Behavior of varying-alpha cosmologies

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We determine the behavior of a time-varying fine structure "constant" $\alpha(t)$ during the early and late phases of universes dominated by the kinetic energy of changing $\alpha(t)$, radiation, dust, curvature, and lambda, respectively. We show that after leaving an initial vacuum-dominated phase during which α increases, α remains constant in universes such as our own during the radiation era, and then increases slowly, proportional to a logarithm of cosmic time, during the dust era. If the universe becomes dominated by a negative curvature or a positive cosmological constant then α tends rapidly to a constant value. The effect of an early period of de Sitter or power-law inflation is to drive α to a constant value. Various cosmological consequences of these results are discussed with reference to recent observational studies of the value of α from quasar absorption spectra and to the existence of life in expanding universes.

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I. INTRODUCTION

One of the problems that cosmologists have faced in their attempts to assess the astronomical consequences of a time variation in the fine structure constants α has been the absence of an exact theory describing cosmological models in the presence of varying α. Until recently, it has not been possible to analyze the behavior of varying-α cosmologies in the same self-consistent way that one can explore universes with varying G using the Brans-Dicke or more general scalar-tensor theories of gravity. However, we have recently extended the generalization of Maxwell's equations developed by Bekenstein so that this can be done self-consistently. In a recent paper [1] we reported numerical studies of the cosmological evolution of varying-α cosmologies with zero curvature, nonzero cosmological constant, and matter density matching observations. They reveal important properties of varying-α cosmologies that are shared by other theories in which "constants" vary via the propagation of a causal scalar field obeying 2nd-order differential equations. Their structure can be compared with that of varying speed of light theories developed in Refs. [2–8] and with Kaluza-Klein-like theories in which constants like α vary at the same rate as the mean size of any extra dimensions of space [24].

Recent observations motivate the formulation and detailed investigation of varying-α cosmological theories. The new observational many-multiplet technique of Webb and co-workers, [9,10], exploits the extra sensitivity gained by studying relativistic transitions to different ground states using absorption lines in quasar spectra at medium redshift. It has provided the first evidence that the fine structure constant might change with cosmological time [9–11]. The trend of these results is that the value of α was lower in the past, with $\Delta\alpha/\alpha = -0.72\pm 0.18\times 10^{-5}$ for $z\approx 0.5$–3.5. Other investigations have claimed preferred nonzero values of $\Delta\alpha<0$ to best fit the cosmic microwave background and big bang nucleosynthesis (BBN) data at $z\approx 10^3$ and $z\approx 10^{10}$ respectively [12,13], but these need to be much larger than those needed to reconcile the observations of [9–11].

In this paper we present a detailed analytic and numerical study of the behavior of the cosmological solutions of the varying theory presented in [1]. We shall confine our attention to universes containing dust and radiation but analyze the effects of negative spatial curvature and a positive cosmological constant. Extensions to general perfect-fluid cosmologies can easily be made if required.

II. A SIMPLE VARYING-ALPHA THEORY

The idea that the charge on the electron, or the fine structure constant, might vary in cosmological time was proposed in 1948 by Teller [14], who suggested that $\alpha\propto(\ln t)^{-1}$ was implied by Dirac's proposal that $G\propto t^{-1}$ and the numerical coincidence that $\alpha^{-1}\sim\ln(hc/Gm_{pr}^2)$, where m_{pr} is the proton mass. Later, in 1967, Gamow [15] suggested $\alpha\propto t$ as an alternative to Dirac's time variation of the gravitation constant, G, as a solution of the large numbers coincidence problem but in 1963 Stanyukovich had also considered varying α [16] in this context. It had the advantage of not producing a terrestrial surface temperature above 100 °C in the pre-Cambrian era when life was known to exist. However, this power-law variation in the recent geological past was soon ruled out by other evidence.

There are a number of possible theories allowing for the variation of the fine structure constant, α. In the simplest cases one takes c and $\hbar$ to be constants and attributes variations in α to changes in e or the permittivity of free space (see [3] for a discussion of the meaning of this choice). This is done by letting e take on the value of a real scalar field which varies in space and time (for more complicated cases, resorting to complex fields undergoing spontaneous symmetry breaking, see the case of fast tracks discussed in [7]). Thus $e_0\rightarrow e=e_0\epsilon(x^\mu)$, where ϵ is a dimensionless scalar field and e_0 is a constant denoting the present value of e. This operation implies that some well established assumptions, like charge conservation, must give way [22]. Nevertheless, the principles of local gauge invariance and causality are maintained, as is the scale invariance of the ϵ field (under a suitable choice of dynamics). In addition there is no conflict with local Lorentz invariance or covariance.

With this setup in mind, the dynamics of our theory is then constructed as follows. Since e is the electromagnetic coupling, the ϵ field couples to the gauge field as ϵA_μ in the Lagrangian and the gauge transformation which leaves the action invariant is $\epsilon A_\mu \rightarrow \epsilon A_\mu + \chi_{,\mu}$, rather than the usual $A_\mu \rightarrow A_\mu + \chi_{,\mu}$. The gauge-invariant electromagnetic field tensor is therefore

$$F_{\mu\nu} = \frac{1}{\epsilon}[(\epsilon A_\nu)_{,\mu} - (\epsilon A_\mu)_{,\nu}], \tag{1}$$

which reduces to the usual form when ϵ is constant. The electromagnetic part of the action is still

$$S_{em} = -\int d^4x \sqrt{-g} F^{\mu\nu} F_{\mu\nu}, \tag{2}$$

and the dynamics of the ϵ field are controlled by the kinetic term

$$S_\epsilon = -\frac{1}{2}\frac{\hbar c}{l^2}\int d^4x \sqrt{-g}\frac{\epsilon_{,\mu}\epsilon^{,\mu}}{\epsilon^2}, \tag{3}$$

as in dilaton theories. Here, l is the characteristic length scale of the theory, introduced for dimensional reasons. This constant length scale gives the scale down to which the electric field around a point charge is accurately Coulombic. The corresponding energy scale, $\hbar c/l$, has to lie between a few tens of MeV and the Planck scale, $\sim 10^{19}$ GeV, to avoid conflict with experiment.

