

## Classification of stringlike solutions in dilaton gravity

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The static stringlike solutions of the Abelian Higgs model coupled to dilaton gravity are analyzed and compared to the nondilatonic case. Except for a special coupling between the Higgs Lagrangian and the dilaton, the solutions are flux tubes that generate a nonasymptotically flat geometry. Any point in parameter space corresponds to two branches of solutions with two different asymptotic behaviors. Unlike the nondilatonic case, where one branch is always asymptotically conic, in the present case the asymptotic behavior changes continuously along each branch.

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### I. INTRODUCTION

Of all the topological defects [1], which may have been formed during phase transitions in the early universe, cosmic strings [2,3] are those which have attracted the most attention from a cosmological point of view.

They were introduced into cosmology some 20 years ago by Kibble [4], Zel'dovich [5] and Vilenkin [6], and were considered for a long time as possible sources for density perturbations and hence for structure formation in the universe. Indeed, the latest data from the BOOMERANG and MAXIMA experiments [7–9] disagree with the predictions [10,11] for the cosmic microwave background anisotropies based on topological defect models (see also Ref. [12]). This seems to point to the conclusion that if cosmic strings were formed in the early universe they could not have been responsible for structure formation. However, cosmic strings are still cosmologically relevant and enjoy wide interest in cosmology.

The most common field-theoretical model, which is used in order to describe the generation of cosmic strings during a phase transition, is the Abelian Higgs model. This model is defined by the action

$$S = \int d^4x \sqrt{|g|} \left( \frac{1}{2} D_\mu \Phi^* D^\mu \Phi - \frac{\lambda}{4} (\Phi^* \Phi - v^2)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{16\pi G} \mathcal{R} \right) \quad (1.1)$$

where  $\mathcal{R}$  is the Ricci scalar,  $F_{\mu\nu}$  the Abelian field strength,  $\Phi$  is a complex scalar field with vacuum expectation value  $v$  and  $D_\mu = \nabla_\mu - ieA_\mu$  is the usual gauge covariant derivative. We use units in which  $\hbar = c = 1$  and a “mostly minus” metric.

The simplest form of a cosmic string is just a direct generalization of the Nielsen-Olesen flux tube [13], which takes into account its gravitational effect. Because of the cylindrical symmetry, we use a line element of the form

$$ds^2 = N^2(r) dt^2 - dr^2 - L^2(r) d\varphi^2 - K^2(r) dz^2 \quad (1.2)$$

and the usual Nielsen-Olesen ansatz for the +1 flux unit:

$$\Phi = v f(r) e^{i\varphi}, \quad A_\mu dx^\mu = \frac{1}{e} (1 - P(r)) d\varphi. \quad (1.3)$$

Since the flux tube is very concentrated along the symmetry axis, it is quite natural to expect that it will generate a spacetime geometry with the asymptotic behavior of the general static and cylindrically symmetric vacuum solution of Einstein equations [14], the so-called Kasner solution:

$$ds^2 = (kr)^{2a} dt^2 - (kr)^{2c} dz^2 - dr^2 - \beta^2 (kr)^{2(b-1)} r^2 d\varphi^2 \quad (1.4)$$

where  $k$  sets the length scale while  $\beta$  represents the asymptotic structure, as will be discussed below. The parameters  $(a, b, c)$  must satisfy the Kasner conditions

$$a + b + c = a^2 + b^2 + c^2 = 1. \quad (1.5)$$

More information about the solutions can be obtained by inspection of the full system of field equations with the appropriate energy-momentum tensor, which we will not write here (see however Sec. IV or Refs. [1,15,16]). Here we will only give a condensed summary. For cylindrical symmetry, the components of the energy-momentum tensor of the flux tube  $T_\nu^\mu$  have the property of  $T_0^0 = T_z^z$ . This means that the solution will have a symmetry under boosts along the string axis, i.e.,  $a = c$ . The Kasner conditions (1.5) then leave only two options, which are indeed realized as solutions of the full Einstein-Higgs system [16].

The standard conic cosmic string solution [6,15] is characterized by an asymptotic behavior given by Eq. (1.4) with

$$a = c = 0, \quad b = 1 \quad (1.6)$$

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which is evidently locally flat. In this case, the parameter  $\beta$  represents a conic angular deficit [17,18], which is also related to the mass distribution of the source.

In addition to the cosmic string solutions, there exists a second possibility:

$$a = c = 2/3, \quad b = -1/3 \quad (1.7)$$

which is the same behavior as that of the Melvin solution [19]. Equation (1.7) is therefore referred to as the Melvin branch. Note however that the magnetic field in this solution inherits the exponential decrease (with  $r$ ) of the original Nielsen-Olesen flux tube. Therefore, it is much more concentrated than in the original Melvin solution where it decreases only according to a power law.

The main difference between the two branches lies in the Tolman mass, which is zero for the conic cosmic string solutions, but nonzero for the Melvin solutions [16]. Moreover, the central magnetic field is generally larger for the Melvin solutions.

The stringlike solutions of the Abelian Higgs model are the simplest and most studied ones (see e.g. [15,16,20–23]). It is, however, probable that when high energy corrections are taken into account, gravity is not purely tensorial. A minimal modification suggested by string theory [24] is the introduction of a scalar degree of freedom, the dilaton  $\phi$ , turning gravity into a scalar-tensor theory in the spirit of Jordan-Brans-Dicke (JBD) theory. Studies of cosmic string solutions in the framework of JBD theory and its extensions already exist in the literature [25–27], and the typical characteristics of the stringlike solutions are found to be quite different from those of the gauge strings of pure tensorial gravity. The main difference is the absence of asymptotically conic solutions due to the long range effect of the massless JBD field. Unlike the Brans-Dicke field, which is postulated not to couple to matter, the dilaton may do so and may further change the situation. This is represented by the following action [28,29], which is written in the so called ‘‘JBD-string frame [30]:’’

$$S = \int d^4x \sqrt{|g|} \left[ e^{2a\phi} \left( \frac{1}{2} D_\mu \Phi^* D^\mu \Phi - \frac{\lambda}{4} (\Phi^* \Phi - v^2)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) + \frac{1}{16\pi G} e^{-2\phi} (\mathcal{R} - 4 \nabla_\mu \phi \nabla^\mu \phi) \right]. \quad (1.8)$$

