frac{1}{2}v_1^a v_2^b({}_1\partial_a\ {}_2\partial_b DN_L{}^{(0,-1)}\right.$$
$$\left.+\ {}_2\partial_{ab}Y_L{}^{(-2,-1)})+16v_1^a\left(v_1^b-\frac{3}{4}v_2^b\right){}_2\partial_b G_L^a+16v_1^a\left(v_2^b-\frac{3}{4}v_1^b\right){}_2\partial_c({}_bG_L^{ac})\right\}+1\leftrightarrow 2, \tag{9.3g}$$

$$\mathrm{SI(38NC)}=\frac{m_1^3}{c^6}\left[-\frac{2}{15}v_1^{ab}\ {}_1\partial_{ab}Y_L{}^{(-3,0)}+\frac{8}{5}v_1^2Y_L{}^{(-5,0)}+\frac{32}{225}v_1^{ab}\ {}_1\partial_{ab}\hat{y}_1^L\right]+\frac{m_1^2m_2}{c^6}\left\{v_1^a v_2^b({}_1\partial_{ab}DN_L{}^{(0,-1)}\right.$$
$$+\ {}_2\partial_a\ {}_1\partial_b Y_L{}^{(-2,-1)})-32v_1^j\left(v_1^k-\frac{3}{4}v_2^k\right){}_2\partial_k(G_L^j)-32v_1^j\left(v_2^k-\frac{3}{4}v_1^k\right){}_2\partial_i({}_kG_L^{ij})+32v_1^j$$
$$\left.\times\left(v_1^k-\frac{3}{4}v_2^k\right){}_2\partial_k(I_{L(j)})+32v_1^j\left(v_2^k-\frac{3}{4}v_1^k\right){}_2\partial_i({}_kI^i_{L(j)})\right\}+1\leftrightarrow 2, \tag{9.3h}$$

$$\mathrm{SII(4NC)}=\frac{m_1^3}{c^6}\partial_t^2\left[\frac{1}{14}S_{ij}{}^{(-5,0)}\right]+\frac{m_1^2m_2}{c^6}\partial_t^2\left[-\frac{1}{28}\hat{y}_2^{ij}\mathbf{y}_2^2r_{12}^{-2}-\frac{1}{56}D^2M_{ij}{}^{(0,-1)}-\frac{2}{7}\ {}_2\partial_b(Q^b_{ij})\right]+1\leftrightarrow 2, \tag{9.3i}$$

$$\mathrm{VI(25NC)}=\frac{m_1^3}{c^6}\frac{d}{dt}\left[\frac{2}{63}v_1^l\ {}_1\partial_{al}Y_{aij}{}^{(-3,0)}-\frac{8}{21}v_1^aY_{aij}{}^{(-5,0)}-\frac{32}{945}v_1^k\ {}_1\partial_{ak}\hat{y}_1^{aij}\right]$$
$$+\frac{m_1^2m_2}{c^6}\frac{d}{dt}\left\{-\frac{5}{21}v_1^l\ {}_1\partial_{al}DN_{aij}{}^{(0,-1)}-\frac{5}{21}v_1^l\ {}_1\partial_a\ {}_2\partial_l Y_{aij}{}^{(-2,-1)}\right.$$
$$+\frac{160}{21}\left(v_2^l-\frac{3}{4}v_1^l\right){}_2\partial_k({}_lG^{ak}_{aij})+\frac{160}{21}\left(v_1^l-\frac{3}{4}v_2^l\right){}_2\partial_l(G^a_{aij})$$
$$\left.-\frac{160}{21}\left(v_2^l-\frac{3}{4}v_1^l\right){}_2\partial_k({}_lI^k_{aij(a)})-\frac{160}{21}\left(v_1^l-\frac{3}{4}v_2^l\right){}_2\partial_l I_{aij(a)}\right\}+1\leftrightarrow 2, \tag{9.3j}$$

VI(26NC+27NC+28NC+29NC)

$$=\frac{m_1^2}{c^6}\frac{d}{dt}\left[-\frac{2}{63}v_1^s\ {}_1\partial_{sa}Y_{aij}{}^{(-3,0)}-\frac{4}{7}v_1^aY_{aij}{}^{(-5,0)}+\frac{32}{945}v_1^k\ {}_1\partial_{ak}\hat{y}_1^{aij}\right]+\frac{m_1^2m_2}{c^6}\frac{d}{dt}\left\{\frac{10}{21}v_2^a\hat{y}_2^{aij}r_{12}^{-2}\right.$$
$$+\frac{5}{21}v_2^aD^2N_{aij}{}^{(0,-1)}+\frac{5}{21}v_1^s\ {}_1\partial_{as}DN_{aij}{}^{(0,-1)}+\frac{5}{21}v_1^s\ {}_1\partial_s\ {}_2\partial_a Y_{aij}{}^{(-2,-1)}+\frac{5}{21}v_2^k\ {}_1\partial_a\ {}_2\partial_k Y_{aij}{}^{(-2,-1)}$$
$$-\frac{5}{21}v_2^k\ {}_1\partial_k\ {}_2\partial_a Y_{aij}{}^{(-2,-1)}+\frac{80}{21}v_1^a\ {}_2\partial_l(G^l_{aij})-\frac{40}{21}v_1^l\ {}_2\partial_l(G^a_{aij})$$
$$-\frac{40}{21}v_1^l\ {}_2\partial_k({}_lG^{ka}_{aij})-\frac{40}{21}\frac{d}{dt}[{}_2\partial_a(G_{aij})+\ {}_2\partial_k({}_aG^k_{aij})]+\frac{40}{21}v_1^l\ {}_2\partial_l I_{aij(a)}+\frac{40}{21}v_1^l\ {}_2\partial_k({}_lI^k_{aij(a)})$$
$$\left.-\frac{40}{21}v_1^l\ {}_2\partial_a(I_{aij(l)})-\frac{40}{21}v_1^l\ {}_2\partial_k({}_aI^k_{aij(l)})\right\}+1\leftrightarrow 2. \tag{9.3k}$$

We have similar expressions (involving VI-type terms) for the current moments. The elementary integrals parametrizing the NC terms include some generalizations of the integrals already introduced in Sec. VIII,

$$Y_L{}^{(n,p)}=-\frac{1}{2\pi}\mathop{\mathrm{FP}}_{B=0}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_Lr_1^nr_2^p, \tag{9.4a}$$

$$S_L{}^{(n,p)}=-\frac{1}{2\pi}\mathop{\mathrm{FP}}_{B=0}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_L|\mathbf{x}|^2r_1^nr_2^p, \tag{9.4b}$$

$$N_L{}^{(n,p)} = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L r_1^n r_2^p \ln \tilde{r}_1 , \tag{9.4c}$$

$$M_L{}^{(n,p)} = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L |\mathbf{x}|^2 r_1^n r_2^p \ln \tilde{r}_1 . \tag{9.4d}$$

As usual the Hadamard partie finie Pf is to be added when the integral diverges near the particles. The logarithms in Eqs. (9.4c) and (9.4d) contain the constant r_0 through the notation $\tilde{r}_1 = r_1/r_0$. In addition we have the more involved integrals

$$G_L^P = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L \partial_{aP}\left(\frac{1}{r_1}\right) {}_a g, \tag{9.5a}$$

$${}_bG_L^{aP} = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L \partial_{aP}\left(\frac{1}{r_1}\right) {}_b g, \tag{9.5b}$$

$$K_L^P = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L \partial_{P}\left(\frac{1}{r_1}\right) g, \tag{9.5c}$$

$$U_L^P = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L \partial_{P}\left(\frac{1}{r_1}\right) {}_k g_k , \tag{9.5d}$$

$$F_L^P = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L \partial_{aP}\left(\frac{1}{r_1}\right) {}_a f, \tag{9.5e}$$

$$F_L^{P12} = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L \partial_{aP}\left(\frac{1}{r_1}\right) {}_a f^{12}, \tag{9.5f}$$

$$F_L^{P21} = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L \partial_{aP}\left(\frac{1}{r_1}\right) {}_a f^{21}, \tag{9.5g}$$

$$H_L^P = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L \partial_{iP}(r_1)\, {}_i g, \tag{9.5h}$$

$$Q_L^P = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L |\mathbf{x}|^2 \partial_{aP}\left(\frac{1}{r_1}\right) {}_a g, \tag{9.5i}$$

$${}_kI^i_{L(j)} = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L \partial_j\left[\partial_i\left(\frac{1}{r_1}\right) {}_k g\right], \tag{9.5j}$$

$$I_{L(j)} = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L \partial_j\left[\partial_i\left(\frac{1}{r_1}\right) {}_i g\right]. \tag{9.5k}$$

The notation is, e.g., ${}_af = {}_1\partial_a f$, $g_b = {}_2\partial_b g$, ${}_ag_b = {}_1\partial_a\, {}_2\partial_b g$ (notably ${}_kg_k = Dg$). The last two integrals are related to some previous ones by

$${}_kI^i_{L(j)} = -({}_1\partial_j + {}_2\partial_j)\, {}_kG^i_L , \tag{9.6a}$$

$$I_{L(j)} = -({}_1\partial_j + {}_2\partial_j) G_L . \tag{9.6b}$$

B. Computation of the elementary integrals

The techniques developed in Secs. VII and VIII can be used to compute many of these integrals. Concerning $S_L^{(n,p)}$ we need only the particular case $l=2$ and $(n,p)=(-5,0)$. It is computed by the same methods as used for $Y_{ij}{}^{(-3,0)}$; we find

$$S_{ij}{}^{(-5,0)} = \left[\frac{14}{3}\ln\left(\frac{u_1}{r_0}\right) + \frac{8}{5}\right] y_1^{\langle ij\rangle} . \tag{9.7}$$

Next the group of integrals constituted by the $N_L{}^{(n,p)}$'s and $M_L{}^{(n,p)}$'s is obtained in a fashion similar to the one employed for $Y_L{}^{(-2,-1)}$ in Sec. VIII, i.e., basically by application of the Riesz formula. The logarithms in these integrals are included by differentiating with respect to the complex parameter B. The relevant results are

$$N_{ij}{}^{(0,-1)}=y_1{}^{\langle ij\rangle}\left\{\frac{8}{105}r_{12}^2\left[\ln\tilde{r}_{12}-\frac{247}{210}\right]-\frac{2}{105}y_1^2-\frac{4}{35}(y_1y_2)+\frac{4}{105}y_2^2\right\}+y_1^{\langle i}y_2^{j\rangle}\left\{\frac{4}{35}r_{12}^2\left[\ln\tilde{r}_{12}-\frac{59}{70}\right]+\frac{2}{105}y_1^2-\frac{4}{35}(y_1y_2)\right\}+y_2^{\langle ij\rangle}\left\{\frac{1}{7}r_{12}^2\left[\ln\tilde{r}_{12}-\frac{9}{14}\right]-\frac{1}{35}y_1^2\right\}, \tag{9.8a}$$

$$N_{ijk}{}^{(0,-1)}=y_1^{\langle ijk\rangle}\left\{\frac{16}{315}r_{12}^2\left[\ln\tilde{r}_{12}-\frac{811}{630}\right]-\frac{4}{945}y_1^2-\frac{88}{945}(y_1y_2)+\frac{4}{105}y_2^2\right\}+y_1^{\langle ij}y_2^{k\rangle}\left\{\frac{8}{105}r_{12}^2\left[\ln\tilde{r}_{12}-\frac{601}{630}\right]+\frac{2}{63}y_1^2-\frac{4}{35}(y_1y_2)+\frac{4}{105}y_2^2\right\}+y_1^{\langle i}y_2^{jk\rangle}\left\{\frac{2}{21}r_{12}^2\left[\ln\tilde{r}_{12}-\frac{95}{126}\right]+\frac{1}{35}y_1^2-\frac{2}{21}(y_1y_2)\right\}+y_2^{\langle ijk\rangle}\left\{\frac{1}{9}r_{12}^2\left[\ln\tilde{r}_{12}-\frac{11}{18}\right]-\frac{1}{63}y_1^2\right\}, \tag{9.8b}$$

$$N_{ij}{}^{(2,-1)}=y_1^{\langle ij\rangle}\left\{\frac{4}{315}r_{12}^4\left[\ln\tilde{r}_{12}-\frac{887}{1260}\right]-\frac{13}{945}y_1^4-\frac{4}{315}y_1^2y_2^2-\frac{2}{315}y_2^4+\frac{4}{945}y_1^2(y_1y_2)+\frac{4}{315}(y_1y_2)^2\right\}+y_1^{\langle i}y_2^{j\rangle}\left\{\frac{2}{63}r_{12}^4\left[\ln\tilde{r}_{12}-\frac{127}{252}\right]-\frac{16}{945}y_1^4-\frac{2}{35}y_1^2y_2^2+\frac{8}{315}y_1^2(y_1y_2)+\frac{4}{63}y_2^2(y_1y_2)\right\}+y_2^{\langle ij\rangle}\left\{\frac{1}{18}r_{12}^4\left[\ln\tilde{r}_{12}-\frac{13}{36}\right]-\frac{59}{1260}y_1^4+\frac{1}{63}y_1^2y_2^2+\frac{1}{7}y_1^2(y_1y_2)-\frac{1}{9}(y_1y_2)^2\right\}, \tag{9.8c}$$

$$D^2N_{ij}{}^{(2,-1)}=\frac{8}{5}y_1^{\langle ij\rangle}\left[\ln\tilde{r}_{12}+\frac{7}{15}\right]-\frac{16}{5}y_1^{\langle i}y_2^{j\rangle}\left[\ln\tilde{r}_{12}-\frac{6}{5}\right]+\frac{68}{5}y_2^{\langle ij\rangle}\left[\ln\tilde{r}_{12}+\frac{71}{170}\right], \tag{9.8d}$$

$$DN_{ij}{}^{(0,-1)}=-\frac{8}{15}y_1^{\langle ij\rangle}\left[\ln\tilde{r}_{12}+\frac{7}{15}\right]-\frac{4}{15}y_1^{\langle i}y_2^{j\rangle}\left[\ln\tilde{r}_{12}+\frac{37}{15}\right]-\frac{6}{5}y_2^{\langle ij\rangle}\left[\ln\tilde{r}_{12}+\frac{2}{15}\right], \tag{9.8e}$$

$$D^2N_{ij}{}^{(0,-1)}=\frac{1}{r_{12}^2}\left\{\frac{4}{3}y_1^{\langle ij\rangle}-\frac{8}{3}y_1^{\langle i}y_2^{j\rangle}+\frac{10}{3}y_2^{\langle ij\rangle}\right\}, \tag{9.8f}$$

$$N_{ij}{}^{(0,1)}=y_1^{\langle ij\rangle}\left\{\frac{4}{315}r_{12}^4\left[\ln\tilde{r}_{12}-\frac{1937}{1260}\right]+\frac{11}{945}y_1^4+\frac{4}{945}y_1^2(y_1y_2)+\frac{44}{945}(y_1y_2)^2-\frac{2}{135}y_1^2y_2^2-\frac{4}{105}y_2^2(y_1y_2)+\frac{2}{315}y_2^4\right\}+y_1^{\langle i}y_2^{j\rangle}\left\{\frac{4}{315}r_{12}^4\left[\ln\tilde{r}_{12}-\frac{1517}{1260}\right]-\frac{1}{135}y_1^4-\frac{4}{189}y_1^2(y_1y_2)+\frac{4}{105}(y_1y_2)^2-\frac{8}{315}y_2^2(y_1y_2)\right\}+y_2^{\langle ij\rangle}\left\{\frac{1}{126}r_{12}^4\left[\ln\tilde{r}_{12}-\frac{253}{252}\right]-\frac{1}{3780}y_1^4-\frac{1}{105}y_1^2(y_1y_2)+\frac{1}{63}(y_1y_2)^2-\frac{2}{315}y_1^2y_2^2\right\}, \tag{9.8g}$$

$$D^2M_{ij}{}^{(0,-1)}=y_1^{\langle ij\rangle}\left\{-\frac{88}{15}\ln\tilde{r}_{12}+r_{12}^{-2}\left[-\frac{676}{225}y_1^2+\frac{2072}{225}(y_1y_2)-\frac{1096}{225}y_2^2\right]\right\}+y_1^{\langle i}y_2^{j\rangle}\left\{-\frac{16}{5}\ln\tilde{r}_{12}+r_{12}^{-2}\left[-\frac{184}{25}y_1^2+\frac{1024}{75}(y_1y_2)-\frac{224}{25}y_2^2\right]\right\}+y_2^{\langle ij\rangle}\left\{-\frac{48}{5}\ln\tilde{r}_{12}+r_{12}^{-2}\left[-\frac{536}{75}y_1^2+\frac{304}{25}(y_1y_2)-\frac{42}{25}y_2^2\right]\right\}. \tag{9.8h}$$

The remaining integrals, defined by Eq. (9.5), are more difficult, but we have been able to obtain all of them using several different methods, adapted to the computation of each of these integrals separately. We shall not present all the details of these computations but simply outline some examples. Consider the integral K_L defined by Eq. (9.5c) with $p=0$, i.e.,

$$K_L=-\frac{1}{2\pi}\underset{B=0}{\mathrm{FP}}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_L\frac{g}{r_1}. \tag{9.9}$$

Using the fact that g/r_1 is a Laplacian,

$$\frac{g}{r_1}=\Delta\left[\frac{r_1+r_{12}}{2}g-\frac{r_1}{4}-\frac{r_2}{2}\right], \tag{9.10}$$

we can integrate by parts and transform K_L into an integral containing an explicit B-factor,

$$K_L = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \left\{ B(B+2l+1) \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B |\mathbf{x}|^{-2} \hat{x}_L \times \left[\frac{r_1 + r_{12}}{2} g - \frac{r_1}{4} - \frac{r_2}{2} \right] \right\}. \tag{9.11}$$

From a previous argument, the value of the integral depends only on the possible occurrence of a pole $\sim 1/B$ at infinity. As the pole is easily computed from expanding the integrand at infinity, we obtain in this way the expression of K_L. Next, from the formula

$$\partial_a \left(\frac{1}{r_1} \right) {}_a g = -\frac{1}{2} \Delta_1 \left(\frac{1}{r_1} g \right) + \frac{g}{2} \Delta_1 \left(\frac{1}{r_1} \right) + \frac{1}{2r_1} \Delta_1 g, \tag{9.12}$$

where one should be careful about considering $\Delta_1 r_1^{-1}$ in the sense of distributions [i.e., $\Delta_1 r_1^{-1} = -4\pi\delta_1$], we deduce G_L from the Laplacian of K_L. Indeed, as a consequence of Eq. (9.12),

$$G_L = -\frac{1}{2} \Delta_1 K_L + (\hat{x}_L g)_1 + \frac{1}{2r_{12}} Y_L{}^{(-2,0)}, \tag{9.13}$$

and we can easily show that $Y_L{}^{(-2,0)}$ is actually zero. Alternatively, one can prove also that

$$G_L = \frac{1}{2} Y_L{}^{(-3,0)} - \frac{1}{2r_{12}} Y_L{}^{(-3,1)} + \frac{1}{2r_{12}} Y_L{}^{(-2,0)}. \tag{9.14}$$

This provides a check of the computation.

