

# Complete high temperature expansions for one-loop finite temperature effects

Peter N. Meisinger and Michael C. Ogilvie

*Department of Physics, Washington University, St. Louis, Missouri 63130*

(Received 6 August 2001; published 15 February 2002)

We develop exact, simple closed form expressions for partition functions associated with relativistic bosons and fermions in odd spatial dimensions. These expressions, valid at high temperature, include the effects of a nontrivial Polyakov loop and generalize well-known high temperature expansions. The key technical point is the proof of a set of Bessel function identities which resum low temperature expansions into high temperature expansions. The complete expressions for these partition functions can be used to obtain one-loop finite temperature contributions to effective potentials, and thus free energies and pressures.

DOI: 10.1103/PhysRevD.65.056013

PACS number(s): 11.10.Wx, 11.15.Bt

## I. INTRODUCTION

The applications of finite temperature field theory are numerous and diverse [1–3]. For many applications, a high-temperature expansion of one-loop contributions to thermodynamic functions is necessary. A typical one-loop contribution to a  $(d+1)$ -dimensional effective potential from a bosonic degree of freedom and its antiparticle has the form

$$V = \frac{2}{\beta} \int \frac{d^d k}{(2\pi)^d} \ln[1 - e^{-\beta\omega_k}] \quad (1)$$

where the relativistic energy  $\omega_k$  is given by  $\sqrt{k^2 + M^2}$ . In many cases, the mass  $M$  is a function of other quantities, most notably the vacuum expectation values of fields. When the total effective potential attains its minimum,  $V$  may be identified as a contribution to the total free energy, and  $-V$  as a contribution to the pressure. A more general case is obtained when there is a non-trivial, but spatially uniform, Polyakov loop as well as a non-zero chemical potential  $\mu$ . In this case we have

$$V_B(\theta - i\beta\mu) = \frac{1}{\beta} \int \frac{d^d k}{(2\pi)^d} \ln[1 - e^{-\beta\omega_k + i\theta + \beta\mu}] \\ + \frac{1}{\beta} \int \frac{d^d k}{(2\pi)^d} \ln[1 - e^{-\beta\omega_k - i\theta - \beta\mu}]. \quad (2)$$

Note that the effect of a non-trivial Polyakov loop is to add a phase factor  $\exp(\pm i\theta)$  to  $\exp(-\beta\omega)$ . We will evaluate  $V_B$  for arbitrary  $\theta$ . In principle, the effect of  $\mu$  can be included by a careful analytic continuation  $\theta \rightarrow \theta - i\beta\mu$ , but we do not consider it here.

We will develop a high temperature expansion for  $V_B(\theta)$ , valid for  $d$  odd, as well as a similar expression for the corresponding fermionic quantity  $V_F(\theta)$ . This derivation is simple and exact, and generalizes the results of Dolan and Jackiw [4], who gave approximate high-temperature expressions for  $V_B(\theta=0)$  and  $V_F(\theta=0)$  valid up to order  $M^4 \ln(\beta M)$  in four dimensions. The work of Dolan and Jackiw was extended by Haber and Weldon [5,6] who gave a complete expression for the Bosonic case  $V_B$  as an infinite

sum over hypergeometric functions. Their work included a non-zero chemical potential. Later work by Actor showed that similar high-temperature expansions could be obtained using zeta-function techniques [7,8]. In both cases, higher-order correction terms are given by infinite series in  $\beta M$ . Our expressions effectively resum these corrections into a simpler form. Analytical results for the case of a nontrivial Polyakov loop,  $\theta \neq 0$ , were first given in the case  $M=0$  by Gross, Pisarski and Yaffe and by Weiss [9–11]. Our work generalizes their results to the case  $M \neq 0$ . The higher-order terms in our expressions are manifestly periodic in  $\theta$ . This periodicity is important in the application of these results to our recent work with Miller on models of the deconfinement transition [12]. In this work, the eigenvalues of the Polyakov loop serve as the order parameters for deconfinement, a point of view also emphasized recently by Pisarski [13].