Our generalization of the scalar theory proposed by Bekenstein [17] described in Ref. [1] includes the gravitational effects of ψ and gives the field equations

$$G_{\mu\nu} = 8\pi G(T^{matter}_{\mu\nu} + T^{\psi}_{\mu\nu} + T^{em}_{\mu\nu} e^{-2\psi}). \tag{4}$$

The stress tensor of the ψ field is derived from the Lagrangian $\mathcal{L}_\psi = -(\omega/2)\partial_\mu\psi\partial^\mu\psi$ and the ψ field obeys the equation of motion

$$\Box\psi = \frac{2}{\omega}e^{-2\psi}\mathcal{L}_{em} \tag{5}$$

where we have defined the coupling constant $\omega = (\hbar c)/l^2$. This constant is of order ~ 1 if, as in [1], the energy scale is similar to the Planck scale. It is clear that $\mathcal{L}_{em}$ vanishes for a sea of pure radiation since then $\mathcal{L}_{em} = (E^2 - B^2)/2 = 0$. We therefore expect the variation in α to be driven by electrostatic and magnetostatic energy-components rather than electromagnetic radiation.

In order to make quantitative predictions we need to know how much of the non-relativistic matter contributes to the RHS of Eq. (5). This is parametrized by $\zeta \equiv \mathcal{L}_{em}/\rho$, where ρ is the energy density, and for baryonic matter $\mathcal{L}_{em} = E^2/2$. For protons and neutrons ζ_p and ζ_n can be *estimated* from the electromagnetic corrections to the nucleon mass, 0.63 MeV and -0.13 MeV, respectively [29]. This correction contains the $E^2/2$ contribution (always positive), but also terms of the form $j_\mu a^\mu$ (where j_μ is the quarks' current) and so cannot be used directly. Hence we take a guiding value $\zeta_p \approx \zeta_n \sim 10^{-4}$. Furthermore the cosmological value of ζ (denoted ζ_m) has to be weighted by the fraction of matter that is nonbaryonic, a point ignored in the literature [17,18]. Hence, ζ_m depends strongly on the nature of the dark matter and can take both positive and negative values depending on which of Coulomb-energy or magnetostatic energy dominates the dark matter of the Universe. It could be that $\zeta_{CDM} \approx -1$ (superconducting cosmic strings, for which $\mathcal{L}_{em} \approx -B^2/2$), or $\zeta_{CDM} \ll 1$ (neutrinos). BBN predicts an approximate value for the baryon density of $\Omega_B \approx 0.03$ with a Hubble parameter of $h_0 \approx 0.6$, implying $\Omega_{CDM} \approx 0.3$. Thus depending on the nature of the dark matter ζ_m can be virtually anything between -1 and $+1$. The uncertainties in the underlying quark physics and especially the constituents of the dark matter make it difficult to impose more certain bounds on ζ_m.

We should not confuse this theory with other similar variations. Bekenstein's theory [17] does not take into account the stress energy tensor of the dielectric field in Einstein's equations, and their application to cosmology. Dilaton theories predict a global coupling between the scalar and all other matter fields. As a result they predict variations in other constants of nature, and also a different dynamics for all the matter coupled to electromagnetism. An interesting application of our approach has also recently been made to braneworld cosmology in [19].

III. THE COSMOLOGICAL EQUATIONS

Assuming a homogeneous and isotropic Friedmann metric with expansion scale factor $a(t)$ and curvature parameter k in Eq. (4), we obtain the field equations ($c \equiv 1$)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho_m(1 + |\zeta_m|\exp[-2\psi]) + \rho_r\exp[-2\psi] + \frac{\omega}{2}\dot{\psi}^2\right) - \frac{k}{a^2} + \frac{\Lambda}{3}, \tag{6}$$

where Λ is the cosmological constant. For the scalar field we have the propagation equation

$$\ddot{\psi} + 3H\dot{\psi} = -\frac{2}{\omega}\exp[-2\psi]\zeta_m\rho_m, \tag{7}$$

where $H \equiv \dot{a}/a$ is the Hubble expansion rate. Note that the sign of the evolution of ψ is dependent on the sign of ζ_m. Since the observational data is consistent with a *smaller* value of α in the past, we will in this paper confine our study to *negative* values of ζ_m, in line with our recent discussion in Ref. [1]. The conservation equations for the noninteracting radiation and matter densities are

$$\dot{\rho}_m + 3H\rho_m = 0, \tag{8}$$

$$\dot{\rho}_r + 4H\rho_r = 2\dot{\psi}\rho_r, \tag{9}$$

and so $\rho_m \propto a^{-3}$ and $\rho_r e^{-2\psi} \propto a^{-4}$. If additional noninteracting perfect fluids satisfying the equation of state $p=(\gamma-1)\rho$ are added to the universe then they contribute density terms $\rho \propto a^{-3\gamma}$ to the RHS of Eq. (6) as usual. This theory enables the cosmological consequences of varying e to be analyzed self-consistently rather than by changing the constant value of e in the standard theory to another constant value, as in the original proposals made in response to the large numbers coincidences (see Ref. [20] for a full discussion).