The parameter  $a$  serves as a general coupling constant between the dilaton and matter fields. Newton’s constant is added explicitly in order to keep track of the dimensionality of the various fields. This makes the action in the ‘‘Einstein-Pauli frame [30]’’ contain the standard Einstein-Hilbert Lagrangian:

$$S = \int d^4x \sqrt{|g|} \left( \frac{1}{2} e^{2(a+1)\phi} D_\mu \Phi^* D^\mu \Phi - \frac{\lambda}{4} e^{2(a+2)\phi} (\Phi^* \Phi - v^2)^2 - \frac{1}{4} e^{2a\phi} F_{\mu\nu} F^{\mu\nu} \right) + \frac{1}{16\pi G} \int d^4x \sqrt{|g|} (\mathcal{R} + 2 \nabla_\mu \phi \nabla^\mu \phi). \quad (1.9)$$

It is related to the former by the metric redefinition:

$$g_{\mu\nu}^{(s)} = e^{2\phi} g_{\mu\nu}^{(E)} \quad (1.10)$$

where  $g_{\mu\nu}^{(s)}$  is the metric tensor used in Eq. (1.8) and  $g_{\mu\nu}^{(E)}$  is used in Eq. (1.9). From now on we will use the Einstein frame for a clearer physical picture.

This system was studied by Gregory and Santos [28]. Their analysis was done to first order in the parameter  $\gamma = 8\pi G v^2$  and is therefore valid only for weak gravitational fields. They concentrated in the one branch which contains for  $a = -1$  asymptotically conic solutions. However, since for  $a \neq -1$  no asymptotically conic solutions exist, there is no natural criterium to prefer one branch over the other, even if one is ‘‘more flat’’ than the other. Moreover, the weak-field approximation breaks down both near the core of the string and far away from the string thus calling for a further investigation of the system.

In this paper, we expand the study of cosmic strings in dilaton gravity in two directions. (1) We analyze both branches and do not limit ourselves to weak gravitational fields. (2) We modify the dilaton coupling to matter fields in such a way that it will be uniform and the original Einstein-Higgs system (1.1) is recovered in a certain limit.

## II. FIELD EQUATIONS AND ASYMPTOTIC SOLUTIONS

The field equations for the Einstein-Higgs-dilaton action (1.9) are straightforward to obtain and we give them already for flux tubes with cylindrical symmetry. Their structure becomes more transparent if we use dimensionless quantities. As a length scale we use  $1/\sqrt{\lambda v^2}$  (the ‘‘correlation length’’ in the superconductivity terminology). We therefore change to the dimensionless length coordinate  $x = \sqrt{\lambda v^2} r$  and we use the metric component  $\sqrt{\lambda v^2} L$  which we still denote by  $L$ . We also introduce the ‘‘Bogomolnyi parameter’’  $\alpha = e^2/\lambda$  in addition to  $\gamma = 8\pi G v^2$ , which has been already defined above. In terms of these new quantities we get, for fixed  $a$ , a two parameter system of five coupled nonlinear ordinary differential equations (the prime denotes  $d/dx$ ):

$$\frac{(e^{2(a+1)\phi} N^2 L f')'}{N^2 L} + \left( e^{2(a+2)\phi} (1 - f^2) - e^{2(a+1)\phi} \frac{P^2}{L^2} \right) f = 0 \quad (2.1)$$

$$\frac{L}{N^2} \left( e^{2a\phi} \frac{N^2 P'}{L} \right)' - \alpha e^{2(a+1)\phi} f^2 P = 0 \quad (2.2)$$

$$\frac{(LNN')'}{N^2L} = \gamma \left( e^{2a\phi} \frac{P'^2}{2\alpha L^2} - \frac{1}{4} e^{2(a+2)\phi} (1-f^2)^2 \right) \quad (2.3)$$

$$\begin{aligned} \frac{(N^2L')'}{N^2L} = & -\gamma \left( e^{2a\phi} \frac{P'^2}{2\alpha L^2} + e^{2(a+1)\phi} \frac{P^2 f^2}{L^2} \right. \\ & \left. + \frac{1}{4} e^{2(a+2)\phi} (1-f^2)^2 \right) \end{aligned} \quad (2.4)$$

$$\begin{aligned} \frac{(N^2L\phi')'}{N^2L} = & \gamma \left( \frac{a+1}{2} e^{2(a+1)\phi} \left( \frac{f^2 P^2}{L^2} + f'^2 \right) + a e^{2a\phi} \frac{P'^2}{2\alpha L^2} \right. \\ & \left. + \frac{a+2}{4} e^{2(a+2)\phi} (1-f^2)^2 \right). \end{aligned} \quad (2.5)$$

The sixth function, the metric component  $K(x)$ , turns out to be equal to  $N(x)$  due to the high symmetry. We have also to keep in mind the existence of the constraint which comes from the  $(rr)$  Einstein equation, and gets the following form:

$$\begin{aligned} \frac{N'}{N} \left( 2 \frac{L'}{L} + \frac{N'}{N} \right) = & \phi'^2 + \gamma \left( e^{2(a+1)\phi} \frac{f'^2}{2} + e^{2a\phi} \frac{P'^2}{2\alpha L^2} \right. \\ & \left. - e^{2(a+1)\phi} \frac{P^2 f^2}{2L^2} - \frac{1}{4} e^{2(a+2)\phi} (1-f^2)^2 \right). \end{aligned} \quad (2.6)$$

The field equations are supplemented by the following boundary conditions, that should be satisfied by the scalar and gauge fields:

$$\begin{aligned} f(0) = 0, \quad \lim_{x \rightarrow \infty} f(x) = 1 \\ P(0) = 1, \quad \lim_{x \rightarrow \infty} P(x) = 0. \end{aligned} \quad (2.7)$$

These are just the usual flux tube boundary conditions borrowed from the flat space version. Moreover, regularity of the geometry on the symmetry axis  $x=0$  will be guaranteed by the ‘‘initial conditions’’:

$$\begin{aligned} L(0) = 0, \quad L'(0) = 1 \\ N(0) = 1, \quad N'(0) = 0. \end{aligned} \quad (2.8)$$

The presence of the dilaton requires additional conditions, which we impose on the dilaton field on the axis:

$$\phi(0) = 0, \quad \phi'(0) = 0. \quad (2.9)$$

The condition  $\phi'(0) = 0$  follows directly from the equations of motion and the constraint (2.6), using also Eqs. (2.7), (2.8). As for the condition  $\phi(0) = 0$ , notice that the equations of motion (2.1)–(2.6) are invariant under the following transformation:

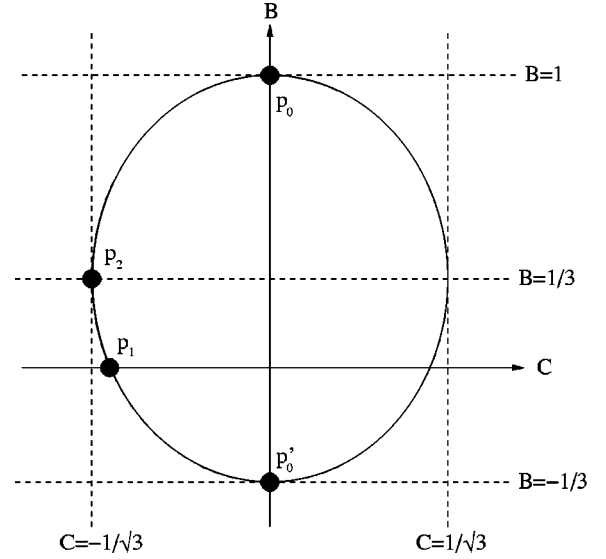


FIG. 1. The ellipse—Eq. (2.13). The origin is a focal point. The special points marked in the figure correspond to solutions discussed in the paper.

$$\begin{aligned} \phi & \rightarrow \phi + \phi_0 \\ x & \rightarrow x e^{-\phi_0} \\ L & \rightarrow L e^{-\phi_0} \\ \gamma & \rightarrow \gamma e^{-2(a+1)\phi_0} \end{aligned} \quad (2.10)$$

where  $\phi_0$  is a constant. Thus, we can trivially get the solution for  $\phi(0) = \phi_0$  from the solution for  $\phi(0) = 0$  by simple rescaling.

Since we are looking for stringlike solutions to the system, we may easily get the asymptotic behavior of the metric components from the assumption that the right-hand sides of the Einstein equations vanish exponentially fast, as in the case of pure tensorial gravity. Consequently, the system reduces to a simple set of equations, which are easily integrated to power law solutions:

$$N(x) \sim \kappa x^A, \quad L(x) \sim \beta x^B, \quad e^{\phi(x)} \sim \delta x^C. \quad (2.11)$$

The 3 parameters  $A$ ,  $B$  and  $C$  are subjected to two conditions, which may be viewed as a generalization of the Kasner conditions, Eq. (1.5):

$$2A + B = 1, \quad A^2 + 2AB = C^2. \quad (2.12)$$

These two conditions leave a one-parameter family of solutions, which may be visualized by the ellipse in the  $B-C$  (or equivalently  $A-C$ ) plane (Fig. 1):

$$\frac{9}{4} \left( B - \frac{1}{3} \right)^2 + 3C^2 = 1. \quad (2.13)$$

The 2 branches of the Einstein-Higgs system (the cosmic string branch and the Melvin branch) are represented in this picture by the top ( $p_0$ ) and bottom ( $p_0'$ ) points of the ellipse,

respectively. These points correspond to an asymptotically constant dilaton ( $C=0$ ), and as we will show later, only the first is realized as an actual solution for the special value of the coupling constant  $a = -1$ .

The parameters  $A$ ,  $B$  and  $C$  are related to the matter distribution through the following three quantities: The Tolman mass (per unit length)  $M$ , or rather its dimensionless representative  $m = GM$

$$m = \frac{\gamma}{2} \int_0^\infty dx N^2 L \left( e^{2a\phi} \frac{P'^2}{2\alpha L^2} - \frac{1}{4} e^{2(a+2)\phi} (1-f^2)^2 \right) \\ = \frac{1}{2} \lim_{x \rightarrow \infty} (L N N') \quad (2.14)$$

and two others

$$w = \frac{\gamma}{2} \int_0^\infty dx N^2 L \left( e^{2a\phi} \frac{P'^2}{2\alpha L^2} + e^{2(a+1)\phi} \frac{P^2 f^2}{L^2} \right. \\ \left. + \frac{1}{4} e^{2(a+2)\phi} (1-f^2)^2 \right) \\ = -\frac{1}{2} \lim_{x \rightarrow \infty} (N^2 L') - 1 \quad (2.15)$$

which is related to the string angular deficit, and

$$D = \gamma \int_0^\infty dx N^2 L \left[ \frac{a+1}{2} e^{2(a+1)\phi} \left( \frac{f^2 P^2}{L^2} + f'^2 \right) \right. \\ \left. + a e^{2a\phi} \frac{P'^2}{2\alpha L^2} + \frac{a+2}{4} e^{2(a+2)\phi} (1-f^2)^2 \right] \\ = \lim_{x \rightarrow \infty} (N^2 L \phi') \quad (2.16)$$

which may be interpreted as a dilaton charge. Note that  $w$  is manifestly positive definite and  $D$  is positive definite for  $a \geq 0$ . The Tolman mass is also non-negative since it is proportional to the power  $A$  which is non-negative due to Eq. (2.12). We further notice that, as in the non-dilatonic case [16],  $w$  may be expressed in terms of the mass parameter  $m$  and the magnetic field on the axis, which we represent by the dimensionless parameter  $\mathcal{B}$ :

$$w = \frac{1}{2} \mathcal{B} - m, \quad \text{where} \quad \mathcal{B} = -\frac{\gamma}{\alpha} \lim_{x \rightarrow 0} \left( \frac{P'(x)}{L(x)} \right). \quad (2.17)$$

The parameters  $A$ ,  $B$  and  $C$  are easily found to be expressible in terms of  $m$ ,  $D$  and  $w$  or as

$$A = \frac{2m}{6m+1-\mathcal{B}}, \quad B = \frac{2m+1-\mathcal{B}}{6m+1-\mathcal{B}}, \quad C = \frac{D}{6m+1-\mathcal{B}}. \quad (2.18)$$