Before beginning the derivation, we give some examples of its application. As a first example, consider a scalar boson in the fundamental representation of an  $SU(N)$  gauge group. The Polyakov loop is an  $N \times N$  unitary matrix given in general by

$$P(\vec{x}) = \mathcal{T} \exp \left[ i \int_0^\beta d\tau A_0(\vec{x}, \tau) \right] \quad (3)$$

where  $\mathcal{T}$  on the right-hand side indicates Euclidean time ordering. Here we assume that the Polyakov loop can be made spatially uniform by an appropriate choice of gauge. A global unitary transformation then puts  $P$  into the diagonal form

$$P_{jk} = \delta_{jk} \exp(i\theta_j) \quad (4)$$

and the free energy is

$$\sum_j V_B(\theta_j). \quad (5)$$

As a second example, consider the case of the gauge bosons themselves, which lie in the adjoint representation of the gauge group. The Polyakov loop in the adjoint representation is an  $(N^2 - 1) \times (N^2 - 1)$  matrix. The partition function for the  $N^2 - 1$  particles is

$$s \frac{1}{2} \sum_{j,k=1}^N \left( 1 - \frac{1}{N} \delta_{jk} \right) V_B(\theta_j - \theta_k) \quad (6)$$

where the  $\delta_{jk}$  removes a singlet contribution, and the factor of  $1/2$  corrects for overcounting since  $V_B$  has both a particle and antiparticle contribution. The factor  $s$  accounts for spin degeneracy; in  $3+1$  dimensions  $s=2$ , a consequence of the two possible polarization states of gauge bosons.

For our third and final example, consider the evaluation of fermionic free energy, which can be reduced to the general bosonic problem. A typical fermionic contribution of particle and antiparticle has the form

$$V_F(\theta - i\beta\mu) = -\frac{1}{\beta} \int \frac{d^d k}{(2\pi)^d} \ln[1 + e^{-\beta\omega_k + i\theta + \beta\mu}] - \frac{1}{\beta} \int \frac{d^d k}{(2\pi)^d} \ln[1 + e^{-\beta\omega_k - i\theta - \beta\mu}] \quad (7)$$

which is easily written as

$$V_F(\theta) = -V_B(\pi + \theta). \quad (8)$$

For fermions in the fundamental representation of  $SU(N)$ , the free energy is

$$s \sum_j V_F(\theta_j) = -s \sum_j V_B(\pi + \theta_j) \quad (9)$$

where the factor  $s$  again accounts for spin degeneracy.

In Sec. II, we review the derivation of low temperature expansions for  $V_B(\theta)$  and  $V_F(\theta)$ . Section III derives the Bessel function identities which convert these low temperature expansions to high temperature expansions. Section IV applies these identities to the case of three spatial dimensions. A final section gives brief conclusions. There are two Appendixes.

## II. LOW TEMPERATURE EXPANSION IN $d$ DIMENSIONS

A low-temperature expansion for  $V_B(\theta)$  can be generated for arbitrary spatial dimension  $d$  by expanding the logarithm and integrating term by term, first over the surface of a  $d$ -dimensional sphere, and then over a radial degree of freedom  $k$  [7,8]. The result, given in terms of modified Bessel functions [14], is

$$V_B(\theta) = \frac{1}{\beta} \int \frac{d^d k}{(2\pi)^d} \ln[1 - e^{-\beta\omega_k + i\theta}] + \frac{1}{\beta} \int \frac{d^d k}{(2\pi)^d} \ln[1 - e^{-\beta\omega_k - i\theta}] \quad (10)$$

$$= -\frac{M^{d/2+1/2}}{2^{d/2-3/2} \pi^{d/2+1/2} \beta^{d/2+1/2}} \sum_{n=1}^{\infty} \frac{1}{n^{d/2+1/2}} \times K_{(d+1)/2}(n\beta M) \cos(n\theta) \quad (11)$$

which is derived in detail in Appendix A. Each term in the series represents the contribution of  $n$  particles or antiparticles, with a corresponding phase factor of  $\exp(\pm in\theta)$ . If the one-loop finite temperature functional determinant is represented as a functional integral over a space-time variable  $x_\mu$ , the phase factors are associated with paths which wind non-trivially in the Euclidean time direction.

For fermions, we have

$$V_F(\theta) = \frac{M^{d/2+1/2}}{2^{d/2-3/2} \pi^{d/2+1/2} \beta^{d/2+1/2}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{d/2+1/2}} \times K_{(d+1)/2}(n\beta M) \cos(n\theta). \quad (12)$$

In a path integral representation, the factors of  $(-1)^n$  are a consequence of fermionic antiperiodic boundary conditions. We next derive a set of identities which resum these series for  $d$  odd.

## III. BESSEL FUNCTION IDENTITIES

We will derive a set of identities for sums of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} K_p(nr) \cos(n\phi) \quad (13)$$

for  $p$  a non-negative integer. Our starting point is the identity

$$\begin{aligned} \sum_{p=1}^{\infty} K_0(pr) \cos(p\phi) &= \frac{1}{2} \left[ \gamma + \ln \left( \frac{r}{4\pi} \right) \right] \\ &+ \frac{\pi}{2} \sum_l' \left[ \frac{1}{\sqrt{r^2 + (\phi - 2\pi l)^2}} - \frac{1}{2\pi|l|} \right] \end{aligned} \quad (14)$$

which may be found in [14]; we provide a derivation in Appendix B which provides some physical insight into its origin. The notation  $\sum_l'$  is used to indicate that singular terms, here the  $1/|l|$  term, are omitted when  $l=0$ .