We have been unable to solve these equations in general except for a few special cases. However, as with the Friedmann equation of general relativity, it is possible to determine the overall pattern of cosmological evolution in the presence of matter, radiation, curvature, and positive cosmological constant by matched approximations. We shall consider the form of the solutions to these equations when the universe is successively dominated by the kinetic energy of the scalar field ψ, pressure-free matter, radiation, negative spatial curvature, and positive cosmological constant. Our analytic expressions are checked by numerical solutions of Eqs. (6) and (7).

A. The dust-dominated era

We consider first the behavior of dust-filled universes far from the initial singularity. We assume that $k=0=\Lambda=\rho_\gamma$, so the Friedmann equation (6) reduces to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho_m(1+|\zeta_m|\exp[-2\psi]) + \frac{\omega}{2}\dot{\psi}^2\right), \tag{10}$$

and seek a self-consistent approximate solution in which the scale factor behaves as

$$a = t^{2/3}, \tag{11}$$

$$\frac{d}{dt}(\dot{\psi}a^3) = N\exp[-2\psi], \tag{12}$$

where

$$N \equiv -\frac{2\zeta_m}{\omega}\rho_m a^3 \tag{13}$$

is a positive constant since we have confined ourselves to $\zeta_m < 0$. If we put

$$x = \ln(t)$$

then Eq. (12) becomes

$$\psi'' + \psi' = N\exp[-2\psi] \tag{14}$$

with $N \geqslant 0$ and the prime indicating d/dx. This equation has awkward behavior. For any power-law behavior of the scale factor other than (11) a simple exact solution of Eq. (12) exists. However, the late-time dust solutions are exceptional, reflecting the coupling of the charged matter to the variations in ψ, and are approximated by the following asymptotic series:

$$\psi = \frac{1}{2}\ln[2Nx] + \sum_{n=1}^{\infty} a_n x^{-n}. \tag{15}$$

To see this, substitute this in the evolution equation (14) for ψ; then it becomes

$$-\frac{1}{2x^2} + \sum_{n=1} n(n+1)a_n x^{-n-2} + \frac{1}{2x} - \sum_{n=1} n a_n x^{-n-1} = \frac{1}{2x}\exp\left[-2\sum_{n=1} a_n x^{-n}\right]. \tag{16}$$

Now we can pick the a_n to cancel out all the terms in x^{-r}, $r \geqslant 2$, on the left-hand side. This requires

$$a_2 = a_1 = -\frac{1}{2}, \quad a_3 = 2a_2, \quad a_4 = 3a_3 = 3\times 2a_2, \quad \text{etc.}$$

Hence

$$\sum_{n=1} a_n x^{-n} = -\frac{1}{2}\left\{\frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{2\times 3}{x^4} + \frac{2\times 3\times 4}{x^5} + \dots + \frac{(r-1)!}{x^r} + \cdots\right\}.$$

All that is left of Eq. (16) is

$$\frac{1}{2x} = \frac{1}{2x}\exp\left[-2\sum_{n=1} a_n x^{-n}\right] \to \frac{1}{2x}$$

as $x \to \infty$. So, at late times, as $x=\ln(t)$ becomes large, we have

$$\psi = \frac{1}{2}\ln\{2N[\ln(t)]\} - \frac{1}{2}\left\{\frac{1}{\ln(t)} + \frac{1}{[\ln(t)]^2} + \frac{2}{[\ln(t)]^3} + \frac{2\times 3}{[\ln(t)]^4} + \frac{2\times 3\times 4}{[\ln(t)]^5} + \cdots + \frac{(r-1)!}{(\ln(t))^r} + \cdots\right\}; \tag{17}$$

also, since $\alpha = \exp[2\psi]$ we have, as $t \to \infty$,

$$\alpha = 2N\ln(t)\times\exp\left[-\frac{1}{\ln(t)} - \frac{1}{[\ln(t)]^2} - \frac{2}{[\ln(t)]^3} - \frac{2\times 3}{[\ln(t)]^4} - \frac{2\times 3\times 4}{[\ln(t)]^5} - \dots - \frac{(r-1)!}{[\ln(t)]^r} - \cdots\right]. \tag{18}$$

So, to leading order, we have

$$\alpha \sim 2N\ln(t)\exp\left[-\frac{1}{\ln(t)}\right]. \tag{19}$$

The nonanalytic $\exp[1/x]$ behavior shows why Eq. (14), despite looking simple, has awkward behavior. We can simplify the asymptotic series (18) a bit further because we know from the definition of the logarithmic integral function $\mathrm{li}(x)=\int_0^x dt/\ln(t)=\mathrm{Ei}[\ln(x)]$ that as $x\to\infty$

$$\mathrm{li}(x)\sim\exp[x]\sum_{n=0}^{\infty}\frac{n!}{x^{n+1}} \tag{20}$$

so the series we have in Eq. (17) in $\{\cdots\}$ brackets is

$$\sum_{r=1}^{\infty}\frac{(r-1)!}{x^r}\sim\exp[-x]\mathrm{li}(\exp[x]), \tag{21}$$