To compute G_L^s (in the quadrupole case $L = ij$, say) we use a different method. We remark that G_{ij}^s obeys a Laplace equation, with respect to the point 2, with known source:

$$\Delta_2 G_{ij}^s = {}_1\partial_a \left(\frac{1}{r_{12}} \right) {}_1\partial_{as} Y_{ij}. \tag{9.15}$$

Here, Y_{ij} is known from Eq. (7.10). The right-hand side of Eq. (9.15) is expanded, and we obtain a particular solution of this equation by integrating each of the terms. Now G_{ij}^s is necessarily equal to this particular solution plus some solution, regular at the origin, of the homogeneous equation. Taking into account the index structure of G_{ij}^s and the fact that it has the dimension of a length, we find that the homogeneous solution is parametrized by solely two numerical constants a and b. At this stage we have

$$G_{ij}^s = -\frac{1}{30} y_{12}^{\langle ij \rangle s} r_{12}^{-2} + \frac{1}{6} y_{12}^{s\langle i} y_1^{j \rangle} r_{12}^{-2} - \frac{1}{15} y_{12}^{\langle i} \delta^{j \rangle s} \ln \tilde{r}_{12} - \frac{4}{3} y_1^{\langle i} \delta^{j \rangle s} \ln \tilde{r}_{12} + a y_1^{\langle i} \delta^{j \rangle s} + b y_2^{\langle i} \delta^{j \rangle s}. \tag{9.16}$$

Incidentally, this expression already gives the complete result for the gradients ${}_1\partial_s G_{ij}^s$ and ${}_2\partial_s G_{ij}^s$, because the gradients of the homogeneous terms are zero. To compute the constants a and b we need some extra information, which is provided by the contracted product between y_{12}^s and G_{ij}^s. Indeed this contraction is a known quantity thanks to the identity

$$y_{12}^s G_{ij}^s = -(1 + y_{12}^s \, {}_1\partial_s) G_{ij} + \frac{1}{4} \Delta_1 Y_{ij}{}^{(-2,1)}, \tag{9.17}$$

where G_{ij} has just been obtained previously. Here, $Y_{ij}{}^{(-2,1)}$ can be computed from the Riesz formula exactly like for $Y_{ij}{}^{(-2,-1)}$ in Sec. VIII. [When deriving Eq. (9.17) we take account of the fact that $Y_{ij}{}^{(-2,0)} = 0$.] Comparing the result for $y_{12}^s G_{ij}^s$ with the one obtained directly from Eq. (9.16) we find *three* equations for the *two* unknown constants a and b. This overdetermined system fixes uniquely the constants to the values $a = 63/100$ and $b = -257/900$.

The preceding method was successfully applied to several integrals of the type (9.5): that is, we (i) compute the "source" of the Laplace equation satisfied by the integral with respect to the point 2 [the source is computable because Δ_2 applies only on the part of the integrand containing the functions g, f, etc., and we can make use of Eqs. (7.15); with respect to the point 1 this would not work], (ii) compute a particular solution of this equation, (iii) write down the most general form of the homogeneous solution in terms of a few arbitrary coefficients (this works only when the dimension of the integral is a small power of a length so that the number of unknown coefficients is small), (iv) compute the coefficients using the extra information provided by the contraction with respect to $\mathbf{y}_{12}$. Alternatively to (iv) one can use an angular average with respect to $\mathbf{n}_{12}$ [see Eq. (9.29) below].

As a verification let us introduce the new integral

$$R_{ij} = -\frac{1}{2\pi} \underset{B=0}{\mathrm{FP}} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_{ij} \partial_a \left(\frac{1}{r_1} \right) g_a. \tag{9.18}$$

From the easily checked formula

$$(\Delta_1 - \Delta) \left(\frac{g}{r_1} \right) = \frac{1}{r_1^2 r_{12}} - \frac{1}{r_1^2 r_2} + 2\partial_a \left(\frac{1}{r_1} \right) g_a, \tag{9.19}$$

we deduce a relation between R_{ij} and some computable quantities,

$$R_{ij} = \frac{1}{2} \Delta_1 K_{ij} + \frac{1}{2} Y_{ij}{}^{(-2,-1)} + \frac{1}{4\pi} \underset{B=0}{\mathrm{FP}} \times \left\{ B(B+5) \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B |\mathbf{x}|^{-2} \hat{x}_{ij} \frac{g}{r_1} \right\}. \tag{9.20}$$

The value of the last integral comes from the pole at infinity—the same method as before. Having obtained R_{ij}, the verification is that ${}_2\partial_s G_{ij}^s$, which on one hand is computed from Eq. (9.16), on the other hand should be given by the following alternative expression:

$$ {}_2\partial_s G_{ij}^s = -\frac{1}{2} \Delta_1 R_{ij} - (\partial_a (\hat{x}_{ij} g_a))_1, \tag{9.21}$$

which is obtained by some integrations by parts inside the integrand of ${}_2\partial_s G^s_{ij}$. Of course, the value of R_{ij} computed by Eq. (9.20) is such that Eq. (9.21) is also satisfied.

Once G^s_{ij} is known we can deduce another needed integral, i.e., ${}_2\partial_s K^{as}_{ij}$, from the identity

$$\partial_{as}\left(\frac{1}{r_1}\right)(g_s+{}_s g)=-\frac{1}{2}\Delta\left[\partial_a\left(\frac{1}{r_1}\right)g\right]+\frac{1}{2}\partial_a\left(\Delta\frac{1}{r_1}\right)g-\frac{1}{4}{}_1\partial_a\left(\frac{1}{r_1}\Delta g\right), \tag{9.22}$$

which implies

$${}_2\partial_s K^{as}_{ij}=-G^a_{ij}-\frac{1}{4}{}_1\partial_a Y_{ij}{}^{(-2,-1)}-(\partial_a(\hat{x}_{ij}g))_1+\frac{1}{4\pi}\mathrm{FP}_{B=0}\left\{B(B+5)\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B|\mathbf{x}|^{-2}\hat{x}_{ij}\partial_a\left(\frac{1}{r_1}\right)g\right\}. \tag{9.23}$$

Again the last integral causes no problem. Next, from both R_{ij} and ${}_2\partial_s K^{bs}_{ij}$, we can further deduce ${}_2\partial_a({}_b G^a_{ij})$. Indeed the other identity

$$\partial_a\left(\frac{1}{r_1}\right){}_b g_a={}_1\partial_b\left[\partial_a\left(\frac{1}{r_1}\right)g_a\right]+\partial_{ab}\left(\frac{1}{r_1}\right)g_a, \tag{9.24}$$

implies

$${}_2\partial_a({}_b G^a_{ij})={}_1\partial_b R_{ij}+{}_2\partial_s K^{bs}_{ij}. \tag{9.25}$$

Some other integrals are connected directly to the simpler Y-type integrals. For instance, the integral (9.5d) is given by

$$U^{ab}_{ij}=\frac{3}{16}{}_1\partial_{ab}Y_{ij}{}^{(-2,-1)}-\frac{1}{8}\delta_{ab}Y_{ij}{}^{(-4,-1)}-\frac{1}{2r_{12}}{}_1\partial_{ab}Y_{ij}{}^{(-1,-1)} \tag{9.26}$$

(using the facts that $Y_{ij}{}^{(-2,0)}=0=Y_{ij}{}^{(-4,0)}$). Once the value of this integral is obtained, we can check that its trace $U^{aa}_{ij}=\delta_{ab}U^{ab}_{ij}$ is especially simple: $U^{aa}_{ij}=-y_1^{\langle ij\rangle}/r^2_{12}$. This is in perfect agreement with

$$U^{aa}_{ij}=-\frac{1}{2\pi}\mathrm{FP}_{B=0}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_{ij}\Delta\left(\frac{1}{r_1}\right){}_k g_k=2(\hat{x}_{ij}\,{}_k g_k)_1, \tag{9.27}$$

the final reduction being obtained thanks to the known formula (see, e.g., [51])

$${}_k g_k=\frac{1}{2}\left(\frac{1}{r_1r_2}-\frac{1}{r_1r_{12}}-\frac{1}{r_2r_{12}}\right). \tag{9.28}$$

Still another method is useful in our computation. All the integrals are certain functions of the two points $\mathbf{y}_1$ and $\mathbf{y}_2$, and it is advantageous to consider their angular average with respect to the relative direction $\mathbf{n}_{12}$ between the points, with the vector $\mathbf{y}_1$ being fixed. As it turns out, the average is much easier to compute (using some methods similar as before) than the integral itself. On the other hand, once we have obtained a result, we can compute its average, so the comparison leads to an interesting check of the calculation. Let us see on the example of G_L how one performs this angular average. From Eq. (9.5a) we write

$$\int\frac{d\Omega_{12}}{4\pi}G_L=-\frac{1}{2\pi}\mathrm{FP}_{B=0}\int d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_L\partial_a\left(\frac{1}{r_1}\right)\int\frac{d\Omega_{12}}{4\pi}{}_a g, \tag{9.29}$$

in which we commuted the angular average (where $d\Omega_{12}$ denotes the solid angle element in the direction $\mathbf{n}_{12}$) with the integral sign and the terms depending only on $\mathbf{y}_1$. This is correct because $\mathbf{y}_1$ is kept fixed in the process; for instance, the average of $\mathbf{y}_2$ is $\mathbf{y}_1$, which is obtained by writing $\mathbf{y}_2=\mathbf{y}_1-r_{12}\mathbf{n}_{12}$ and averaging over $\mathbf{n}_{12}$ with fixed r_{12} and $\mathbf{y}_1$. In practice, computing the average (9.29) is not too complicated because the average of ${}_a g$ is rather simple,

$$\int\frac{d\Omega_{12}}{4\pi}{}_a g=\begin{cases}\left(\dfrac{r_1}{6r^2_{12}}-\dfrac{1}{2r_{12}}\right)n_1^a & \text{when } r_1\leqslant r_{12},\\[2ex] \left(-\dfrac{1}{2r_1}+\dfrac{r_{12}}{6r_1^2}\right)n_1^a & \text{when } r_1>r_{12}.\end{cases} \tag{9.30}$$

A more complicated example, that was useful for us, is

$$\int\frac{d\Omega_{12}}{4\pi}{}_a g_b=\begin{cases}-\dfrac{r_1^2}{20r^4_{12}}n_1^{ab}+\left(\dfrac{r_1^2}{60r^4_{12}}-\dfrac{1}{6r^2_{12}}\right)\delta^{ab} & \text{when } r_1\leqslant r_{12},\\[2ex] \left(-\dfrac{1}{4r_1^2}+\dfrac{r_{12}}{5r_1^3}\right)n_1^{ab}+\left(-\dfrac{1}{3r_1r_{12}}+\dfrac{1}{4r_1^2}-\dfrac{r_{12}}{15r_1^3}\right)\delta^{ab} & \text{when } r_1>r_{12}.\end{cases} \tag{9.31}$$

According to Eq. (9.30), we must split the integration over $d^3\mathbf{x}$ into two "near-zone" and "far-zone" contributions,

$$\int \frac{d\Omega_{12}}{4\pi} G_L = -\frac{1}{2\pi}\int_{r_1<r_{12}} d^3\mathbf{x}\,\hat{x}_L\left(-\frac{1}{6r_1r_{12}^2}+\frac{1}{2r_1^2r_{12}}\right) - \frac{1}{2\pi}\underset{B=0}{\mathrm{FP}}\int_{r_1>r_{12}} d^3\mathbf{x}|\tilde{\mathbf{x}}|^B\hat{x}_L\left(\frac{1}{2r_1^3}-\frac{r_{12}}{6r_1^4}\right). \tag{9.32}$$

The finite part at $B=0$ is necessary only for the far-zone integral. Both integrals in Eq. (9.32) are now evaluated using standard methods. In the case $l=2$ we find

$$\int \frac{d\Omega_{12}}{4\pi} G_{ij} = y_1^{\langle ij\rangle}\left(\ln\tilde{r}_{12}+\frac{1}{30}\right). \tag{9.33}$$

This is in agreement with the average of G_{ij} computed directly with the result calculated from Eq. (9.13) or Eq. (9.14). This method of averaging has been applied for checking many other integrals. Even, in several cases, the method has been employed in order to determine some unknown coefficients. However, for this purpose the method is less powerful than the method of contraction with the vector $\mathbf{y}_{12}$, since the latter method yields in general a redundant determination of the coefficients.

The complete list of the results for the elementary integrals is as follows:

$$G_{ij} = y_1^{\langle ij\rangle}\left[\ln\tilde{r}_{12}-\frac{23}{60}\right]+\frac{1}{3}y_1^{\langle i}y_2^{j\rangle}+\frac{1}{12}y_2^{\langle ij\rangle}, \tag{9.34a}$$

$${}_2\partial_b(G_{ij}) = -y_1^{\langle ij\rangle}y_{12}^b r_{12}^{-2}+\frac{1}{2}\delta^{b\langle i}y_1^{j\rangle}-\frac{1}{6}\delta^{b\langle i}y_{12}^{j\rangle}, \tag{9.34b}$$

$$\begin{aligned}{}_2\partial_b(G_{ij}^c) = &-\frac{1}{15}y_{12}^{\langle ij\rangle bc}r_{12}^{-4}+\frac{1}{30}y_{12}^{\langle ij\rangle}\delta^{bc}r_{12}^{-2}+\frac{1}{15}(y_{12}^{b\langle i}\delta^{j\rangle c}+y_{12}^{c\langle i}\delta^{j\rangle b})r_{12}^{-2}+\frac{1}{3}y_{12}^{bc\langle i}y_1^{j\rangle}r_{12}^{-4}\\ &-\frac{1}{6}y_{12}^{\langle i}y_1^{j\rangle}\delta^{bc}r_{12}^{-2}+\frac{4}{3}y_{12}^b y_1^{\langle i}\delta^{j\rangle c}r_{12}^{-2}-\frac{1}{6}y_{12}^c y_1^{\langle i}\delta^{j\rangle b}r_{12}^{-2}+\frac{1}{15}\delta^{b\langle i}\delta^{j\rangle c}\left[\ln\tilde{r}_{12}-\frac{257}{60}\right],\end{aligned} \tag{9.34c}$$

$$\begin{aligned}{}_2\partial_a({}_bG_{ij}^{ac}) = &\frac{1}{60}y_{12}^{\langle ij\rangle bc}r_{12}^{-4}-\frac{1}{20}y_{12}^{\langle ij\rangle}\delta^{bc}r_{12}^{-2}+\frac{7}{30}y_{12}^{b\langle i}\delta^{j\rangle c}r_{12}^{-2}+\frac{1}{15}y_{12}^{c\langle i}\delta^{j\rangle b}r_{12}^{-2}+\frac{1}{2}y_{12}^{\langle i}y_1^{j\rangle}\delta^{bc}r_{12}^{-2}\\ &-\frac{1}{2}y_{12}^c y_1^{\langle i}\delta^{j\rangle b}r_{12}^{-2}+\frac{3}{4}y_{12}^{bc}y_1^{\langle ij\rangle}r_{12}^{-4}-\frac{1}{4}\delta^{bc}y_1^{\langle ij\rangle}r_{12}^{-2}+\frac{1}{15}\left[\ln\tilde{r}_{12}-\frac{317}{60}\right]\delta^{b\langle i}\delta^{j\rangle c},\end{aligned} \tag{9.34d}$$

$${}_2\partial_a({}_bG_{ij}^a) = \frac{2}{15}y_{12}^{\langle ij\rangle b}r_{12}^{-2}-\frac{1}{2}y_{12}^{b\langle i}y_1^{j\rangle}r_{12}^{-2}+\frac{2}{3}\delta^{b\langle i}y_1^{j\rangle}+\delta^{b\langle i}y_{12}^{j\rangle}\left[-\frac{1}{15}\ln\tilde{r}_{12}-\frac{103}{900}\right], \tag{9.34e}$$

$${}_2\partial_s(G_{ij}^s) = \left(\frac{7}{6}y_1^{\langle ij\rangle}-\frac{4}{3}y_1^{\langle i}y_2^{j\rangle}+\frac{1}{6}y_2^{\langle ij\rangle}\right)r_{12}^{-2}, \tag{9.34f}$$

$$G_{ij}^s = -\frac{1}{30}y_{12}^{\langle ij\rangle s}r_{12}^{-2}+\frac{1}{6}y_{12}^{s\langle i}y_1^{j\rangle}r_{12}^{-2}+y_{12}^{\langle i}\delta^{j\rangle s}\left[-\frac{1}{15}\ln\tilde{r}_{12}+\frac{257}{900}\right]+y_1^{\langle i}\delta^{j\rangle s}\left[-\frac{4}{3}\ln\tilde{r}_{12}+\frac{31}{90}\right], \tag{9.34g}$$