Using the recursion formula

$$\frac{d}{dz} K_\nu(z) = -K_{\nu-1}(z) - \frac{\nu}{z} K_\nu(z), \quad (15)$$

it follows immediately that

$$\frac{d}{dz} \sum_{p=1}^{\infty} \frac{z}{p} K_1(pz) \cos(p\phi) = -z \sum_{p=1}^{\infty} K_0(pz) \cos(p\phi). \quad (16)$$

This in turn implies that

$$\begin{aligned} \sum_{p=1}^{\infty} \frac{1}{p} K_1(pz) \cos(p\phi) &= -\frac{1}{z} \int dz z \left[ \sum_{p=1}^{\infty} K_0(pz) \cos(p\phi) \right] \\ &+ \frac{C(\phi)}{z} \end{aligned} \quad (17)$$

where  $C(\phi)$  is an unknown function to be determined. Integration yields immediately

$$\begin{aligned} \sum_{p=1}^{\infty} \frac{1}{p} K_1(pz) \cos(p\phi) &= -\frac{1}{4}z \left[ \ln\left(\frac{z}{4\pi}\right) + \gamma - \frac{1}{2} \right] \\ &\quad - \frac{\pi}{2z} \sum_l' \left[ \sqrt{z^2 + (\phi - 2\pi l)^2} - \frac{z^2}{4\pi|l|} \right] + \frac{C(\phi)}{z}. \end{aligned} \quad (18)$$

The function  $C(\phi)$  is determined from the behavior of  $K_\nu(z)$  for  $z \rightarrow 0$ ,

$$K_\nu(z) \rightarrow \frac{1}{2} \Gamma(\nu) \left(\frac{2}{z}\right)^\nu, \quad (19)$$

in combination with the standard result [14]

$$\sum_{p=1}^{\infty} \frac{\cos(p\phi)}{p^2} = \frac{1}{4} \phi^2 - \frac{\pi}{2} \phi + \frac{\pi^2}{6}, \quad (20)$$

which is valid for  $0 \leq \phi < 2\pi$ . The right hand side of Eq. (20) is a rescaling of the second Bernoulli polynomial; it can be extended to all real values if  $\phi$  is replaced by  $\phi \bmod 2\pi$  on the right hand side of the equation. This implies the leading behavior of the sum as  $z \rightarrow 0$  is given by

$$\begin{aligned} \lim_{z \rightarrow 0} z \sum_{p=1}^{\infty} \frac{1}{p} K_1(pz) \cos(p\phi) &= \sum_{p=1}^{\infty} \frac{\cos(p\phi)}{p^2} \\ &= \frac{1}{z} \left[ \frac{1}{4} \phi^2 - \frac{\pi}{2} \phi + \frac{\pi^2}{6} \right], \end{aligned} \quad (21)$$

giving us finally

$$\begin{aligned} \sum_{p=1}^{\infty} \frac{1}{p} K_1(pz) \cos(p\phi) &= -\frac{1}{4}z \left[ \ln\left(\frac{z}{4\pi}\right) + \gamma - \frac{1}{2} \right] \\ &\quad + \frac{1}{z} \left[ \frac{1}{4} \phi_+^2 - \frac{\pi}{2} \phi_+ + \frac{\pi^2}{6} \right] \\ &\quad - \frac{\pi}{2z} \sum_l' \left[ \sqrt{z^2 + (\phi - 2\pi l)^2} \right. \\ &\quad \left. - |\phi - 2\pi l| - \frac{z^2}{4\pi|l|} \right] \end{aligned} \quad (22)$$

where we have introduced the notation  $\phi_+$  to represent  $\phi \bmod 2\pi$ . When discussing fermions, we will also use  $\phi_-$  to similarly represent an angle chosen to lie between  $-\pi$  and  $\pi$ . Note that the last part of this expression is automatically periodic due to the sum over  $l$ .