and so asymptotically

$$\psi=\frac{1}{2}\ln[2Nx]-\frac{1}{2}\exp[-x]\mathrm{li}(\exp[x]). \tag{22}$$

Hence, as $t\to\infty$,

$$\psi=\frac{1}{2}\ln[2N\ln(t)]-\frac{1}{2t}\mathrm{li}(t)=\frac{1}{2}\ln[2N\ln(t)]-\frac{1}{2t}\mathrm{Ei}[\ln(t)] \tag{23}$$

and so asymptotically

$$\alpha=\exp[2\psi]=2N\exp[-t^{-1}\mathrm{li}(t)]\ln t. \tag{24}$$

This asymptotic behavior is confirmed by solving Eqs. (6)–(9) numerically for $\rho_m\gg\rho_r,\rho_\psi$. By using a range of initial values for ψ we produce the plot in Fig. 1, in which the asymptotic solution is clearly approached.

We need to check that the original assumption of $a=t^{2/3}$ in the Friedmann equation (6) is self-consistent. The relevant terms are

$$\rho_m(1+|\zeta_m|\exp[-2\psi])+\frac{\omega}{2}\dot{\psi}^2. \tag{25}$$

The $\exp[-2\psi]=\alpha^{-1}$ falls off as $t\to\infty$ so the $\rho_m(1+|\zeta_m|\times\exp[-2\psi])\propto a^{-3}$ term dominates as expected. For the kinetic term $\dot{\psi}^2$ we have

$$\dot{\psi}=\frac{1}{t}\times O\left(\frac{1}{\ln(t)}\right) \tag{26}$$

and so again the $\dot{\psi}^2$ term falls off faster than t^{-2} as $t\to\infty$ and the $a=t^{2/3}$ behavior is an ever-improving approximation at late times. If we examine the form of the solution (24) we see that α always *increases* with time as a logarithmic power until it grows sufficiently for the exponential term on the right-hand side of Eq. (7) to affect the solution significantly and slow the rate of increase by the series terms. The rate at which α grows is controlled by the total density of matter in the model, which is directly proportional to the constant N, defined by Eq. (13). The higher the density of matter (and hence N) the faster the growth in α. However, because of the logarithmic time variation, the dependence on ρ_m, ω, and ζ_m is weak. The self-consistency of the usual $a=t^{2/3}$ dust evolution for the scale factor leaves the standard cosmological tests unaffected. This is just as one expects for the very variations indicated by the observations of [11].

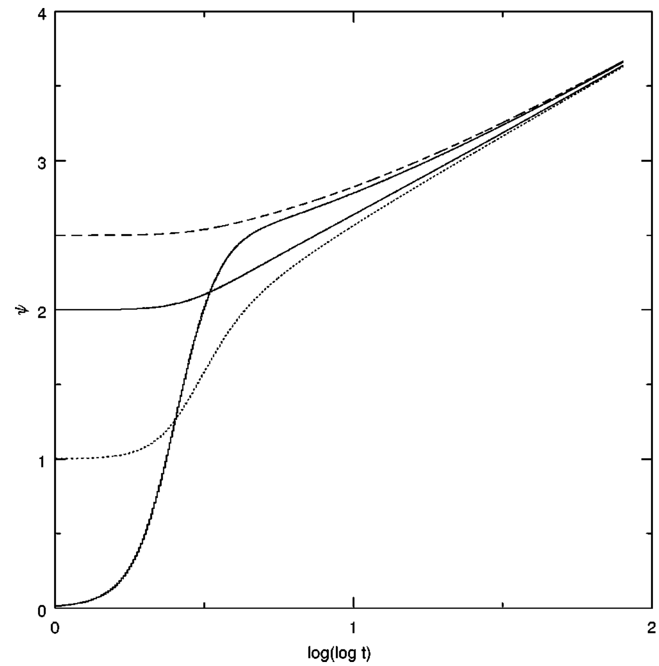

FIG. 1. Numerical solution to the equations in the dust-dominated epoch. ψ is plotted against log(log$t$), with initial conditions ψ=0, 1, 2, 2.5. The numerical solution clearly approaches the asymptotic solution, Eq. (19), in the expected manner. The time is plotted in Planck units of 10^{-43} s.

B. The radiation-dominated era

In the radiation era we assume $k=\Lambda=0$ and take $a=t^{1/2}$ as the leading order solution to Eq. (6). We must now solve

$$\frac{d}{dt}(\dot{\psi}a^3)=N\ \exp[-2\psi]. \tag{27}$$

There is a simple particular exact solution

$$\psi=\frac{1}{2}\ln(8N)+\frac{1}{4}\ln(t). \tag{28}$$

Consider a perturbation of this solution by $f(t)$:

$$\psi=\frac{1}{2}\ln(8N)+\frac{1}{4}\ln(t)+f(t).$$

Inserting this in Eq. (27) we then get

$$\ddot{f}+\frac{3}{2t}\dot{f}=\frac{1}{8t^2}(\exp[-2f]-1). \tag{29}$$

Let us first consider the case of a large perturbation, $\exp(-2f)\ll1$. The RHS of Eq. (29) then reduces to $-1/(8t^2)$, and through a straightforward integration we get

$$\dot{f} = -\frac{1}{4t} + Ct^{-3/2} \tag{30}$$

with C an arbitrary constant. As t increases this will approach $-1/(4t)$ which has the same absolute value and is opposite in sign to the derivative of the exact solution (28). Thus for values of ψ much higher than this solution $\dot{\psi}$ is zero. ψ will stay constant until the perturbation f becomes small and ψ approaches the exact solution (28).

