No one loop back reaction in chaotic inflation

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We use an invariant operator to study the quantum gravitational back reaction to scalar perturbations during chaotic inflation. Our operator is the inverse covariant d'Alembertian expressed as a function of the local value of the inflaton. In the slow roll approximation this observable gives $-1/(2H^2)$ for an arbitrary homogeneous and isotropic geometry; hence it is a good candidate for measuring the local expansion rate even when the spacetime is not perfectly homogeneous and isotropic. Corrections quadratic in the scalar creation and annihilation operators of the initial value surface are included using the slow-roll and long wavelength approximations. The result is that all terms which could produce a significant secular back reaction cancel from the operator, before one even takes its expectation value. Although it is not relevant to the current study, we also develop a formalism for using stochastic samples to study back reaction.

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I. INTRODUCTION

Cosmological perturbations generated during an inflationary era are almost certainly responsible for the density ripples out of which the presently observed stars, galaxies and clusters have grown. The basic inflationary mechanism that generated the seeds for galaxy formation and growth of large-scale structure is quantum mechanical particle production powered by the accelerated expansion of the Universe. Now that the latest balloon experiments [1] have all but validated this picture, we are left with the duty of testing further the consequences of particle production in the early Universe. An outstanding question is whether quantum pair creation has any effect on the background in which it takes place.

The simplest system in which the problem of back reaction can be studied is scalar-driven inflation. Since dynamical scalars mix with the gravitational potentials they cause, gravitational back reaction can in principle occur at one loop in scalar-driven inflation. Indeed, previous results by the present authors and collaborators [2–5] suggested that the one loop back reaction of scalar perturbations could slow down the expansion rate of the Universe in the case of the so-called "chaotic" inflationary models.

In [2,3], spatial averaging and a fixed gauge were employed in order to compute the effective energy-momentum tensor for cosmological perturbations. The present authors studied the same physical problem [4,5], this time taking expectation values of the metric and using those expectation values to form invariants, and came to conclusions identical to those of [2,3]. Nevertheless, these works have been criticized in two grounds.

The first objection [6] concerns the use of expectation values of the gauge-fixed metric to build physical observables. The concern is that certain variations in the gauge fixing condition could change the expectation value of the metric in ways which cannot be subsumed into a coordinate transformation. Therefore, forming the expectation value of the metric into coordinate invariant quantities would not purge these quantities of gauge dependence. Back-reaction should then be studied with an operator which is itself an invariant, before taking the expectation value.

The second objection [7] states that using expectation values invites a Schrödinger cat paradox. This is because the process of superadiabatic amplification leaves superhorizon modes in highly squeezed states whose behavior is essentially classical. That is, the various portions of the wave function no longer interfere with one another. Quantum mechanics determines the random choice of where we are in the wave function, but the evolution of each portion is approximately classical. Under these conditions, averaging over the full wave function does not give a good representation of the physics. In the context of the Schrödinger cat paradox Linde observes: "When you open the box a week later what you find is either a very hungry cat or else a smelly piece of meat, not the average of the two." He and his colleagues would prefer that back reaction be studied stochastically [8], where each mode is assigned a random, $\mathbb{C}$-number value as it experiences horizon crossing and evolves classically thereafter.

In the present study of the one-loop back reaction effect we have tried to address these objections. To avoid potential problems from using the gauge fixed metric we have instead computed the functional inverse of the covariant d'Alembertian (see also [9] for a more detailed discussion):

$$\Box_c \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) - \frac{1}{6}R. \tag{1}$$

This operator—$\mathcal{A} \equiv \Box_c^{-1}[g](t,\vec{x})$—averages over the past light cone, very much as astronomers do when compiling a Hubble diagram. In the slow roll approximation it gives

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$-\frac{1}{2}H^{-2}$ for an arbitrary homogeneous and isotropic universe [9]. The scalar $\mathcal{A}$ is therefore a reasonable candidate for measuring the local expansion rate when the universe is not precisely homogeneous and isotropic. By measuring $\mathcal{A}$ on 3-surfaces upon which the dynamical scalar is constant we promote it into a full invariant, $\mathcal{A}_{\rm inv}$.

At one loop, and to leading order in the standard infrared expansion, corrections to the operator $\mathcal{A}$ depend upon a variable we call $\Phi(x)$. This is the accumulated Newtonian potential from infrared modes. $\Phi(x)$ grows with time, since more and more modes redshift into the infrared regime as inflation progresses, so dependence upon any positive power of $\Phi(x)$ would give a secular effect. However, when full invariance is enforced by evaluating $\mathcal{A}$ on 3-surfaces of constant scalar, all terms involving only the undifferentiated Newtonian potential cancel. This result was anticipated by Unruh [6] who noted that the linearized mode functions become pure gauge in the long wavelength limit. The fact that more and more modes redshift to the infrared regime is not pure gauge, but Unruh's observation means that physical effects must involve derivatives of $\Phi(x)$. Such terms do contribute, but they are small. We conclude that there is no significant deviation from the background expansion rate at this order, in marked contrast with the results of [2–4].

Section II describes the perturbative background in which we work. It also illustrates the slow roll expansion through which one can perform most operations of temporal calculus in an arbitrary inflationary background. Section III describes the formalism by which the field equations are solved perturbatively. It concludes with the long wavelength approximation which effectively removes spatial dependence and produces a one dimensional problem that can be treated in the slow roll expansion. Section IV applies the technology to determine the various field operators to quadratic order in the creation and annihilation operators on the initial value surface. The operator $\mathcal{A}$ is evaluated to the same order in Sec. V. The distinction between expectation values and stochastic samples is irrelevant, at this order, in view of the cancellation of all potentially significant one loop corrections to the $\mathcal{A}_{\rm inv}$ operator. However, the issue should re-emerge at higher loops where secular back reaction can derive from the coherent superposition of interactions within the observer's past light cone. We have therefore developed a formalism for the perturbative study of stochastic effects, which is presented in Section VI. Our results are summarized and discussed in Section VII.

II. THE PERTURBATIVE BACKGROUND

The system under study is that of general relativity with a general, minimally coupled scalar:

$$\mathcal{L}=\frac{1}{16\pi G}R\sqrt{-g}-\frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu}\sqrt{-g}-V(\varphi)\sqrt{-g}. \tag{2}$$

This section concerns the homogeneous and isotropic backgrounds g_0 and φ_0 about which perturbation theory will be formulated. Three classes of identities turn out to be interesting for our purposes:

(1) Those which are exact and valid for any potential $V(\varphi)$;

(2) Those which are valid in the slow roll approximation but still for any potential; and

(3) Those which are valid for the slow roll approximation with the potential $V(\varphi)=\frac{1}{2}m^2\varphi^2$.

We shall develop them in this order, identifying the point at which each further specialization and approximation is made.

Among the exact identities is the relation between co-moving and conformal coordinates:

$$ds_0^2=-dt^2+a^2d\vec{x}\cdot d\vec{x}=a^2(\eta)\{-d\eta^2+d\vec{x}\cdot d\vec{x}\}, \tag{3}$$

where conformal time η is defined in the usual way:

$$dt=a(\eta)d\eta. \tag{4}$$

The Hubble "constant" is the logarithmic co-moving time derivative of the background scale factor:

$$H\equiv\frac{\dot{a}}{a}=\frac{a'}{a^2}, \tag{5}$$

where a dot denotes differentiation with respect to (background) co-moving time and a prime stands for differentiation with respect to conformal time.

Two of Einstein's equations are nontrivial in this background:

$$3H^2=\frac{1}{2}\kappa^2\left\{\frac{1}{2}\dot{\varphi}_0^2+V(\varphi_0)\right\}, \tag{6}$$

$$-2\dot{H}-3H^2=\frac{1}{2}\kappa^2\left\{\frac{1}{2}\dot{\varphi}_0^2-V(\varphi_0)\right\}, \tag{7}$$

where $\kappa^2\equiv16\pi G$ is the loop counting parameter of perturbative quantum gravity. By adding the two Einstein equation one can solve for the time derivative of the scalar, which we assume to be negative:

$$\dot{\varphi}_0=-\frac{2}{\kappa}\sqrt{-\dot{H}}. \tag{8}$$

Successful models of inflation require the following two conditions which define the *slow roll approximation*:

$$|\ddot{\varphi}_0|\ll H|\dot{\varphi}_0|, \tag{9}$$

$$\dot{\varphi}_0^2\ll V(\varphi_0). \tag{10}$$

It follows that there are two small parameters. Although these are traditionally expressed as ratios of the potential and its derivatives the more useful quantities for our work are ratios of the Hubble constant and its derivatives:

$$\frac{-\dot{H}}{H^2}\ll1,\quad\frac{|\ddot{H}|}{-H\dot{H}}\ll1. \tag{11}$$

For models of interest to us the rightmost of these parameters is negligible with respect to the leftmost one.

The slow roll approximation gives useful expansions for simple calculus operations. For example, ratios of derivatives of the field are

$$\frac{\varphi_0''}{\varphi_0'}=aH\left(1+\frac{1}{2}\frac{\ddot{H}}{H\dot{H}}\right), \tag{12}$$

$$\frac{\varphi_0'''}{\varphi_0'}=2a^2H^2\left(1+\frac{\dot{H}}{2H^2}+\cdots\right). \tag{13}$$

Successive partial integration also defines useful slow roll expansions:

$$\int dtH^{\alpha}a^{\beta}=\frac{1}{\beta}H^{\alpha-1}a^{\beta}\left\{1+\frac{(\alpha-1)}{\beta}\left(\frac{-\dot{H}}{H^2}\right)+\cdots\right\}, \tag{14}$$

$$\int dtH^{\alpha}=\frac{1}{\alpha+1}\frac{H^{\alpha+1}}{\dot{H}}\left\{1+\frac{1}{\alpha+2}\frac{H\ddot{H}}{\dot{H}^2}+\cdots\right\}. \tag{15}$$

III. EINSTEIN SCALAR DURING INFLATION

The purpose of this section is to describe our procedure for expressing the metric and scalar fields at any spacetime point as functionals of the unconstrained initial value operators. These unconstrained initial value operators are the only true degrees of freedom in any quantum theory and expressing the dynamical variables as functionals of them is what it means to solve the equations of motion in the Heisenberg picture. As a simple example, if the dynamical variable is the position $q(t)$ of a one-dimensional particle then the equation of motion,

$$\ddot{q}(t)+\omega^2 q(t)=0, \tag{16}$$

has as its ''solution,''

$$q(t)=q_0\cos(\omega t)+\frac{\dot{q}_0}{\omega}\sin(\omega t). \tag{17}$$

In this case the unconstrained initial value operators are q_0 and $\dot{q}_0$.

Obtaining such a solution perturbatively entails two expansions. In the first stage one writes the dynamical variables as background plus perturbations and expands the equations of motion in powers of the perturbed quantities. To continue with our particle example, suppose the full equation of motion is $E[x](t)=0$ and that we are perturbing around some classical solution $X(t)$. If we define the perturbed position as $q(t)$ then $x(t)\equiv X(t)+q(t)$. Since the background is a solution the perturbative equations of motion necessarily begin at linear order,

$$E[X+q](t)=\ddot{q}(t)+\omega^2 q(t)+\Delta E[q](t), \tag{18}$$

where $\Delta E[q](t)$ contains terms of order q^2 and higher.