Application of this technique a second time gives

$$\begin{aligned} \sum_{p=1}^{\infty} \frac{1}{p^2} K_2(pz) \cos(p\phi) &= \frac{1}{16}z^2 \left[ \ln\left(\frac{z}{4\pi}\right) + \gamma - \frac{3}{4} \right] - \frac{1}{2} \left[ \frac{1}{4} \phi_+^2 - \frac{\pi}{2} \phi_+ + \frac{\pi^2}{6} \right] \\ &\quad + \frac{2}{z^2} \left[ -\frac{1}{48} \phi_+^4 + \frac{\pi}{12} \phi_+^3 - \frac{\pi^2}{12} \phi_+^2 + \frac{\pi^4}{90} \right] \\ &\quad + \frac{\pi}{2z^2} \sum_l' \left\{ \frac{1}{3} [z^2 + (\phi - 2\pi l)^2]^{3/2} - \frac{1}{3} |\phi - 2\pi l|^3 \right. \\ &\quad \left. - \frac{1}{2} |\phi - 2\pi l| z^2 - \frac{z^4}{16\pi|l|} \right\} \end{aligned} \quad (23)$$

which is needed for the case  $d=3$ . We have used the standard result [14]

$$\sum_{p=1}^{\infty} \frac{\cos(p\phi)}{p^4} = -\frac{1}{48} \phi_+^4 + \frac{\pi}{12} \phi_+^3 - \frac{\pi^2}{12} \phi_+^2 + \frac{\pi^4}{90}. \quad (24)$$

Formulas appropriate for  $d=5,7,\dots$  can also be derived in the same manner.

#### IV. HIGH-TEMPERATURE EXPANSIONS FOR $d=3$

We now can write complete expressions for  $V_B(\theta)$  and  $V_F(\theta)$  in three spatial dimensions:

$$\begin{aligned} V_B(\theta) &= -\frac{M^2}{\pi^2 \beta^2} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(n\beta M) \cos(n\theta) \\ &= -\frac{2}{\pi^2 \beta^4} \left[ \frac{\pi^4}{90} - \frac{1}{48} \theta_+^4 + \frac{\pi}{12} \theta_+^3 - \frac{\pi^2}{12} \theta_+^2 \right] \\ &\quad + \frac{M^2}{2\pi^2 \beta^2} \left[ \frac{1}{4} \theta_+^2 - \frac{\pi}{2} \theta_+ + \frac{\pi^2}{6} \right] \\ &\quad - \frac{1}{2\pi \beta^4} \sum_l' \left\{ \frac{1}{3} [(\beta M)^2 + (\theta - 2\pi l)^2]^{3/2} \right. \\ &\quad \left. - \frac{1}{3} |\theta - 2\pi l|^3 - \frac{1}{2} |\theta - 2\pi l| \beta^2 M^2 - \frac{(\beta M)^4}{16\pi|l|} \right\} \\ &\quad - \frac{M^4}{16\pi^2} \left[ \ln\left(\frac{\beta M}{4\pi}\right) + \gamma - \frac{3}{4} \right]. \end{aligned} \quad (25)$$

Parts of this complete expression have been known for some time. For  $\theta=0$ , the leading behavior is

$$V_B(\theta=0) \approx -2 \frac{\pi^2}{90\beta^4} + \frac{M^2}{12\beta^2} - \frac{M^3}{6\pi\beta} - \frac{M^4}{16\pi^2} \left[ \ln\left(\frac{\beta M}{4\pi}\right) + \gamma - \frac{3}{4} \right], \quad (26)$$

an expression first derived by Dolan and Jackiw [4]. The leading  $T^4$  behavior for  $\theta \neq 0$  was first derived by Gross, Pisarski and Yaffe and Weiss for  $SU(N)$  gauge bosons and for massless fermions [9–11].

Each of the four terms deserves comment. The first is the blackbody free energy for two degrees of freedom, and depends only on the temperature and the angle  $\theta$ . The second term, which is the leading correction due to the mass  $M$ , often appears in discussions of symmetry restoration at high temperatures with  $\theta=0$ . For example, suppose we are calculating the effective potential for a complex scalar field  $\Phi$ . The mass  $M^2$  is given by the second derivative of the classical potential,  $\partial^2 U / \partial \Phi^* \partial \Phi$ , and depends on the expectation value of the field  $\Phi$ . If  $U$  has the form  $U = -m^2 \Phi^* \Phi + \lambda (\Phi^* \Phi)^2$ , then  $M^2 = -m^2 + 4\lambda \Phi^* \Phi$ . For  $m^2 > 0$ , the  $U(1)$  symmetry is spontaneously broken at low temperature. At high temperature, the  $M^2/12\beta^2$  term generates a positive mass for the  $\Phi$  field of order  $T$ , restoring the symmetry. The third term in  $V_B(\theta)$  is linear in  $T$ , and non-analytic in  $M^2$  for  $\theta=0$ . It is closely associated with the  $n=0$  Matsubara mode, which is the most infrared singular contribution to a finite temperature functional determinant. This term is responsible for non-analytic behavior in finite temperature perturbation theory via the summation of ring diagrams. For example, in a scalar theory it gives rise to the  $\lambda^{3/2}$  contribution to the free energy; in QED, the contribution is  $e^3$  [1]. Note how subtractions occur in the  $l \neq 0$  parts of this term to keep these parts subleading. The last term is logarithmic in the dimensionless combination  $\beta M$  and independent of  $\theta$ . In calculations of effective potentials, it typically combines with zero-temperature logarithms in such a way that the temperature  $T$  sets the scale of running coupling constants at high  $T$ .