To establish the stability of the exact solution we need to consider small perturbations around it. For small f we have

$$\ddot{f} + \frac{3}{2t}\dot{f} + \frac{1}{4t^2}f = 0. \tag{31}$$

Hence,

$$f = \frac{1}{t}\{A\sin[\sqrt{3}\ln(t)] + B\cos[\sqrt{3}\ln(t)]\}. \tag{32}$$

Thus, we have

$$\psi \to \frac{1}{2}\ln(8N) + \frac{1}{4}\ln(t) + \frac{1}{t}\{A\sin[\sqrt{3}\ln(t)] + B\cos[\sqrt{3}\ln(t)]\}, \tag{33}$$

$$\alpha = e^{2\psi} \to 8Nt^{1/2}\exp\left[\frac{2}{t}\{A\sin[\sqrt{3}\ln(t)] + B\cos[\sqrt{3}\ln(t)]\}\right] \to 8Nt^{1/2} \tag{34}$$

as $t\to\infty$.

We need to check that the $\dot{\psi}^2$ term does not dominate as $t\to\infty$. We have

$$\dot{\psi} \sim \frac{1}{4t} + \frac{1}{t^2}\times \text{oscillations}. \tag{35}$$

Thus the $\dot{\psi}^2$ term is of the *same order* of t as the radiation density term if we assume $a\sim t^{1/2}$. Also, the matter density term $\rho_m(1+|\zeta_m|\exp[-2\psi])\sim\rho_m\exp[-2\psi]\sim a^{-3}\exp[-2\psi]\sim t^{-3/2}\times t^{-1/2}\sim t^{-2}$ is of the same order of time variation as the radiation-density term because of the variation in α. The assumption $a=t^{1/2}$ is still good asymptotically but there is an algebraic constraint from the Friedmann equation (6).

Evaluating the terms in Eq. (6), we have

$$\frac{1}{4t^2} = \frac{8\pi G}{3}\left(\frac{M}{t^{3/2}}\left[1+\frac{S}{8Nt^{1/2}}\right]\right) + \frac{\Gamma}{t^2} + \frac{\omega}{32t^2} \tag{36}$$

where $\rho_m = Ma^{-3}$, $\rho_\gamma\exp[-2\psi]=\Gamma a^{-4}$, $N=-2M\zeta_m\omega^{-1}$, S is a constant, and we have $\rho_m/\omega\sim 0.01\%$. So, to $O(t^{-2})$, we have the algebraic constraint

$$\frac{1}{4} = \frac{8\pi G}{3}\left[\frac{3\omega}{32} + \Gamma\right].$$

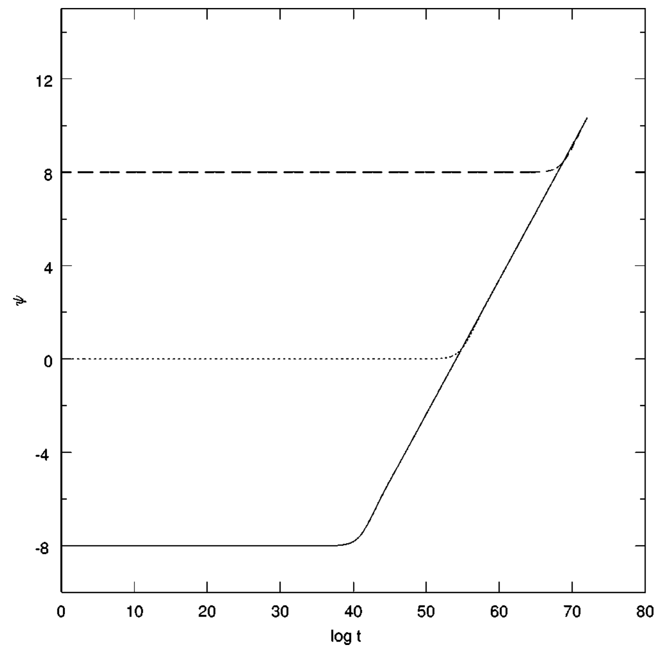

FIG. 2. Numerical solution to the equations in the radiation-dominated epoch given different initial conditions. The particular exact solution, Eq. (28), is eventually reached in all cases. The time is plotted in units of the Planck time.

This generalizes the familiar general relativity $(\omega=0)$ radiation universe case where we have $\Gamma = 3/32\pi G$.

Again, the asymptotic behavior in Eqs. (33), (34), and the approach to the exact solution (28), can be confirmed by numerical solutions to Eqs. (6)–(9) in the case of radiation domination. The results from runs with initial values for $\psi=-8$, 0, 8, $\dot{\psi}=0$ and the same value for N are shown in Fig. 2. The particular solution (28) is clearly an attractor. It is also seen that if the system starts off with values higher than $1/2\ln(8N)$, ψ will stay constant until it reaches the value of the solution, as predicted above. In cosmological models containing matter and radiation with densities given by those observed in our universe this is the case, as seen in the computations shown in Ref. [1]. Hence, during the radiation era α remains approximately constant until the dust era begins.

This analysis can easily be extended to other equations of state. If the Friedmann equation contains a perfect fluid with equation of state $p=(\gamma-1)\rho$ with $\gamma\neq 0$, 1, 2 then there is a late time solution of Eqs. (6) and (7) of the form

$$a = t^{2/3\gamma}, \tag{37}$$

$$\psi = \frac{1}{2}\ln\left[\frac{N\gamma^2}{(\gamma-1)(2-\gamma)}\right] + \left(\frac{\gamma-1}{\gamma}\right)\ln(t) \tag{38}$$

which reduces to Eq. (28) when $\gamma=4/3$. This solution exists only for fluids with $1<\gamma<2$.