The first of the two relations (2.12) is satisfied identically; the second translates in terms of  $m$ ,  $D$  and  $\mathcal{B}$  to

$$D^2 = 4m(3m+1-\mathcal{B}). \quad (2.19)$$

### III. STRINGLIKE SOLUTIONS

Regarding the complexity and nonlinearity of the system, it is quite evident that a detailed structure of the solutions can be obtained only numerically. We have performed a numerical analysis of the system and found that the solutions are paired, as in the non-dilatonic case, and have the characteristics that will be described below. The solutions have been found numerically by first discretizing the radial coordinate and then applying a relaxation procedure to the set of nonlinear and coupled algebraic equations,  $\mathcal{F}(\bar{f}) = 0$ , in the set of scalar functions evaluated at the grid points,  $\bar{f}$ . If the initial guess  $\bar{f}_i$  is good enough then the iteration obtained by  $\bar{f}_{i+1} = \bar{f}_i + \Delta \bar{f}_i$ , where  $\Delta \bar{f}_i$  is found by solving the linear equation

$$\nabla \mathcal{F}(\bar{f}_i) \Delta \bar{f}_i = -\mathcal{F}(\bar{f}_i) \quad (3.1)$$

(Gauss elimination with scaling and pivoting will do), will converge towards a solution for the discretized variables. We constructed a starting guess satisfying the boundary and initial conditions and still having several free parameters. Changing gradually these parameters, we eventually stumbled into the region of convergence and a solution was found. Having obtained one solution of this type, the rest could easily be generated by moving around in the parameter space in sufficiently small steps. Representative plots of the solutions are given in Figs. 2, 3 and 4.

The simplest picture emerges in the case  $a = -1$ . The two branches of solutions for this particular value of  $a$  are particularly interesting. For fixed  $\alpha$ , both solutions exist up to some critical  $\gamma$ ; see Fig. 5. One is an ordinary cosmic string, as in Einstein gravity with the additional feature of a non-constant dilaton. This solution has  $A=0$ ,  $B=1$ ,  $C=0$  independent of  $\alpha$  and  $\gamma$ . The metric is very similar to the usual gauge string, but there are small deviations, due to the presence of the dilaton. It becomes identical to the metric of the Einstein gravity gauge string only in the Bogomolnyi limit ( $\alpha=2$ ), where Eqs. (2.1)–(2.6) reduce to

$$P' = L(f^2 - 1) \quad (3.2)$$

$$f' = \frac{P f}{L} \quad (3.3)$$

$$L' - \frac{\gamma}{2} P(1-f^2) = 1 - \frac{\gamma}{2} \quad (3.4)$$

with  $N=1$  and  $\phi=0$ , i.e. first order differential equations for  $L$ ,  $f$  and  $P$ . A typical solution of this kind is depicted in Fig. 2(a).

The second branch in the case  $a = -1$  has for all values of  $\alpha$  and  $\gamma$  below the critical curve (Fig. 5),  $C = -\frac{1}{2}$  and  $B = 0$ . Accordingly,  $A = \frac{1}{2}$ . This solution is quite different from

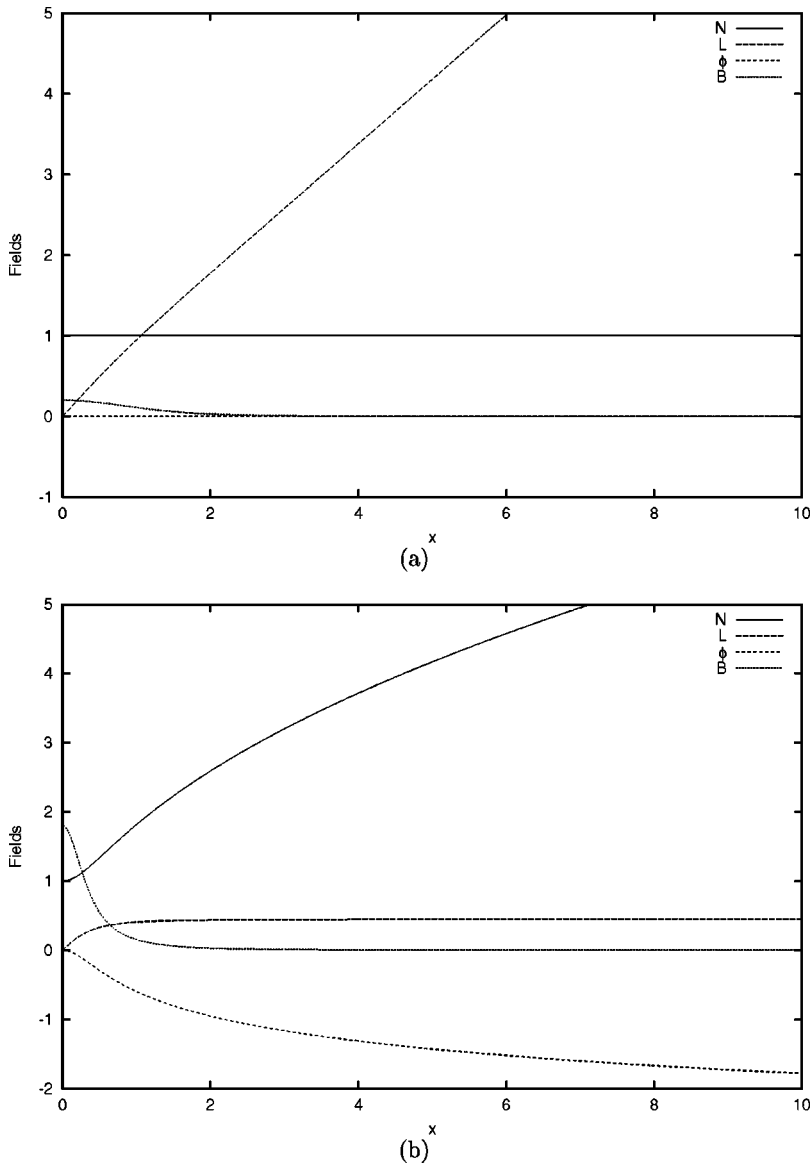


FIG. 2. The solutions for  $a = -1$ ,  $\alpha = 2$ ,  $\gamma = 0.4$ . (a) Cosmic string branch. (b) Dilatonic Melvin branch.

its non-dilatonic analog, the main difference being the asymptotic spatial geometry which is now cylindrical.

At the critical curve (Fig. 5), the two types of solutions become identical. Asymptotically the corresponding spatial geometry is conic with deficit angle  $\delta\varphi = 2\pi$ . The critical curve itself can be approximated by a power law

$$\gamma \approx c_1 \alpha^{c_2} \tag{3.5}$$

where  $c_1 \approx 1.66$  and  $c_2 \approx 0.275$ . This is quite, although not exactly, similar to the result obtained in tensorial gravity [16].