From the basic result for  $V_B$ , we can build other results. Consider a complex scalar field in the fundamental representation of  $SU(N)$ . The partition function in a constant background Polyakov loop is given by

$$V_{FT} = \sum_j V_B(\theta_j) = -\frac{\pi^2 N}{45\beta^4} + \frac{2}{\pi^2 \beta^4} \sum_j \left[ \frac{1}{48} \theta_j^4 - \frac{\pi}{12} \theta_j^3 + \frac{\pi^2}{12} \theta_j^2 \right] + \frac{NM^2}{12\beta^2} + \frac{M^2}{2\pi^2 \beta^2} \sum_j \left[ \frac{1}{4} \theta_j^2 - \frac{\pi}{2} \theta_j \right] - \frac{1}{2\pi\beta^4} \sum_{j,l}' \left\{ \frac{1}{3} [(\beta M)^2 + (\theta_j - 2\pi l)^2]^{3/2} - \frac{1}{3} |\theta_j - 2\pi l|^3 - \frac{1}{2} |\theta_j - 2\pi l| \beta^2 M^2 - \frac{(\beta M)^4}{16\pi|l|} \right\} - \frac{NM^4}{16\pi^2} \left[ \ln\left(\frac{\beta M}{4\pi}\right) + \gamma - \frac{3}{4} \right] \quad (27)$$

where we assume all  $\theta$ 's are chosen to lie between 0 and  $2\pi$  in accordance with the convention for  $\theta_+$ . A similar expression holds for bosons in the adjoint representation.

For fermions, we have similarly

$$V_F(\theta) = \frac{2}{\pi^2 \beta^4} \left[ -\frac{7}{720} \pi^4 + \frac{1}{24} \pi^2 \theta_-^2 - \frac{1}{48} \theta_-^4 \right] + \frac{M^2}{2\pi^2 \beta^2} \left[ \frac{1}{12} \pi^2 - \frac{1}{4} \theta_-^2 \right] + \frac{1}{2\pi\beta^4} \sum_l' \left\{ \frac{1}{3} [(\beta M)^2 + (\theta - (2l-1)\pi)^2]^{3/2} - \frac{1}{3} |\theta - (2l-1)\pi|^3 - \frac{1}{2} |\theta - (2l-1)\pi| \beta^2 M^2 - \frac{(\beta M)^4}{16\pi|l|} \right\} + \frac{M^4}{16\pi^2} \left[ \ln\left(\frac{\beta M}{4\pi}\right) + \gamma - \frac{3}{4} \right] \quad (28)$$

with  $\theta_-$  now used. For fermions in the fundamental representation of  $SU(N)$ , we may write

$$V_{FT} = -\frac{7\pi^2 N}{180\beta^4} + \sum_j \frac{1}{12\pi^2 \beta^4} [2\pi^2 \theta_j^2 - \theta_j^4] + \frac{NM^2}{12\beta^2} - \sum_j \frac{M^2}{4\pi^2 \beta^2} \theta_j^2 + \frac{1}{\pi\beta^4} \sum_{j,l}' \left\{ \frac{1}{3} [(\beta M)^2 + (\theta_j - (2l-1)\pi)^2]^{3/2} - \frac{1}{3} |\theta_j - (2l-1)\pi|^3 - \frac{1}{2} |\theta_j - (2l-1)\pi| \beta^2 M^2 - \frac{(\beta M)^4}{16\pi|l|} \right\} + \frac{NM^4}{8\pi^2} \left[ \ln\left(\frac{\beta M}{4\pi}\right) + \gamma - \frac{3}{4} \right] \quad (29)$$

where the angles  $\theta_j$  must now be chosen to lie between  $-\pi$  and  $\pi$ .

## V. CONCLUSIONS

We have found complete, simple expressions for  $V_B(\theta)$  and  $V_F(\theta)$  in the high-temperature limit which generalize previously known expressions. Not only are the expressions simple, their derivation is direct and relatively elementary. Our formulas reflect in a direct way periodicity in  $\theta$ , a property which is lost when analytically continuing power series in  $\beta\mu$  to  $\beta\mu + i\theta$ .

