Production of light pseudoscalars in external electromagnetic fields by the Schwinger mechanism

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We calculate the probability of the decay of external inhomogeneous electromagnetic fields to neutral pseudoscalar particles that have a coupling to two photons. We also point out that our estimate for axion emission in a previous paper was incorrect.

I. INTRODUCTION

The Schwinger mechanism is a nonperturbative process by which an infinite number of zero frequency photons can decay into electron-positron pairs [1]. In this paper we show that this mechanism can be generalized to study the production of other kinds of light particles from intense electromagnetic (EM) fields. The light particle that we consider is a pseudoscalar (PS) having a coupling to two photons.

In Sec. II we derive the formula for the decay of classical background fields into PS particles. This is achieved by integrating out the particle fields from the total Lagrangian to obtain the effective action of the classical background fields. The imaginary part of the effective Lagrangian is related to the probability of decay of classical background fields into particles. In Sec. III we derive, from the usual coupling of the PS to two photons, the specific interaction Lagrangian that should be used in the general formalism of Sec. II in order to account for vacuum decay into PS. For static EM fields, we show that a necessary condition is that the fields are inhomogeneous. In Secs. IV, V, and VI, we explicitly calculate the PS production in a variety of situations. Specifically we consider a dipole magnetic field, a cylindrical capacitor, and a spherical capacitor. Section VII is devoted to the conclusions.

II. DECAY OF CLASSICAL BACKGROUND FIELDS INTO PARTICLES

We start with the action for the pseudoscalar ϕ (mass m) coupled to the background **E** and **B** fields of the general form

$$S[\phi,\mathbf{E},\mathbf{B}] = \int d^4x \frac{1}{2} \phi(x)[-\partial^2 - m^2 + f(x)]\phi(x) \quad (1)$$

where $f(x)$ is some scalar function of **E** and **B** fields. From Eq. (1) we obtain the effective action for the background **E** and **B** fields formally as

$$e^{iS_{eff}[\mathbf{E},\mathbf{B}]} = \int \mathcal{D}\phi \, e^{iS[\phi,\mathbf{E},\mathbf{B}]}. \quad (2)$$

The effective Lagrangian for the **E** and **B** fields can be related to the Green's function of ϕ in external **E** and **B** fields as follows. Differentiate Eq. (2) by m^2

$$i\frac{\partial S_{eff}[\mathbf{E},\mathbf{B}]}{\partial m^2} = -\frac{\int \mathcal{D}\phi \, \phi^2 e^{iS[\phi,\mathbf{E},\mathbf{B}]}}{\int \mathcal{D}\phi \, e^{iS[\phi,\mathbf{E},\mathbf{B}]}}$$

$$= -\frac{1}{2} \int d^4x \, G(x,x;\mathbf{E},\mathbf{B})$$

$$= -\frac{1}{2} \int d^4x \int \frac{d^4p}{(2\pi)^4} G(p;\mathbf{E},\mathbf{B}). \quad (3)$$

The effective Lagrangian of the background fields is therefore formally given by the expression

$$\mathcal{L}_{eff}[\mathbf{E},\mathbf{B}] = \frac{i}{2} \int dm^2 \int \frac{d^4p}{(2\pi)^4} G(p;\mathbf{E},\mathbf{B}). \quad (4)$$

The probability of external **E** and **B** fields to decay into quanta of ϕ is related to the imaginary part of $\mathcal{L}_{eff}$ as follows:

$$P = 1 - |\langle 0|e^{iS_{eff}[\mathbf{E},\mathbf{B}]}|0\rangle|^2$$

$$= 1 - \exp\left[-2 \, \text{Im} \int d^3x \, dt \mathcal{L}_{eff}[\mathbf{E},\mathbf{B}]\right]. \quad (5)$$

In the case that this probability is small, we can write the probability density w (per unit volume and unit time) approximately as

$$w = 2 \, \text{Im} \, \mathcal{L}_{eff}[\mathbf{E},\mathbf{B}]. \quad (6)$$

We now give the general procedure for obtaining the effective action of the background fields by calculating the Green's function of ϕ in background **E** and **B** fields following the method of Duff and Brown [2].