C. The curvature-dominated era

In our earlier study [1] we showed that the evolution of α stops when the universe becomes dominated by the cosmological constant. This behavior also occurs when an open universe becomes dominated by negative spatial curvature. In a curvature-dominated era we assume that Eq. (6) has the Milne universe solution with

$$a=t. \tag{39}$$

We must now solve Eq. (27) again. It has the form

$$\frac{d}{dt}(\dot{\psi}t^3)=N\exp[-2\psi]. \tag{40}$$

We seek a solution of the form

$$\psi=\frac{1}{2}+f(t). \tag{41}$$

Hence, for small f

$$\ddot{f}+\frac{3}{t}\dot{f}+\frac{2N}{t^2}f=0. \tag{42}$$

Solutions exist with $f\propto t^n$ and

$$n=-1\pm\sqrt{1-2N}. \tag{43}$$

Since $N>0$ we see that the real part of n is always decaying and so

$$\psi\rightarrow \mathrm{const} \tag{44}$$

as $t\rightarrow\infty$. Thus, as $t\rightarrow\infty$ we have

$$\alpha\sim\alpha_\infty \exp[2At^{-1\pm\sqrt{1-2N}}], \tag{45}$$

where α_∞ and A are constants.

Again we need to check that the $\dot{\psi}^2$ term does not come to dominate. We have $\dot{\psi}^2\sim t^{2(n-1)}$ as $t\rightarrow\infty$ and this always falls faster than $ka^{-2}\propto t^{-2}$ since $n\leqslant 0$, so our approximation is always good. Thus we have shown that in open Friedmann universes α rapidly approaches a constant value after the universe becomes curvature dominated. The rate of approach is controlled by the matter density through the constant N in Eq. (45).

This behavior is again confirmed by numerical solution. Figure 3 shows how alpha changes through the dust epoch and how the change comes to an end as curvature takes over the expansion.

D. The lambda-dominated era

We can prove what was displayed in the numerical results of [1], and again in Fig. 4 for the Λ-dominated era when the value of Λ matches that inferred from recent high redshift supernova observations [21]. At late times we assume the scale factor to take the form

$$a=\exp[\lambda t] \tag{46}$$

where $\lambda\equiv\sqrt{\Lambda/3}$ and so Eq. (6) becomes

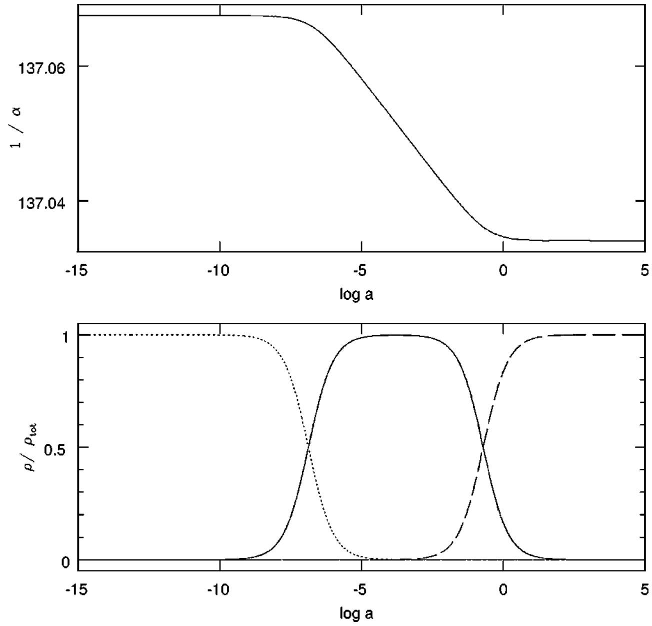

FIG. 3. The top plot shows evolution of α from radiation domination through matter domination and into curvature domination where the change in α comes to an end. The lower plot shows radiation (dotted), matter (solid) and curvature (dashed) densities as fractions of the total energy density .

$$\frac{d}{dt}(\dot{\psi}e^{3\lambda t})=N\exp[-2\psi]. \tag{47}$$

Linearizing in ψ, we have

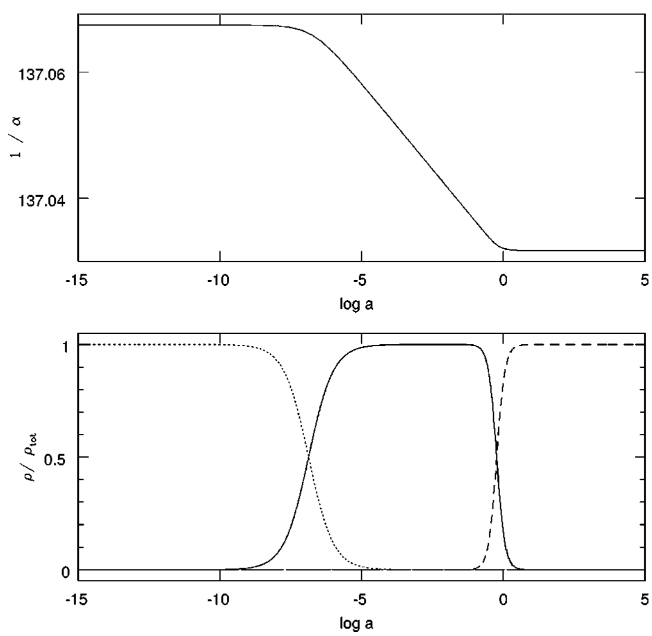

FIG. 4. The top figure shows numerical evolution of α from radiation domination through matter domination and into lambda domination where the change in α comes to an end. The lower plot shows radiation (dotted), matter (solid) and lambda (dashed) densities as fractions of the total energy density.