The  $\alpha$  and  $\gamma$  independence of  $A$ ,  $B$  and  $C$  for  $a = -1$  is however the only case of such behavior. The generic one is such that for a given value of  $a$ , the powers of the asymptotic form of the metric components and the dilaton change with  $\alpha$  and  $\gamma$ . Put differently, we have now a situation where the values of  $A$ ,  $B$  and  $C$  depend upon the three parameters  $a$ ,  $\alpha$

and  $\gamma$ . Therefore, the universal asymptotic form of the metric tensor, which corresponds to the two extremal points on the ellipse, is now replaced by a motion along the ellipse whose specific details depend on the value of  $a$ . Additional values of  $a$  were studied in detail and here we give few further results for two more special representative points, so in all there are

$$a = -\sqrt{3}, \quad -1, \quad 0. \tag{3.6}$$

These values were selected for the following reasons: Concentrating on the Einstein-dilaton-Maxwell system,  $a = -1$  corresponds to certain compactified superstring theories [31],  $a = 0$  corresponds to Jordan-Brans-Dicke theory and  $a = -\sqrt{3}$  corresponds to ordinary Kaluza-Klein theory.

In both cases  $a = -\sqrt{3}$ ,  $a = 0$  the same two-branch picture emerges. One branch always contains the upper point of the ellipse ( $p_0$ ) at the limit  $\gamma \rightarrow 0$ , such that for small  $\gamma$  the

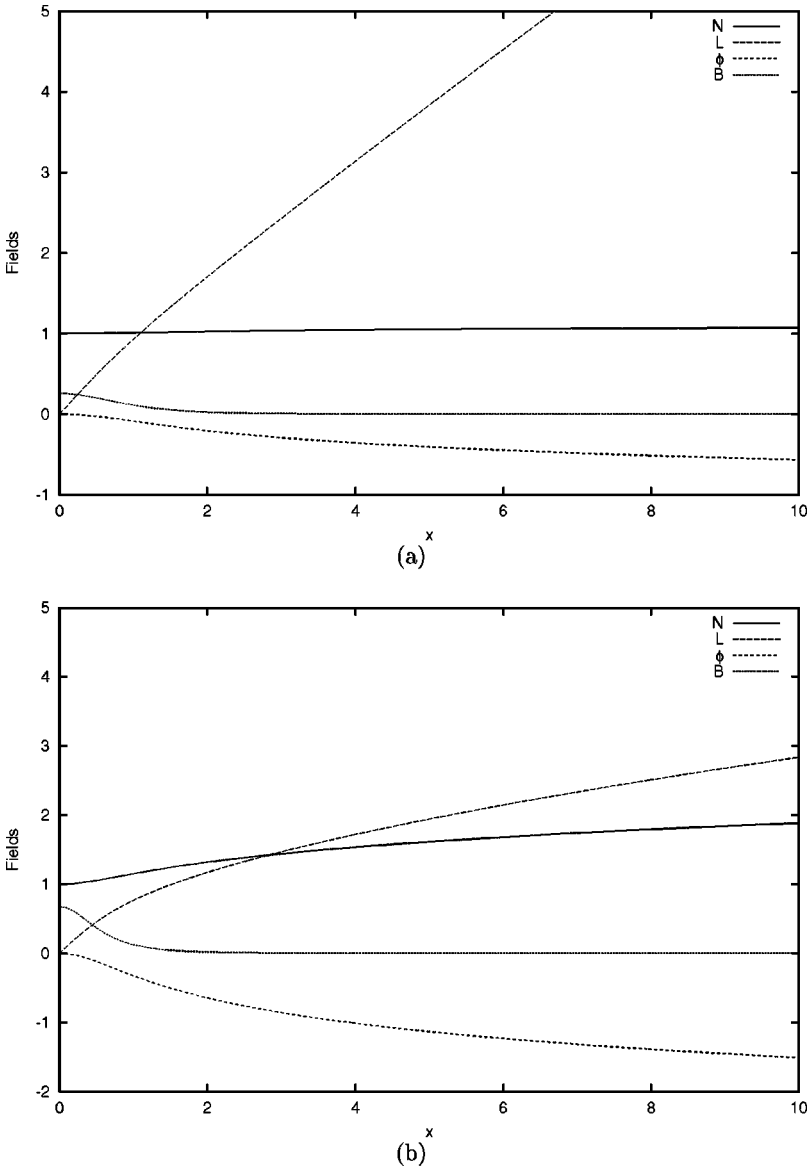


FIG. 3. The solutions for  $a = -\sqrt{3}$ ,  $\alpha = 2$ ,  $\gamma = 0.4$ . (a) Cosmic string branch. (b) Dilatonic Melvin branch.

“almost asymptotically conic” solutions of Gregory and Santos [28] are obtained. For large  $\gamma$  the deviations from asymptotically conic geometry are more pronounced. At the same time a gravitational Newtonian potential appears, so this kind of string exerts force on non-relativistic test particles far outside its core.

The other branch exhibits also an  $\alpha$  and  $\gamma$  dependence of the powers  $A$ ,  $B$  and  $C$ , but has its  $\gamma \rightarrow 0$  limit at  $a$ -dependent points. This  $a$ -dependence can be understood if we note that the  $\gamma \rightarrow 0$  limit corresponds to two types of asymptotic behavior. One is the asymptotically conic cosmic string behavior and the other is that of the dilatonic Melvin universe [29,32], which in our coordinate system corresponds to an asymptotic behavior with the following powers:

$$A = \frac{2}{a^2 + 3}, \quad B = \frac{a^2 - 1}{a^2 + 3}, \quad C = \frac{2a}{a^2 + 3}. \quad (3.7)$$

The situation therefore is the following. First we fix  $a$ . One branch (the “upper” or “cosmic string” branch) has the point  $p_0$ , i.e.,  $A = C = 0$ ,  $B = 1$  as a starting point for  $\gamma \rightarrow 0$ , and as  $\alpha$  and  $\gamma$  change, the powers  $B$  and  $C$  move along the ellipse while  $A$  changes accordingly [see Eq. (2.12)]. The other branch (the “lower” or “dilatonic Melvin” branch) starts, for  $\gamma \rightarrow 0$ , with the above values Eq. (3.7) for the powers  $A$ ,  $B$  and  $C$  and the motion along the ellipse is done in the opposite direction so the two branches approach each other and “meet” at a point in between. In the meeting point, the solutions not only have the same asymptotic behavior, but they become actually identical.