The second stage of perturbation theory consists of solving for the perturbed quantities as a series in powers of the unconstrained initial value operators. This is done by first solving the linearized equations so as to make the unconstrained variables agree with their full initial value data on the initial value surface. The linearized solution in our particle example would be

$$q_{\rm lin}(t)=\sqrt{\frac{\hbar}{2m\omega}}\{ae^{-i\omega t}+a^{\dagger}e^{i\omega t}\}, \tag{19}$$

where we have chosen to organize the initial value data into the annihilation operator, $a\equiv\sqrt{m\omega/2\hbar}(q_0+i\dot{q}_0/\omega)$, and its conjugate.

One then ''integrates'' the full equations of motion using the retarded Green's functions of the linearized theory. The retarded Green's function of our particle example is

$$G_{\rm ret}(t;t')=\frac{\theta(t-t')}{\omega}\sin[\omega(t-t')], \tag{20}$$

and the integrated equation of motion is

$$q(t)=q_{\rm lin}(t)-\frac{1}{\omega}\int_0^t dt'\sin[\omega(t-t')]\Delta E[q](t'). \tag{21}$$

The perturbative solution derives from successive substitution of this integrated form for the perturbed quantities on the righthand side of the equation,

$$q(t)=q_{\rm lin}(t)-\frac{1}{\omega}\int_0^t dt'\sin[\omega(t-t')]\Delta E[q_{\rm lin}](t')+\cdots. \tag{22}$$

It will be seen that the perturbative operator expansion which results bears a close relation to the diagrammatic expansion of quantum field theoretic amplitudes. The place of external lines is taken by the linearized solution; propagators are replaced by the corresponding retarded Green's functions; and, except for some factors of i, there is no change at all in the vertices. Since no loop diagrams appear there is no need of Faddeev-Popov ghosts. Owing to this correspondence many of the results we shall obtain for the operator expansion have already appeared in our previous perturbative expansion for expectation values [4]. We shall retain most of the conventions used in that paper.

In a gauge theory such as gravity there is the additional complication of constraints which require some of the dynamical variables to be nonlinear functionals of the unconstrained initial value operators even on the initial value surface. We shall deal with this by working in a gauge with a nonlocal field redefinition such that there are no nonlinear corrections on the initial value surface. Of course the nonlinear corrections required by the initial value constraints return when our field redefinition is inverted to give the perturbed metric and scalar fields.

Another complication is that one cannot generally obtain explicit solutions to the linearized field equations in the time dependent backgrounds of chaotic inflation. We therefore explain how to construct plane wave solutions as series expansions around the ultraviolet and infrared limiting cases. Since the putative physical effect derives from modes which have redshifted into the infrared regime it is the long wavelength approximation—coupled with the slow roll expansion—which allows us to obtain quantitative results.

Finally, it should be noted that we are not actually including all the physical degrees of freedom. Since gravitons cannot give a significant back reaction at one loop we have suppressed the initial value operators associated with them, leaving only scalar degrees of freedom. Of course we still work out how the full metric depends upon this scalar initial value data.

A. Perturbative field equations

The field equations derived from the Einstein-scalar action (2) are most usefully expressed in the form

$$F \equiv \frac{1}{a}(\Box\varphi - V'(\varphi)) = 0, \tag{23}$$

$$E^{\mu\nu} \equiv -\frac{a}{\kappa}\left(G^{\mu\nu} - \frac{\kappa^2}{2}T^{\mu\nu}\right) = 0. \tag{24}$$

Here $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ is the Einstein tensor and $T^{\mu\nu}$ is the scalar stress tensor whose covariant expression has the form

$$T_{\mu\nu} \equiv \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi + V(\varphi)\right). \tag{25}$$

Our dynamical variables are the scalar perturbation ϕ and the conformally rescaled pseudo-graviton $\psi_{\mu\nu}$:

$$\varphi(\eta,\vec{x}) \equiv \varphi_0(\eta) + \phi(\eta,\vec{x}) \tag{26}$$

$$g_{\mu\nu}(\eta,\vec{x}) \equiv a^2(\eta)[\eta_{\mu\nu} + \kappa\psi_{\mu\nu}(\eta,\vec{x})], \tag{27}$$

where $\eta_{\mu\nu}$ is the (spacelike) Lorentz metric which is used to raise and lower pseudo-graviton indices. We define the quantities ΔF and $\Delta E^{\mu\nu}$ to include all terms of second and higher orders in $\psi_{\mu\nu}$ and ϕ. One can read off the quadratic terms of ΔF and $\Delta E^{\mu\nu}$ from Tables 1 and 3 of [4]: multiply the terms of Table 1 by $1/a$ and then drop the "external" $\psi_{\mu\nu}$; vary the terms of Table 3 with respect to ϕ or $\psi_{\mu\nu}$ and then multiply by $1/a$.

The linearized equations are vastly simplified by imposing the gauge condition,

$$F_\mu \equiv a\left(\psi^\nu_{\mu,\nu} - \frac{1}{2}\psi_{,\mu} - 2\frac{a'}{a}\psi_{\mu 0} + \eta_{\mu 0}\kappa\varphi_0'\phi\right) = 0, \tag{28}$$

where a comma denotes ordinary differentiation and $\psi \equiv \eta^{\mu\nu}\psi_{\mu\nu}$. It is also useful to $3+1$ decompose and to slightly rearrange the perturbed fields as follows:

$$f \equiv a\phi, \quad z \equiv a\psi_{00}, \quad v_i \equiv a\psi_{0i}, \quad h_{ij} \equiv a(\psi_{ij} - \delta_{ij}\psi_{00}). \tag{29}$$

The various differential operators of the linearized equations can also be given a simple common form,

$$\hat{\mathcal{D}}_I \equiv \nabla^2 + \mathcal{D}_I \equiv \nabla^2 - \partial_0^2 + \frac{\theta_I''(\eta)}{\theta_I(\eta)}, \tag{30}$$

where $I = A,B,C$ and

$$\theta_A \equiv a, \quad \theta_B \equiv a^{-1}, \quad \theta_C \equiv \frac{1}{a}\sqrt{\frac{H^2}{-\dot{H}}}. \tag{31}$$

Making use of the gauge condition results in the following expansions for F and the various components of $E^{\mu\nu}$ [4]:

$$F = -\kappa\varphi_0''z + \left(\hat{\mathcal{D}}_B + \frac{\varphi_0'''}{\varphi_0'}\right)f + \Delta F, \tag{32}$$

$$E^{00} = \hat{\mathcal{D}}_B z - \kappa\varphi_0'' f + \frac{1}{4}D_A h + \Delta E^{00}, \tag{33}$$

$$E^{0i} = -\frac{1}{2}\hat{\mathcal{D}}_B v_i + \Delta E^{0i}, \tag{34}$$

$$E^{ij} = \frac{1}{2}\hat{\mathcal{D}}_A\left(h_{ij} - \frac{1}{2}\delta_{ij}h\right) + \Delta E^{ij}. \tag{35}$$

The mixing in Eq. (35) between h_{ij} and its trace $h \equiv h_{kk}$ can be removed by taking linear combinations. The same is true for the mixing in Eq. (33) between h and the variables f and z. However, the mixing between z and f in Eqs. (32), (33) cannot be removed algebraically for a general background. To diagonalize this sector we must make a nonlocal field redefinition,

$$x(\eta,\vec{x}) \equiv z'(\eta,\vec{x}) + \frac{\kappa}{2}\varphi_0'(\eta)f(\eta,\vec{x}), \tag{36}$$

$$y(\eta,\vec{x}) \equiv \frac{\kappa}{2}\varphi_0'(\eta)z(\eta,\vec{x}) + \varphi_0'(\eta)\left(\frac{f(\eta,\vec{x})}{\varphi_0'(\eta)}\right)'. \tag{37}$$

When all this is done the fully diagonalized equations take the form

$$\hat{\mathcal{D}}_A x = -(\Delta E^{00} + \Delta E^{ii})' - \frac{\kappa}{2}\varphi_0'\Delta F, \tag{38}$$

$$\hat{\mathcal{D}}_C y = -\frac{\kappa}{2}\varphi_0'(\Delta E^{00} + \Delta E^{ii}) - \varphi_0'\left(\frac{\Delta F}{\varphi_0'}\right)', \tag{39}$$

$$\hat{\mathcal{D}}_B v_i = 2\Delta E^{0i}, \tag{40}$$

$$\hat{\mathcal{D}}_A h_{ij} = -2(\Delta E^{ij} - \delta_{ij}\Delta E^{kk}). \tag{41}$$

Finally, one must check that the gauge conditions can be consistently imposed as constraints. Their $3+1$ decomposition in the diagonal variables takes the form

$$F_0=-2x+v_{i,i}-\frac{a}{2}\left(\frac{h}{a}\right)', \quad F_i=-\frac{1}{a}(av_i)'-h_{ij,j}-\frac{1}{2}h_{,i}. \tag{42}$$

Using the "commutator" identities,

$$\hat{\mathcal{D}}_B a\partial_0\frac{1}{a}=a\partial_0\frac{1}{a}\hat{\mathcal{D}}_A, \quad \hat{\mathcal{D}}_A\frac{1}{a}\partial_0 a=\frac{1}{a}\partial_0 a\hat{\mathcal{D}}_B, \tag{43}$$

and the gauge-fixed field equations (38)–(41), it is easy to show that the constraints obey

$$\hat{\mathcal{D}}_B F_0=2\left(\Delta E^{\mu 0}{}_{,\mu}+\frac{a'}{a}\Delta E^{ii}+\frac{\kappa}{2}\varphi_0'\Delta F\right), \tag{44}$$

$$\hat{\mathcal{D}}_A F_i=-2\left(\Delta E^{\mu i}{}_{,\mu}+\frac{a'}{a}\Delta E^{0i}\right). \tag{45}$$

The quantities on the right-hand side of Eqs. (44), (45) vanish as a consequence of the background Bianchi identities in the usual way. Therefore, the constraints are preserved by the gauge fixed evolution equations and full equivalence between the invariant field equations and those of the gauge-fixed theory will hold if the initial value operators are constrained to make F_μ and its first conformal time derivative vanish at $\eta=\eta_0$.

B. Unconstrained plane waves

It is useful to consider plane wave solutions to the linearized, gauge-fixed field equations before suppressing the physical gravitons and using the constraints and the residual gauge freedom to purge unphysical initial value data. That is, we seek the kernel of the differential operators $\mathcal{D}_I-k^2=-\partial_0^2-k^2+\theta_I''/\theta_I$, where the functions $\theta_I(\eta)$ were defined in relation (31). Although explicit solutions cannot be obtained for an arbitrary inflationary background, the limiting cases of $k=0$ and $\theta_I=0$ serve as the basis of series solutions for the "infrared" ($k^2\ll\theta_I''/\theta_I$) and "ultraviolet" ($k^2\gg\theta_I''/\theta_I$) regimes. In the ultraviolet regime the particle interpretation is the same as for flat space so its normalization is used to define that of the infrared regime. The final step of this subsection is superposing plane waves multiplied by the creation and annihilation operators of the linearized, gauge-fixed action. The commutator functions between such fields give the retarded Green's functions needed to integrate the perturbative field equations.