$$\ddot{\psi}+3\lambda\dot{\psi}=N\exp[-3\lambda t]. \tag{48}$$

Hence,

$$\psi=\psi_0+A\exp[-3\lambda t]-\frac{Nt}{3\lambda}\exp[-3\lambda t]\rightarrow\psi_0 \tag{49}$$

as $t\rightarrow\infty$, where A,ψ_0 are arbitrary constants. Thus α approaches a constant with double-exponential rapidity during a Λ-dominated phase of the universe. The dominant term controlling the late-time approach to the constant solution is proportional to the matter density via the constant N.

E. Inflationary universes

The behavior found for lambda-dominated universes enables us to understand what would transpire during a period of de Sitter inflation during the early stages of a varying-α cosmology. It is straightforward to extend these conclusions to any cosmology undergoing power-law inflation. Suppose the varying-α Friedmann model contains a perfect fluid with $p=(\gamma-1)\rho$ and $0<\gamma<2/3$. The expansion scale factor will increase with $a(t)\propto t^{2/3\gamma}$, while ψ will be governed, to leading order, by

$$(\dot{\psi}t^{2/\gamma})=0. \tag{50}$$

Hence, for large expansion

$$\psi=\psi_0+Dt^{-(2-\gamma)/\gamma}\rightarrow\psi_0 \tag{51}$$

and so ψ and α approach a constant with power-law (exponential) rapidity during any period of power-law (de Sitter) inflation. If we evaluate the kinetic term $O(\dot{\psi}^2)$ in the Friedmann equation and the terms $O(N\exp[-2\psi])$ in the ψ conservation equation, we see that the assumption of $a(t)\propto t^{2/3\gamma}$ is an increasingly good approximation as inflation proceeds. Similar behavior would be displayed by a quintessence field that violated the strong-energy condition and came to dominate the expansion of the universe at late times. It would turn off the time variation of the fine structure constant in the same manner as the curvature of lambda terms discussed above. Note that the ψ field itself is not a possible source of inflationary behavior in these models. We are assuming that the inflation is contributed, as usual, by some other scalar matter field with a self-interaction potential. However, if this field was charged then these conclusions could be altered as the coupling of the inflationary scalar field to the ψ field would be more complicated.

F. The very early universe ($t\rightarrow 0$)

As $t\rightarrow 0$ we expect (just as in Brans-Dicke theory) to encounter a situation where the kinetic energy of ψ dominates the evolution of $a(t)$. This is equivalent to the solution approaching a vacuum solution of Eqs. (6), (7) with $\rho_m=\rho_r=0$, as $t\rightarrow 0$. In the flat case with $\Lambda=0$ (the $k\neq 0$ and $\Lambda\neq 0$ cases can be solved straightforwardly and the models with $\rho_r\neq 0$ can also be solved exactly in parametric form) we have

$$\left(\frac{\dot{a}}{a}\right)^2=\frac{4\pi G\omega}{3}\dot{\psi}^2, \tag{52}$$

$$\ddot{\psi}+3H\dot{\psi}=0. \tag{53}$$

Thus the exact vacuum solution is

$$\psi=\psi_0+\frac{1}{\sqrt{12\pi G\omega}}\ln(t), \tag{54}$$

$$a=t^{1/3}. \tag{55}$$

During this phase the fine structure constant *increases* as a power law of the comoving proper time as t increases:

$$\alpha=\exp[2\psi]\propto t^{1/\sqrt{3\pi G\omega}}. \tag{56}$$

Note that the matter and radiation density terms fall off more slowly than $\dot{\psi}^2\propto t^{-2}$ as $t\rightarrow 0$ and $\exp[-2\psi]\propto t^{-1/(\sqrt{3\pi G\omega})}$. They will eventually dominate the evolution at some later time and the vacuum approximation will break down. As in Brans-Dicke cosmology [23,25] we expect the general solutions of the cosmological equations to approach this vacuum solution as $t\rightarrow 0$ and to approach the other late-time asymptotes discussed above as $t\rightarrow\infty$.

IV. DISCUSSION

The overall pattern of cosmological evolution is clear from the results of the last section even though it is not possible to solve the Friedmann equation exactly in most cases. There are five distinct phases:

(a) Near the initial singularity the kinetic part of the scalar field ψ will dominate the expansion and the universe behaves like a general relativistic Friedmann universe containing a massless scalar or stiff perfect fluid field, with $a=t^{1/3}$. During this "vacuum phase," the fine structure constant increases as a power law in time.

(b) As the universe ages the radiation density will eventually become larger than the kinetic energy of the ψ field. In this radiation dominated epoch, the fine structure constant will approach a specific solution, $\alpha\propto t^{1/2}$, asymptotically. In reality, however, if the initial value of α is much larger than the specific solution, we will have a potentially very long transient period of constant evolution, and the universe may become dust dominated while α is still constant.