The effective Lagrangian can be calculated by this method if the background fields contained in $f(x)$ in the interaction Lagrangian

$$\mathcal{L}_I(x) = \frac{1}{2} f(x) \phi^2(x) \qquad (7)$$

can be expanded in a Taylor series near some reference point $\bar{x}$. Expanding $f(x)$ near $x = \bar{x}$,

$$f(x) = \alpha(\bar{x}) + \beta_\mu(\bar{x})(x-\bar{x})^\mu$$
$$+ \gamma^2_{\mu\nu}(\bar{x})(x-\bar{x})^\mu(x-\bar{x})^\nu + \cdots$$

$$\alpha(\bar{x}) = f(\bar{x}), \quad \beta_\mu(\bar{x}) = \left(\frac{\partial f}{\partial x^\mu}\right)_{x=\bar{x}},$$

$$\gamma^2_{\mu\nu}(\bar{x}) = \frac{1}{2}\left(\frac{\partial}{\partial x^\mu}\frac{\partial f}{\partial x^\nu}\right)_{x=\bar{x}}. \qquad (8)$$

The equation for the Green's function for the ϕ field is given by

$$[\partial_x^2 + m^2 - \alpha - \beta_\mu(x-\bar{x})^\mu - \gamma^2_{\mu\nu}(x-\bar{x})^\mu(x-\bar{x})^\nu]G(x,\bar{x})$$
$$= \delta^4(x-\bar{x}). \qquad (9)$$

In momentum space

$$(x-\bar{x})^\mu \rightarrow -i\frac{\partial}{\partial p_\mu} \qquad (10)$$

and the equation for the Green's function in momentum space is

$$\left[-p^2 + m^2 - \alpha + i\beta_\mu\frac{\partial}{\partial p_\mu} + \gamma^2_{\mu\nu}\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial p_\nu}\right]G(p) = 1. \qquad (11)$$

We choose as an ansatz for the solution $G(p)$ the form

$$G(p) = i\int_0^\infty ds\, e^{-is(m^2-i\epsilon)} e^{ip_\mu A^{\mu\nu}p_\nu + B^\mu p_\mu + C} \qquad (12)$$

where $A(s), B(s)$ and $C(s)$ are to be determined. They must satisfy the boundary condition in the case of vanishing external fields, i.e. when $\alpha, \beta, \gamma \rightarrow 0$

$$A^{\mu\nu} \rightarrow sg^{\mu\nu}, \quad B^\mu \rightarrow 0, \quad C \rightarrow 0 \qquad (13)$$

and in this limit we should obtain

$$G(p) = i\int_0^\infty ds\, e^{-ism^2 + isp^2} = \frac{1}{m^2 - p^2} \qquad (14)$$

i.e., the free particle Green's function.

To solve for A, B, and C we insert ansatz (12) in Eq. (11). We have

$$i\int_0^\infty ds[-p^2 + m^2 - \alpha + i\beta\cdot(2iA\cdot p + B)$$

$$+ (2ip\cdot A + B)\cdot\gamma^2\cdot(2iA\cdot p + B) + 2i\ \text{tr}(\gamma^2\cdot A)]$$

$$\times\exp\{-ism^2 + ip\cdot A\cdot p + B\cdot p + C\} = 1. \qquad (15)$$

Equation (15) has the general form

$$\int_0^\infty ds\, g(s) e^{-h(s)} = 1 \qquad (16)$$

whose solution is

$$g(s) = \frac{\partial h(s)}{\partial s} \qquad (17)$$

with $h(0) = 0$ and $h(\infty) = \infty$. Using the form of the solution (17) for Eq. (15),

$$i[-p^2 + m^2 - \alpha + i\beta\cdot(2iA\cdot p + B)$$
$$+ (2ip\cdot A + B)\cdot\gamma^2\cdot(2iA\cdot p + B) + 2i\ \text{tr}(\gamma^2\cdot A)]$$
$$= im^2 - ip\cdot\frac{\partial A}{\partial s}\cdot p - \frac{\partial B}{\partial s}\cdot p - \frac{\partial C}{\partial s}, \qquad (18)$$

and comparing equal powers of p on both sides we get the following linear differential equations for A, B, and C:

$$\frac{\partial A}{\partial s} = 1 + 4A\cdot\gamma^2\cdot A$$

$$\frac{\partial B}{\partial s} = 2iA\cdot\beta + 4A\cdot\gamma^2\cdot B$$

$$\frac{\partial C}{\partial s} = i\alpha + \beta\cdot B - iB\cdot\gamma^2\cdot B + 2\ \text{tr}(\gamma^2\cdot A). \qquad (19)$$