$$G_{ijk} = y_1^{\langle ijk\rangle}\left[\ln\tilde{r}_{12}-\frac{307}{840}\right]+\frac{3}{8}y_1^{\langle ij}y_2^{k\rangle}+\frac{1}{8}y_1^{\langle i}y_2^{jk\rangle}+\frac{1}{24}y_2^{\langle ijk\rangle}, \tag{9.34h}$$

$$\begin{aligned}G_{ijk}^s = &\left(\frac{1}{56}y_{12}^{s\langle ijk\rangle}-\frac{1}{10}y_{12}^{s\langle ij}y_1^{k\rangle}+\frac{1}{4}y_{12}^{s\langle i}y_1^{jk\rangle}\right)r_{12}^{-2}+\delta^{s\langle i}y_{12}^{jk\rangle}\left[\frac{1}{35}\ln\tilde{r}_{12}-\frac{2843}{29400}\right]+\delta^{s\langle i}y_{12}^{j}y_1^{k\rangle}\left[-\frac{1}{5}\ln\tilde{r}_{12}+\frac{1739}{2100}\right]\\ &+\delta^{s\langle i}y_1^{jk\rangle}\left[-2\ln\tilde{r}_{12}+\frac{97}{420}\right],\end{aligned} \tag{9.34i}$$

$$
{}_2\partial_b({}_sG^b_{ijk})=\left(-\frac{123}{280}y_1^{\langle ijk\rangle}+\frac{61}{280}y_1^{\langle ij}y_2^{k\rangle}+\frac{37}{280}y_1^{\langle i}y_2^{jk\rangle}+\frac{5}{56}y_2^{\langle ijk\rangle}\right)y_{12}^s r_{12}^{-2}+y_1^{\langle ij}\delta^{k\rangle s}\left[-\frac{1}{7}\ln\tilde{r}_{12}+\frac{699}{980}\right]
+y_1^{\langle i}y_2^{j}\delta^{k\rangle s}\left[\frac{3}{35}\ln\tilde{r}_{12}+\frac{2963}{14700}\right]+y_2^{\langle ij}\delta^{k\rangle s}\left[\frac{2}{35}\ln\tilde{r}_{12}+\frac{313}{3675}\right], \tag{9.34j}
$$

$$
{}_2\partial_b({}_aG^b_{aij})=r_{12}^{-2}y_1^{\langle ij\rangle}\left\{-\frac{1}{5}r_{12}^2\ln\tilde{r}_{12}+\frac{147}{200}y_1^2-\frac{463}{300}(y_1y_2)+\frac{109}{120}y_2^2\right\}+r_{12}^{-2}y_1^{\langle i}y_2^{j\rangle}\left\{\frac{3}{25}r_{12}^2\ln\tilde{r}_{12}+\frac{167}{750}y_1^2-\frac{449}{750}(y_1y_2)
+\frac{22}{125}y_2^2\right\}+r_{12}^{-2}y_2^{\langle ij\rangle}\left\{\frac{2}{25}r_{12}^2\ln\tilde{r}_{12}+\frac{577}{3000}y_1^2-\frac{79}{500}(y_1y_2)+\frac{197}{3000}y_2^2\right\}, \tag{9.34k}
$$

$$
{}_2\partial_b({}_sG^{bu}_{ijk})=-\frac{1}{56}y_{12}^{\langle ijk\rangle su}r_{12}^{-4}+\frac{3}{140}\delta^{us}y_{12}^{\langle ijk\rangle}r_{12}^{-2}-\frac{1}{28}\delta^{s\langle i}y_{12}^{jk\rangle u}r_{12}^{-2}-\frac{31}{280}\delta^{u\langle i}y_{12}^{jk\rangle s}r_{12}^{-2}+\frac{1}{20}y_1^{\langle i}y_{12}^{jk\rangle su}r_{12}^{-4}-\frac{3}{20}\delta^{su}y_1^{\langle i}y_{12}^{jk\rangle}r_{12}^{-2}
+\frac{1}{5}\delta^{s\langle i}y_1^{j}y_{12}^{k\rangle u}r_{12}^{-2}+\frac{7}{10}\delta^{u\langle i}y_1^{j}y_{12}^{k\rangle s}r_{12}^{-2}+\frac{3}{4}\delta^{su}y_1^{\langle ij}y_{12}^{k\rangle}r_{12}^{-2}-\frac{3}{4}\delta^{s\langle i}y_1^{jk\rangle}y_{12}^{u}r_{12}^{-2}+\frac{3}{4}y_1^{\langle ijk\rangle}y_{12}^{su}r_{12}^{-4}-\frac{1}{4}y_1^{\langle ijk\rangle}\delta^{su}r_{12}^{-2}
+\delta^{s\langle i}y_{12}^{j}\delta^{k\rangle u}\left[\frac{1}{35}\ln\tilde{r}_{12}+\frac{1361}{14700}\right]+\delta^{s\langle i}y_1^{j}\delta^{k\rangle u}\left[\frac{1}{5}\ln\tilde{r}_{12}-\frac{2159}{2100}\right], \tag{9.34l}
$$

$$
{}_2\partial_b({}_sG^{ab}_{aij})=-\frac{11}{75}y_{12}^{\langle ij\rangle s}r_{12}^{-2}+\frac{1}{300}(y_1y_{12})y_{12}^{\langle ij\rangle s}r_{12}^{-4}-\frac{23}{300}y_1^{s}y_{12}^{\langle ij\rangle}r_{12}^{-2}+\frac{71}{75}y_1^{\langle i}y_{12}^{j\rangle s}r_{12}^{-2}+\frac{7}{10}y_1^{s\langle i}y_{12}^{j\rangle}r_{12}^{-2}-\frac{1}{4}y_1^{\langle ij\rangle s}r_{12}^{-2}
+\frac{3}{4}(y_1y_{12})y_1^{\langle ij\rangle}y_{12}^{s}r_{12}^{-4}-\frac{3}{10}y_1^2y_1^{\langle i}y_{12}^{j\rangle s}r_{12}^{-4}+\frac{2}{25}(y_1y_{12})\delta^{s\langle i}y_{12}^{j\rangle}r_{12}^{-2}-\frac{7}{10}(y_1y_{12})\delta^{s\langle i}y_1^{j\rangle}r_{12}^{-2}+\frac{1}{10}y_1^2\delta^{s\langle i}y_1^{j\rangle}r_{12}^{-2}
+\delta^{s\langle i}y_{12}^{j\rangle}\left[\frac{1}{25}\ln\tilde{r}_{12}+\frac{51}{500}\right]+\delta^{s\langle i}y_1^{j\rangle}\left[\frac{7}{25}\ln\tilde{r}_{12}-\frac{2029}{1500}\right], \tag{9.34m}
$$

$$
U^{ab}_{ij}=y_{12}^{ab}\left(-\frac{7}{30}y_1^{\langle ij\rangle}-\frac{1}{30}y_1^{\langle i}y_2^{j\rangle}+\frac{1}{60}y_2^{\langle ij\rangle}\right)r_{12}^{-4}+\delta^{ab}\left(-\frac{3}{10}y_1^{\langle ij\rangle}+\frac{1}{10}y_1^{\langle i}y_2^{j\rangle}-\frac{1}{20}y_2^{\langle ij\rangle}\right)r_{12}^{-2}+\frac{1}{15}\delta^{a\langle i}y_{12}^{j\rangle b}r_{12}^{-2}
+\frac{1}{15}\delta^{b\langle i}y_{12}^{j\rangle a}r_{12}^{-2}+\delta^{a\langle i}\delta^{j\rangle b}\left[\frac{2}{5}\ln\tilde{r}_{12}-\frac{97}{150}\right], \tag{9.34n}
$$

$$
{}_2\partial_s(K^{as}_{ij})=\left(-\frac{9}{10}y_1^{\langle ij\rangle}-\frac{1}{30}y_1^{\langle i}y_2^{j\rangle}-\frac{1}{15}y_2^{\langle ij\rangle}\right)y_{12}^{a}r_{12}^{-2}+\delta^{a\langle i}y_1^{j\rangle}\left[-\frac{17}{15}\ln\tilde{r}_{12}-\frac{851}{900}\right]+\delta^{a\langle i}y_2^{j\rangle}\left[-\frac{1}{5}\ln\tilde{r}_{12}-\frac{13}{300}\right], \tag{9.34o}
$$

$$
{}_2\partial_b(Q^b_{ij})=y_1^{\langle ij\rangle}\left\{-\frac{8}{5}\ln\tilde{r}_{12}-\frac{11731}{4200}+r_{12}^{-2}\left[\frac{243}{70}y_1^2-\frac{22}{7}(y_1y_2)+\frac{88}{105}y_2^2\right]\right\}+y_1^{\langle i}y_2^{j\rangle}\left\{-\frac{8}{15}\ln\tilde{r}_{12}-\frac{5603}{6300}
+r_{12}^{-2}\left[-\frac{134}{105}y_1^2-\frac{8}{21}(y_1y_2)+\frac{34}{105}y_2^2\right]\right\}+y_2^{\langle ij\rangle}\left\{-\frac{1}{5}\ln\tilde{r}_{12}-\frac{1777}{4200}+r_{12}^{-2}\left[\frac{29}{210}y_1^2-\frac{1}{7}(y_1y_2)+\frac{6}{35}y_2^2\right]\right\}, \tag{9.34p}
$$

$$
{}_2\partial_s(H^s_{ij})=y_1^{\langle ij\rangle}\left[\frac{8}{15}\ln\tilde{r}_{12}-\frac{227}{1800}\right]+y_1^{\langle i}y_2^{j\rangle}\left[\frac{4}{15}\ln\tilde{r}_{12}-\frac{33}{900}\right]+y_2^{\langle ij\rangle}\left[\frac{1}{5}\ln\tilde{r}_{12}-\frac{97}{1800}\right], \tag{9.34q}
$$

$$
{}_2\partial_b(H^{bs}_{ij})=\frac{1}{30}y_{12}^{s\langle ij\rangle}r_{12}^{-2}-\frac{1}{6}y_{12}^{s\langle i}y_1^{j\rangle}r_{12}^{-2}-\frac{1}{2}y_{12}^{s}y_1^{\langle ij\rangle}r_{12}^{-2}+y_{12}^{\langle i}\delta^{j\rangle s}\left[\frac{1}{15}\ln\tilde{r}_{12}-\frac{107}{900}\right]+y_1^{\langle i}\delta^{j\rangle s}\left[-\frac{2}{3}\ln\tilde{r}_{12}+\frac{19}{45}\right], \tag{9.34r}
$$

$$
{}_2\partial_s(H_{ij}^{abs}) = -\frac{3}{40}y_{12}^{ab\langle ij\rangle}r_{12}^{-4} + \frac{7}{120}\delta^{ab}y_{12}^{\langle ij\rangle}r_{12}^{-2} + \frac{7}{60}(\delta^{a\langle i}y_{12}^{j\rangle b} + \delta^{b\langle i}y_{12}^{j\rangle a})r_{12}^{-2} + \frac{1}{3}y_{12}^{ab\langle i}y_1^{j\rangle}r_{12}^{-4} + \frac{1}{3}\delta^{ab}y_{12}^{\langle i}y_1^{j\rangle}r_{12}^{-2} + \frac{1}{3}(\delta^{a\langle i}y_1^{j\rangle}y_{12}^{b} + \delta^{b\langle i}y_1^{j\rangle}y_{12}^{a})r_{12}^{-2} + \delta^{a\langle i}\delta^{j\rangle b}\left[\frac{8}{15}\ln\tilde{r}_{12} - \frac{241}{900}\right], \tag{9.34s}
$$

$$
{}_2\partial_s(F_{ij}^{s}) = y_1^{\langle ij\rangle}\left[\ln\tilde{r}_{12} + \frac{2}{3}\ln 2 + \frac{29}{360}\right] - \frac{1}{12}y_1^{\langle i}y_2^{j\rangle} + \frac{1}{24}y_2^{\langle ij\rangle}, \tag{9.34t}
$$

$$
{}_2\partial_b(F_{ij}^{bs}) = -\frac{1}{30}y_{12}^{s\langle ij\rangle}r_{12}^{-2} - \frac{1}{2}y_{12}^{s\langle i}y_1^{j\rangle}r_{12}^{-2} - \frac{1}{2}y_{12}^{s}y_1^{\langle ij\rangle}r_{12}^{-2} + y_{12}^{\langle i}\delta^{j\rangle s}\left[-\frac{1}{15}\ln\tilde{r}_{12} - \frac{43}{900}\right] + y_1^{\langle i}\delta^{j\rangle s}\left[-2\ln\tilde{r}_{12} - \frac{4}{3}\ln 2 + \frac{28}{90}\right], \tag{9.34u}
$$

$$
{}_2\partial_s(F_{ij}^{abs}) = -\frac{1}{120}y_{12}^{ab\langle ij\rangle}r_{12}^{-4} + \frac{23}{120}\delta^{ab}y_{12}^{\langle ij\rangle}r_{12}^{-2} + \frac{23}{60}(\delta^{a\langle i}y_{12}^{j\rangle b} + \delta^{b\langle i}y_{12}^{j\rangle a})r_{12}^{-1} + \frac{2}{3}y_{12}^{ab\langle i}y_1^{j\rangle}r_{12}^{-4} + \frac{2}{3}\delta^{ab}y_{12}^{\langle i}y_1^{j\rangle}r_{12}^{-2} + \frac{2}{3}(y_{12}^{b}\delta^{a\langle i} + y_{12}^{a}\delta^{b\langle i})y_1^{j\rangle}r_{12}^{-2} + \delta^{a\langle i}\delta^{j\rangle b}\left[\frac{32}{15}\ln\tilde{r}_{12} + \frac{4}{3}\ln 2 - \frac{409}{900}\right], \tag{9.34v}
$$

$$
{}_2\partial_s(F_{ij}^{s\,12}) = y_1^{\langle ij\rangle}\left[-\frac{7}{15}\ln\tilde{r}_{12} - \frac{2}{3}\ln 2 - \frac{79}{100}\right] + y_1^{\langle i}y_2^{j\rangle}\left[\frac{4}{15}\ln\tilde{r}_{12} + \frac{148}{225}\right] + y_2^{\langle ij\rangle}\left[\frac{1}{5}\ln\tilde{r}_{12} + \frac{13}{300}\right], \tag{9.34w}
$$

$$
{}_2\partial_b(F_{ij}^{bs12}) = -\frac{2}{15}y_{12}^{s\langle ij\rangle}r_{12}^{-2} + y_{12}^{s\langle i}y_1^{j\rangle}r_{12}^{-2} - y_{12}^{s}y_1^{\langle ij\rangle}r_{12}^{-2} + y_1^{\langle i}\delta^{j\rangle s}\left[\frac{4}{3}\ln 2 + \frac{13}{45}\right] + y_{12}^{\langle i}\delta^{j\rangle s}\left[\frac{2}{5}\ln\tilde{r}_{12} + \frac{44}{75}\right], \tag{9.34x}
$$

$$
{}_2\partial_s(F_{ij}^{abs12}) = -\frac{7}{20}y_{12}^{ab\langle ij\rangle}r_{12}^{-4} + \frac{1}{20}\delta^{ab}y_{12}^{\langle ij\rangle}r_{12}^{-2} - \frac{17}{30}(\delta^{a\langle i}y_{12}^{j\rangle b} + \delta^{b\langle i}y_{12}^{j\rangle a}) + \frac{5}{3}(y_{12}^{b}\delta^{a\langle i} + y_{12}^{a}\delta^{b\langle i})y_1^{j\rangle}r_{12}^{-2} + \frac{5}{3}y_{12}^{ab\langle i}y_1^{j\rangle}r_{12}^{-4} - \frac{1}{3}\delta^{ab}y_{12}^{\langle i}y_1^{j\rangle}r_{12}^{-2} - \frac{5}{2}y_{12}^{ab}y_1^{\langle ij\rangle}r_{12}^{-4} + \frac{3}{2}\delta^{ab}y_1^{\langle ij\rangle}r_{12}^{-2} + \delta^{a\langle i}\delta^{j\rangle b}\left[\frac{4}{15}\ln\tilde{r}_{12} - \frac{4}{3}\ln 2 - \frac{229}{450}\right], \tag{9.34y}
$$

$$
{}_2\partial_s(F_{ij}^{s\,21}) = y_1^{\langle ij\rangle}\left[\frac{11}{15}\ln\tilde{r}_{12} - \frac{2}{3}\ln 2 - \frac{787}{900}\right] + y_1^{\langle i}y_2^{j\rangle}\left[-\frac{4}{5}\ln\tilde{r}_{12} + \frac{196}{225}\right] + y_2^{\langle ij\rangle}\left[\frac{1}{15}\ln\tilde{r}_{12} - \frac{77}{900}\right]. \tag{9.34z}
$$

Inserting these elementary integrals into the expressions of non-compact terms [see Eq. (9.3)], and reducing to the case of circular orbits, we obtain the results reported in the Appendix.

X. POINT-MASS REGULARIZATION AMBIGUITIES

The computation of the multipole moments we performed so far has been carried out with standard techniques: standard Hadamard regularization [see Sec. V], and Schwartz distributions [see, e.g., Eqs. (9.2)]. The result we obtained depends on three arbitrary constants: the two Hadamard regularization constants u_1 and u_2 introduced in Eq. (5.3), and the constant r_0 entering the definition of the source multipole moments through the analytic-continuation factor $|\tilde{\mathbf{x}}|^B = |\mathbf{x}/r_0|^B$ [see Eqs. (2.5)]. The constant r_0 is not a problem since we know that in this formalism the multipole expansion of the field exterior to any source is actually independent of r_0 [21]. Indeed we shall check in Sec. XII that r_0 disappears from the final expression of the energy flux [the constant r_0 in the source moments is cancelled by the same constant present in the contribution of "tails of tails" in the wave zone; see Eq. (11.8) below]. However, it will turn out that the constants u_1 and u_2, which encode some arbitrariness of the Hadamard regularization, lead *a priori* to two undetermined purely numerical parameters in the expression of the 3PN quadrupole moment. In addition, we shall argue that because of some delicate problems linked with the use of the Hadamard regularization at the 3PN order, we should consider *a priori* a third undetermined parameter in the quadrupole moment. However, the important point is that these three parameters combine to yield *one and only one* undetermined constant, that we shall call θ, in the third time-derivative of the moment which is needed to compute the physical energy flux for circular orbits. Furthermore, we shall find that the constant θ enters the energy flux at the same level as the constant λ coming from the equations of motion (see below), so that the energy flux depends *in fine* merely on one combination of θ and λ.