(c) After dust domination begins, α slowly approaches an asymptotic solution, $\alpha=2N\ln(t)\times\exp[-t^{-1}\mathrm{li}(t)]$, where $\mathrm{li}(t)$ is the logarithmic integral function. If the universe has zero curvature and no cosmological constant this will approach the late time solution $\alpha\propto\ln(t)$.

(d) If the universe is open then this increase will be brought to an end when the universe becomes dominated by spatial curvature and α will approach a constant. If the curvature is positive the universe will eventually reach an expansion maximum and contract so long as there are no fluids present which violate the strong energy condition. The behavior of closed universes also offers a good approximation to the evolution of bound spherically symmetric density inhomogeneities of large scale in background universes and will be discussed in a separate paper.

(e) If there is a positive cosmological constant, the change in α will be halted when the cosmological constant starts to accelerate the universe. If any other quintessential perfect fluid with equation of state satisfying $p < -\rho/3$ is present in the universe then it will also ultimately halt the change in α when it begins to dominate the expansion of the universe.

To obtain a more holistic picture of the evolution it is useful to string these different parts together. To a good approximation we know that in the vacuum phase from the Planck time t_p until t_v we have

$$a \propto t^{1/3}; \quad \alpha \propto t^A; \quad A = \frac{1}{\sqrt{3\pi G\omega}}. \tag{57}$$

In the radiation era we have α constant until the growth kicks in at a time t_{growth}. The fine structure constant then increases as

$$a \propto \alpha \propto t^{1/2} \tag{58}$$

until t_{eq} when the radiation era ends and dust takes over. However, in universes like our own, this growth era is never reached. Then, in the dust era,

$$\alpha \propto \ln t \tag{59}$$

until the curvature or lambda eras begin at t_c or t_Λ, after which α remains constant until the present, t_0. So, matching these phases of evolution together we can express $\alpha(t_0)$ in terms of $\alpha(t_p)$.

When the universe is open with $\Lambda = 0$,

$$\alpha(t_0) = \alpha(t_p)\left(\frac{t_v}{t_p}\right)^A \left(\frac{t_{eq}}{t_{growth}}\right)^{1/2} \left(\frac{\ln(t_c/t_p)}{\ln(t_{eq}/t_p)}\right), \tag{60}$$

where we have used our logarithmic formula to express ages in Planck time units.

When the universe is flat with $\Lambda > 0$,

$$\alpha(t_0) = \alpha(t_p)\left(\frac{t_p}{t_v}\right)^A \left(\frac{t_{eq}}{t_{growth}}\right)^{1/2} \left(\frac{\ln(t_\Lambda/t_p)}{\ln(t_{eq}/t_p)}\right) \tag{61}$$

and t_c has been replaced by t_Λ.

For the radiation era we consider two extreme cases. We look at a constant α scenario with $t_{growth} = t_{eq}$ and a scenario where it grows throughout the radiation era, $t_{growth} = t_v$.

Typically, $t_c/t_p \sim t_\Lambda/t_p \sim 10^{59}$ and $t_{eq}/t_p \sim 10^{53}$, so in both cases for constant α evolution in the radiation epoch we get

$$\alpha(t_0) = \alpha(t_p)\left(\frac{t_v}{t_p}\right)^A \left(\frac{59}{53}\right) \sim 1.11\alpha(t_p)\left(\frac{t_v}{t_p}\right)^A. \tag{62}$$

We approximate the value for $t_v \sim t_p \sim 1$, so for continuous growth through the radiation epoch we get

$$\alpha(t_0) = \alpha(t_p)\left(\frac{t_v}{t_p}\right)^A (10^{53})^{1/2} \left(\frac{59}{53}\right) \sim 10^{26}\alpha(t_p)\left(\frac{t_v}{t_p}\right)^A. \tag{63}$$

Hence there are very different possibilities for the change in α depending on the evolution in the radiation era.

We have proved this sequence of phases by an exhaustive numerical and analytical study. The ensuing scenario finds two interesting applications, with which we conclude.

In [1] we found that our theory could fit simultaneously the varying α results reported in [10,9,11] and the evidence for an accelerating universe presented in [21]. We noted the curious fact that there is a coincidence between the redshift at which the universe starts accelerating and the redshift where variations in α have been observed but below which α must stabilize to be in accord with geochemical evidence [26,27]. This may be explained dynamically in our theory by the fact that the onset of lambda domination suppresses variations in α. Therefore α remains almost constant in the radiation era, undergoes a small logarithmic time increase in the matter era, but approaches a constant value when the universe starts accelerating because of the presence of a positive cosmological constant. Hence, we comply with geological, nucleosynthesis, and microwave background radiation constraints on time variations in α, while fitting simultaneously the observed accelerating universe and the recent high-redshift evidence for small α variations in quasar spectra.

We have also noted that within this theory the usual anthropic arguments for a lambda free universe may be reversed [28]. Usually, the anthropic principle is used to justify the near flatness and $\Lambda \approx 0$ nature of our universe since large curvature and lambda prevent the formation of galaxies and stars from small perturbations. We have shown that it might be anthropically *disadvantageous* for a universe to lie too close to flatness or for the cosmological constant to lie too close to zero. This constraint occurs because ‘‘constants’’ change throughout the dust-dominated period when the curvature and lambda do not influence the expansion of the universe. The onset of a period of lambda or curvature domination has the property of dynamically stabilizing the constants, thereby creating favorable conditions for the emergence of structures. If the universe were exactly flat and lambda were exactly zero then α would continue to grow to a value that appears to make living complexity impossible [28].

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