There are no asymptotically conic solutions except for  $a = -1$ . Actually, this property can be understood analytically. For this, we note that a useful relation is obtained by adding the equation for the dilaton, Eq. (2.5), to the equation for the metric component  $N$ , Eq. (2.3), and integrating from 0 to  $x$ , using the boundary conditions (on the axis):

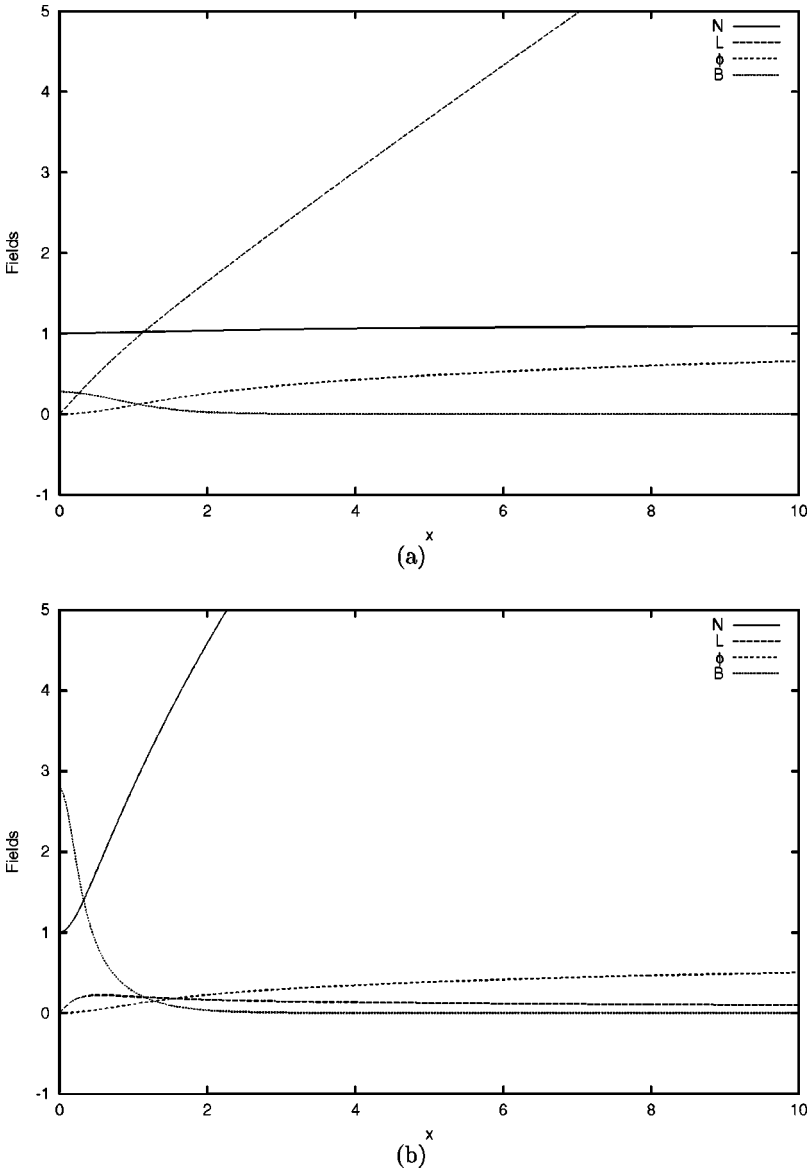


FIG. 4. The solutions for  $a=0$ ,  $\alpha=2$ ,  $\gamma=0.4$ . (a) Cosmic string branch. (b) Dilatonic Melvin branch.

$$\begin{aligned}
 & N^2 L (\phi' + N'/N) \\
 &= \gamma(a+1) \int_0^x d\bar{x} N^2 L \left[ \frac{1}{2} e^{2(a+1)\phi} \left( \frac{f^2 P^2}{L^2} + f'^2 \right) \right. \\
 & \quad \left. + e^{2a\phi} \frac{P'^2}{2\alpha L^2} + \frac{1}{4} e^{2(a+2)\phi} (1-f^2)^2 \right] \quad (3.8)
 \end{aligned}$$

where we denote by  $\bar{x}$  the integration variable. In the case  $a = -1$ , this equation is easily integrated again and gives in this case:

$$N = e^{-\phi}. \quad (3.9)$$

It follows that the dilaton charge is minus twice the Tolman mass,  $D = -2m$ . This imposes an additional condition on the

powers of the asymptotic metric components for any  $\alpha$ ,  $\gamma$  which selects two points on the ellipse. The first is

$$A = C = 0, \quad B = 1 \quad (3.10)$$

which is evidently locally flat.

It is also easy to prove the converse, i.e., that there are no asymptotically conic solutions for  $a \neq -1$ . In order to do it, we start again with Eq. (3.8) and notice also that for asymptotically conic solutions we need  $A=0$  and  $B=1$ , which yield  $C=0$ . Thus,  $N'/N$  and  $\phi'$  vanish asymptotically faster than  $1/x$  and the left-hand side of Eq. (3.8) vanishes asymptotically. But the right hand side of this equation is an integral of a sum of 4 positive-definite terms and it can vanish only if  $a = -1$ .

It also follows that in this asymptotically conic solutions  $D = m = 0$ , while the central magnetic field is given by

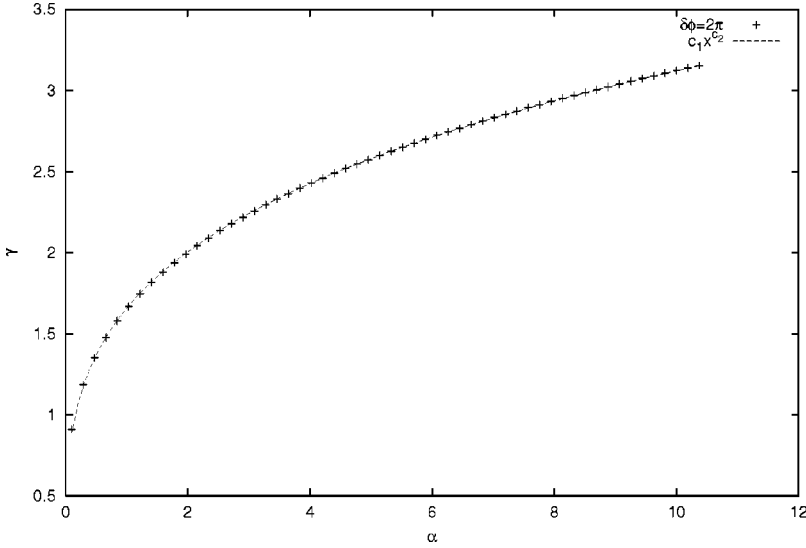


FIG. 5. The curve where  $\delta\varphi=2\pi$  for the  $a = -1$  case. The curve is fitted to a power law with  $c_1 \approx 1.66$  and  $c_2 \approx 0.275$ .

$$B = 1 - \kappa^2 \beta = 1 - \kappa^2 \left( 1 - \frac{\delta\varphi}{2\pi} \right) \quad (3.11)$$

where  $\delta\varphi$  is the deficit angle.