The equations of motion of compact objects at the 3PN order have been investigated using the ADM-Hamiltonian formulation of general relativity [34,35], and by integrating the field equations in harmonic coordinates [40,43]. In both approaches the compact objects are modeled by point-like particles described by delta-functions, and the self-field of the particles is removed by a Hadamard regularization. It was shown that the regularization permits the determination of the full equations of motion at the 3PN order except for one undetermined coefficient, λ in the harmonic-coordinate approach and $\omega_{\rm static}$ in the ADM-Hamiltonian. Very likely the unknown coefficient accounts for a physical incompleteness of the point-mass regularization. Actually two unknown coefficients were originally introduced in [34,35], but one of them was shown later [36,37] to be fixed to a unique value by requiring, in an *ad hoc* manner, the global Poincaré invariance of the Hamiltonian. On the other hand, in the harmonic-coordinate approach [40,43] a new Hadamard-type regularization was developed in order to account for the mathematical ambiguities of the standard Hadamard regularization [41,42]. A characteristic of this regularization is the systematic use of a theory of generalized functions. The regularization is defined in a Lorentz-invariant way, but was ultimately shown to yield incomplete results for the equations of motion, in the sense that there remained the unknown numerical coefficient λ. The complete physical equivalence between the harmonic-coordinate [40,43] and ADM-Hamiltonian [34–37] formalisms has been established [38,44]. Indeed a unique ''contact'' transformation of the particles motion which changes the harmonic-coordinate Lagrangian (as given in Ref. [44]) into the ADM-Hamiltonian obtained in Ref. [37] exists. The equivalence holds if and only if the harmonic-coordinate constant λ is related to the ADM-Hamiltonian static ambiguity by

$$\lambda = -\frac{3}{11}\,\omega_{\rm static} - \frac{1987}{3080}. \tag{10.1}$$

Recently, the value $\omega_{\rm static}=0$ has been obtained by means of a different regularization (dimensional) within the ADM-Hamiltonian approach [39]. This result would mean that $\lambda = -1987/3080$. Note that a feature of the harmonic-coordinate equations of motion derived in [40,43,44] is the dependence, in addition to λ, on two arbitrary constants r'_1 and r'_2 parametrizing some logarithmic terms. However, contrary to λ which is a true physical ambiguity, the constants r'_1 and r'_2 can be removed by a coordinate transformation and therefore represent merely some unphysical gauge constants. For instance these constants cancel out in the center-of-mass invariant energy of circular binaries [40].

A. Hadamard-regularization constants

The first problem in the present calculation lies in the *a priori* unknown relation between the Hadamard regularization constants u_1 and u_2 introduced by Eqs. (5.3) and the two gauge constants r'_1 and r'_2 which parametrize the harmonic-coordinate equations of motion. Let us investigate more precisely the dependence of the quadrupole moment on the constants u_1 and u_2. Inspection of our computation shows that these constants come only from the cubic and non-compact terms obtained in Sec. VIII and IX. More precisely, we find that the whole computation depends on u_1, u_2 only through the elementary integrals $Y_L{}^{(-3,0)}$ and $S_L{}^{(-5,0)}$, which parametrize the ''self'' parts, proportional to m_1^3 or m_2^3, of the cubic and non-compact terms (recall also that $Y_L{}^{(-5,0)}$ is zero). See for instance the expressions (9.3) of NC terms. The relevant $Y_L{}^{(-3,0)}$ and $S_L^{(-5,0)}$ were obtained in Eqs. (8.15a) and (9.7). The dependence on u_1 and u_2 therein is

$$Y_{ij}{}^{(-3,0)} = 2\ln\left(\frac{u_1}{r_0}\right) y_1^{\langle ij\rangle} + \cdots, \tag{10.2a}$$

$$S_{ij}{}^{(-5,0)} = \frac{14}{3}\ln\left(\frac{u_1}{r_0}\right) y_1^{\langle ij\rangle} + \cdots. \tag{10.2b}$$

The dots indicate the terms independent of u_1 and u_2. We take all the cubic and NC terms given by Eqs. (8.2) and (9.3) [only the mass quadrupole is to be considered], plug into them the results (10.2) and find after summation the following part of the quadrupole moments depending on these constants (for general orbits):

$$I_{ij}[u_1,u_2] = \left(-\frac{44}{3}\,\frac{G^2 m_1^3}{c^6}\ln\left(\frac{u_1}{r_0}\right) a_1^{\langle i} y_1^{j\rangle} + 1 \leftrightarrow 2 \right) + \cdots. \tag{10.3}$$

By $I_{ij}[u_1,u_2]$ we mean the quadrupole obtained from summing all the terms computed in the previous sections, i.e., depending on the Hadamard-regularization constants u_1, u_2 (as well as, of course, the constant r_0). On the other hand, we found that many of the ''interaction'' terms, proportional to $m_1^2 m_2$ or $m_1 m_2^2$, depend on time-dependent logarithms of the ratio $\tilde{r}_{12} = r_{12}/r_0$, where r_0 is the constant dealing with the behavior of the moments at infinity. See for instance the elementary integrals (9.8). The effect of the result (10.3) is to ''replace'' a part of the latter logarithms of $\tilde{r}_{12}$ by some corresponding logarithms of the ratio r_{12}/u_1 (and ditto with u_2). The remaining logarithms stay as they are as logarithms of the ratio $\tilde{r}_{12}$. Thus we can re-write the dependence of the quadrupole on u_1 and u_2 through the logarithms of r_{12}/u_1 and r_{12}/u_2 in the form

$$I_{ij}[u_1,u_2] = \left[\frac{44}{3}\,\frac{G^2 m_1^3}{c^6}\ln\left(\frac{r_{12}}{u_1}\right) a_1^{\langle i} y_1^{j\rangle} + 1 \leftrightarrow 2 \right] + \cdots. \tag{10.4}$$

All the other logarithms, present in the dots of Eq. (10.4), are of the type $\ln(r_{12}/r_0)$. In this paper we assumed nothing about the values of u_1 and u_2. In particular we did not assume any relation between u_1, u_2 and the gauge constants r'_1, r'_2 that parametrize the final equations of motion in harmonic coordinates [40,43]. However, when computing the energy flux we shall need to obtain the third time-derivative of the quadrupole moment, and for that purpose we shall replace the accelerations by their expressions obtained from the 3PN equations of motion, depending on r'_1, r'_2. As a result the third time-derivative of the moment will depend on u_1, u_2 as well as on r'_1, r'_2. Therefore, we definitely need to control the relation between u_1, u_2 and r'_1, r'_2; then we shall have the quadrupole moment expressed solely in terms of r'_1 and r'_2 and we shall check that the latter constants can be removed by the same coordinate transformation as in the equations of motion, and thus that the final expression of the physical energy flux must be independent of these constants. From Eq. (10.4) we can write

$$I_{ij}[u_1,u_2]=I_{ij}[r'_1,r'_2]+\frac{44}{3}\frac{G^2m_1^3}{c^6}\ln\left(\frac{r'_1}{u_1}\right)a_1^{\langle i}y_1^{j\rangle}+1\leftrightarrow 2. \tag{10.5}$$

The notation for $I_{ij}[r'_1,r'_2]$ is clear: we mean the sum of all the contributions obtained in the previous sections, but computed with r'_1,r'_2 in place of the regularization constants u_1,u_2.

We shall now look for the most general $\ln(r'_1/u_1)$ that is allowed by physical requirements. In this connection recall the spirit of the regularization: the constants u_1 and u_2 reflect some incompleteness of the process, that may or may not be fixed in a given computation, and therefore they should be kept completely arbitrary unless there are some physical arguments to restrict their form. In particular, when used in different computations, these regularization constants have no reason *a priori* to be the same. For instance, in the present computation of the moments, the constants u_1 and u_2 are *a priori* different from the constants s_1 and s_2 which were originally used in the 3PN equations of motion [see Eq. (2.3) in [43]]. They are *a fortiori* different from the constants r'_1 and r'_2 chosen to parametrize the final equations of motion [Eq. (7.16) in [43]]. See also the discussion in Sec. VII in Ref. [43], where we determined the general form of the relation between s_1,s_2 and r'_1,r'_2 by imposing the polynomial mass dependence of the equations of motion, the correct perturbative limit, and the existence of a conserved energy. Here we shall basically do the same in order to restrict the form of the relation between u_1, u_2 and r'_1, r'_2. Note that *a priori* the logarithms $\ln(r'_1/u_1)$ and $\ln(r'_2/u_2)$ can depend on the masses m_1 and m_2. To determine just what combination of masses is allowed we make (similarly to the equations of motion) two physical requirements: (i) that the quadrupole moment be a polynomial function of the two masses m_1,m_2 when taken separately, (ii) that the perturbative limit (corresponding to $\nu\to 0$) not be affected by this possible dependence over the masses. Because of the factor m_1^3 in front of the log-term in Eq. (10.5), and because the acceleration a_1^i brings another factor m_2, the most general solution for this logarithm in order to satisfy the requirement (i) is to be composed of: a pure numerical constant (say ξ), plus a pure constant (say κ) times the mass ratio m/m_1, plus a constant times m/m_2, next five terms involving the mass ratios m^2/m_1^2, $m^2/m_1/m_2$, m^3/m_1^3, $m^3/m_1^2/m_2$ and $m^4/m_1^3/m_2$. Each of these terms must be such that it does not violate the perturbative limit [our requirement (ii)]. This means that they should involve, in a center-of-mass frame, a factor ν^2 at least. We readily find that the only two admissible terms in this respect are the first two in the previous list (with constants ξ and κ). So we end up with the most general admissible solution

$$\ln\left(\frac{r'_1}{u_1}\right)=\xi+\kappa\frac{m_1+m_2}{m_1}\quad(\text{and } \textit{idem} \text{ with } 1\leftrightarrow 2), \tag{10.6}$$

where ξ and κ denote some arbitrary purely numerical constants (for instance rational fractions). This result is similar to the one obtained in the 3PN equations of motion, concerning the relation between s_1, s_2 and r'_1, r'_2. See Eqs. (7.9) in Ref. [43], where the determination of the constant analogous to ξ was possible from the requirement of existence of a conserved energy (and Lagrangian) for the equations of motion.

We now check that the logarithms of r_{12}/r'_1 and r_{12}/r'_2 in the quadrupole moment, which are of the form

$$I_{ij}[r'_1,r'_2]=\left(\frac{44}{3}\frac{G^2m_1^3}{c^6}\ln\left(\frac{r_{12}}{r'_1}\right)a_1^{\langle i}y_1^{j\rangle}+1\leftrightarrow 2\right)+\cdots, \tag{10.7}$$

can be eliminated by the *same* coordinate transformation as found in Ref. [43] for the logarithms in the harmonic-coordinate equations of motion. [As concerns the logarithms of r_{12}/r_0 in the moment they cannot be eliminated by a change of coordinates but will match precisely with corresponding logarithms present in the ''tails of tails'' at infinity.] We look for a coordinate change of the type considered in Sec. VI A of [43]: namely $\delta x^\mu=\xi^\mu$, where $\xi_\mu=\eta_{\mu\nu}\xi^\nu$ is a 3PN gauge vector given by

$$\xi_\mu=\frac{G^3m^3}{c^6}\partial_\mu\left(\frac{\epsilon_1}{r_1}+\frac{\epsilon_2}{r_2}\right). \tag{10.8}$$

We have factorized out m^3 (where $m=m_1+m_2$) so that ϵ_1 and ϵ_2, which are constants or mere functions of time t, will be dimensionless. The corresponding change of the particle's trajectories is given to this order by the regularized value of the gauge vector at the location of the particle (see Sec. VI A in [43]). We obtain

$$\delta_\xi y_1^i=-\epsilon_2\frac{G^3m^3}{c^6r_{12}^3}y_{12}^i, \tag{10.9a}$$

$$\delta_\xi y_2^i=\epsilon_1\frac{G^3m^3}{c^6r_{12}^3}y_{12}^i. \tag{10.9b}$$

Since the quadrupole moment starts at the Newtonian level with the usual $m_1y_1{}^{\langle ij\rangle}+1\leftrightarrow 2$, we easily find its coordinate change as

$$\begin{aligned}\delta_\xi I_{ij}&=2m_1y_1^{\langle i}\delta_\xi y_1^{j\rangle}+1\leftrightarrow 2\\&=-2m_1\epsilon_2\frac{G^3m^3}{c^6r_{12}^3}y_1^{\langle i}y_{12}^{j\rangle}+1\leftrightarrow 2.\end{aligned} \tag{10.10}$$

By comparing this with Eq. (10.7) (using the Newtonian particles acceleration), we find that the gauge transformation required to eliminate the logarithms is

$$\epsilon_2 = -\frac{22}{3}\frac{m_1^2 m_2}{m^3}\ln\left(\frac{r_{12}}{r'_1}\right), \tag{10.11a}$$

$$\epsilon_1 = -\frac{22}{3}\frac{m_1 m_2^2}{m^3}\ln\left(\frac{r_{12}}{r'_2}\right), \tag{10.11b}$$

in complete agreement with Eq (7.2) in Ref. [43]. In summary, not only will these logarithms disappear when considering physical quantities associated with the equations of motion (such as the invariant energy), but they will also cancel from physical quantities associated with the wave field at infinity, viz. the invariant energy flux we compute in Sec. XII.

B. Special features of the regularization

We now discuss some subtleties of the Hadamard regularization which motivate the introduction in the quadrupole moment, in addition to ξ and κ considered in Eq. (10.6), of still another constant (however, see below for the definition of a single constant θ).

Non-distributivity of the Hadamard partie finie. By "non-distributivity" we mean the fact that the regularization of a product of two functions F and G, singular in the sense of Eq. (5.1), does not equal, in general, the product of the regularized functions: $(FG)_1 \neq (F)_1(G)_1$. For instance, with $U = Gm_1/r_1 + Gm_2/r_2$ the Newtonian potential, we have $(U^n)_1 = [(U)_1]^n$ for $n=1,2,3$, but $(U^4)_1 = [(U)_1]^4 + 2[(U)_1]^2[(U)_2]^2$. An immediate consequence is that the product of a singular function F with a delta-function does not equal, in general, the product of its regularized value with the delta-function: $F\delta_1 \neq (F)_1\delta_1$. Here we are assuming that the three-dimensional integral of the product of F with $\delta_1 \equiv \delta(\mathbf{x}-\mathbf{y}_1)$ gives back the regularized value $(F)_1$. Notice that only at the 3PN order does the non-distributivity play a role. Up to the 2PN order, the distributivity holds for all the functions encountered in the problem (hence the computation of the moments as was done in [13] is correct).

The non-distributivity at 3PN has an important bearing on the choice of the stress-energy tensor for describing point-particles. In this paper, we adopted the most naive choice for the stress-energy tensor. See Eq. (5.4) above, which is equivalent, at 3PN order, to

$$T^{\mu\nu} = \frac{m_1 v_1^\mu v_1^\nu}{\sqrt{(gg_{\rho\sigma})_1 v_1^\rho v_1^\sigma/c^2}}\,\delta(\mathbf{x}-\mathbf{y}_1) + 1 \leftrightarrow 2. \tag{10.12}$$

Namely, we assumed that the whole factor of the delta-function consists of a regularized value at point 1. But because $F\delta_1 \neq (F)_1\delta_1$, we could obtain a different result by choosing another stress-energy tensor, defined by replacing the factor of the delta-function in Eq. (10.12), or part of it, by a function depending on any field point $\mathbf{x}$ and such that its regularized value when $\mathbf{x}\to\mathbf{y}_1$ is the same. In fact, a specific form of the stress-energy tensor of point-particles, compatible with the Hadamard regularization, was advocated in Ref. [42] and used to compute the 3PN equations of motion [43]. This form, given by Eq. (5.11) in Ref. [42], reads

$$\hat{T}^{\mu\nu} = \frac{m_1 v_1^\mu v_1^\nu}{\sqrt{-(g_{\rho\sigma})_1 v_1^\rho v_1^\sigma/c^2}}\,\frac{\delta(\mathbf{x}-\mathbf{y}_1)}{\sqrt{-g(\mathbf{x},t)}} + 1 \leftrightarrow 2. \tag{10.13}$$

Choosing one or the other form of stress-energy tensor does make a difference in our computation. Consider for instance the term $\mathrm{SI}(1) = \int d^3\mathbf{x}\hat{x}^{ij}\sigma$. We find that the result for this term, when computed using the tensor (10.13), i.e., using $c^2\hat{\sigma} = \hat{T}^{00} + \hat{T}^{ii}$, differs from the original result by the amount

$$\Delta\mathrm{SI}(1) = \frac{G^2 m_1^3}{c^6}\left[\frac{2}{3}a_1^{\langle i}y_1^{j\rangle} - \frac{1}{5}v_1^{\langle i}v_1^{j\rangle}\right] + 1 \leftrightarrow 2. \tag{10.14}$$

There is also a modification $\Delta\mathrm{SII}(1)$ but which is of the same structure (with different numerical coefficients).