As a practical matter, it is natural to ask how accurate both the low- and high-temperature expansions are. The low temperature expansion for  $V_B(\theta)$  is an infinite series in  $n$ ; using the first 10 terms in the series gives an accuracy better

than 1 part in  $10^3$  over the entire range 0 to  $2\pi$  for temperatures  $T \leq 0.25M$ . The high temperature expansion also involves an infinite series, in the parameter  $l$ . In comparison, the high temperature expansion is within 5% of the exact answer for all values of  $\theta$  at  $T=0.5M$  when terms up to  $|l|=10$  are included. The accuracy improves substantially as  $T$  increases. Both expansions are more accurate when restricted to  $\theta=0$ .

Our primary interest in these results lies in their application to the study of systems where a non-trivial Polyakov loop is expected to occur. The foremost physical example is QCD at finite temperature. The high-temperature form of  $V_B(\theta)$  suggests that the Bernoulli polynomials appear naturally in the free energy of  $SU(N)$  gauge theories with a non-trivial Polyakov loop, essentially as polynomials in the Polyakov loop eigenvalues. In our recent work with Miller [12], we have used this observation to construct a phenomenological free energy for the quark-gluon plasma which reproduces much of the thermodynamic observed in lattice simulations. One can also apply the results obtained here to the Savvidy model at finite temperature [15,16]. Savvidy originally proposed a model of the QCD vacuum in which gluons moved in a constant chromomagnetic field [17]. Using low-temperature expansions, we have shown that a confining state, where the Polyakov loop expectation value is

zero, minimizes the free energy. Using the Bessel function identities proven here, we have recently [18] developed a high-temperature expansion for this model which shows that a non-zero Polyakov line is favored at high temperature.

## APPENDIX A: DERIVATION OF LOW TEMPERATURE EXPANSIONS

We begin by expanding the logarithms and performing the angular integrations:

$$\begin{aligned} V_B(\theta) &= \frac{1}{\beta} \int \frac{d^d k}{(2\pi)^d} \ln[1 - e^{-\beta\omega_k + i\theta}] \\ &\quad + \frac{1}{\beta} \int \frac{d^d k}{(2\pi)^d} \ln[1 - e^{-\beta\omega_k - i\theta}] \\ &= -\frac{4\pi^{d/2}}{\Gamma(d/2)(2\pi)^d \beta} \int dk k^{d-1} \\ &\quad \times \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\beta\omega_k} \cos(n\theta). \end{aligned} \quad (\text{A1})$$

The standard substitution  $k = M \sinh(t)$  gives

$$\begin{aligned} V_B(\theta) &= -\frac{4\pi^{d/2}}{\Gamma(d/2)(2\pi)^d \beta} \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} M^d \int_0^{\infty} dt \cosh t \sinh t^{d-1} e^{-n\beta M \cosh t} \\ &= \frac{4\pi^{d/2}}{\Gamma(d/2)(2\pi)^d} \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^2 \beta^2} M^d \frac{d}{dM} \int_0^{\infty} dt \sinh t^{d-1} e^{-n\beta M \cosh t} \\ &= \frac{4\pi^{d/2}}{\Gamma(d/2)(2\pi)^d} \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^2 \beta^2} M^d \frac{d}{dM} \left[ \frac{\Gamma(d/2)}{\sqrt{\pi}} \left( \frac{2}{n\beta M} \right)^{(d-1)/2} K_{(d-1)/2}(n\beta M) \right]. \end{aligned} \quad (\text{A2})$$

This can in turn be reduced using standard recursion relations for modified Bessel functions:

$$\begin{aligned} V_B(\theta) &= \frac{4\pi^{(d-1)/2}}{(2\pi)^d \beta} M^d \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} \frac{d}{dz} \left[ \left( \frac{2}{z} \right)^{\nu} K_{\nu}(z) \right]_{z=n\beta M, \nu=(d-1)/2} \\ &= \frac{4\pi^{(d-1)/2}}{(2\pi)^d \beta} M^d \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} \left[ \left( \frac{2}{z} \right)^{\nu} \left( \frac{dK_{\nu}(z)}{dz} - \frac{\nu}{z} K_{\nu}(z) \right) \right]_{z=n\beta M, \nu=(d-1)/2} \\ &= \frac{4\pi^{(d-1)/2}}{(2\pi)^d \beta} M^d \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} \left[ -\left( \frac{2}{z} \right)^{\nu} K_{\nu+1}(z) \right]_{z=n\beta M, \nu=(d-1)/2} \\ &= -\frac{4\pi^{(d-1)/2}}{(2\pi)^d \beta} M^d \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} \left( \frac{2}{n\beta M} \right)^{(d-1)/2} K_{(d+1)/2}(n\beta M) \\ &= -\frac{M^{d/2+1/2}}{2^{d/2-3/2} \pi^{d/2+1/2} \beta^{d/2+1/2}} \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^{d/2+1/2}} K_{(d+1)/2}(n\beta M). \end{aligned} \quad (\text{A3})$$