There is as usual a second point on the ellipse, namely

$$A = -C = 1/2, \quad B = 0. \quad (3.12)$$

As indicated above, this represents the same asymptotic behavior as the dilatonic Melvin solution (with  $a = -1$ ). This solution has non-vanishing dilaton charge and Tolman mass. The central magnetic field satisfies the equation

$$B = 1 + 2m. \quad (3.13)$$

#### IV. UNIFORM DILATON COUPLING

The model analyzed so far has a dilaton which couples (in the Einstein-Pauli frame) with different strengths to various matter fields. As a consequence, we do not return to the pure tensorial (Einstein) gravity in the limit of vanishing dilaton coupling. The only case of identical solutions of the two systems is the cosmic string solutions with  $a = -1$  and  $\alpha = 2$ . It seems therefore natural to consider a uniform coupling of the matter fields to the dilaton. This uniform coupling will be described by the following action:

$$S = \int d^4x \sqrt{|g|} \left[ e^{2a\phi} \left( \frac{1}{2} D_\mu \Phi^* D^\mu \Phi - \frac{\lambda}{4} (\Phi^* \Phi - v^2)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) + \frac{1}{16\pi G} (\mathcal{R} + 2\nabla_\mu \phi \nabla^\mu \phi) \right]. \quad (4.1)$$

The corresponding field equations for the cylindrically symmetric case will be a simplified version of Eqs. (2.1)–(2.6):

$$\frac{(e^{2a\phi} N^2 L f')'}{N^2 L} + e^{2a\phi} \left( (1-f^2) - \frac{P^2}{L^2} \right) f = 0 \quad (4.2)$$

$$\frac{L}{N^2} \left( e^{2a\phi} \frac{N^2 P'}{L} \right)' - \alpha e^{2a\phi} f^2 P = 0 \quad (4.3)$$

$$\frac{(LNN')'}{N^2 L} = \gamma e^{2a\phi} \left( \frac{P'^2}{2\alpha L^2} - \frac{1}{4} (1-f^2)^2 \right) \quad (4.4)$$

$$\frac{(N^2 L')'}{N^2 L} = -\gamma e^{2a\phi} \left( \frac{P'^2}{2\alpha L^2} + \frac{P^2 f^2}{L^2} + \frac{1}{4} (1-f^2)^2 \right) \quad (4.5)$$

$$\frac{(N^2 L \phi')'}{N^2 L} = \gamma a e^{2a\phi} \left( \frac{1}{2} \left( \frac{f^2 P^2}{L^2} + f'^2 \right) + \frac{P'^2}{2\alpha L^2} + \frac{1}{4} (1-f^2)^2 \right) \quad (4.6)$$

with the constraint

$$\frac{N'}{N} \left( 2 \frac{L'}{L} + \frac{N'}{N} \right) = \phi'^2 + \gamma e^{2a\phi} \left( \frac{f'^2}{2} + \frac{P'^2}{2\alpha L^2} - \frac{P^2 f^2}{2L^2} - \frac{1}{4} (1-f^2)^2 \right). \quad (4.7)$$

The asymptotic behavior of the metric components and the dilaton field are obtained as before and the results are identical to Eqs. (2.11)–(2.13).

Analysis of the new system shows that the solutions do not change much except in the case  $a = -1$ , where an asymptotically conic solution does not exist any more, see Fig. 6. Actually, there are no asymptotically conic solutions in this system for any combination of parameters and in this respect the solutions are more similar to the string-like solutions in JBD theory. It is quite simple to show that the only case where asymptotically conic solutions exist, in the case