On the other hand, some terms in our computation would be different if the regularization would be distributive. For instance, if for computing the term SI(16NC) we take into account the nondistributivity (as we did), we find the result (6.9): namely,

$$\mathrm{SI}(16\mathrm{NC}) = \frac{4m_1}{c^6}v_1^{ab}(x^{\langle i}x^{j\rangle}U_{ab}^{(\mathrm{NC})})_1 + 1 \leftrightarrow 2. \tag{10.15}$$

If instead we incorrectly assume that the partie finie is distributive, then we get

$$\mathrm{SI}(16\mathrm{NC})_{\mathrm{distr}} = \frac{4m_1}{c^6}y_1^{\langle i}y_1^{j\rangle}v_1^{ab}(U_{ab}^{(\mathrm{NC})})_1 + 1 \leftrightarrow 2. \tag{10.16}$$

The difference between the two results is not zero:

$$\Delta\mathrm{SI}(16\mathrm{NC}) = -\frac{2}{15}\frac{G^2 m_1^3}{c^6}v_1^{\langle i}v_1^{j\rangle} + 1 \leftrightarrow 2. \tag{10.17}$$

The same happens with the other terms VI(10NC) and VI(12NC); each time the structure of the difference is the same as in Eq. (10.14) or Eq. (10.17).

Violation of the Leibniz rule by the distributional derivative. In Ref. [41] a new kind of distributional derivative of singular functions of the type F was introduced. It was found that it is impossible to define a derivative satisfying the Leibniz rule for the derivation of the product, but that a mathematical structure exists when we replace the Leibniz rule by the weaker rule of "integration by parts." The latter rule can be seen as an integrated version of the Leibniz rule (see Sec. VII A in [41]). More precisely, two different distributional derivatives were proposed in [41]: a "particular" derivative, and a "correct" one. Both derivatives reduce to the derivative of the standard distribution theory [32] when applied to smooth test functions with compact support. The particular derivative is simpler to use in practical computations, but the correct one is more satisfying because successive derivatives to any order commute.

Previously we performed numerous simplifications, with the help of the Leibniz rule, to arrive at the form of multipole moments given by Eqs. (4.2). Thus we made some errors because of the violation of the Leibniz rule by the distributional derivative. The strategy adopted in Ref. [43] was to keep track of all these error terms and to compute them using the particular and correct derivatives of [41]. In the present paper we shall proceed differently. We simply give an example. When simplifying the moment to arrive at the simple-looking term SI(39) in Eq. (4.2a), we ''forgot'' to include the error term

$$\Delta \mathrm{SI}(39) = \frac{2}{3\pi G c^6} \mathop{\mathrm{FP}}_{B=0} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_{ij} [\Delta(U^4) - 4U^3 \Delta U - 12 U^2 \partial_a U \partial_a U]. \tag{10.18}$$

Clearly this term would be zero for any derivative satisfying the Leibniz rule (in a distributional sense). However, computing it by means of for instance the ''particular'' derivative [defined by Eq. (7.7) in Ref. [41]], we find that it is not zero, but

$$\Delta \mathrm{SI}(39) = \frac{64}{3} \frac{G^2 m_1^3}{c^6} a_1^{\langle i} y_1^{j\rangle} + 1 \leftrightarrow 2. \tag{10.19}$$

Again this result has the same type of structure as found previously. We have checked that all the terms coming from the violation of the Leibniz rule have the same structure, either of type $m_1^3 a_1^{\langle i} y_1^{j\rangle}$ like in Eq. (10.19) or of type $m_1^3 v_1^{\langle i} v_1^{j\rangle}$.

Cubically non-linear self-interaction terms. We take the example of the self contribution in the term SI(35NC). This term is

$$\mathrm{SI}(35\mathrm{NC}) = -\frac{4}{\pi G c^6} \mathop{\mathrm{FP}}_{B=0} \int d^3\mathbf{x} |\tilde{\mathbf{x}}|^B \hat{x}_L \hat{Z}_{ij}^{(\mathrm{NC})} \partial_{ij}^2 U. \tag{10.20}$$

The ''self'' part of this term corresponds to that part in $\hat{Z}_{ij}^{(\mathrm{NC})}$ which is proportional to m_1^2, in the sense that

$$\hat{Z}_{ij}^{(\mathrm{NC})} = G^2 m_1^2 \left\{ a_1^{(i} {}_1\partial_{j)} \ln r_1 + \frac{1}{8} v_1^2 {}_1\partial_{ij} \ln r_1 + \frac{1}{32} \delta^{ij} v_1^{km} {}_1\partial_{km} \ln r_1 + \frac{1}{2} \frac{v_1^{ij}}{r_1^2} - \frac{11}{32} \delta^{ij} \frac{v_1^2}{r_1^2} \right\} + \mathcal{O}(m_2) \tag{10.21}$$

[see Eq. (9.1d)], and that part of U due to 1 itself, i.e., $U = G m_1/r_1 + \mathcal{O}(m_2)$. The resulting term, proportional to m_1^3, is ill defined in distribution theory because the delta-function, coming from the distributional derivative of $1/r_1$ as given by Eq. (9.2b), is multiplied by the terms in Eq. (10.21) which are singular at point 1. The partie finie pseudo-functions and their derivatives proposed in Ref. [41] permit us to give a mathematical meaning to such ill-defined terms. The ''particular'' derivative of $1/r_1$ reads $\partial_{ij}^2(1/r_1) = \partial_{ij}^2(1/r_1)_{\mathrm{ordinary}} + D_{ij}[1/r_1]$, where the purely distributional part is

$$D_{ij}\left[\frac{1}{r_1}\right] = -\frac{4\pi}{3}\left(\delta^{ij} + \frac{15}{2} \hat{n}_1^{ij}\right) \delta_1. \tag{10.22}$$

[Compare this with the result (9.2b) of distribution theory.] We easily compute the effect of this new derivative on the self part of the term (10.20). Once again we find the same type of structure as before:

$$\Delta \mathrm{SI}(35\mathrm{NC}) = \frac{G^3 m_1^3}{c^6} \left[-\frac{64}{3} a_1^{\langle i} y_1^{j\rangle} + \frac{38}{15} v^{\langle i} v_1^{j\rangle} \right] + 1 \leftrightarrow 2. \tag{10.23}$$

Similarly we checked that all other self-interaction contributions take the same form with simply different numerical coefficients.

C. Definition of the θ-ambiguity

As we have seen the structure of the possible terms associated with the previous subtleties in the Hadamard regularization are limited to only two types, either $m_1^3 a_1^{\langle i} y_1^{j\rangle}$ or $m_1^3 v_1^{\langle i} v_1^{j\rangle}$. The first type was already considered in Eqs. (10.5) and (10.6), where it yielded the arbitrary constant ξ. Thus, modulo a redefinition of ξ, we do not need to consider this term. The other type, given by $m_1^3 v_1^{\langle i} v_1^{j\rangle}$, was not considered earlier. Therefore, motivated by the previous discussion, we shall from now on add such a term to the multipole moment, with a new constant in front, say ζ. In summary, we consider three types of ''ambiguous'' terms (in the sense of [34,35]), parametrized by the two constants ξ, κ of Eq. (10.5), and the ζ. The quadrupole moment we finally consider in this paper is thus

$$I_{ij} = I_{ij}[r_1', r_2'] + \Delta I_{ij}, \tag{10.24}$$

where $I_{ij}[r_1', r_2']$ denotes the computation we have done in Sections VI–IX (i.e., the sum of all the terms, defined for general orbits, and given for circular orbits in the Appendix), when expressed by means of the same regularization constants r_1', r_2' as the ones appearing in the 3PN equations of motion (we know that these constants are pure gauge). Now the undetermined part reads as

$$\Delta I_{ij} = \frac{44}{3} \frac{G^2 m_1^3}{c^6} \left[\left(\xi + \kappa \frac{m}{m_1} \right) a_1^{\langle i} y_1^{j\rangle} + \zeta v_1^{\langle i} v_1^{j\rangle} \right] + 1 \leftrightarrow 2. \tag{10.25}$$

In a center-of-mass frame we get

$$\Delta I_{ij} = \frac{44}{3} \frac{G^2 m^3 \nu^2}{c^6} [(\xi + 2\kappa) a^{\langle i} x^{j\rangle} + \zeta v^{\langle i} v^{j\rangle}] \tag{10.26}$$

(where $x^i = y_1^i - y_2^i$, $v^i = dx^i/dt$ and $a^i = dv^i/dt$). The constants ξ, κ and ζ will be left unspecified in the present paper. It could be possible that the more sophisticated regularization procedure of [41,42] determines some of these constants. However, the point for our purpose is that we are going to show that the physical energy flux for circular orbits depends only on one parameter. Indeed, the flux depends on the third time-derivative of the quadrupole, and by a straightforward computation (using the Newtonian equations of motion) we find that, in the case of circular orbits, the third time-derivative of Eq. (10.26) is

$$\Delta I_{ij}^{(3)} = \frac{352}{3}\frac{Gm^2\nu^2}{r^3}\gamma^3\theta x^{\langle i}v^{j\rangle}, \tag{10.27}$$

where $\theta = \xi + 2\kappa + \zeta$ is a single unknown constant. Therefore, the ambiguous part of the physical 3PN flux, as concerns this effect, depends in fact only on θ. It is given (for circular orbits) by

$$\Delta\mathcal{L} = \frac{2G}{5c^5} I_{ij}^{(3)}\Delta I_{ij}^{(3)} = \frac{32c^5}{5G}\nu^2\gamma^5\left\{-\frac{88}{3}\theta\nu\gamma^3\right\}. \tag{10.28}$$

In addition to θ, the flux will depend also on the constant λ coming from the equations of motion [40,43]. However, we shall find that, in the case of circular orbits, both θ and λ enter the flux at the same level, so the flux depends only on one combination of these constants: $\lambda - \frac{2}{3}\theta$, from the end result (12.9) below. Further work, supplementing the Hadamard self-field regularization by suitable extensions and alternative methods, may be required to determine the constants θ and λ.

XI. THE BINARY'S MULTIPOLE MOMENTS

The computation of the moments is now almost complete. The remaining terms are as follows.

(i) The "odd" terms: SI(11), SI(12). These terms involve the fifth (odd) power of $1/c$ (2.5PN order). They appear because of the expansion of retardations in the potentials (3.6); they are pure functions of time, parametrized by $Q_{ij}(t)$ and $Q(t)$ [see Eq. (3.9)]. The sum of the two odd terms has been computed in Eqs. (4.9) and (4.12) of Ref. [26]. With the present notation, in the quadrupole case, we have

$$\mathrm{SI}(11)+\mathrm{SI}(12) = \frac{G}{c^5}\left\{-\frac{8}{7}Q^{(3)}_{k\langle i}Q_{j\rangle k} - \frac{10}{7}Q^{(3)}_{\langle ij\rangle}Q - \frac{2}{21}Q_{\langle ij\rangle}Q^{(3)}\right\}. \tag{11.1}$$

These terms do not contribute to the flux for circular orbits.

(ii) The "divergence" terms: SI(2), SI(8), SI(9), SI(10), SI(39), SI(40), SI(41), SI(42), SI(43), SI(44), SI(45), SI(46), SI(47), SI(48), SI(49), SI(50), SII(8), SII(10), SII(12), SII(14), SIII(3), VI(6), VI(30), VI(31), VI(32), VI(33), VI(34), VI(35), VII(7), TI(9). The integrand of these terms is made of the product of $|\tilde{\mathbf{x}}|^B$ and a pure divergence $\partial_i A$ or ΔA. Their computation makes use of the same techniques as those employed in Secs. VII–IX, but with the notable simplification that because of the divergence one can perform an integration by parts, and that as a result the elementary integrals contain explicitly a factor B (due to the differentiation of $|\tilde{\mathbf{x}}|^B$) so their computation is quite easy. See the results in the Appendix.

(iii) Four particular terms that we have left out because their sum is in fact zero:

$$\mathrm{SI}(22)+\mathrm{SI}(23)+\mathrm{SI}(32)+\mathrm{SII}(11) = 0. \tag{11.2}$$

We sum up all the terms given in the Appendix, plus the undetermined correction given by Eq. (10.26), and obtain the expressions of the 3PN mass-type quadrupole moment, 2PN mass-type octupole moment and 2PN current-type quadrupole moment of the compact binary moving on a circular orbit. (Note that most of the investigation of this paper is valid for general orbits, but we are interested in inspiraling binaries whose orbit is quickly circularized by radiation reaction.) The 3PN mass quadrupole reads

$$I_{ij} = \mu\left(A\hat{x}_{ij} + B\frac{r^2}{c^2}\hat{v}_{ij} + \frac{48}{7}\frac{r}{c}x_{\langle i}v_{j\rangle}\nu\gamma^2\right) + \mathcal{O}(7), \tag{11.3}$$

where the third term is the 2.5PN odd term, and where

$$\begin{aligned} A = {} & 1 + \gamma\left(-\frac{1}{42} - \frac{13}{14}\nu\right) + \gamma^2\left(-\frac{461}{1512} - \frac{18395}{1512}\nu - \frac{241}{1512}\nu^2\right) \\ & + \gamma^3\left\{\frac{395899}{13200} - \frac{428}{105}\ln\left(\frac{r}{r_0}\right) + \left[\frac{139675}{33264} - \frac{44}{3}(\xi + 2\kappa) \right.\right. \\ & \left.\left. - \frac{44}{3}\ln\left(\frac{r}{r_0'}\right)\right]\nu + \frac{162539}{16632}\nu^2 + \frac{2351}{33264}\nu^3\right\}, \end{aligned} \tag{11.4a}$$

$$\begin{aligned} B = {} & \frac{11}{21} - \frac{11}{7}\nu + \gamma\left(\frac{1607}{378} - \frac{1681}{378}\nu + \frac{229}{378}\nu^2\right) \\ & + \gamma^2\left\{-\frac{357761}{19800} + \frac{428}{105}\ln\left(\frac{r}{r_0}\right) + \left[-\frac{75091}{5544} + \frac{44}{3}\zeta\right]\nu \right. \\ & \left. + \frac{35759}{924}\nu^2 + \frac{457}{5544}\nu^3\right\}. \end{aligned} \tag{11.4b}$$

The mass parameters are $m = m_1 + m_2$, $\delta m = m_1 - m_2$, $\mu = m_1 m_2/m$, and $\nu = \mu/m$. The post-Newtonian parameter is $\gamma = Gm/(rc^2) = \mathcal{O}(2)$ [see Eq. (5.8)]. The logarithms depend either on the constant r_0 associated with the finite part at infinity (recall $|\tilde{\mathbf{x}}|^B = |\mathbf{x}/r_0|^B$) or on the "logarithmic barycenter" r_0' of the regularization constants r_1' and r_2' (see Sec. X), defined by $m\ln r_0' = m_1 \ln r_1' + m_2 \ln r_2'$. We shall investigate in Sec. XII the fate of these constants r_0 and r_0'. In addition the moment depends on the unknown constants ξ, κ, and ζ introduced in Eq. (10.26). The 2PN mass-octupole and 2PN current-quadrupole are free of any of such constants and given by

$$I_{ijk}=\mu\frac{\delta m}{m}\hat{x}_{ijk}\left[-1+\gamma\nu+\gamma^2\left(\frac{139}{330}+\frac{11923}{660}\nu+\frac{29}{110}\nu^2\right)\right]+\mu\frac{\delta m}{m}x_{\langle i}v_{jk\rangle}\frac{r^2}{c^2}\left[-1+2\nu+\gamma\left(-\frac{1066}{165}+\frac{1433}{330}\nu-\frac{21}{55}\nu^2\right)\right]+\mathcal{O}(5),\tag{11.5a}$$

$$J_{ij}=\mu\frac{\delta m}{m}\varepsilon_{ab\langle i}x_{j\rangle a}v_b\left[-1+\gamma\left(-\frac{67}{28}+\frac{2}{7}\nu\right)+\gamma^2\left(-\frac{13}{9}+\frac{4651}{252}\nu+\frac{1}{168}\nu^2\right)\right]+\mathcal{O}(5).\tag{11.5b}$$

The higher multipole moments which are needed in the 3PN energy flux are the 1PN current octupole, 1PN mass 2^4-pole, Newtonian current 2^4-pole, and Newtonian mass 2^5-pole. For these moments we simply report the expressions already obtained in Ref. [13]:

$$I_{ijkl}=\mu\hat{x}_{ijkl}\left[1-3\nu+\gamma\left(\frac{3}{110}-\frac{25}{22}\nu+\frac{69}{22}\nu^2\right)\right]+\frac{78}{55}\mu x_{\langle ij}v_{kl\rangle}\frac{r^2}{c^2}(1-5\nu+5\nu^2)+\mathcal{O}(3),\tag{11.6a}$$

$$J_{ijk}=\mu\varepsilon_{ab\langle i}x_{jk\rangle a}v_b\left[1-3\nu+\gamma\left(\frac{181}{90}-\frac{109}{18}\nu+\frac{13}{18}\nu^2\right)\right]+\frac{7}{45}\mu(1-5\nu+5\nu^2)\varepsilon_{ab\langle i}v_{jk\rangle b}x_a\frac{r^2}{c^2}+\mathcal{O}(3),\tag{11.6b}$$

$$I_{ijklm}=\mu\frac{\delta m}{m}(-1+2\nu)\hat{x}_{ijklm}+\mathcal{O}(1),\tag{11.6c}$$

$$J_{ijkl}=\mu\frac{\delta m}{m}(-1+2\nu)\varepsilon_{ab\langle i}x_{jkl\rangle a}v_b+\mathcal{O}(1).\tag{11.6d}$$

As proved in Refs. [26,21] the multipole moments I_L and J_L are not the only source moments entering the radiation field. However, the other moments, denoted W_L, X_L, Y_L and Z_L, parametrize a (linearized) gauge transformation in the exterior field, and as a result make a contribution to the non-linear radiation field at a quite high post-Newtonian order: 2.5PN. It is always possible to re-express the radiation field in terms of solely two sets of moments, denoted M_L and S_L, given by some non-linear functionals of the moments I_L, J_L, W_L, X_L, Y_L and Z_L, but differing from I_L and J_L starting at the 2.5PN order (see Sec. VI in [21] for a discussion). From Eqs. (4.20)–(4.24) in [26] the 3PN quadrupole M_{ij} is related to I_{ij} by

$$M_{ij}=I_{ij}-\frac{4G}{c^5}[W^{(2)}I_{ij}-W^{(1)}I^{(1)}_{ij}]+\mathcal{O}(7),\tag{11.7a}$$

$$W=\frac{1}{3}\int d^3\mathbf{x}x_i\sigma_i=\frac{1}{3}m_1(y_1v_1)+\mathcal{O}(2)+1\leftrightarrow 2.\tag{11.7b}$$

For the other moments there is no correction to be made at this order [for instance $S_{ij}=J_{ij}+\mathcal{O}(5)$]. Actually we observe that W is zero for circular orbits, and thus we shall from now on replace all the moments I_L and J_L by the corresponding M_L and S_L.