**APPENDIX B: PROOF OF A BESSEL FUNCTION IDENTITY**

In this appendix, we prove the Bessel function identity

$$\sum_{m=1}^{\infty} K_0(mr) \cos(m\phi) = \frac{1}{2} \left[ \ln \left( \frac{r}{4\pi} \right) + \gamma \right] + \frac{\pi}{2} \sum_l' \left[ \frac{1}{\sqrt{r^2 + (\phi - 2\pi l)^2}} - \frac{1}{2\pi|l|} \right]. \quad (\text{B1})$$

Using a standard integral representation [14]

$$K_0(pz) = \int_0^{\infty} dt \frac{\cos(pt)}{\sqrt{t^2 + z^2}}, \quad (\text{B2})$$

we have

$$\begin{aligned} \sum_{m=1}^{\infty} \cos(m\phi) K_0(mr) &= \sum_{m=1}^{\infty} \cos(m\phi) \int_0^{\infty} dk_x \frac{\cos(k_x r)}{\sqrt{k_x^2 + m^2}} = \frac{1}{2} \sum_{m=1}^{\infty} \cos(m\phi) \int dk_x \frac{1}{\sqrt{k_x^2 + m^2}} e^{ik_x r} \\ &= \frac{1}{4\pi} \sum_{m \neq 0} \int dk_x dk_y \frac{1}{k_x^2 + k_y^2 + m^2} e^{ik_x r + im\phi}. \end{aligned} \quad (\text{B3})$$

We introduce a regulating mass  $\mu$ , which will be taken to zero at the end of the calculation, obtaining

$$\frac{1}{4\pi} \sum_{m \neq 0} \int dk_x dk_y \frac{1}{k_x^2 + k_y^2 + m^2 + \mu^2} e^{ik_x r + im\phi}. \quad (\text{B4})$$

We add and subtract the divergent  $m=0$  term

$$\begin{aligned} &\frac{1}{4\pi} \sum_m \int dk_x dk_y \frac{1}{k_x^2 + k_y^2 + m^2 + \mu^2} e^{ik_x r + im\phi} - \frac{1}{4\pi} \int dk_x dk_y \frac{1}{k_x^2 + k_y^2 + \mu^2} e^{ik_x r} \\ &= \frac{1}{4\pi} \sum_m \int dk_x dk_y dk_z \frac{1}{k_x^2 + k_y^2 + k_z^2 + \mu^2} \delta(k_z - m) e^{ik_x r + ik_z \phi} - \frac{1}{4\pi} \int dk_x dk_y \frac{1}{k_x^2 + k_y^2 + \mu^2} e^{ik_x r}. \end{aligned} \quad (\text{B5})$$

Using the Poisson summation technique in the form  $\sum_m \delta(k_z - m) = \sum_n \exp(-2\pi i n k_z)$ , we obtain

$$\frac{1}{4\pi} \sum_n \int dk_x dk_y dk_z \frac{1}{k_x^2 + k_y^2 + k_z^2 + \mu^2} e^{ik_x r + ik_z(\phi - 2\pi n)} - \frac{1}{4\pi} \int dk_x dk_y \frac{1}{k_x^2 + k_y^2 + \mu^2} e^{ik_x r} \quad (\text{B6})$$

which reads in a compact notation

$$\frac{\pi}{2} \sum_n \int \frac{d^3 k}{(2\pi)^3} \frac{4\pi}{k^2 + \mu^2} e^{ik_x r + ik_z(\phi - 2\pi n)} - \frac{1}{4\pi} \int d^2 k \frac{1}{k^2 + \mu^2} e^{ik_x r}. \quad (\text{B7})$$

The first integral gives a sum of screened Coulomb, or Yukawa, potentials

$$\frac{\pi}{2} \sum_n \left[ \frac{e^{-\mu \sqrt{r^2 + (\phi - 2\pi n)^2}}}{\sqrt{r^2 + (\phi - 2\pi n)^2}} \right] - \frac{1}{4\pi} \int d^2 k \frac{1}{k^2 + \mu^2} e^{ik_x r} \quad (\text{B8})$$

and both terms appear to be problematic as  $\mu \rightarrow 0$ . The first term can be made finite in this limit by subtracting the contribution at  $r = \phi = 0$  for  $n \neq 0$ . Using the notation  $\sum_n'$  to denote a summation over all  $n$  with the omission of the singular term when  $n = 0$ , we have

$$\frac{\pi}{2} \sum_n' \left[ \frac{e^{-\mu \sqrt{r^2 + (\phi - 2\pi n)^2}}}{\sqrt{r^2 + (\phi - 2\pi n)^2}} - \frac{e^{-\mu 2\pi|n|}}{2\pi|n|} \right] + \left[ \frac{\pi}{2} \sum_n' \frac{e^{-\mu 2\pi|n|}}{2\pi|n|} - \frac{1}{4\pi} \int d^2 k \frac{1}{k^2 + \mu^2} e^{ik_x r} \right]. \quad (\text{B9})$$