The following two solutions comprise a useful basis set in the infrared limit ($k=0$):[1]

$$Q_{10,I}(\eta)=\theta_I(\eta), \quad Q_{20,I}(\eta)=\theta_I(\eta)\int_{-\infty}^{\eta}d\eta'\frac{1}{\theta_I^2(\eta')}. \tag{46}$$

One of these is a "growing" mode, the other a "decaying" mode. For example, the decaying C mode is $Q_{10,C}(\eta)$. The growing C mode can be evaluated in the slow roll expansion:

$$Q_{20,C}(\eta)=\frac{\sqrt{-\dot{H}}}{H^2}\left[1+\mathcal{O}\left(\frac{-\dot{H}}{H^2}\right)\right]. \tag{47}$$

We can find the inverse of $\mathcal{D}_I$ on any function $f(\eta)$ by two simple integrations:

$$(\mathcal{D}_I^{-1}f)(\eta)=-\theta_I(\eta)\int_{\eta_0}^{\eta}d\eta'\frac{1}{\theta_I^2(\eta')}\int_{\eta_0}^{\eta}d\eta''\theta_I(\eta'')f(\eta''). \tag{48}$$

By integrating the full equation, $\mathcal{D}_I Q=k^2 Q$, one obtains a relation,

$$Q_{i,I}(\eta,k)=Q_{i0,I}(\eta)+k^2(\mathcal{D}_I^{-1}Q_{i,I})(\eta,k), \quad i=1,2, \tag{49}$$

whose iteration results in a convergent series expansion in powers of k^2. Note that constancy of the Wronskian and the vanishing of corrections at $\eta=\eta_0$ allows us to evaluate it using the zeroth order solutions,

$$Q_{2,I}'(\eta,k)Q_{1,I}(\eta,k)-Q_{1,I}'(\eta,k)Q_{2,I}(\eta,k)=1. \tag{50}$$

The useful basis elements for the ultraviolet limit ($\theta_I=0$) are $e^{-ik\eta}$ and its conjugate. In this limit we know the absolute normalization from correspondence with flat space. Coupling this with the harmonic oscillator Green's function gives the integrated mode equation,

$$Q_I(\eta,k)=\frac{e^{-ik\eta}}{\sqrt{2k}}+\frac{1}{k}\int_{-\infty}^{\eta}d\bar{\eta}\sin[k(\eta-\bar{\eta})]\frac{\theta_I''(\bar{\eta})}{\theta_I(\bar{\eta})}Q_I(\bar{\eta},k), \tag{51}$$

whose iteration yields an asymptotic expansion in powers of $1/k$. The Wronskian for the ultraviolet mode functions is

$$Q_I'(\eta,k)Q_I^*(\eta,k)-Q_I'^*(\eta,k)Q_I(\eta,k)=-i. \tag{52}$$

Any quantum operator $\psi_I(\eta,\vec{x})$ which is annihilated by $\hat{\mathcal{D}}_I$ can be expressed as a superposition of plane waves with operator coefficients,

$$\psi_I(\eta,\vec{x})=\int\frac{d^3k}{(2\pi)^3}\{Q_I(\eta,k)e^{i\vec{k}\cdot\vec{x}}\Psi(\vec{k})+Q_I^*(\eta,k)e^{-i\vec{k}\cdot\vec{x}}\Psi^\dagger(\vec{k})\}. \tag{53}$$

If the conjugate momentum is $\psi_I'(\eta,\vec{x})$ then the Wronskian (52) implies that the coefficients have the algebra of canonically normalized creation and annihilation operators,

$$[\Psi(\vec{k}),\Psi^\dagger(\vec{k}')]=\delta ij(2\pi)^3\delta^3(\vec{k}-\vec{k}'). \tag{54}$$

By taking the commutator of two such fields we obtain a sequence of lovely expressions for the retarded Green's function of the differential operator $\hat{\mathcal{D}}_I$,

[1]Here our notation differs slightly from [4] in that the signs of θ_C and of the second solutions Eq. (46) have been inverted.

$$G_I(x;x') = -i\theta(\Delta\eta)[\psi_I(\eta,\vec{x}),\psi_I(\eta',\vec{x}')], \quad (55)$$

$$= -i\theta(\Delta\eta)\int \frac{d^3k}{(2\pi)^3} \times e^{i\vec{k}\cdot\Delta\vec{x}}\{Q_I(\eta,k)Q_I^*(\eta',k) - (\eta\leftrightarrow\eta')\}, \quad (56)$$

$$= \theta(\Delta\eta)\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\Delta\vec{x}} \times\{Q_{1,I}(\eta,k)Q_{2,I}(\eta',k) - (\eta\leftrightarrow\eta')\}, \quad (57)$$

where $\Delta\eta \equiv \eta - \eta'$ and $\Delta\vec{x} \equiv \vec{x} - \vec{x}'$. The crucial final expression—in terms of the infrared mode functions—follows from the fact that that it obeys the same differential equation—$\hat{\mathcal{D}}_I G_I(x;x') = \delta^4(x-x')$—and retarded boundary conditions as the first two.

The preceding analysis suffices to define the retarded Green's functions needed to integrate equations (40) and (41),

$$v_i(\eta,\vec{x}) = v_i^{\rm lin}(\eta,\vec{x}) + 2\int_{\eta_0}^{\eta} d\eta' \int d^3x' \, G_B(x;x') \times \Delta E^{0i}(x'), \quad (58)$$

$$h_{ij}(\eta,\vec{x}) = h_{ij}^{\rm lin}(\eta,\vec{x}) - 2(\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl})\int_{\eta_0}^{\eta} d\eta' \times \int d^3x' \, G_A(x;x')\Delta E^{kl}(x'). \quad (59)$$

It would be straightforward to give plane wave expansions for $v_i^{\rm lin}$ and $h_{ij}^{\rm lin}$. However, we will not bother since these linearized fields vanish when physical graviton degrees of freedom are suppressed and the various constraints and residual gauge conditions are imposed.

The same general technology can be applied to integrate the more complicated (f,z) system,

$$\begin{pmatrix} \hat{\mathcal{D}}_B + \dfrac{\varphi_0'''}{\varphi_0'} & -\kappa\varphi_0'' \\ -\kappa\varphi_0'' & \hat{\mathcal{D}}_B \end{pmatrix} \begin{pmatrix} f \\ z \end{pmatrix} = -\begin{pmatrix} \Delta F \\ \Delta E^{00} + \Delta E^{ii} \end{pmatrix}. \quad (60)$$

The desired retarded Green's functions,

$$\begin{pmatrix} \hat{\mathcal{D}}_B + \dfrac{\varphi_0'''}{\varphi_0'} & -\kappa\varphi_0'' \\ -\kappa\varphi_0''\hat{\mathcal{D}}_B & \end{pmatrix} \begin{pmatrix} G_{ff}(x;x') & G_{fz}(x;x') \\ G_{zf}(x;x') & G_{zz}(x;x') \end{pmatrix} = \delta^4(x-x')\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (61)$$

also follow from commutators of the linearized fields,

$$\begin{pmatrix} G_{ff}(x;x') & G_{fz}(x;x') \\ G_{zf}(x;x') & G_{zz}(x;x') \end{pmatrix} = -i\theta(\Delta\eta)\begin{pmatrix} [f(x),f(x')] & [f(x),z(x')] \\ [z(x),f(x')] & [z(x),z(x')] \end{pmatrix}. \quad (62)$$

Unfortunately it is the fields $x(\eta,\vec{x})$ and $y(\eta,\vec{x})$ which are annihilated by $\hat{\mathcal{D}}_B$ and $\hat{\mathcal{D}}_C$, so it is they that possess simple plane wave expansions in terms of the B and C mode functions.

At linearized order the gauge-fixed action reveals the conjugate momenta to f and z to be simply their conformal time derivatives. Definitions (36) and (37) give the following nonzero equal-time commutation algebra for x and y,

$$[x(\eta,\vec{x}),x'(\eta,\vec{y})] = -i\nabla^2\delta^3(\vec{x}-\vec{y}) = [y(\eta,\vec{x}),y'(\eta,\vec{y})]. \quad (63)$$

If we still employ canonically normalized creation and annihilation operators,

$$[X(\vec{k}),X(\vec{k}')] = (2\pi)^3\delta^3(\vec{k}-\vec{k}') = [Y(\vec{k}),Y(\vec{k}')], \quad (64)$$

then the additional factor of $-\nabla^2$ in (63) implies the following plane wave expansions:

$$x(\eta,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} k\{Q_B(\eta,k)e^{i\vec{k}\cdot\vec{x}}X(\vec{k}) + Q_B^*(\eta,k)e^{-i\vec{k}\cdot\vec{x}}X^\dagger(\vec{k})\}, \quad (65)$$

$$y(\eta,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} k\{Q_C(\eta,k)e^{i\vec{k}\cdot\vec{x}}Y(\vec{k}) + Q_C^*(\eta,k)e^{-i\vec{k}\cdot\vec{x}}Y^\dagger(\vec{k})\}. \quad (66)$$

Since the transformation between (f,z) and (x,y) involves conformal time derivatives, its inverse cannot be local in time for the off shell fields. However, for the on-shell solutions we can use the linearized equations to obtain the following expressions:

$$f = \frac{1}{\nabla^2}\left[-\frac{\kappa}{2}\varphi_0' x + y' + \frac{\varphi_0''}{\varphi_0'} y\right]. \quad (67)$$

$$z = \frac{1}{\nabla^2}\left[x' - \frac{\kappa}{2}\varphi_0' y\right], \quad (68)$$

It is simplest to exploit these relations in evaluating the various commutators of the (f,z) retarded Greens functions. One then replaces products of the mode functions with the B and C-type Green's functions. The answer is

$$G_{ff}(x;x')=\frac{1}{\nabla^2}\Bigg\{-\frac{\kappa^2}{4}\varphi_0'(\eta)\varphi_0'(\eta')G_B(x;x') - \frac{1}{\varphi_0'(\eta)\varphi_0'(\eta')}\partial_0\partial_0'\varphi_0'(\eta)\varphi_0'(\eta')G_C(x;x') + \delta^4(x-x')\Bigg\}, \tag{69}$$

$$G_{fz}(x;x')=\frac{\kappa}{2\nabla^2}\Bigg\{\varphi_0'(\eta)\partial_0'G_B(x;x') + \frac{\varphi_0'(\eta')}{\varphi_0'(\eta)}\partial_0\varphi_0'(\eta)G_C(x;x')\Bigg\}, \tag{70}$$

$$G_{zf}(x;x')=\frac{\kappa}{2\nabla^2}\Bigg\{\varphi_0'(\eta')\partial_0G_B(x;x') + \frac{\varphi_0'(\eta)}{\varphi_0'(\eta')}\partial_0'\varphi_0'(\eta')G_C(x;x')\Bigg\}, \tag{71}$$

$$G_{zz}(x;x')=\frac{1}{\nabla^2}\Bigg\{-\partial_0\partial_0'G_B(x;x') - \frac{\kappa^2}{4}\varphi_0'(\eta)\varphi_0'(\eta')G_C(x;x')+\delta^4(x-x')\Bigg\}. \tag{72}$$

Finally, we substitute the relation, $G_I(x;x')=\hat{\mathcal{D}}_I^{-1}\delta^4(x-x')$,

$$G_{ff}(x;x')=\frac{1}{\nabla^2}\Bigg[1-\frac{\kappa^2}{4}\varphi_0'\hat{\mathcal{D}}_B^{-1}\varphi_0'+\frac{1}{\varphi_0'}\partial_0\varphi_0'\hat{\mathcal{D}}_C^{-1}\varphi_0'\partial_0\frac{1}{\varphi_0'}\Bigg]\times\delta^4(x-x'), \tag{73}$$

$$G_{fz}(x;x')=\frac{\kappa}{2\nabla^2}\Bigg[-\varphi_0'\hat{\mathcal{D}}_B^{-1}\partial_0+\frac{1}{\varphi_0'}\partial_0\varphi_0'\hat{\mathcal{D}}_C^{-1}\varphi_0'\Bigg]\times\delta^4(x-x'), \tag{74}$$

$$G_{zf}(x;x')=\frac{\kappa}{2\nabla^2}\Bigg[\partial_0\hat{\mathcal{D}}_B^{-1}\varphi_0'-\varphi_0'\hat{\mathcal{D}}_C^{-1}\varphi_0'\partial_0\frac{1}{\varphi_0'}\Bigg]\times\delta^4(x-x'), \tag{75}$$

$$G_{zz}(x;x')=\frac{1}{\nabla^2}\Bigg[1+\partial_0\hat{\mathcal{D}}_B^{-1}\partial_0-\frac{\kappa^2}{4}\varphi_0'\hat{\mathcal{D}}_C^{-1}\varphi_0'\Bigg]\times\delta^4(x-x'). \tag{76}$$

C. Integrated field equations

In this subsection we first purge unphysical initial value data by imposing the constraints and making use of the residual gauge freedom. Then gravitons are suppressed to leave what we call the physical linearized fields. Finally, the various field equations are integrated using the retarded Green's functions of the previous section.