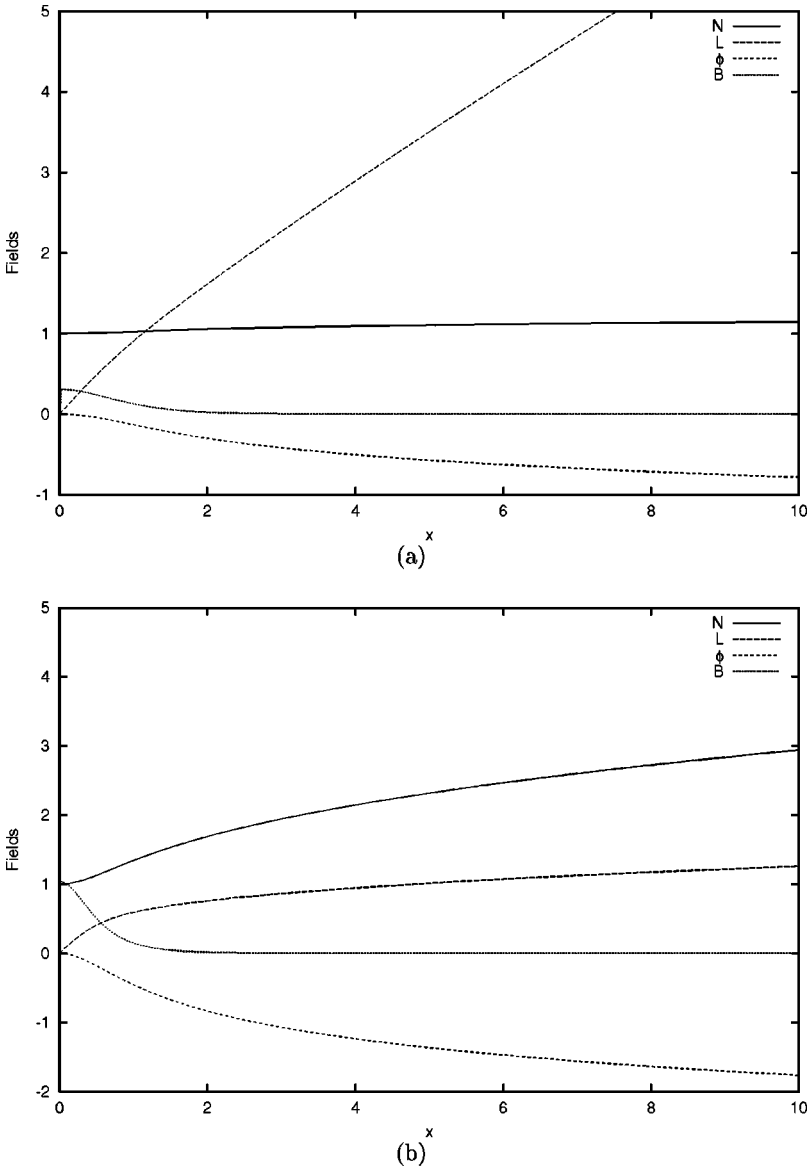


FIG. 6. The solutions for uniform coupling with  $a = -1$ ,  $\alpha = 2$ ,  $\gamma = 0.4$ . (a) Cosmic string branch. (b) Dilatonic Melvin branch.

of this uniform coupling, is  $a = 0$ . The way to do it is to obtain an equation analogous to Eq. (3.8) and to see that it can be satisfied by asymptotically constant  $N$  and  $\phi$  only for  $a = 0$ . This is just the Einstein-Higgs system with a massless real scalar.

**V. CONCLUSION**

We have analyzed in detail, using both numerical and analytic methods, the cosmic string solutions in dilaton gravity. Contrary to previous works, we did not limit ourselves to one branch in the weak-field approximation. We found it most natural to work in the Einstein frame, but all our results are trivially transformed into the string frame.

We have shown that the solutions generally come in pairs. That is, any point in parameter space gives rise to two solutions with completely different asymptotic behaviors for the dilaton and for the geometry. Generally, neither of the two

branches describes asymptotically flat solutions. This should be contrasted with the non-dilatonic case, where one branch always describes asymptotically flat solutions. Only for a very special coupling to the dilaton, which is related to certain compactified superstring theories, do asymptotically flat solutions appear.

Our aim in this paper was to consider cosmic strings in the simplest dilaton-gravity models parametrized by a phenomenological coupling constant  $a$ . In the special case of fundamental string theory, which is related to our models for  $a = -1$ , there is an infinity of curvature corrections (controlled by the reciprocal string tension  $\alpha'$ ) and an infinity of string loop corrections (controlled by the string coupling which is proportional to  $e^\phi$ ). However, since our solutions for  $a = -1$  are everywhere non-singular and with finite dilaton, we expect that they will be only slightly modified in the full (yet unknown) quantum string theory.

The asymptotic behaviors of the solutions in the two branches seemed at first to depend in an extremely complicated way on the parameters of the theory. However, a universal picture emerged, as explained in Sec. III.

These results were obtained by coupling the dilaton “uniformly” to matter in the string frame. For comparison, we

re-analyzed the whole problem using a uniform coupling in the Einstein frame which may seem to be more natural. Among other things, the correct tensorial (Einstein) gravity solutions now come out in the limit of vanishing dilaton coupling. But otherwise the solutions in the two models are quite similar.

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- [1] A. Vilenkin and E.P.S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, England, 1994).
- [2] A. Vilenkin, Phys. Rep. **121**, 263 (1985).
- [3] M. Hindmarsh and T.W.B. Kibble, Rep. Prog. Phys. **58**, 477 (1995).
- [4] T.W.B. Kibble, J. Phys. A **9**, 1387 (1976).
- [5] Ya.B. Zel’dovich, Mon. Not. R. Astron. Soc. **192**, 663 (1980).
- [6] A. Vilenkin, Phys. Rev. D **23**, 852 (1981).
- [7] A.E. Lange *et al.*, Phys. Rev. D **63**, 042001 (2001).
- [8] A.T. Lee *et al.*, Astrophys. J. Lett. **561**, L1 (2001).
- [9] A.H. Jaffe *et al.*, Phys. Rev. Lett. **86**, 3475 (2001).
- [10] R. Durrer, A. Gangui, and M. Sakellariadou, Phys. Rev. Lett. **76**, 579 (1996).
- [11] A.E. Lewin and A. Albrecht, Phys. Rev. D **64**, 023514 (2001).
- [12] R. Durrer, M. Kunz, and A. Melchiorri, astro-ph/0110348.
- [13] H.B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).
- [14] D. Kramer, H. Stephani, E. Herlt, and M. MacCallum, *Exact Solutions of Einstein’s Field Equations* (Cambridge University Press, Cambridge, England, 1980).
- [15] D. Garfinkle, Phys. Rev. D **32**, 1323 (1985).
- [16] M. Christensen, A.L. Larsen, and Y. Verbin, Phys. Rev. D **60**, 125012 (1999).
- [17] L. Marder, Proc. R. Soc. London **A252**, 45 (1959).
- [18] W.B. Bonnor, J. Phys. A **12**, 847 (1979).
- [19] M.A. Melvin, Phys. Lett. **8**, 65 (1964).
- [20] B. Linet, Phys. Lett. A **124**, 240 (1987).
- [21] P. Laguna-Castillo and R.A. Matzner, Phys. Rev. D **36**, 3663 (1987).
- [22] D. Garfinkle and P. Laguna, Phys. Rev. D **39**, 1552 (1989).
- [23] P. Laguna and D. Garfinkle, Phys. Rev. D **40**, 1011 (1989).
- [24] M. B. Green, J. H. Schwarz, and E. Witten, *Superstrings Theory* (Cambridge University Press, Cambridge, England, 1987).
- [25] C. Gundlach and M. Ortiz, Phys. Rev. D **42**, 2521 (1990).
- [26] L.O. Pimentel and A.N. Morales, Rev. Mex. Fis. **36**, S199 (1990).
- [27] M.E.X. Guimaraes, Class. Quantum Grav. **14**, 435 (1997).
- [28] R. Gregory and C. Santos, Phys. Rev. D **56**, 1194 (1997).
- [29] C. Santos, Class. Quantum Grav. **18**, 1835 (2001).
- [30] V. Faraoni, E. Gunzig, and P. Nardone, Fundam. Cosm. Phys. **20**, 121 (1999).
- [31] A. Sen, Nucl. Phys. **B388**, 457 (1992).
- [32] F. Dowker, J. Gauntlett, D. Kastor, and J. Traschen, Phys. Rev. D **49**, 2909 (1994).