Finally we need to relate the moments M_L, S_L to the "radiative" moments, say U_L (mass-type) and V_L (current-type), which play the role of observables associated with the radiation field at infinity. Since such a relation has already been worked out at the 3.5PN level in Ref. [27], we simply report the main result, which concerns the mass-quadrupole radiative moment U_{ij}, that is

$$U_{ij}(t)=M^{(2)}_{ij}(t)+\frac{2Gm}{c^3}\int_0^{+\infty}d\tau M^{(4)}_{ij}(t-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{11}{12}\right]+\frac{2G^2m^2}{c^6}\int_0^{+\infty}d\tau M^{(5)}_{ij}(t-\tau)\left[\ln^2\left(\frac{c\tau}{2r_0}\right)+\frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right)+\frac{124627}{44100}\right]+\frac{1}{c^5}\{\cdots\}+\frac{1}{c^7}\{\cdots\}+\mathcal{O}(8).\tag{11.8}$$

This formula is valid through 3.5PN order, modulo the odd-order 2.5PN and 3.5PN terms that we do not show because they do not contribute to the flux for circular orbits. The only contributions coming from Eq. (11.8) are the 1.5PN tail, and 3PN "tail of tail" integrals. In the flux we shall derive below the terms at the orders 2.5PN and 3.5PN are due to the tail integrals in higher multipole moments (see Ref. [27] for details).

XII. THE ENERGY FLUX OF CIRCULAR COMPACT BINARIES

For general sources, the total energy flux (or gravitational luminosity $\mathcal{L}$) to the 3PN order is composed of an "instantaneous" contribution—i.e., a functional of the multipole moments M_L and S_L at the same instant—and a "tail" contribution. We shall now follow the study in [27] of the occurrence of non-linear effects in $\mathcal{L}$ up to 3.5PN order. Following the equation (4.18) in Ref. [27] we split $\mathcal{L}$ into an instantaneous part, a tail part, a tail square part, and a tail of tail part:

$$\mathcal{L}=\mathcal{L}_{\text{inst}}+\mathcal{L}_{\text{tail}}+\mathcal{L}_{(\text{tail})^2}+\mathcal{L}_{\text{tail(tail)}}.\tag{12.1}$$

As all the parts involving tails have already been computed for circular binaries [27], we need only to compute the instantaneous part which is given by

$$\mathcal{L}_{\rm inst}=\frac{G}{c^5}\left\{\frac{1}{5}M^{(3)}_{ij}M^{(3)}_{ij}+\frac{1}{c^2}\left[\frac{1}{189}M^{(4)}_{ijk}M^{(4)}_{ijk}+\frac{16}{45}S^{(3)}_{ij}S^{(3)}_{ij}\right]+\frac{1}{c^4}\left[\frac{1}{9072}M^{(5)}_{ijkm}M^{(5)}_{ijkm}+\frac{1}{84}S^{(4)}_{ijk}S^{(4)}_{ijk}\right]+\frac{1}{c^6}\left[\frac{1}{594000}M^{(6)}_{ijkmn}M^{(6)}_{ijkmn}+\frac{4}{14175}S^{(5)}_{ijkm}S^{(5)}_{ijkm}\right]+\mathcal{O}(8)\right\} \tag{12.2}$$

[see, e.g., (4.15) in [27]]. To obtain $\mathcal{L}_{\rm inst}$ we compute the time derivatives of the multipole moments. At this stage we need a new input, namely the 3PN equations of motion of circular binaries which are crucial in the differentiation of the 3PN quadrupole moment. As recently obtained [40,43] the 3PN orbital frequency of the circular motion reads as

$$\omega^2=\frac{Gm}{r^3}\left(1+[-3+\nu]\gamma+\left[6+\frac{41}{4}\nu+\nu^2\right]\gamma^2+\left\{-10+\left[22\ln\left(\frac{r}{r'_0}\right)+\frac{41\pi^2}{64}-\frac{67759}{840}+\frac{44\lambda}{3}\right]\nu+\frac{19}{2}\nu^2+\nu^3\right\}\gamma^3+\mathcal{O}(\gamma^4)\right). \tag{12.3}$$

The inverse of this formula gives the post-Newtonian parameter γ as a function of the frequency related parameter $x=(Gm\omega/c^3)^{2/3}$,

$$\gamma=x\left(1+\left[1-\frac{\nu}{3}\right]x+\left[1-\frac{65}{12}\nu\right]x^2+\left\{1+\left[-\frac{22}{3}\ln\left(\frac{r}{r'_0}\right)-\frac{41\pi^2}{192}-\frac{10151}{2520}-\frac{44\lambda}{9}\right]\nu+\frac{229}{36}\nu^2+\frac{1}{81}\nu^3\right\}x^3+\mathcal{O}(x^4)\right). \tag{12.4}$$

Note that Eq. (12.3) or Eq. (12.4) involves the same constant r'_0 as in the 3PN mass quadrupole moment (11.3) and (11.4).

Taking all the expressions of the multipole moments found in Sec. XI, computing their time-derivatives according to the latter circular-orbit 3PN equations of motion, and inserting them into Eq. (12.2) we then arrive at the following instantaneous part of the flux:

$$\mathcal{L}_{\rm inst}=\frac{32c^5}{5G}\gamma^5\nu^2\left(1+\left(-\frac{2927}{336}-\frac{5}{4}\nu\right)\gamma+\left(\frac{293383}{9072}+\frac{380}{9}\nu\right)\gamma^2+\left\{\frac{53712289}{1108800}-\frac{1712}{105}\ln\left(\frac{r}{r_0}\right)+\left[-\frac{332051}{720}+\frac{110}{3}\ln\left(\frac{r}{r'_0}\right)+\frac{123\pi^2}{64}+44\lambda-\frac{88}{3}\theta\right]\nu-\frac{383}{9}\nu^2\right\}\gamma^3+\mathcal{O}(\gamma^4)\right), \tag{12.5}$$

where we recall that $\theta=\xi+2\kappa+\zeta$. Next we simply add the known other contributions. The tail one is due to such terms as the 1.5PN integral appearing in Eq. (11.8) [and other equations corresponding to higher multipole moments]. The result is derived to the 3.5PN order in Eq. (5.5a) in Ref. [27]:

$$\mathcal{L}_{\rm tail}=\frac{32c^5}{5G}\gamma^5\nu^2\left\{4\pi\gamma^{3/2}+\left(-\frac{25663}{672}-\frac{109}{8}\nu\right)\pi\gamma^{5/2}+\left(\frac{90205}{576}+\frac{3772673}{12096}\nu+\frac{32147}{3024}\nu^2\right)\pi\gamma^{7/2}+\mathcal{O}(\gamma^4)\right\}. \tag{12.6}$$

Second, the tail of tail comes from the 3PN term in Eq. (11.8), and the tail square from the square of the 1.5PN term. The sum of these parts reads, following Eq. (5.9) in [27],

$$\mathcal{L}_{({\rm tail})^2+{\rm tail(tail)}}=\frac{32c^5}{5G}\gamma^5\nu^2\left(\left\{-\frac{1712}{105}\left[C+\ln\left(\frac{4\omega r_0}{c}\right)\right]+\frac{16\pi^2}{3}-\frac{116761}{3675}\right\}\gamma^3+\mathcal{O}(\gamma^4)\right), \tag{12.7}$$

where C denotes the Euler constant ($C=0.577\ldots$), and where the constant r_0 is the same as the r_0 occurring in the mass quadrupole moment (11.3) and (11.4). Thus, the energy flux, complete up to the 3.5PN order, reads

$$\mathcal{L}=\frac{32c^5}{5G}\gamma^5\nu^2\left(1+\left(-\frac{2927}{336}-\frac{5}{4}\nu\right)\gamma+4\pi\gamma^{3/2}+\left(\frac{293383}{9072}+\frac{380}{9}\nu\right)\gamma^2+\left(-\frac{25663}{672}-\frac{109}{8}\nu\right)\pi\gamma^{5/2}+\left\{\frac{129386791}{7761600}+\frac{16\pi^2}{3}-\frac{1712}{105}C-\frac{856}{105}\ln(16\gamma)+\left[-\frac{332051}{720}+\frac{110}{3}\ln\left(\frac{r}{r'_0}\right)+\frac{123\pi^2}{64}+44\lambda-\frac{88}{3}\theta\right]\nu-\frac{383}{9}\nu^2\right\}\gamma^3+\left(\frac{90205}{576}+\frac{3772673}{12096}\nu+\frac{32147}{3024}\nu^2\right)\pi\gamma^{7/2}+\mathcal{O}(\gamma^4)\right). \tag{12.8}$$

We observe that the constants r_0 have canceled out between the instantaneous flux $\mathcal{L}_{\rm inst}$ and the part $\mathcal{L}_{({\rm tail})^2+{\rm tail(tail)}}$. This cancellation is to be expected for any source: see a proof in [27] [Eqs. (4.14) there] where it is shown that the tails of tails at the 3PN order depend on r_0 through the effective quadrupole moment $M^{\rm eff}_{ij}=M_{ij}+\frac{214}{105}\ln r_0(G^2m^2/c^6)M^{(2)}_{ij}$. Using our explicit result (11.4) for $I_{ij}=M_{ij}+\mathcal{O}(5)$ we find that indeed the r_0's cancel out. The fact that we have recovered the expected dependence on r_0 of the source quadrupole moment is a good check of the computation.

On the other hand, the point-mass regularization constant r'_0 still remains in the flux (12.8). This is because the energy flux is not yet expressed in a coordinate-independent way, as the post-Newtonian parameter γ depends on the distance between the masses in harmonic coordinates. To find a truly coordinate-independent result we must replace γ by its expression given by Eq. (12.4) in terms of the frequency-related parameter x. With this change of variable, at long last we obtain our end result:

$$\mathcal{L}=\frac{32c^5}{5G}x^5\nu^2\left\{1+\left(-\frac{1247}{336}-\frac{35}{12}\nu\right)x+4\pi x^{3/2}+\left(-\frac{44711}{9072}+\frac{9271}{504}\nu+\frac{65}{18}\nu^2\right)x^2+\left(-\frac{8191}{672}-\frac{535}{24}\nu\right)\pi x^{5/2}+\left(\frac{6643739519}{69854400}+\frac{16\pi^2}{3}-\frac{1712}{105}\times C-\frac{856}{105}\ln(16x)+\left[-\frac{11497453}{272160}+\frac{41\pi^2}{48}+\frac{176}{9}\lambda-\frac{88}{3}\theta\right]\nu-\frac{94403}{3024}\nu^2-\frac{775}{324}\nu^3\right)x^3+\left(-\frac{16285}{504}+\frac{176419}{1512}\nu+\frac{19897}{378}\nu^2\right)\pi x^{7/2}+\mathcal{O}(x^4)\right\}. \tag{12.9}$$

In the above expression the constant r_0' has cleanly disappeared. Of course, this was to be expected because we have seen that r_0' is pure-gauge; nevertheless this cancellation constitutes a satisfactory test of the algebra. However, the result still depends on one physical undetermined numerical coefficient, which is a linear combination of the equation-of-motion-related constant λ and the multipole-moment-related constant θ. On the other hand, our final expression (12.9) is in perfect agreement, in the test-mass limit $\nu\to 0$, with the result of black-hole perturbation theory which is already known to a very high post-Newtonian order [46,47].

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APPENDIX: RESULTS FOR ALL THE TERMS

For the mass quadrupole we factorize out a factor $\mu=m\nu$ in front of all the terms. We denote $v^i=\sqrt{(Gm/r^3)}w^i$, so for instance $\hat{w}_{ij}=(r^3/Gm)v^{\langle i}v^{j\rangle}$ (and $\hat{x}_{ij}=x^{\langle i}x^{j\rangle}$). For the current quadrupole all the terms have to be multiplied by $\delta m/mL_{\langle i}x_{j\rangle}$, where $\delta m=m_1-m_2$ and $L_i=\mu\epsilon_{ijk}x^jv^k$ is the angular momentum. For the mass octupole we factorize out $\mu\,\delta m/m=\nu\,\delta m$. For simplicity the constants r_0 and r_0' in the logarithms are set to 1. In the case of the 3PN mass quadrupole, to the sum of all these terms one must add the undetermined contribution given by Eq. (10.26) in the text.

1. The 3PN mass quadrupole

Miscellaneous:

$$\mathrm{SI}(22+23+32)+\mathrm{SII}(11)=0, \tag{A1a}$$

$$\mathrm{VI}(16+20)+\mathrm{VII}(6)=\mathrm{VI}(19). \tag{A1b}$$

Compact term at Newtonian order:

$$\mathrm{SI}(1)=\left[1+\frac{\gamma}{2}(1-5\nu)-\frac{\gamma^2}{8}(13-61\nu+25\nu^2)+\frac{\gamma^3}{16}\times(149-573\nu+354\nu^2-29\nu^3)\right]\hat{x}_{ij}. \tag{A2}$$

Compact terms at 1PN:

$$\mathrm{SII}(1)=\frac{1}{56}\gamma\{[-8+24\nu+\gamma(20-52\nu-20\nu^2)+\gamma^2\times(-23-17\nu+160\nu^2-55\nu^3)]\hat{x}_{ij}+[8-24\nu+\gamma(4-28\nu+44\nu^2)+\gamma^2(-13-9\nu+238\nu^2+35\nu^3)]\hat{w}_{ij}\}, \tag{A3a}$$

$$\mathrm{VI}(1)=\frac{1}{21}\gamma\{[-8+24\nu+\gamma(28-92\nu+20\nu^2)+\gamma^2\times(-75+259\nu-176\nu^2+13\nu^3)]\hat{x}_{ij}+[8-24\nu+\gamma(-4+12\nu+4\nu^2)+\gamma^2(15-157\nu+414\nu^2+7\nu^3)]\hat{w}_{ij}\}. \tag{A3b}$$

Compact terms at 2PN:

$$\mathrm{SI}(3)=2\gamma^2[2-8\nu+4\nu^2+\gamma(-7+28\nu-12\nu^2-4\nu^3)]\hat{x}_{ij}, \tag{A4a}$$

$$\mathrm{SIII}(1)=\frac{\gamma^2}{126}\{[2-10\nu+10\nu^2+\gamma(-11+55\nu-56\nu^2+3\nu^2)]\hat{x}_{ij}+[-2+10\nu-10\nu^2+\gamma(5-23\nu+16\nu^2+7\nu^3)]\hat{w}_{ij}\}, \tag{A4b}$$

$$\mathrm{VI}(2)=-\frac{8}{21}\gamma^2\{[2-8\nu+4\nu^2+\gamma(-7+28\nu-12\nu^2-4\nu^3)]\hat{x}_{ij}+[-2+8\nu-4\nu^2+\gamma(1-2\nu-8\nu^2+8\nu^3)]\hat{w}_{ij}\}, \tag{A4c}$$

$$\mathrm{VI}(3)=\frac{8}{21}\gamma^2\nu\{[-2+4\nu+\gamma(5-6\nu-4\nu^2)]\hat{x}_{ij}+[2-4\nu+\gamma(1-8\nu+8\nu^2)]\hat{w}_{ij}\}, \tag{A4d}$$

$$\mathrm{VII}(1)=\frac{8}{189}\gamma^2\{[2-10\nu+10\nu^2+\gamma(-13+69\nu-84\nu^2+17\nu^3)]\hat{x}_{ij}+[-2+10\nu-10\nu^2+\gamma(7-37\nu+44\nu^2-7\nu^3)]\hat{w}_{ij}\}, \tag{A4e}$$

$$\mathrm{TI}(1)=\frac{1}{54}\gamma^2\{[2-10\nu+10\nu^2+\gamma(-13+69\nu-84\nu^2+17\nu^3)]\hat{x}_{ij}+[-2+10\nu-10\nu^2+\gamma(7-37\nu+44\nu^2-7\nu^3)]\hat{w}_{ij}\}. \tag{A4f}$$