Since our gauge condition involves derivatives it is preserved by residual gauge transformations which obey second order differential equations.[2] One can therefore freely impose four residual conditions and their first conformal time derivatives on the initial value surface. We choose these to null $x(\eta_0,\vec{x})$, $x'(\eta_0,\vec{x})$, $v_i(\eta_0,\vec{x})$ and $v_i'(\eta_0,\vec{x})$.

At the end of sec. III A we showed that the four constraints are also annihilated by second order differential operators. It therefore follows that the initial values of the four constraints and their first conformal time derivatives are the only extra conditions to be imposed in addition to the gauge-fixed field equations. With our residual gauge choice these eight conditions imply that $h_{ij}(\eta,\vec{x})$ is transverse and traceless at linearized order. Suppressing gravitons makes it vanish completely at linearized order.

The only linearized fields which remain are f and z. Since x vanishes, they can depend only upon the operators $Y(\vec{k})$ and $Y^\dagger(\vec{k})$,

$$f_{\rm ph}(\eta,\vec{x})=\frac{-1}{\varphi_0'(\eta)}\partial_0\varphi_0'(\eta)\int\frac{d^3k}{(2\pi)^3}\frac{1}{k}\times\{Q_C(\eta,k)e^{i\vec{k}\cdot\vec{x}}Y(\vec{k})+\text{H.c.}\}, \tag{77}$$

$$z_{\rm ph}(\eta,\vec{x})=\frac{\kappa}{2}\varphi_0'(\eta)\int\frac{d^3k}{(2\pi)^3}\frac{1}{k}\times\{Q_C(\eta,k)e^{i\vec{k}\cdot\vec{x}}Y(\vec{k})+\text{H.c.}\}. \tag{78}$$

We can obviously eliminate $f_{\rm ph}$ in favor of $z_{\rm ph}$,

$$f_{\rm ph}(\eta,\vec{x})=-\frac{2}{\kappa}\frac{z_{\rm ph}'(\eta,\vec{x})}{\varphi_0'(\eta)}=\frac{z_{\rm ph}'(\eta,\vec{x})}{a(\eta)\sqrt{-\dot{H}}}. \tag{79}$$

This is the well-known constraint between the (dimensionless) Newtonian potential $\Phi(\eta,\vec{x})\equiv\kappa z(\eta,\vec{x})/2a(\eta)$ and the scalar field fluctuation [11].

We can now write down the integrated field equations whose iteration produces the final perturbative expansion of the various fields:

$$f(\eta,\vec{x})=f_{\rm ph}(\eta,\vec{x})-\int_{\eta_0}^{\eta}d\eta'\int d^3x'\{G_{ff}(x;x')\Delta F(x') + G_{fz}(x;x')[\Delta E^{00}(x')+\Delta E^{ii}(x')]\}, \tag{80}$$

[2] The analysis is quite similar to that carried out for de Sitter background in [10].

$$z(\eta,\vec{x})=z_{\rm ph}(\eta,\vec{x})-\int_{\eta_0}^{\eta}d\eta'\int d^3x'\{G_{zf}(x;x')\Delta F(x')+G_{zz}(x;x')[\Delta E^{00}(x')+\Delta E^{ii}(x')]\}, \tag{81}$$

$$v_i(\eta,\vec{x})=2\int_{\eta_0}^{\eta}d\eta'\int d^3x'G_B(x;x')\Delta E^{0i}(x'), \tag{82}$$

$$h_{ij}(\eta,\vec{x})=-2(\delta_{ik}\delta_{jl}-\delta_{ij}\delta_{kl})\times\int_{\eta_0}^{\eta}d\eta'\int d^3x'G_A(x;x')\Delta E^{kl}(x'). \tag{83}$$

Note that v_i and h_{ij} vanish at $\eta=\eta_0$, along with their first derivatives. On the other hand, both f and z suffer perturbative correction on the initial value surface. This derives from the existence of a gravitational interaction even on the initial value surface, as required by the constraint equations of the ungauged formalism.

D. The long wavelength approximation

Except for suppressing the gravitons, all the results obtained to this point have been exact. Unfortunately they are also largely useless because we lack explicit forms for the various mode functions and retarded Green's functions in a general inflationary background. Nor could we perform the required integrations if we did possess such expressions. What makes calculations possible is the long wavelength approximation which effectively removes spatial variation. The resulting problem of temporal calculus is still formidable, but it can be treated using the slow roll expansion discussed in Sec. II.

The back reaction we seek to study is the response to the superadiabatic amplification of modes which experience horizon crossing. This is an intrinsicly infrared phenomenon so one might suspect that it can be studied effectively in the limit where spatial derivatives are dropped and the mode functions approach their leading infrared forms. This limit defines the long wavelength approximation and it effects three sorts of simplifications on the perturbative apparatus of relations (80)–(83).

The first simplification is to ignore spatial derivatives in the source terms, ΔF and $\Delta E^{\mu\nu}$. An immediate consequence of this, and of the absence of dynamical gravitons, is that ΔE^{0i} vanishes and ΔE^{ij} is proportional its trace,

$$\Delta E^{0i}|_{l.w.}\rightarrow 0,\quad \Delta E^{ij}|_{l.w.}\rightarrow\frac{1}{3}\delta_{ij}\Delta E^{kk}. \tag{84}$$

It follows that v_i remains zero for all time and that only the trace part of h_{ij} ever becomes nonzero,

$$v_i(\eta,\vec{x})|_{l.w.}\rightarrow 0, \tag{85}$$

$$h_{ij}(\eta,\vec{x})|_{l.w.}\rightarrow\frac{4}{3}\delta_{ij}\int_{\eta_0}^{\eta}d\eta'\int d^3x'G_A(x;x')\times\Delta E^{kk}(x'). \tag{86}$$

The second simplification is that one can replace the mode functions of the linearized solutions by their infrared limiting forms. These forms were worked out in [4] by matching the leading ultraviolet term of the ultraviolet expansion to the leading term of the infrared expansion at the time $\eta_*(k)$ of horizon crossing: $k=H(\eta_*)a(\eta_*)$. Since the physical fields contain only C modes the result we require is

$$Q_C(\eta,k)|_{l.w.}\rightarrow\frac{\sqrt{-\dot{H}(\eta)}}{H^2(\eta)}\cdot\frac{H^2(\eta_*)}{\sqrt{-\dot{H}(\eta_*)}}\frac{e^{-ik\eta_*}}{\sqrt{2k}}. \tag{87}$$

Note also that momentum integrations are cut off at $k=H(\eta)a(\eta)$. The physical justification for this is that only modes below the cutoff have undergone the superadiabatic amplification that is the basis for the effect whose back reaction we seek to evaluate. However, one must take care that the resulting time dependence is a genuine infrared effect and not simply the result of allowing more and more modes to contribute to what would be an ultraviolet divergence without the cutoff. One consequence is that time derivatives of the fields should not be allowed to act on the momentum cutoff.

The third simplification is that we expand the inverses of $\hat{\mathcal{D}}_I\equiv\mathcal{D}_I+\nabla^2$ in powers of ∇^2,

$$\hat{\mathcal{D}}_I^{-1}=\mathcal{D}_I^{-1}-\mathcal{D}_I^{-2}\nabla^2+\cdots. \tag{88}$$

Only the zeroth order term is required for h_{ij},

$$h_{ij}|_{l.w.}\rightarrow\frac{4}{3}\delta_{ij}\mathcal{D}_A^{-1}\Delta E^{kk}, \tag{89}$$

but one must go to first order for the various (f,z) Green's functions on account of their prefactors of $1/\nabla^2$. The Appendix demonstrates that the leading results are

$$G_{ff}(x;x')|_{l.w.}\rightarrow\left[\varphi_0'\mathcal{D}_B^{-1}\frac{1}{\varphi_0'}+2\varphi_0'\mathcal{D}_B^{-1}\frac{\varphi_0''}{\varphi_0'^2}\mathcal{D}_C^{-1}\varphi_0'\partial_0\frac{1}{\varphi_0'}\right]\times\delta^4(x-x'), \tag{90}$$

$$G_{fz}(x;x')|_{l.w.}\rightarrow\kappa\varphi_0'\mathcal{D}_B^{-1}\frac{\varphi_0''}{\varphi_0'^2}\mathcal{D}_C^{-1}\varphi_0'\delta^4(x-x'), \tag{91}$$

$$G_{zf}(x;x')|_{l.w.}\rightarrow\kappa\varphi_0'\mathcal{D}_C^{-1}\frac{\varphi_0''}{\varphi_0'^2}\mathcal{D}_B^{-1}\varphi_0'\delta^4(x-x'), \tag{92}$$

$$G_{zz}(x;x')|_{l.w.}\rightarrow\left[\varphi_0'\mathcal{D}_C^{-1}\frac{1}{\varphi_0'}+2\varphi_0'\mathcal{D}_C^{-1}\frac{\varphi_0''}{\varphi_0'^2}\mathcal{D}_B^{-1}\partial_0\right]\times\delta^4(x-x'). \tag{93}$$

At this point we can make contact with the similar calculation that was done in [4]. In the infrared limit, the general retarded propagators (90)–(93) correspond exactly to the result that can be inferred from Eqs. (131), (132) of [4]. The "C" and "D" in Eqs. (131) and (132) from that paper (where $\Omega \equiv a$) correspond respectively to the change induced in the z and in the f fields that obtains from our equations (80), (81). The amputated 1-point functions in those equations correspond to our source terms according to the rule: $\widetilde{\alpha} \to -(\kappa/3) a \Delta E^{kk}$, $\widetilde{\gamma} \to -\kappa a \Delta E^{00}$ and $\widetilde{\delta} \to -\kappa a \Delta F$.

IV. QUADRATIC CORRECTIONS

The purpose of this section is to apply the technology of Sec. III to obtain the scalar and metric at quadratic order in the initial value operators $Y(\vec{k})$ and $Y^\dagger(\vec{k})$, and to leading order in the slow roll and long wavelength expansions. In Ref. [4] we computed the one loop expectation values of the scalar and metric fields, also to leading order in the slow roll and long wavelength expansions. Much of that calculation can be used to get the quadratic order operator expansions of the same quantities. In particular, all the vertices are catalogued in that reference.