The second term in brackets can be evaluated by summing the series

$$\frac{\pi}{2} \sum_n' \frac{e^{-\mu 2\pi|n|}}{2\pi|n|} = -\frac{1}{2} \ln[1 - e^{-2\pi\mu}] \quad (\text{B10})$$

and using the identity

$$-\frac{1}{4\pi} \int d^2k \frac{1}{k^2 + \mu^2} e^{ik_x r} = -\frac{1}{2} K_0(\mu r) \quad (\text{B11})$$

so we have

$$\frac{\pi}{2} \sum_n' \left[ \frac{e^{-\mu\sqrt{r^2 + (\phi - 2\pi n)^2}}}{\sqrt{r^2 + (\phi - 2\pi n)^2}} - \frac{e^{-\mu 2\pi|n|}}{2\pi|n|} \right] + \left[ -\frac{1}{2} \ln[1 - e^{-2\pi\mu}] - \frac{1}{2} K_0(\mu r) \right]. \quad (\text{B12})$$

In the limit  $\mu \rightarrow 0$ , the second term in brackets gives

$$-\frac{1}{2} \ln[1 - e^{-2\pi\mu}] - \frac{1}{2} K_0(\mu r) \rightarrow -\frac{1}{2} \ln[2\pi\mu] + \frac{1}{2} \ln\left(\frac{\mu r}{2}\right) - \frac{1}{2} \psi(1) = \frac{1}{2} \ln\left(\frac{r}{4\pi}\right) + \frac{1}{2} \gamma \quad (\text{B13})$$

so we finally obtain

$$\sum_{m=1}^{\infty} K_0(mr) \cos(m\phi) = \frac{\pi}{2} \sum_n' \left[ \frac{1}{\sqrt{r^2 + (\phi - 2\pi n)^2}} - \frac{1}{2\pi|n|} \right] + \frac{1}{2} \ln\left(\frac{r}{4\pi}\right) + \frac{1}{2} \gamma. \quad (\text{B14})$$

- 
- |   |   |
|---|---|
| <p>[1] J. I. Kapusta, <i>Finite Temperature Field Theory</i> (Cambridge University Press, Cambridge, England, 1989).</p> <p>[2] M. Le Bellac, <i>Thermal Field Theory</i> (Cambridge University Press, Cambridge, England, 1996).</p> <p>[3] A. Das, <i>Finite Temperature Field Theory</i> (World Scientific, Singapore, 1997).</p> <p>[4] L. Dolan and R. Jackiw, Phys. Rev. D <b>9</b>, 3320 (1974).</p> <p>[5] H. E. Haber and H. A. Weldon, Phys. Rev. D <b>25</b>, 502 (1982).</p> <p>[6] H. E. Haber and H. A. Weldon, J. Math. Phys. <b>23</b>, 1852 (1982).</p> <p>[7] A. Actor, Nucl. Phys. <b>B265</b>, 689 (1986).</p> <p>[8] A. Actor, Fortschr. Phys. <b>35</b>, 793 (1987).</p> <p>[9] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. <b>53</b>, 43 (1981).</p> | <p>[10] N. Weiss, Phys. Rev. D <b>24</b>, 475 (1981).</p> <p>[11] N. Weiss, Phys. Rev. D <b>25</b>, 2667 (1982).</p> <p>[12] P. N. Meisinger, T. R. Miller, and M. C. Ogilvie, Phys. Rev. D <b>65</b>, 034009 (2002).</p> <p>[13] R. D. Pisarski, Phys. Rev. D <b>62</b>, 111501(R) (2000).</p> <p>[14] I. Gradshteyn and I. Ryzhik, <i>Table of Integrals, Series, and Products</i>, 5th ed. (Academic, San Diego, 1994).</p> <p>[15] A. O. Starinets, A. S. Vshivtsev, and V. C. Zhukovsky, Phys. Lett. B <b>322</b>, 403 (1994).</p> <p>[16] P. N. Meisinger and M. C. Ogilvie, Phys. Lett. B <b>407</b>, 297 (1997).</p> <p>[17] G. K. Savvidy, Phys. Lett. <b>71B</b>, 133 (1977).</p> <p>[18] P. N. Meisinger and M. C. Ogilvie (in preparation).</p> |
|---|---|