Compact terms at 3PN:

$$\mathrm{SI}(13)=16\gamma^3\nu^2\hat{x}_{ij}, \tag{A5a}$$

$$\mathrm{SI}(14)=8\gamma^3(1-5\nu+5\nu^2)\hat{x}_{ij}, \tag{A5b}$$

$$\mathrm{SI}(15)=-2\gamma^3\nu(1-4\nu+2\nu^2)\hat{x}_{ij}, \tag{A5c}$$

$$\mathrm{SI}(16\mathrm{C})=0. \tag{A5d}$$

$$\mathrm{SI}(16\mathrm{NC})=(-1+9\nu-17\nu^2)\gamma^3\hat{x}_{ij}+\frac{2}{15}\gamma^3\nu\hat{w}_{ij}, \tag{A5e}$$

$$\mathrm{SII}(2)=\frac{4}{7}\gamma^3(-1+2\nu)(1-4\nu+\nu^2)(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A5f}$$

$$\mathrm{SIV}(1)=\frac{2}{2079}\gamma^3(-1+7\nu-14\nu^2+7\nu^3)(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A5g}$$

$$\mathrm{VI}(7)=\frac{8}{21}\gamma^3\nu(1-4\nu+2\nu^2)(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A5h}$$

$$\mathrm{VI}(8)=-\frac{16}{21}\gamma^3(1-5\nu+5\nu^2)(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A5i}$$

$$\mathrm{VI}(9)=-\frac{8}{21}\gamma^3\nu(1+\nu)(-1+2\nu)(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A5j}$$

$$\mathrm{VI}(10\mathrm{C})=\frac{32}{21}\gamma^3\nu(-1+2\nu)(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A5k}$$

$$\mathrm{VI}(10\mathrm{NC})=-\frac{4}{21}\nu\left(\frac{8}{5}-11\nu\right)\gamma^3(\hat{w}_{ij}-\hat{x}_{ij}), \tag{A5l}$$

$$\mathrm{VI}(11)=\frac{16}{21}\gamma^3\nu(1-4\nu+2\nu^2)(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A5m}$$

$$\mathrm{VI}(12\mathrm{C})=0, \tag{A5n}$$

$$\mathrm{VI}(12\mathrm{NC})=-\frac{4}{21}\left(1-\frac{38}{5}\nu+17\nu^2\right)\gamma^3(\hat{w}_{ij}-\hat{x}_{ij}), \tag{A5o}$$

$$\mathrm{VI}(13)=\frac{16}{21}\gamma^3\nu(1-4\nu+2\nu^2)(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A5p}$$

$$\mathrm{VII}(2)=-\frac{32}{189}\gamma^3(-1+2\nu)(1-4\nu+\nu^2)(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A5q}$$

$$\mathrm{VII}(3)=\frac{32}{189}\gamma^3\nu(1-4\nu+2\nu^2)(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A5r}$$

$$\mathrm{VIII}(1)=\frac{16}{2079}\gamma^3(-1+7\nu-14\nu^2+7\nu^3)(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A5s}$$

$$\mathrm{TI}(3)=-\frac{4}{27}\gamma^3(-1+2\nu)(1-4\nu+\nu^2)\times(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A5t}$$

$$\mathrm{TI}(4)=\frac{4}{27}\gamma^3\nu(1-4\nu+2\nu^2)(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A5u}$$

$$\mathrm{TII}(1)=\frac{2}{297}\gamma^3(-1+7\nu-14\nu^2+7\nu^3)(\hat{x}_{ij}-\hat{w}_{ij}). \tag{A5v}$$

Y-terms at 2PN:

$$\mathrm{SI}(4)=\frac{8}{3}\gamma^2\nu\{(2-4\gamma\nu)\hat{w}_{ij}+[1-3\nu+\gamma(-3+8\nu+3\nu^2)]\hat{x}_{ij}\}, \tag{A6a}$$

$$\mathrm{SI}(5\mathrm{C})=\frac{1}{3}\gamma^2\{[4-8\nu+\gamma(-2+16\nu^2)]\hat{w}_{ij}+[2-10\nu-12\nu^2+\gamma(-7+35\nu+34\nu^2+12\nu^3)]\hat{x}_{ij}\}, \tag{A6b}$$

$$\mathrm{SI}(6)=\frac{1}{3}\gamma^2\nu\{[-2+\gamma(-1+6\nu)]\hat{w}_{ij}+[2-6\nu+\gamma(-5+11\nu+12\nu^2)]\hat{x}_{ij}\}, \tag{A6c}$$

$$\mathrm{SI}(7)=\frac{4}{3}\gamma^2\nu\{[2+\gamma(-1-2\nu)]\hat{w}_{ij}+[-2+6\nu+\gamma(7-21\nu)]\hat{x}_{ij}\}, \tag{A6d}$$

$$\mathrm{SII}(7)=\frac{2}{3}\gamma^2(-1+3\nu)\{[-2+\gamma(-1+6\nu)]\hat{w}_{ij}+[2+\gamma(-5-4\nu)]\hat{x}_{ij}\}, \tag{A6e}$$

$$\mathrm{VI}(4)=\frac{4}{63}\gamma^2\{[28-110\nu+24\nu^2+\gamma(-14+15\nu+160\nu^2-48\nu^3)]\hat{w}_{ij}+[-28+110\nu-24\nu^2+\gamma(98-373\nu+22\nu^2+24\nu^3)]\hat{x}_{ij}\}, \tag{A6f}$$

$$\mathrm{VI}(5)=\frac{1}{42}\gamma^2(-12+62\nu-24\nu^2)\{[2+\gamma(1-6\nu)]\hat{w}_{ij}+[-2+\gamma(5+4\nu)]\hat{x}_{ij}\}, \tag{A6g}$$

$$\mathrm{TI}(2)=\frac{2}{63}\gamma^2(2-7\nu+\nu^2)\{[-2+\gamma(-1+6\nu)]\hat{w}_{ij}+[2+\gamma(-5-4\nu)]\hat{x}_{ij}\}. \tag{A6h}$$

Y-terms at 3PN:

$$\mathrm{SI}(31)=\frac{8}{3}\gamma^3\nu[(1-2\nu-3\nu^2)\hat{x}_{ij}-\nu\hat{w}_{ij}], \tag{A7a}$$

$$\mathrm{SI}(33\mathrm{C})=-8\gamma^3\nu^2\hat{x}_{ij}, \tag{A7b}$$

$$\mathrm{SI}(35\mathrm{C})=-\frac{8}{3}\gamma^3[-(1-6\nu+3\nu^2)\hat{x}_{ij}+(-2+6\nu)\hat{w}_{ij}], \tag{A7c}$$

$$\mathrm{SI}(37\mathrm{C})=\frac{16}{3}\gamma^3\nu[(1-3\nu)\hat{x}_{ij}+2\hat{w}_{ij}], \tag{A7d}$$

$$\mathrm{SI}(38\mathrm{C})=\frac{32}{3}\gamma^3\nu[(-1+3\nu)\hat{x}_{ij}+\hat{w}_{ij}], \tag{A7e}$$

$$\mathrm{VI}(16+20)+\mathrm{VII}(6)=\frac{2}{63}\gamma^3(12-49\nu+10\nu^2+24\nu^3)\times(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A7f}$$

$$\mathrm{VI}(19)=\frac{2}{63}\gamma^3(12-49\nu+10\nu^2+24\nu^3)\times(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A7g}$$

$$\mathrm{VI}(21)=\frac{8}{63}\gamma^3\nu(-1+2\nu)(5+12\nu)\times(-\hat{x}_{ij}+\hat{w}_{ij}), \tag{A7h}$$

$$\mathrm{VI}(25\mathrm{C})=\frac{16}{63}\gamma^3(14-55\nu+12\nu^2)(-\hat{x}_{ij}+\hat{w}_{ij}), \tag{A7i}$$

$$\mathrm{VI}(26\mathrm{C})=\frac{4}{63}\gamma^3\nu(5-70\nu+24\nu^2)(-\hat{x}_{ij}+\hat{w}_{ij}), \tag{A7j}$$

$$\mathrm{VI}(27\mathrm{C})=\frac{4}{63}\gamma^3(-5+21\nu)(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A7k}$$

$$\mathrm{VI}(28\mathrm{C})=0, \tag{A7l}$$

$$\mathrm{VI}(29\mathrm{C})=\frac{4}{63}\gamma^3\nu(-1+12\nu)(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A7m}$$

$$\mathrm{TI}(6)=\frac{1}{189}\gamma^3(-16-21\nu+174\nu^2+56\nu^3)\times(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A7n}$$

$$\mathrm{TI}(7)=\frac{16}{63}\gamma^3\nu(2-7\nu+\nu^2)(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A7o}$$

$$\mathrm{TI}(8)=\frac{1}{27}\gamma^3\nu(13-46\nu+8\nu^2)(-\hat{x}_{ij}+\hat{w}_{ij}). \tag{A7p}$$

S-terms at 3PN:

$$\mathrm{SII}(3)=\frac{2}{21}\gamma^3\nu(1+2\nu+12\nu^2)(\hat{w}_{ij}-\hat{x}_{ij}), \tag{A8a}$$

$$\mathrm{SII}(4\mathrm{C})=\frac{1}{42}\gamma^3(1-2\nu)(1+2\nu-12\nu^2)(\hat{w}_{ij}-\hat{x}_{ij}), \tag{A8b}$$

$$\mathrm{SII}(5)=\frac{1}{126}\gamma^3\nu(-11+6\nu+36\nu^2)(\hat{w}_{ij}-\hat{x}_{ij}), \tag{A8c}$$

$$\mathrm{SII}(6)=-\frac{2}{63}\gamma^3\nu(-11+6\nu+36\nu^2)(\hat{w}_{ij}-\hat{x}_{ij}), \tag{A8d}$$

$$\mathrm{SIII}(2)=-\frac{2}{63}\gamma^3(3-22\nu+36\nu^2)(\hat{w}_{ij}-\hat{x}_{ij}), \tag{A8e}$$

$$\mathrm{VII}(4)=\frac{8}{189}\gamma^3(3-2\nu-34\nu^2+8\nu^3)(\hat{w}_{ij}-\hat{x}_{ij}), \tag{A8f}$$

$$\mathrm{VII}(5) = -\frac{2}{63}\gamma^3(-1+6\nu-18\nu^2+8\nu^3)(\hat{w}_{ij} - \hat{x}_{ij}), \tag{A8g}$$

$$\mathrm{TII}(2) = \frac{2}{2079}\gamma^3(8-67\nu+134\nu^2-12\nu^3)(\hat{w}_{ij} - \hat{x}_{ij}). \tag{A8h}$$

T-terms at 3PN:

$$\mathrm{SI}(17) = \frac{1}{9}\gamma^3\nu[(7-36\nu)\hat{w}_{ij}+(-15+36\nu+54\nu^2)\hat{x}_{ij}], \tag{A9a}$$

$$\mathrm{SI}(18) = \frac{1}{3}\gamma^3\nu[-3\hat{w}_{ij}+(-1-12\nu+18\nu^2)\hat{x}_{ij}], \tag{A9b}$$

$$\mathrm{SI}(19C) = \gamma^3\left[-\frac{1}{6}(1-\nu)\hat{w}_{ij}+\frac{1}{12}(2-7\nu+70\nu^2 + 12\nu^3)\hat{x}_{ij}\right], \tag{A9c}$$

$$\mathrm{SI}(21C) = \frac{1}{12}\gamma^3\nu[(-2-8\nu)\hat{w}_{ij}+(-1+22\nu+12\nu^2)\hat{x}_{ij}], \tag{A9d}$$

$$\mathrm{SI}(24) = \frac{1}{18}\gamma^3\nu[11\hat{w}_{ij}+(-15+36\nu+54\nu^2)\hat{x}_{ij}], \tag{A9e}$$

$$\mathrm{SI}(25) = -\frac{2}{9}\gamma^3\nu[-\hat{w}_{ij}+(-3-36\nu+54\nu^2)\hat{x}_{ij}], \tag{A9f}$$

$$\mathrm{SII}(9) = \frac{2}{9}\gamma^3(1-9\nu^2)(-\hat{w}_{ij}+\hat{x}_{ij}), \tag{A9g}$$

$$\mathrm{VI}(14) = -\frac{2}{63}\gamma^3(20-71\nu+36\nu^3)(\hat{w}_{ij}-\hat{x}_{ij}), \tag{A9h}$$

$$\mathrm{VI}(15) = -\frac{2}{63}\gamma^3\nu(17-98\nu+36\nu^2)(\hat{w}_{ij}-\hat{x}_{ij}), \tag{A9i}$$

$$\mathrm{VI}(17) = \frac{1}{42}\gamma^3\nu(-43+94\nu+36\nu^2)(\hat{w}_{ij}-\hat{x}_{ij}), \tag{A9j}$$

$$\mathrm{VI}(18) = \frac{1}{42}\gamma^3(4-3\nu-72\nu^2+36\nu^3)(\hat{w}_{ij}-\hat{x}_{ij}), \tag{A9k}$$

$$\mathrm{TI}(5) = \frac{1}{189}\gamma^3(4-9\nu-21\nu^2+13\nu^3)(\hat{w}_{ij}-\hat{x}_{ij}). \tag{A9l}$$

Cubic terms:

$$\mathrm{SI}(26+27+28+29+30+34+36) = \gamma^3\nu\left[\left(\frac{74}{15}-\frac{152}{3}\nu\right)\hat{x}_{ij}+\left(\frac{128}{15}\ln r - \frac{1024}{225}\right)\hat{w}_{ij}\right], \tag{A10a}$$

$$\mathrm{SII}(13) = \frac{16}{225}\gamma^3[(45-75\nu)\ln r-9-25\nu] \times(-\hat{x}_{ij}+\hat{w}_{ij}), \tag{A10b}$$

$$\mathrm{VI}(22+23+24) = \frac{8}{525}\gamma^3(60\ln r-7-265\nu+275\nu^2) \times(\hat{x}_{ij}-\hat{w}_{ij}). \tag{A10c}$$

Non-compacts terms:

$$\mathrm{SI}(5NC) = -\frac{1}{2}\gamma^2[4+10\nu+\gamma(2-6\nu - 47\nu^2)]\hat{x}_{ij}, \tag{A11a}$$

$$\mathrm{SI}(19NC) = \left(\frac{1}{6}+\frac{71}{36}\nu-\frac{7}{3}\nu^2\right)\gamma^3\hat{x}_{ij}+\left(\frac{2}{15}\nu\ln r - \frac{1}{6}-\frac{77}{450}\nu\right)\gamma^3\hat{w}_{ij}, \tag{A11b}$$

$$\mathrm{SI}(20) = \left[\left(\frac{4}{15}+\frac{4}{3}\nu\right)\ln r-\frac{77}{225}-\frac{25}{6}\nu + 6\nu^2\right]\gamma^3\hat{x}_{ij} + \left[\left(-\frac{4}{15}+\frac{4}{15}\nu\right)\ln r+\frac{77}{225} - \frac{77}{225}\nu\right]\gamma^3\hat{w}_{ij}, \tag{A11c}$$

$$\mathrm{SI}(21NC) = \left[\left(\frac{8}{15}-\frac{2}{3}\nu\right)\ln r+\frac{26}{225}-\frac{1}{36}\nu-\nu^2\right] \times\gamma^3\hat{x}_{ij}+\left[\left(-\frac{8}{15}+\frac{8}{15}\nu\right)\ln r-\frac{26}{225} + \frac{397}{450}\nu\right]\gamma^3\hat{w}_{ij}, \tag{A11d}$$

$$\mathrm{SI}(33\mathrm{NC})=\left(-\frac{164}{15}\nu+\frac{62}{3}\nu^2\right)\gamma^3\hat{x}_{ij}+\left(-\frac{8}{15}\nu\ln r-\frac{176}{225}\nu\right)\gamma^3\hat{w}_{ij}, \tag{A11e}$$

$$\mathrm{SI}(35\mathrm{NC})=\left[\left(-\frac{16}{15}-\frac{64}{3}\nu\right)\ln r+\frac{1028}{225}+\frac{1148}{45}\nu-\frac{145}{3}\nu^2\right]\gamma^3\hat{x}_{ij}+\left[\left(\frac{16}{15}-\frac{16}{5}\nu\right)\ln r-\frac{1028}{225}+\frac{568}{75}\nu\right]\gamma^3\hat{w}_{ij}, \tag{A11f}$$

$$\mathrm{SI}(37\mathrm{NC})=\left(-\frac{62}{15}\nu-\frac{17}{3}\nu^2\right)\gamma^3\hat{x}_{ij}+\left(-\frac{4}{15}\nu\ln r+\frac{512}{225}\nu\right)\gamma^3\hat{w}_{ij}, \tag{A11g}$$

$$\mathrm{SI}(38\mathrm{NC})=\left(\frac{124}{15}\nu+\frac{34}{3}\nu^2\right)\gamma^3\hat{x}_{ij}+\left(\frac{8}{15}\nu\ln r+\frac{1376}{225}\nu\right)\gamma^3\hat{w}_{ij}, \tag{A11h}$$

$$\mathrm{SII}(4\mathrm{NC})=\left[\left(\frac{16}{35}-\frac{2}{3}\nu\right)\ln r+\frac{139}{1050}-\frac{164}{315}\nu+\frac{11}{7}\nu^2\right]\gamma^3(\hat{w}_{ij}-\hat{x}_{ij}), \tag{A11i}$$

$$\mathrm{VI}(25\mathrm{NC})=\left[\left(-\frac{48}{35}+\frac{8}{15}\nu\right)\ln r-\frac{968}{1575}+\frac{10792}{1575}\nu-\frac{236}{63}\nu^2\right]\gamma^3(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A11j}$$

$$\mathrm{VI}(26\mathrm{NC})+\mathrm{VI}(27\mathrm{NC})+\mathrm{VI}(28\mathrm{NC})+\mathrm{VI}(29\mathrm{NC})=\left[\left(\frac{32}{105}-\frac{8}{15}\nu\right)\ln r+\frac{2644}{1575}-\frac{488}{175}\nu-\frac{124}{63}\nu^2\right]\gamma^3(\hat{x}_{ij}-\hat{w}_{ij}). \tag{A11k}$$

Terms at 2.5PN:

$$\mathrm{SI}(11+12)=\frac{48}{7}\gamma^{5/2}\nu x_{\langle i}w_{j\rangle}. \tag{A12}$$