As an example, consider vertex No. 9 of Table 3 from [4]. That vertex corresponds to the following cubic term in the Lagrangian:

$$\mathcal{L}_{no.\ 9} = -\frac{\kappa}{4} V_{,\varphi\varphi}(\varphi_0) a^4 \phi^2 \psi = -\frac{\kappa}{4} V_{,\varphi\varphi}(\varphi_0) a f^2 (2z + h). \tag{94}$$

Variation with respect to ϕ and $\psi_{\mu\nu}$ gives the following source terms:

$$\Delta F_{no.\ 9} = -\kappa V_{,\varphi\varphi}(\varphi_0) a f \left(z + \frac{1}{2} h \right),$$

$$\Delta E^{\mu\nu}_{no.\ 9} = -\frac{\kappa}{4} V_{,\varphi\varphi}(\varphi_0) a f^2 \eta^{\mu\nu}. \tag{95}$$

The lowest order perturbative corrections from these terms come when f, z and h are replaced by the associated physical linearized fields: $f_{\rm ph}$, $z_{\rm ph}$ and 0, respectively. Recall also that $f_{\rm ph} = z'/a\sqrt{-\dot{H}}$.

The preceding are all exact results. Because we are also making the long wavelength and slow roll approximations, $z_{\rm ph}$ can be written as

$$z_{\rm ph}(\eta,\vec{x})|_{l.w.+s.r.} \to a(\eta) \left[\frac{-\dot{H}(\eta)}{H^2(\eta)} \right] \int \frac{d^3k}{(2\pi)^3} \frac{H_*}{\sqrt{2k^3}} \times \sqrt{\frac{H_*^2}{-\dot{H}_*}} \{ Y(\vec{k}) + Y^\dagger(\vec{k}) \}. \tag{96}$$

To leading order in the slow roll approximation the time depedence of this expression comes entirely in the initial factor of $a(\eta)$, which allows the following simplification for $f_{\rm ph}$:

$$f_{\rm ph}(\eta,\vec{x})|_{l.w.+s.r} \to \sqrt{\frac{H^2(\eta)}{-\dot{H}(\eta)}} z_{\rm ph}(\eta,\vec{x}). \tag{97}$$

We can also use the slow roll approximation to evaluate derivatives of the scalar potential,

$$\kappa V_{,\varphi}(\varphi_0)|_{s.r.} \to 6H\sqrt{-\dot{H}}, \quad V_{,\varphi\varphi}(\varphi_0)|_{s.r.} \to -3\dot{H}. \tag{98}$$

The source terms finally contributed by vertex no. 9 in the slow roll and long wavelength approximations are

$$\Delta F_{no.\ 9}|_{l.w.+s.r.} \to -3\kappa H^2 \sqrt{\frac{-\dot{H}}{H^2}} a z_{\rm ph}^2, \tag{99}$$

$$\Delta E^{00}_{no.\ 9}|_{l.w.+s.r.} \to +\frac{3}{4} \kappa H^2 a z_{\rm ph}^2, \tag{100}$$

$$\Delta E^{kk}_{no.\ 9}|_{l.w.+s.r.} \to -\frac{9}{4} \kappa H^2 a z_{\rm ph}^2. \tag{101}$$

Summing the contributions from all vertices of Einstein scalar [4] under the same set of approximations gives the following quadratic source terms:

$$\Delta F|_{l.w.+s.r.} \to 3\kappa H^2 \sqrt{\frac{-\dot{H}}{H^2}} a z_{\rm ph}^2, \tag{102}$$

$$\Delta E^{00}|_{l.w.+s.r.} \to -\frac{9}{4} \kappa H^2 a z_{\rm ph}^2, \tag{103}$$

$$\Delta E^{kk}|_{l.w.+s.r.} \to +\frac{27}{4} \kappa H^2 a z_{\rm ph}^2. \tag{104}$$

The next step is to apply the various retarded Green's functions in the long wavelength and slow roll approximations. As an example, we substitute Eq. (104) into Eq. (89) to obtain the leading quadratic correction to h_{ij},

$$h^{(2)}_{ij}|_{l.w.+s.r.} \to -9\kappa \delta_{ij} a \int_{\eta_0}^{\eta} d\eta' a^{-2} \int_{\eta_0}^{\eta'} d\eta'' a^2 H^2 z_{\rm ph}^2, \tag{105}$$

where $z_{\rm ph}$ is given by Eq. (96). Taking the momentum integrals outside, we are left with the following integrations over time:

$$a(t) \int_{t_0}^{t} dt' a^{-3}(t') \int_{t_0}^{t'} dt'' a^3(t'') \frac{\dot{H}^2}{H^2} = \frac{1}{6} a(t) \left(\frac{-\dot{H}}{H^2} \right) + \cdots. \tag{106}$$

The result is therefore

$$h_{ij}|_{l.w.+s.r.} \to \delta_{ij} \left(\frac{H^2}{-\dot{H}} \right) \left\{ -\frac{3}{2} \frac{\kappa z_{\rm ph}^2}{a} + O(z_{\rm ph}^3) \right\}. \tag{107}$$

The analogous reductions for f and z are straightforward but rather tedious, owing to their complicated mixing. The final answer is

$$f|_{l.w.+s.r.} \rightarrow \sqrt{\frac{H^2}{\dot{H}}}\left\{z_{\rm ph} + \frac{5}{4}\,\frac{\kappa z_{\rm ph}^2}{a} + O(z_{\rm ph}^3)\right\}, \qquad (108)$$

$$z|_{l.w.+s.r.} \rightarrow \left\{z_{\rm ph} + \frac{7}{2}\,\frac{\kappa z_{\rm ph}^2}{a} + O(z_{\rm ph}^3)\right\}. \qquad (109)$$

We invert Eq. (29) to recover ϕ and $\psi_{\mu\nu}$. It is useful to multiply by κ to absorb the dimensions, and to express the result in terms of the (dimensionless) Newtonian potential:

$$\Phi(x) \equiv \frac{\kappa z_{\rm ph}(x)}{2a(\eta)}. \qquad (110)$$

Note that, from Eqs. (110) and (96) we have $\Phi \propto H^{-2}$ (since $\dot{H} \sim$ const) so that its time derivative is down from $H\Phi$ by a slow roll parameter,

$$\dot{\Phi}|_{l.w.+s.r.} \rightarrow 2\left(\frac{-\dot{H}}{H^2}\right) H\Phi. \qquad (111)$$

With this terminology, the nonzero perturbed fields are

$$\kappa\phi|_{l.w.+s.r.} \rightarrow \sqrt{\frac{H^2}{-\dot{H}}}\{2\Phi + 5\Phi^2 + O(\Phi^3)\}, \qquad (112)$$

$$\kappa\psi_{00}|_{l.w.s.r.} \rightarrow 2\Phi + 14\Phi^2 + O(\Phi^3), \qquad (113)$$

$$\kappa\psi_{ij}|_{l.w.+s.r.} \rightarrow \delta_{ij}\left\{2\Phi - 6\left(\frac{H^2}{-\dot{H}}\right)\Phi^2 + O(\Phi^3)\right\}. \qquad (114)$$

V. AN INVARIANT MEASURE OF EXPANSION

The purpose of this section is to compute corrections through quadratic order to the operator $\mathcal{A}_{\rm inv}$ we have proposed [9] as an invariant measure of the rate of cosmological expansion. We first expand the scalar $\mathcal{A} \equiv \Box_c^{-1}$ in powers of the pseudo-graviton field and then substitute Eqs. (113), (114) to express the result as a function of the unconstrained initial value operators, to leading order in the slow roll and long wavelength approximations. Full invariance is achieved by evaluating the $\mathcal{A}(\eta,\vec{x})$ in a geometrically specified coordinate system.

A. The scalar observable

The pseudo-graviton expansion is most easily accomplished by first expressing $\Box_c$ in terms of the conformally rescaled metric,

$$\tilde{g}_{\mu\nu}(\eta,\vec{x}) \equiv a^{-2}(\eta) g_{\mu\nu}(\eta,\vec{x}) = \eta_{\mu\nu} + \kappa\psi_{\mu\nu}(\eta,\vec{x}). \qquad (115)$$

We write $\Box_c = a^{-3}\mathcal{D}a$, where $\mathcal{D}$ and its expansion are

$$\mathcal{D} \equiv \frac{1}{\sqrt{-\tilde{g}}}\partial_\mu(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu) - \frac{1}{6}\tilde{R} = \partial^2 + \kappa\mathcal{D}_1 + \kappa^2\mathcal{D}_2 + \cdots. \qquad (116)$$

The first two operators in the expansion are

$$\mathcal{D}_1 = -\psi^{\mu\nu}\partial_\mu\partial_\nu + \left(-\psi^{\mu\alpha}_{\ \ ,\alpha} + \frac{1}{2}\psi^{,\mu}\right)\partial_\mu - \frac{1}{6}(\psi^{\rho\sigma}_{\ \ ,\rho\sigma} - \psi_\rho^{,\rho}), \qquad (117)$$

$$\mathcal{D}_2 = \psi^{\mu\alpha}\psi^\nu_\alpha\partial_\mu\partial_\nu + \left((\psi^{\alpha\beta}\psi^\mu_\alpha)_{,\beta} - \frac{1}{2}\psi^{\alpha\beta,\mu}\psi_{\alpha\beta} + \frac{1}{2}\psi^{\alpha\mu}\psi_{,\alpha}\right) \times \partial_\mu - \frac{1}{6}\tilde{R}_2. \qquad (118)$$

The second order, conformally rescaled Ricci scalar is

$$\tilde{R}_2 \equiv \psi^{\alpha\beta}(\psi_{\alpha\beta\ \ \gamma}^{\ \ ,\gamma} + \psi_{,\alpha\beta} - 2\psi^\gamma_{\ \alpha,\beta\gamma}) + \frac{3}{4}\psi^{\alpha\beta,\gamma}\psi_{\alpha\beta,\gamma} - \frac{1}{2}\psi^{\alpha\beta,\gamma}\psi_{\gamma\beta,\alpha} - \psi^{\alpha\beta}_{\ \ ,\beta}\psi^\gamma_{\alpha,\gamma} + \psi^{\alpha\beta}_{\ \ ,\beta}\psi_{,\alpha} - \frac{1}{4}\psi^{,\alpha}\psi_{,\alpha}. \qquad (119)$$

The next step is to factor ∂^2 out of $\mathcal{D}$:

$$\mathcal{D} = \partial^2\left(1 + \frac{1}{\partial^2}\kappa\mathcal{D}_1 + \frac{1}{\partial^2}\kappa^2\mathcal{D}_2 + O(\kappa^3)\right). \qquad (120)$$

Inverting $\mathcal{D}$ is now straightforward:

$$\frac{1}{\mathcal{D}} = \frac{1}{\partial^2} - \frac{1}{\partial^2}\kappa\mathcal{D}_1\frac{1}{\partial^2} + \frac{1}{\partial^2}\kappa\mathcal{D}_1\frac{1}{\partial^2}\kappa\mathcal{D}_1\frac{1}{\partial^2} - \frac{1}{\partial^2}\kappa^2\mathcal{D}_2\frac{1}{\partial^2} + O(\kappa^3). \qquad (121)$$

All this implies the following expansion for the scalar observable:

$$\mathcal{A}[g] = a^{-1}\frac{1}{\mathcal{D}}a^3 = \mathcal{A}_0 + \kappa\mathcal{A}_1 + \kappa^2\mathcal{A}_2 + O(\kappa^3), \qquad (122)$$

where the first two corrections are

$$\mathcal{A}_1 \equiv -a^{-1}\frac{1}{\partial^2}\mathcal{D}_1\frac{1}{\partial^2}a^3, \qquad (123)$$

$$\mathcal{A}_2 \equiv -a^{-1}\frac{1}{\partial^2}\mathcal{D}_2\frac{1}{\partial^2}a^3 + a^{-1}\frac{1}{\partial^2}\mathcal{D}_1\frac{1}{\partial^2}\mathcal{D}_1\frac{1}{\partial^2}a^3. \qquad (124)$$

The zeroth-order term $\mathcal{A}_0 \equiv a^{-1}\partial^{-2}a^3$ has a very simple expression in terms of the background expansion rate,

$$\mathcal{A}_0 = -a^{-1}\int_{\eta_0}^{\eta} d\eta' \int_{\eta_0}^{\eta'} d\eta'' a^3(\eta'') = \frac{-1}{2H^2}\left\{1 + O\left(\frac{-\dot{H}}{H^2}\right)\right\}. \tag{125}$$

Our proposal is to define the general expansion rate, as an operator, to bear the same relation to the full scalar $\mathcal{A}$ [9]. Astronomers do the same thing when they infer the Hubble constant under the assumption that the relation between luminosity distance and redshift is the same in the actual universe as for a perfectly homogeneous and isotropic one.

The higher order contributions $\mathcal{A}_1$ and $\mathcal{A}_2$ are nominally inhomogeneous but lose their dependence upon space in the long wavelength approximation. The consequent suppression of spatial derivatives and anisotropic components of the pseudo-graviton field reduces $\mathcal{D}_1$ to the following simple form:

$$\mathcal{D}_1|_{l.w.} \to -\psi_{00}\partial_0^2 - \frac{1}{2}(\psi_{00}' + \psi_{ii}')\partial_0 - \frac{1}{6}\psi_{ii}'. \tag{126}$$

Further simplifications result from the fact that (co-moving) time derivatives of Φ are weaker than $H\Phi$ by a slow roll parameter as per Eq. (111). Only the quadratic term of ψ_{ij} possesses the enhancement necessary to survive differentiation at leading order,

$$\kappa\psi_{00}'|_{l.w.+s.r.} \to Ha\{0 + O(\Phi^3)\}, \tag{127}$$

$$\kappa\psi_{ij}'|_{l.w.+s.r.} \to Ha\{-12\Phi^2 + O(\Phi^3)\}\delta_{ij}. \tag{128}$$

Of course subsequent conformal time derivatives are dominated by the factor(s) of a,

$$\kappa\psi_{00}''|_{l.w.+s.r.} \to H^2a^2\{0 + O(\Phi^3)\}, \tag{129}$$

$$\kappa\psi_{ij}''|_{l.w.+s.r.} \to H^2a^2\{-12\Phi^2 + O(\Phi^3)\}\delta_{ij}. \tag{130}$$

The result is that only a few terms in $\mathcal{D}_1$ and $\mathcal{D}_2$ can contribute at leading order,

$$\kappa\mathcal{D}_1|_{l.w.+s.r.} \to -2\Phi\partial_0^2 + \Phi^2\{-14\partial_0^2 + 18Ha\partial_0 + 6H^2a^2t\}, \tag{131}$$

$$\kappa\mathcal{D}_2|_{l.w.+s.r.} \to -4\Phi^2\partial_0^2 + O(\Phi^3). \tag{132}$$

Because $\dot{\Phi} \ll H\Phi$ we can ignore its time dependence with respect to factors of $a(\eta)$. This makes acting the various factors of ∂^{-2} quite simple in the slow roll approximation,

$$\mathcal{A}_1|_{l.w.+s.r.} \to \frac{1}{a}\frac{1}{\partial^2}a^3(-2\Phi - 2\Phi^2 + O(\Phi^3)), \tag{133}$$

$$\to \frac{-1}{2H^2}\{-2\Phi - 2\Phi^2 + O(\Phi^3)\}. \tag{134}$$

Since the two terms in $\mathcal{A}_2$ cancel to leading order, the result is

$$\mathcal{A}(x)|_{l.w.+s.r.} \to \frac{-1}{2H^2}\{1 - 2\Phi - 2\Phi^2 + O(\Phi^3)\}. \tag{135}$$

B. The invariant observable

Even a scalar changes under coordinate transformation. To achieve full invariance we measure $\mathcal{A}[g](x)$ at a point $Y^\mu[\varphi,g](x)$ which is invariantly specified in terms of the operators φ and $g_{\mu\nu}$. Since there is no spatial dependence in the long wavelength approximation we actually need only fix the 3-surface, $Y^0 = \tau$. Following the suggestion of Unruh, we chose the functional $\tau[\varphi](\eta,\vec{x})$ to make the full scalar agree with its background value at conformal time η,

$$\varphi(\tau(\eta,\vec{x}),\vec{x}) = \varphi_0(\eta). \tag{136}$$

The weak field expansion of $\tau[\varphi]$ is [9]

$$\tau[\varphi](\eta,\vec{x}) = \eta - \frac{\phi(\eta,\vec{x})}{\varphi_0'(\eta)} + \frac{\phi(\eta,\vec{x})\phi'(\eta,\vec{x})}{\varphi_0'^2(\eta)} - \frac{\varphi_0''(\eta)}{2\varphi_0'(\eta)}\left(\frac{\phi(\eta,\vec{x})}{\varphi_0'(\eta)}\right)^2 + O(\phi^3). \tag{137}$$

Substitution of our result (112) for the perturbed scalar gives the final expansion to leading order in the slow roll and long wavelength approximations,

$$\tau|_{l.w.+s.r} \to \eta + \left(\frac{H^2}{-\dot{H}}\right)\frac{\Phi}{Ha} - \left[\left(\frac{H^2}{-\dot{H}}\right)^2 - 7\left(\frac{H^2}{-\dot{H}}\right)\right] \times \frac{\Phi^2}{2Ha} + O(\Phi^3). \tag{138}$$

Our invariant expansion operator is just $\mathcal{A}$ evaluated at this point,

$$\mathcal{A}_{\rm inv}[\varphi,g](\eta,\vec{x}) \equiv \mathcal{A}[g](\tau[\varphi](\eta,\vec{x}),\vec{x}). \tag{139}$$

Note that there is no obstacle in perturbation theory to evaluating an operator at a point $\tau[\varphi] \equiv \eta + \delta\tau[\varphi]$ which is itself an operator,

$$\mathcal{A}_{\rm inv} \equiv \mathcal{A} + \mathcal{A}'\delta\tau + \frac{1}{2}\mathcal{A}''\delta\tau^2 + \cdots. \tag{140}$$

The derivatives are straightforward to evaluate with Eqs. (135) and (111),

$$\mathcal{A}'|_{l.w.+s.r.} \to \frac{-Ha}{2H^2}\left(\frac{-\dot{H}}{H^2}\right)\{2 - 8\Phi + O(\Phi^2)\} \tag{141}$$

$$\mathcal{A}''|_{l.w.+s.r.} \rightarrow \frac{-H^2a^2}{2H^2}\left\{2\left(\frac{-\dot{H}}{H^2}\right)+6\left(\frac{-\dot{H}}{H^2}\right)^2+O(\Phi)\right\}. \tag{142}$$

However, putting everything together results in complete cancellation of all corrections to the order we are working,

$$\mathcal{A}_{\rm inv}=\frac{-1}{2H^2}\{1+\mathcal{O}(\Phi^3)\}. \tag{143}$$

Although this result surprised us, it could have been anticipated by noting that, in the slow roll approximation, our scalar degenerates to a local algebraic function of the Ricci scalar [9],

$$\mathcal{A}[g](x)|_{s.r.} \rightarrow \frac{-6}{R(x)}. \tag{144}$$

Einstein's equations completely determine the Ricci scalar, as an operator, from the local matter stress tensor,

$$R(x) = -8\pi G g^{\mu\nu}T_{\mu\nu}, \tag{145}$$

$$=16\pi G\left\{V(\varphi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right\}. \tag{146}$$

We can always choose to work in a coordinate system for which the full scalar agrees with its background value. When this is done one sees that back reaction can only enter through the kinetic term. Since it is certainly from kinetic effects that the pure gravitational result derives [12], there may well be a significant back reaction from the scalar kinetic term as well. Unfortunately, one can never tell whether there is or not when the kinetic term is systematically neglected, which is just what happens when the slow roll and long wavelength approximations are combined. It follows that our approximations must produce a null result—whether or not there really is significant back reaction—not just at quadratic order in the initial value operators, but at all higher orders as well.

VI. STOCHASTIC SAMPLES

The last section has demonstrated that scalar-driven inflation can show no secular back reaction to leading order in the long wavelength and slow roll approximations. Since this holds as a strong operator equation for our expansion observable, $\mathcal{A}_{\rm inv}$, there is no need to choose between expectation values and stochastic samples. However, we believe the case is still quite strong for secular back reaction when the long wavelength approximation is relaxed, although it would have to come from the coherent superposition of interactions at higher than quadratic order in the initial value operators. The purpose of this section is to develop a theoretical framework for studying such an effect through stochastic samples as recommended by Linde and others [7]. We begin by motivating and defining the basic formalism, then we treat the crucial question of the degree of stochastic fluctuation expected in functionals of the fields such as $\mathcal{A}_{\rm inv}[\varphi,g]$.

A. Motivation and basic formalism

The simplest way to motivate stochastic effects is by considering the linearly independent mode functions of the infrared regime, $Q_{1,I}(\eta,k)$ and $Q_{2,I}(\eta,k)$, which were described in Sec. III B. The *full* linearized field operator must involve both of these mode functions in order to avoid commuting with its conjugate momentum. However, after horizon crossing ($k=Ha$) one of the mode functions becomes vastly larger than the other. This is the phenomenon of superadiabatic amplification and it corresponds to the simple physical picture of particle production through infrared virtual quanta becoming trapped in the inflationary Hubble flow. If one retains just the larger mode function then the linearized operator which results is effectively classical in that it commutes with its time derivative. Although the result is still probabilistic, one can simultaneously measure the value of such an operator and its time derivative, and subsequent measurements will show only the classical evolution expected from these initial values. This is the reflection, in the Heisenberg picture, of a "squeezed state."

The expectation value of a functional of squeezed operators can fail to provide a good estimate of what an actual observer sees. For example, the expectation value of the current stress tensor is presumably homogeneous and isotropic. We see nothing of the sort because we exist at the end of a long period of essentially classical evolution from one particular choice, random but definite, of superadiabatically amplified density perturbations.

A better way of treating squeezed operators is to sample the result of making random but definite choices for them from the relevant quantum mechanical wave function, and then evolving classically. It is crucial to understand that taking such a stochastic sample is perfectly consistent with the use of quantum field theory to express the Heisenberg operators as functionals of the unconstrained initial value operators. Nor is there any change in how the observable $\mathcal{A}_{\rm inv}(\eta,\vec{x})$ depends upon the Heisenberg operators. (Of course we do want to avoid making the long wavelength approximation!) What changes is that the scalar creation and annihilation operators—$Y^\dagger(\vec{k})$ and $Y(\vec{k})$—are random numbers up to $k=H(\eta)a(\eta)$, and zero beyond.