Divergence terms:

$$\mathrm{SI}(2)=0, \tag{A13a}$$
$$\mathrm{SI}(8)=0, \tag{A13b}$$
$$\mathrm{SI}(9)=0, \tag{A13c}$$
$$\mathrm{SI}(10)=0, \tag{A13d}$$
$$\mathrm{SI}(39)=0, \tag{A13e}$$
$$\mathrm{SI}(40)=0, \tag{A13f}$$
$$\mathrm{SI}(41)=-\frac{8}{3}\gamma^3(-\hat{x}_{ij}+\hat{w}_{ij}), \tag{A13g}$$
$$\mathrm{SI}(42)=0, \tag{A13h}$$
$$\mathrm{SI}(43)=0, \tag{A13i}$$
$$\mathrm{SI}(44)=\frac{1}{3}\gamma^3(-\hat{x}_{ij}+\hat{w}_{ij}), \tag{A13j}$$
$$\mathrm{SI}(45)=\gamma^3(-\hat{x}_{ij}+\hat{w}_{ij}), \tag{A13k}$$
$$\mathrm{SI}(46)=\frac{1}{3}\gamma^3(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A13l}$$
$$\mathrm{SI}(47)=0, \tag{A13m}$$
$$\mathrm{SI}(48)=-\frac{2}{3}\gamma^3(-\hat{x}_{ij}+\hat{w}_{ij}), \tag{A13n}$$
$$\mathrm{SI}(49)=16\gamma^3(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A13o}$$
$$\mathrm{SI}(50)=\frac{8}{3}\gamma^3(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A13p}$$
$$\mathrm{SII}(8)=0, \tag{A13q}$$
$$\mathrm{SII}(10)=0, \tag{A13r}$$
$$\mathrm{SII}(12)=\frac{27}{35}\gamma^3(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A13s}$$
$$\mathrm{SII}(14)=\frac{72}{35}\gamma^3(-\hat{x}_{ij}+\hat{w}_{ij}), \tag{A13t}$$
$$\mathrm{SIII}(3)=0, \tag{A13u}$$
$$\mathrm{VI}(6)=0, \tag{A13v}$$
$$\mathrm{VI}(30)=0, \tag{A13w}$$
$$\mathrm{VI}(31)=0, \tag{A13x}$$
$$\mathrm{VI}(32)=0, \tag{A13y}$$
$$\mathrm{VI}(33)=\frac{8}{15}\gamma^3(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A13z}$$

$$\mathrm{VI}(34)=0, \tag{A13aa}$$

$$\mathrm{VI}(35)=-\frac{4}{15}\gamma^3(\hat{x}_{ij}-\hat{w}_{ij}), \tag{A13bb}$$

$$\mathrm{VII}(7)=0, \tag{A13cc}$$

$$\mathrm{TI}(9)=0. \tag{A13dd}$$

2. The 2PN current quadrupole

Compact term at Newtonian order:

$$\mathrm{VI}(1)=\frac{1}{8}[-8+4\gamma+\gamma^2(-15+88\nu+3\nu^2)]. \tag{A14}$$

Compact terms at 1PN:

$$\mathrm{VI}(2)=\gamma[-2+2\nu+\gamma(1+\nu-4\nu^2)], \tag{A15a}$$

$$\mathrm{VI}(3)=\gamma\nu[-2+\gamma(-1+4\nu)], \tag{A15b}$$

$$\mathrm{VII}(1)=\frac{\gamma}{28}[2-4\nu+\gamma(-7+16\nu-3\nu^2)], \tag{A15c}$$

$$\mathrm{TI}(1)=\frac{\gamma}{56}[2-4\nu+\gamma(-7+16\nu-3\nu^2)]. \tag{A15d}$$

Compact terms at 2PN:

$$\mathrm{VI}(7)=\gamma^2\nu(1-\nu), \tag{A16a}$$

$$\mathrm{VI}(8)=2\gamma^2(-1+2\nu), \tag{A16b}$$

$$\mathrm{VI}(9)=\gamma^2\nu(1+\nu), \tag{A16c}$$

$$\mathrm{VI}(10\mathrm{C})=-4\gamma^2\nu, \tag{A16d}$$

$$\mathrm{VI}(10\mathrm{NC})=\frac{3}{2}\gamma^2\nu, \tag{A16e}$$

$$\mathrm{VI}(11)=2\gamma^2\nu(1-\nu), \tag{A16f}$$

$$\mathrm{VI}(12\mathrm{C})=0, \tag{A16g}$$

$$\mathrm{VI}(12\mathrm{NC})=\frac{\gamma^2}{2}(1-6\nu), \tag{A16h}$$

$$\mathrm{VI}(13)=2\gamma^2\nu(1-\nu), \tag{A16i}$$

$$\mathrm{VII}(2)=\frac{\gamma^2}{7}(1-3\nu+\nu^2), \tag{A16j}$$

$$\mathrm{VII}(3)=\frac{\gamma^2}{7}\nu(1-\nu), \tag{A16k}$$

$$\mathrm{VIII}(1)=\frac{\gamma^2}{504}(1-3\nu)(-1+\nu), \tag{A16l}$$

$$\mathrm{TI}(3)=\frac{\gamma^2}{7}(1-3\nu+\nu^2), \tag{A16m}$$

$$\mathrm{TI}(4)=\frac{\gamma^2}{7}\nu(1-\nu), \tag{A16n}$$

$$\mathrm{TII}(1)=\frac{\gamma^2}{504}(1-3\nu)(-1+\nu). \tag{A16o}$$

Y-terms at 1PN:

$$\mathrm{VI}(4)=\frac{\gamma}{2}[-2+4\nu+\gamma(1-8\nu^2)], \tag{A17a}$$

$$\mathrm{VI}(5)=-\frac{3}{4}\gamma\nu[2+\gamma(1-6\nu)], \tag{A17b}$$

$$\mathrm{TI}(2)=0. \tag{A17c}$$

Y-terms at 2PN:

$$\mathrm{VI}(16+20)+\mathrm{VII}(6)=\frac{\gamma^2}{3}(1-2\nu-3\nu^2), \tag{A18a}$$

$$\mathrm{VI}(19)=\frac{\gamma^2}{3}(1-2\nu-3\nu^2), \tag{A18b}$$

$$\mathrm{VI}(21)=-\frac{2}{3}\gamma^2\nu(5-6\nu), \tag{A18c}$$

$$\mathrm{VI}(25\mathrm{C})=2\gamma^2(-1+2\nu), \tag{A18d}$$

$$\mathrm{VI}(26\mathrm{C})=\frac{2}{3}\gamma^2\nu(1+3\nu), \tag{A18e}$$

$$\mathrm{VI}(27\mathrm{C})=\frac{\gamma^2}{3}(1-2\nu), \tag{A18f}$$

$$\mathrm{VI}(28\mathrm{C})=0, \tag{A18g}$$

$$\mathrm{VI}(29\mathrm{C})=\frac{2}{3}\gamma^2\nu, \tag{A18h}$$

$$\mathrm{TI}(6)=\frac{\gamma^2}{21}(-2+4\nu-3\nu^2), \tag{A18i}$$

$$\mathrm{TI}(7)=0, \tag{A18j}$$

$$\mathrm{TI}(8)=-\frac{\gamma^2}{21}\nu(7-3\nu). \tag{A18k}$$

S-terms at 2PN:

$$\mathrm{VII}(4)=\frac{1}{42}\gamma^2(-2-3\nu)(1-2\nu), \tag{A19a}$$

$$\mathrm{VII}(5)=-\frac{1}{28}\gamma^2\nu(2+3\nu), \tag{A19b}$$

$$\mathrm{TII}(2)=0. \tag{A19c}$$

T-terms at 2PN:

$$\mathrm{VI}(14)=\frac{1}{6}\gamma^2(1+4\nu-9\nu^2), \tag{A20a}$$

$$\mathrm{VI}(15)=\frac{1}{6}\gamma^2\nu(1-9\nu), \tag{A20b}$$

$$\mathrm{VI}(17)=\frac{1}{8}\gamma^2\nu(4+9\nu), \tag{A20c}$$

$$\mathrm{VI}(18)=\frac{3}{8}\gamma^2\nu(2+3\nu), \tag{A20d}$$

$$\mathrm{TI}(5)=\frac{1}{168}\gamma^2\nu(2+3\nu). \tag{A20e}$$

Cubic terms:

$$\mathrm{VI}(22+23+24)=2\gamma^2\left(\frac{1}{3}-\frac{9}{4}\nu\right). \tag{A21}$$

Non-compact terms:

$$\mathrm{VI(25NC)}=\gamma^2\left(-\frac{2}{3}+\frac{17}{6}\nu\right), \tag{A22a}$$

$$\mathrm{VI(26NC+27NC+28NC+29NC)}=\gamma^2\left(\frac{3}{2}+\frac{19}{6}\nu\right). \tag{A22b}$$

Divergence terms:

$$\mathrm{VI}(6)=0, \tag{A23a}$$

$$\mathrm{VI}(30)=0, \tag{A23b}$$

$$\mathrm{VI}(31)=0, \tag{A23c}$$

$$\mathrm{VI}(32)=0, \tag{A23d}$$

$$\mathrm{VI}(33)=0, \tag{A23e}$$

$$\mathrm{VI}(34)=0, \tag{A23f}$$

$$\mathrm{VI}(35)=0, \tag{A23g}$$

$$\mathrm{VII}(7)=0, \tag{A23h}$$

$$\mathrm{TI}(9)=0. \tag{A23i}$$

3. The 2PN mass octupole

Compact term at Newtonian order:

$$\mathrm{SI}(1)=\frac{1}{8}[-8+\gamma(-4+16\nu)+\gamma^2(13+24\nu+15\nu^2)]\hat{x}_{ijk}. \tag{A24}$$

Compact terms at 1PN:

$$\mathrm{SII}(1)=\frac{1}{180}\gamma\{[30-60\nu+\gamma(-75+120\nu+45\nu^2)]\hat{x}_{ijk}+3[-20+40\nu+\gamma(-10+60\nu-70\nu^2)]x_{\langle i}w_{jk\rangle}\}, \tag{A25a}$$

$$\mathrm{VI}(1)=\frac{1}{90}\gamma\{[30-60\nu+\gamma(-105+240\nu-45\nu^2)]\hat{x}_{ijk}+3[-20+40\nu+\gamma(10-20\nu-10\nu^2)]x_{\langle i}w_{jk\rangle}\}. \tag{A25b}$$

Compact terms at 2PN:

$$\mathrm{SI}(3)=-4\gamma^2(1-3\nu+\nu^2)\hat{x}_{ijk}, \tag{A26a}$$

$$\mathrm{SIII}(1)=\frac{1}{3960}\gamma^2(1-3\nu)(1-\nu)(-105\hat{x}_{ijk}+300x_{\langle i}w_{jk\rangle}), \tag{A26b}$$

$$\mathrm{VI}(2)=\frac{2}{45}\gamma^2(1-3\nu+\nu^2)(15\hat{x}_{ijk}-30x_{\langle i}w_{ij\rangle}), \tag{A26c}$$

$$\mathrm{VI}(3)=\frac{2}{45}\gamma^2\nu(-1+\nu)(-15\hat{x}_{ijk}+30x_{\langle i}w_{jk\rangle}), \tag{A26d}$$

$$\mathrm{VII}(1)=\frac{1}{990}\gamma^2(1-3\nu)(1-\nu)(-105\hat{x}_{ijk}+300x_{\langle i}w_{jk\rangle}), \tag{A26e}$$

$$\mathrm{TI}(1)=\frac{1}{990}\gamma^2(1-3\nu)(1-\nu)(-33\hat{x}_{ijk}+84x_{\langle i}w_{jk\rangle}). \tag{A26f}$$

Y-terms at 2PN:

$$\mathrm{SI}(4)=-\frac{2}{15}\gamma^2\nu[(15-30\nu)\hat{x}_{ijk}+60x_{\langle i}w_{jk\rangle}], \tag{A27a}$$

$$\mathrm{SI(5C)}=\frac{1}{30}\gamma^2[-(15-60\nu-60\nu^2)\hat{x}_{ijk}+3(-10+40\nu)x_{\langle i}w_{jk\rangle}], \tag{A27b}$$

$$\mathrm{SI}(6)=-\frac{1}{30}\gamma^2\nu[(15-30\nu)\hat{x}_{ijk}-30x_{\langle i}w_{jk\rangle}], \tag{A27c}$$

$$\mathrm{SI}(7) = -\frac{2}{15}\gamma^2\nu[(-15-30\nu)\hat{x}_{ijk}+30x_{\langle i}w_{jk\rangle}], \tag{A27d}$$

$$\mathrm{SII}(7) = -\frac{1}{10}\gamma^2(1-2\nu)(-15\hat{x}_{ijk}+30x_{\langle i}w_{jk\rangle}), \tag{A27e}$$

$$\mathrm{VI}(4) = -\frac{1}{450}\gamma^2(45-140\nu+20\nu^2) \times(-15\hat{x}_{ijk}+30x_{\langle i}w_{jk\rangle}), \tag{A27f}$$

$$\mathrm{VI}(5) = \frac{1}{60}\gamma^2(2-9\nu+2\nu^2)(-15\hat{x}_{ijk}+30x_{\langle i}w_{jk\rangle}), \tag{A27g}$$

$$\mathrm{TI}(2) = \frac{1}{990}\gamma^2(8-21\nu+2\nu^2) \times(-15\hat{x}_{ijk}+30x_{\langle i}w_{jk\rangle}). \tag{A27h}$$

Non-compact term:

$$\mathrm{SI}(5\mathrm{NC}) = 2\gamma^2(1+2\nu)\hat{x}_{ijk}. \tag{A28}$$

[1] C. Cutler, T. A. Apostolatos, L. Bildsten, L. S. Finn, E. E. Flanagan, D. Kennefick, D. M. Markovic, A. Ori, E. Poisson, G. J. Sussman, and K. S. Thorne, Phys. Rev. Lett. **70**, 2984 (1993).
[2] C. Cutler, L. S. Finn, E. Poisson, and G. J. Sussman, Phys. Rev. D **47**, 1511 (1993).
[3] L. S. Finn and D. F. Chernoff, Phys. Rev. D **47**, 2198 (1993).
[4] C. Cutler and E. E. Flanagan, Phys. Rev. D **49**, 2658 (1994).
[5] H. Tagoshi and T. Nakamura, Phys. Rev. D **49**, 4016 (1994).
[6] E. Poisson, Phys. Rev. D **52**, 5719 (1995); **55**, 7980 (1997).
[7] E. Poisson and C. M. Will, Phys. Rev. D **52**, 848 (1995).
[8] A. Krolàk, K. D. Kokkotas, and G. Schäfer, Phys. Rev. D **52**, 2089 (1995).
[9] T. Damour, B. R. Iyer, and B. S. Sathyaprakash, Phys. Rev. D **57**, 885 (1998); **62**, 084036 (2000); **63**, 044023 (2001).
[10] R. V. Wagoner and C. M. Will, Astrophys. J. **210**, 764 (1976).
[11] L. Blanchet and G. Schäfer, Mon. Not. R. Astron. Soc. **239**, 845 (1989).
[12] L. Blanchet, T. Damour, B. R. Iyer, C. M. Will, and A. G. Wiseman, Phys. Rev. Lett. **74**, 3515 (1995).
[13] L. Blanchet, T. Damour, and B. R. Iyer, Phys. Rev. D **51**, 5360 (1995).
[14] C. M. Will and A. G. Wiseman, Phys. Rev. D **54**, 4813 (1996).
[15] L. Blanchet, B. R. Iyer, C. M. Will, and A. G. Wiseman, Class. Quantum Grav. **13**, 575 (1996).
[16] A. Gopakumar and B. R. Iyer, Phys. Rev. D **56**, 7708 (1997).
[17] L. Blanchet and T. Damour, Ann. I.H.P. Phys. Theor. **50**, 377 (1989).
[18] T. Damour and B. R. Iyer, Ann. I.H.P. Phys. Theor. **54**, 115 (1991).
[19] L. Blanchet and T. Damour, Phys. Rev. D **46**, 4304 (1992).
[20] L. Blanchet, Phys. Rev. D **51**, 2559 (1995).
[21] L. Blanchet, Class. Quantum Grav. **15**, 1971 (1998).
[22] C. M. Will, Prog. Theor. Phys. Suppl. **136**, 158 (1999).
[23] M. E. Pati and C. M. Will, Phys. Rev. D **62**, 124015 (2000).
[24] R. Epstein and R. V. Wagoner, Astrophys. J. **197**, 717 (1975).
[25] K. S. Thorne, Rev. Mod. Phys. **52**, 299 (1980).
[26] L. Blanchet, Phys. Rev. D **54**, 1417 (1996).
[27] L. Blanchet, Class. Quantum Grav. **15**, 113 (1998).
[28] E. Poisson, Phys. Rev. D **47**, 1497 (1993).
[29] A. G. Wiseman, Phys. Rev. D **48**, 4757 (1993).
[30] L. Blanchet and G. Schäfer, Class. Quantum Grav. **10**, 2699 (1993).
[31] J. Hadamard, *Le Problème de Cauchy et les Équations aux Dérivées Partielles Linéaires Hyperboliques* (Hermann, Paris, 1932).
[32] L. Schwartz, *Théorie des Distributions* (Hermann, Paris, 1978).
[33] A. Sellier, Proc. R. Soc. London, Ser. A **445**, 69 (1964).
[34] P. Jaranowski and G. Schäfer, Phys. Rev. D **57**, 7274 (1998).
[35] P. Jaranowski and G. Schäfer, Phys. Rev. D **60**, 124003 (1999).
[36] P. Jaranowski and G. Schäfer, Ann. Phys. (Leipzig) **9**, 378 (2000).
[37] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **62**, 021501(R) (2000); **63**, 029903(E) (2001).
[38] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **63**, 044021 (2001).
[39] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Lett. B **513**, 147 (2001).
[40] L. Blanchet and G. Faye, Phys. Lett. A **271**, 58 (2000).
[41] L. Blanchet and G. Faye, J. Math. Phys. **41**, 7675 (2000).
[42] L. Blanchet and G. Faye, J. Math. Phys. **42**, 4391 (2001).
[43] L. Blanchet and G. Faye, Phys. Rev. D **63**, 062005 (2001).
[44] V. C. de Andrade, L. Blanchet, and G. Faye, Class. Quantum Grav. **18**, 753 (2001).
[45] M. Sasaki, Prog. Theor. Phys. **92**, 17 (1994).
[46] H. Tagoshi and M. Sasaki, Prog. Theor. Phys. **92**, 745 (1994).
[47] T. Tanaka, H. Tagoshi, and M. Sasaki, Prog. Theor. Phys. **96**, 1087 (1996).
[48] Y. Mino, M. Sasaki, M. Shibata, H. Tagoshi, and T. Tanaka, Prog. Theor. Phys. Suppl. **128**, 1 (1997).
[49] L. Blanchet, G. Faye, B. R. Iyer, and B. Joguet, this issue, Phys. Rev. D **65**, 061501(R) (2002).
[50] T. Damour and B. R. Iyer, Phys. Rev. D **43**, 3259 (1991).
[51] L. Blanchet, G. Faye, and B. Ponsot, Phys. Rev. D **58**, 124002 (1998).
[52] M. Riesz, Acta Math. **81**, 1 (1949).