To avoid problems with continuum normalization we take the 3-manifold to be T^3 with identical co-moving coordinate radii of $H_0^{-1}\equiv H^{-1}(\eta_0)$. (Because the conformal coordinate volume is so restricted during inflation the integral approximation is excellent for mode sums and there is no conflict with any of our previous, continuously normalized, results.) On this manifold the co-moving wave vectors become discrete,

$$\vec{k}=2\pi H_0\vec{n}. \tag{147}$$

Phase space integrals are converted into mode sums in the usual way,

$$\int \frac{d^3k}{(2\pi)^3} f(\vec{k}) \rightarrow H_0^3 \sum_{\vec{n}} f(2\pi H_0 \vec{n}). \tag{148}$$

And the Dirac delta function goes into a Kronecker one,

$$(2\pi)^3 \delta^3(\vec{k}-\vec{k}') \rightarrow H_0^{-3} \delta_{\vec{n},\vec{n}'}. \tag{149}$$

Because the initial scalar state is free, the associated creation and annihilation operators are stochastically realized as independent, complex, Gaussian random variables with standard deviation H_0^{-3}. It is convenient to scale out the dimensions,

$$A_{\vec{n}} \equiv H_0^{3/2} Y(2\pi H_0 \vec{n}), \quad A_{\vec{n}}^* \equiv H_0^{3/2} Y^\dagger(2\pi H_0 \vec{n}), \tag{150}$$

so that the probability density for each mode $\vec{n}$ has a simple expression,

$$\rho(A_{\vec{n}}, A_{\vec{n}}^*) = \frac{1}{2\pi} e^{-A_{\vec{n}} A_{\vec{n}}^*}. \tag{151}$$

Since the modes are independent, the joint probability distribution is just the product of Eq. (151) over the relevant range of $\vec{n}$.

B. Functionals of stochastic variables

Our observable $\mathcal{A}_{\rm inv}(\eta,\vec{x})$ depends in a complicated way upon the scalar and metric fields, which are themselves functionals of the stochastic variables $A_{\vec{n}}$ and $A_{\vec{n}}^*$. The nature of this dependence determines the crucial issues of whether or not $\mathcal{A}_{\rm inv}(\eta,\vec{x})$ is well represented by its expectation value and whether or not a definite sign can be inferred for corrections to the cosmological expansion rate. Of course we do not yet have a replacement for the long wavelength approximation which shows secular back reaction. However, we do here develop a technique for separating nonlinear functionals of the stochastic variables into a part whose percentage fluctuation becomes negligible for a long period of inflation, plus another part whose fluctuation is not negligible but which has a definite sign.

Recall from probability theory that a functional $f = F[A,A^*]$ of random numbers is itself a random number. Its probability distribution function descends from that of the $A_{\vec{n}}$ and $A_{\vec{n}}^*$ by Fourier transformation,

$$\rho(f) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikf} \langle e^{-ikF[A,A^*]} \rangle, \tag{152}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikf} \left(\prod_{\vec{n}} \int \frac{dA_{\vec{n}} dA_{\vec{n}}^*}{2\pi} e^{-A_{\vec{n}} A_{\vec{n}}^*} \right) \times e^{-ikF[A,A^*]}. \tag{153}$$

As an example, consider the Newtonian potential (110),

$$\Phi(\eta,\vec{x}) = \frac{\kappa\sqrt{-H_0 \dot{H}(\eta)}}{4\pi} \sum_{\vec{n}} \frac{1}{n} \{ Q_C(\eta, 2\pi H_0 n) \times e^{i2\pi H_0 \vec{n}\cdot\vec{x}} A_{\vec{n}} + \text{c.c.} \}. \tag{154}$$

The various integrations are trivial Gaussians, as they would be for any linear variable. The result is that Φ follows a Gaussian distribution with mean zero,

$$\rho(\Phi) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\Phi^2/2\sigma^2}, \tag{155}$$

and a spatially constant variance equal to

$$\sigma^2(\eta) = \frac{-\kappa^2 H_0 \dot{H}}{8\pi^2} \sum_{\vec{n}} \frac{1}{n^2} \| Q_C(\eta, 2\pi H_0 n) \|^2. \tag{156}$$

Making the slow roll and long wavelength approximations we see that the variance grows as the logarithm of the scale factor,

$$\sigma^2(\eta)|_{l.w.+s.r.} \rightarrow \frac{\kappa^2}{32\pi^3} \left(\frac{-\dot{H}}{H^2} \right)^2 \sum_{\vec{n}} \frac{H_*^2}{n^3} \left(\frac{H_*^2}{-\dot{H}_*} \right), \tag{157}$$

$$\rightarrow \frac{\kappa^2}{8\pi^2} \left(\frac{-\dot{H}}{H^2} \right)^2 \int_1^a \frac{dn}{n} \left(\frac{H_*^4}{-\dot{H}_*} \right), \tag{158}$$

$$\rightarrow \frac{\kappa^2 H^2}{8\pi^2} \left(\frac{-\dot{H}}{H^2} \right) \ln[a(\eta)]. \tag{159}$$

The simplest sort of nonlinear functional is just the square of a linear one. Since linear functionals of $A_{\vec{n}}$ and $A_{\vec{n}}^*$ are always Gaussian, their squares always follow a chi-squared distribution whose mean is the variance of the Gaussian and whose variance is twice the square of this. For example, the variable $\Phi^2(\eta,\vec{x})$ follows a chi-squared with mean σ^2 and standard deviation $\sqrt{2}\sigma^2$. Although the fluctuations of Φ^2 are of the same order as its mean, the sign is definite. Note also that, for a long period of inflation, only an incredibly fortuitous sequence of choices for the stochastic variables $A_{\vec{n}}$ and $A_{\vec{n}}^*$ would result in Φ^2 having a value significantly below some constant times our estimate (159). So that if some reliable approximation scheme should wind up giving the effective expansion rate as

$$H_{\rm eff}(\eta,\vec{x}) = H(\eta)\{1 + \Phi(\eta,\vec{x}) - \Phi^2(\eta,\vec{x})\}, \tag{160}$$

then the conclusion would be that there is only a vanishingly small probability to observe anything except a secular slowing of inflation. This is an example of how stochastic samples might show the same qualitative results as expectation values while breaking exact homogeneity and isotropy and altering the numerical coefficient of the order of magnitude estimate.

It might be thought that quantitative results are not obtainable for more complicated nonlinear functionals. That useful statements can still be made derives from the following fact: *any secular back reaction effect must result from the coherent superposition of contributions from an enormously large number of modes*. This allows us to exploit the same sorts of simplifications that underlie statistical mechanics.

To illustrate the important considerations without becoming too mired in technical detail let us consider quadratic superpositions of the form

$$s=S[A,A^*]\equiv\frac{1}{2}\sum_{\vec{m},\vec{n}}S_{\vec{m}\vec{n}}(A_{\vec{m}}+A_{\vec{m}}^*)(A_{\vec{n}}+A_{\vec{n}}^*). \quad (161)$$

The characteristic function is

$$\langle e^{-ikS[A,A^*]}\rangle=\frac{1}{\sqrt{\det(I+2ikS)}}, \quad (162)$$

where S stands for the symmetric matrix $S_{\vec{m}\vec{n}}$ and I is the unit matrix of the same rank. The various moments of s follow by differentiation,

$$\langle s\rangle=\mathrm{Tr}[S],\quad \langle(s-\mathrm{Tr}[S])^2\rangle=2\,\mathrm{Tr}[S^2], \quad (163)$$

$$\langle(s-\mathrm{Tr}[S])^3\rangle=4\,\mathrm{Tr}[S^3],\quad \dots . \quad (164)$$

How much fluctuation one should expect is governed by the relation between traces of powers of the matrix S. We distinguish two cases:

(1)“Local” superpositions which are characterized by $\mathrm{Tr}[S^n]=(\mathrm{Tr}[S])^n$ for $n\geqslant 2$; and

(2)“Nonlocal” superpositions which obey $\mathrm{Tr}[S^2]\ll(\mathrm{Tr}[S])^2$ and $\mathrm{Tr}[S^n]\ll\mathrm{Tr}[S^2](\mathrm{Tr}[S])^{n-2}$ for $n\geqslant 3$.

The local case is just the square of a linear superposition and has already been considered. To recapitulate, it has significant fluctuation but definite sign. The distribution for a nonlocal superposition can be approximated by dropping higher traces of S in the exponent of Eq. (153),

$$\rho(s)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{iks-(1/2)\mathrm{Tr}[\ln(I+2ikS)]}, \quad (165)$$

$$\approx\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{iks-ik\mathrm{Tr}[S]-k^2\mathrm{Tr}[S^2]}, \quad (166)$$

$$=\frac{1}{\sqrt{4\pi\mathrm{Tr}[S^2]}}\exp\left[-\frac{(s-\mathrm{Tr}[S])^2}{4\,\mathrm{Tr}[S^2]}\right]. \quad (167)$$

This is just a Gaussian centered on $\mathrm{Tr}[S]$ with variance $2\,\mathrm{Tr}[S^2]$. Since its standard deviation is insignificant compared with the mean, stochastic effects are not important.

Recall that invariant measures of the cosmological expansion rate derive their dependence upon $A_{\vec{n}}$ and $A_{\vec{n}}^*$ from two expansions. In the first the observable is expanded in powers of the perturbed fields, then one substitutes the expansions of the perturbed fields in powers of the stochastic initial value data. Both of these expansions really involve averages over the past light cone of the observation point. However, depending upon the observable and the approximation techniques used to solve for the fields, the terrific expansion of spacetime during inflation may weight the averages heavily towards the most recent times. In that case one gets a local superposition and stochastic effects are important but simple to include. An improved observable can also be defined so that the entire past light cone participates effectively [9], and it seems likely that improving on the long wavelength approximation will have this result anyway. In this case one tends to get a nonlocal superposition because modes with different $\vec{n}$ interfere destructively. The stochastic fluctuation of back reaction would then be negligible and one may as well resort to expectation values.

There is also the possibility of mixing of local and nonlocal, in which case the best strategy is to separate the two effects and treat them as above. One might anticipate that analytic considerations would render the local part obvious but even if not, its dyadic form makes the decomposition a simple linear algebra problem. For example, suppose that at late times we have

$$\mathrm{Tr}[S^n]\to(\alpha\mathrm{Tr}[S])^n, \quad (168)$$

for some positive constant $\alpha<1$. The putative decomposition would be

$$S_{\vec{m}\vec{n}}=u_{\vec{m}}u_{\vec{n}}+\Delta S_{\vec{m}\vec{n}}, \quad (169)$$

where the various traces obey

$$u\cdot u\to\alpha\mathrm{Tr}[S],\quad \mathrm{Tr}[\Delta S]\to(1-\alpha)\mathrm{Tr}[S],$$

$$\mathrm{Tr}[\Delta S^n]\ll(\mathrm{Tr}[S])^n. \quad (170)$$

Simply pick any mode function $v_{\vec{n}}$ with nonzero overlap—it need not be close to $u_{\vec{n}}$—then contract into S^2 and divide by the trace of S,

$$\frac{(S^2v)_{\vec{n}}}{\mathrm{Tr}[S]}=\alpha u\cdot v\,u_{\vec{n}}+\frac{u^t\Delta Sv}{\mathrm{Tr}[S]}u_{\vec{n}}+\frac{u\cdot v}{\mathrm{Tr}[S]}(\Delta Su)_{\vec{n}}+\frac{(\Delta S^2v)_{\vec{n}}}{\mathrm{Tr}[S]}. \quad (171)$$

Only the first term can matter at late times so we recover $u_{\vec{n}}$ by normalizing and then multiplying by the square root of $\alpha\mathrm{Tr}[S]$.

VII. SUMMARY AND DISCUSSION

We have calculated the gravitational back reaction on scalar-driven inflation using an invariant observable, to quadratic order in the initial creation and annihilation operators and to leading order in the long wavelength and slow roll approximations. No effect was found, contrary to previous work by ourselves and others which indicated a secular slowing of the expansion rate at this order [2–4]. It is significant that the inclusion of stochastic effects played no role in this change. Under our approximations, secular back reaction would enter through $\Phi(x)$, the accumulated Newtonian potential from modes which have redshifted into the infrared regime. However, all terms of order Φ and Φ^2 drop out of the Heisenberg operator $\mathcal{A}_{\rm inv}$, before one has to choose between the alternatives of expectation values or stochastic samples.

Another improvement in our analysis which had nothing to do with changing the result was the use of a scalar to measure the expansion rate. The entire difference from previous work is in fact attributable to defining the coordinate system so as to make the quantum scalar vanish on surfaces of simultaneity. When these coordinates are used even the gauge fixed expectation value of the metric fails to show significant back reaction at one loop.

The null result was anticipated by Unruh who noticed that scalar mode solutions become pure gauge in the long wavelength limit [6]. This does not preclude back reaction but it does rule out dependence upon Φ which fails to vanish in the long wavelength limit. For example, spatial and certain temporal derivatives of $\Phi(x)$ can contribute. In fact there are such contributions but they are negligible at quadratic order.

Although we have obtained a discouraging result about the possibility of a simple one-loop effect, our analysis does not invalidate the idea of gravitational back reaction on inflation. It would be fairer to say that what we have learned constrains the form any such effect can take. In particular one should not take the long wavelength limit. It is highly significant, in this regard, that the purely gravitational effect claimed at two loops [12] does not involve the long wavelength approximation in any way because the locally de Sitter background is simple enough that the full propagator can be worked out. In fact the gravitational response to the inflationary production of gravitons comes entirely from the graviton kinetic energy and would vanish in the long wavelength limit.

The long wavelength approximation was also avoided in the effect claimed at three loops for massless, minimally coupled φ^4 theory in a locally de Sitter background [13]. In a subsequent paper [14] we resolve the issue of coincident propagators in this model by a procedure of covariant normal ordering. The resulting theory exhibits a secular back reaction which slows inflation in a manner that is unaltered by either the use of an invariant operator to measure expansion or by the inclusion of stochastic effects.

Although there was no need to consider stochastic effects in the present work, Sec. VI describes a formalism for including them in higher order processes which may show secular back reaction. This formulation differs from the standard one [8] in three ways. First, the focus is perturbative and local whereas previous previous treatments have been concerned with nonperturbative effects on the global geometry. A second difference is that past treatments incorporated stochastic degrees of freedom only as they experienced superadiabatic amplification. Although this is doubtless an excellent approximation for the sorts of global and nonperturbative issues that were being studied, we cannot afford to ignore the non-conservation of stress-energy implicit in continually injecting new degrees of freedom into the system. Therefore, the creation and annihilation operators for any mode we wish to treat stochastically are considered to be nonzero random numbers even on the initial value surface. The final difference is that we enforce all the perturbative field equations of the Einstein-scalar system so that the various modes are in gravitational interaction even on the initial value surface.

An amusing consequence of all this is that stochastic effects provide a nonperturbative proof that back reaction must eventually become significant if nothing else stops inflation first. For it will be noted that, in our version, including stochastic effects at some time η corresponds to a universe that began inflation at η_0 with a random collection of modes excited up to co-moving wave number $k=H(\eta)a(\eta)$. The hypothesis that inflation *never* slows amounts to the assumption that inflation can begin with the inital state populated to arbitrarily high wave number. Of course this is nonsense. Even if one subtracts off the spatially averaged energy density—as should probably be done—what must actually happen for arbitrarily high excitations is that random inhomogeneities produce a gravitational collapse.

Finally, we comment on the degree to which a random stochastic sample should be expected to differ from its mean. Because secular back reaction must manifest itself through the coherent superposition of an enormously large number of independent random variables, one can sometimes employ the methods of statistical mechanics. The exception is when back reaction involves an ordinary function of the local stochastic fields, for example $\Phi^2(x)$. This can happen if the terrific inflationary expansion causes the average over the past light cone to be weighted so that only interactions just before the observation contribute effectively. In that case stochastic fluctuation is not negligible, but the sign of the effect is definite. In the other, "nonlocal" case, stochastic fluctuation is negligible compared with the mean effect and one may as well use expectation values. Our suspicion is that improving upon the long wavelength approximation and improving on the expansion observable will result in secular back reaction of the nonlocal sort. However, if there should be local mixing, it is straightforward to untangle.

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APPENDIX

In this appendix we derive the infrared limits of the generalized Green's functions of Sec. III D. First notice that from Eq. (30) and the background identities we can derive the following useful "commutation" relations:

$$\partial_0\hat{\mathcal{D}}_B-\hat{\mathcal{D}}_B\partial_0=\frac{\kappa^2}{2}\varphi_0'\varphi_0'', \tag{A1}$$

$$\varphi_0'\hat{\mathcal{D}}_B-\hat{\mathcal{D}}_C\varphi_0'=2\varphi_0'\partial_0\frac{\varphi_0''}{\varphi_0'},$$

$$\varphi_0'\hat{\mathcal{D}}_C-\hat{\mathcal{D}}_B\varphi_0'=2\frac{\varphi_0''}{\varphi_0'}\partial_0\varphi_0'. \tag{A2}$$

$$\frac{1}{\varphi_0'}\hat{\mathcal{D}}_B-\hat{\mathcal{D}}_C\frac{1}{\varphi_0'}=-2\frac{\varphi_0''}{\varphi_0'^2}\partial_0,$$

$$\frac{1}{\varphi_0'}\hat{\mathcal{D}}_C-\hat{\mathcal{D}}_B\frac{1}{\varphi_0'}=-2\partial_0\frac{\varphi_0''}{\varphi_0'^2}. \tag{A3}$$

Using Eqs. (30) and (31) for $\hat{\mathcal{D}}_B$ and $\hat{\mathcal{D}}_C$ we have also the relations

$$\hat{\mathcal{D}}_B\frac{1}{\varphi_0'^2}\partial_0\varphi_0'=\partial_0\frac{1}{\varphi_0'}\hat{\mathcal{D}}_C-2\frac{\varphi_0''}{\varphi_0'^2}\nabla^2, \tag{A4}$$

$$\hat{\mathcal{D}}_C\frac{1}{\varphi_0'}\partial_0=\varphi_0'\partial_0\frac{1}{\varphi_0'^2}\hat{\mathcal{D}}_B+2\frac{\varphi_0''}{\varphi_0'^2}\nabla^2. \tag{A5}$$

Dropping spatial derivatives and inverting implies the following "inverse commutation" relations:

$$\frac{1}{\varphi_0'}\partial_0\varphi_0'\mathcal{D}_C^{-1}\varphi_0'=\varphi_0'\mathcal{D}_B^{-1}\partial_0\ ,\ \frac{1}{\varphi_0'}\partial_0\mathcal{D}_B^{-1}\varphi_0'=\mathcal{D}_C^{-1}\varphi_0'\partial_0\frac{1}{\varphi_0'}, \tag{A6}$$

Substituting Eq. (A6) into the right-hand side of Eq. (76) we see that the $1/\nabla^2$ term cancels because

$$\partial_0\mathcal{D}_B^{-1}\partial_0-\frac{\kappa^2}{4}\varphi_0'\mathcal{D}_C^{-1}\varphi_0'$$

$$=\varphi_0'\mathcal{D}_C^{-1}\varphi_0'\partial_0\frac{1}{\varphi_0'^2}\partial_0-\frac{\kappa^2}{4}\varphi_0'\mathcal{D}_C^{-1}\varphi_0', \tag{A7}$$

$$=\varphi_0'\mathcal{D}_C^{-1}\left[-\mathcal{D}_C+\frac{\kappa^2}{4}\varphi_0'^2\right]\frac{1}{\varphi_0'}-\frac{\kappa^2}{4}\varphi_0'\mathcal{D}_C^{-1}\varphi_0'$$

$$=-1. \tag{A8}$$

Similar manipulations reveal that the $1/\nabla^2$ terms cancel as well in the expansions (73)–(75).

The next order terms can be found by expanding $\hat{\mathcal{D}}_I^{-1}$ in powers of the Laplacian. Doing this in expression (76) for $G_{zz}(x;x')$ yields

$$G_{zz}(x;x')=\left[-\partial_0\mathcal{D}_B^{-1}\mathcal{D}_B^{-1}\partial_0+\frac{\kappa^2}{4}\varphi_0'\mathcal{D}_C^{-1}\mathcal{D}_C^{-1}\varphi_0'+O(\nabla^2)\right]\delta^4(x-x'). \tag{A9}$$

The term of order ∇^0 can be simplified with the commutation relations (A1)–(A3),

$$-\partial_0\mathcal{D}_B^{-1}\mathcal{D}_B^{-1}\partial_0+\frac{\kappa^2}{4}\varphi_0'\mathcal{D}_C^{-1}\mathcal{D}_C^{-1}\varphi_0'$$

$$=-\varphi_0'\mathcal{D}_C^{-1}\varphi_0'\partial_0\frac{1}{\varphi_0'^2}\mathcal{D}_B^{-1}\partial_0$$

$$+\frac{\kappa^2}{4}\varphi_0'\mathcal{D}_C^{-1}\frac{1}{\varphi_0'}\frac{1}{\partial_0}\varphi_0'^2\mathcal{D}_B^{-1}\partial_0, \tag{A10}$$

$$=-\varphi_0'\mathcal{D}_C^{-1}\varphi_0'\partial_0\frac{1}{\varphi_0'^2}\mathcal{D}_B^{-1}\partial_0+\varphi_0'\mathcal{D}_C^{-1}\frac{1}{\varphi_0'}\frac{1}{\partial_0}\times(\mathcal{D}_B+\partial_0^2)\mathcal{D}_B^{-1}\partial_0, \tag{A11}$$

$$=-\varphi_0'\mathcal{D}_C^{-1}\left(\frac{1}{\varphi_0'}\partial_0-2\frac{\varphi_0''}{\varphi_0'^2}\right)\mathcal{D}_B^{-1}\partial_0+\varphi_0'\mathcal{D}_C^{-1}\frac{1}{\varphi_0'}+\varphi_0'\mathcal{D}_C^{-1}\frac{1}{\varphi_0'}\partial_0\mathcal{D}_B^{-1}\partial_0, \tag{A12}$$

$$=\varphi_0'\mathcal{D}_C^{-1}\frac{1}{\varphi_0'}+2\varphi_0'\mathcal{D}_C^{-1}\frac{\varphi_0''}{\varphi_0'^2}\mathcal{D}_B^{-1}\partial_0. \tag{A13}$$

The last line, acting of a delta function, gives the long wavelength limit of the zz retarded propagator, Eq. (93). Similar reductions pertain for the other Green's functions of the (f,z) sector, Eqs. (90)–(92).

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