The solutions of these equations which satisfy the boundary conditions (13) are given by

$$A = \frac{1}{2}\gamma^{-1}\cdot\tan(2\gamma s) \qquad (20)$$

$$B = -\frac{i}{2}\gamma^{-2}\cdot[1 - \sec(2\gamma s)]\cdot\beta \qquad (21)$$

$$C = i\alpha s - \frac{1}{2}\text{tr}[\ln\cos(2\gamma s)]$$

$$+ \frac{i}{8}\beta\cdot\gamma^{-3}\cdot[\tan(2\gamma s) - 2\gamma s]\cdot\beta. \qquad (22)$$

These A, B and C determine $G(p)$ when substituted in Eq. (12). The effective Lagrangian is obtained by substituting this $G(p)$ in Eq. (4) and carrying out the integration over m^2,

$$\mathcal{L}_{eff} = -\frac{i}{2}\int_0^\infty\frac{ds}{s}\int\frac{d^4p}{(2\pi)^4}\exp\{-ism^2 + ip\cdot A\cdot p$$

$$+ B\cdot p + C\}. \qquad (23)$$

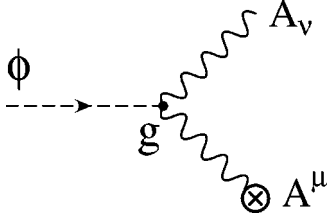The Gaussian integral may be evaluated using

FIG. 1. PS–two-photon interaction.

$$\int d^4 p \, \exp\{ip \cdot A \cdot p + B \cdot p\}$$

$$= -i\pi^2 (\det A)^{-1/2} \exp\left[\frac{i}{4} B \cdot A^{-1} \cdot B\right] \qquad (24)$$

where $\det A$ is the determinant of the matrix A_ν^μ. Using now Eqs. (20)–(22), we have

$$\mathcal{L}_{eff} = -\frac{1}{32\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-is(m^2-\alpha)} \left[\det\left(\frac{2\gamma s}{\sin 2\gamma s}\right)\right]^{1/2} e^{il(s)} \qquad (25)$$

where

$$l(s) = \frac{1}{4} \boldsymbol{\beta} \cdot \boldsymbol{\gamma}^{-3} \cdot [\tan(\boldsymbol{\gamma}s) - \boldsymbol{\gamma}s] \cdot \boldsymbol{\beta}. \qquad (26)$$

The coefficients of the Taylor expansion of the background fields (8) determine the effective action on integrating out the quantum field ϕ. In particular, an imaginary part of $\mathcal{L}_{eff}$ may be non-zero, depending on the signs of the eigenvalues of the γ^2 matrix. When this occurs, we have a non-zero probability (6) that the external EM fields decay in PS particles.

To the effective Lagrangian in Eq. (25) we should add subtractions to render it finite at $s=0$. When this is done, we have that in the limit $\beta \to 0$, $\gamma^2 \to 0$, the effective Lagrangian $\mathcal{L}_{eff} \to 0$, as it should be. In the Appendix A we illustrate the method for the familiar case of production of charged scalar fields in a constant electric field.

The formulas (20)–(22), (25) differ by some signs and factors of i from the solutions displayed in Ref. [3]. There, we presented the formulas for the case that $f(x)$ had only spatial variation and therefore β and γ^2 had only $i=1,2,3$ indices. In [3] we used the metric $(+,+,+)$ while in the present paper we consider spatial as well temporal variation and use the metric $(+,-,-,-)$. This introduces some changes in intermediate formulas but of course the final results we get in the present paper are identical with the final results we got in [3].

III. EFFECTIVE EM-FIELDS–PS-PAIR INTERACTIONS

In this section we show that a coupling of a pseudoscalar to two photons induces an interaction $\mathcal{L}_I$ that may lead to PS production in a background of EM fields.

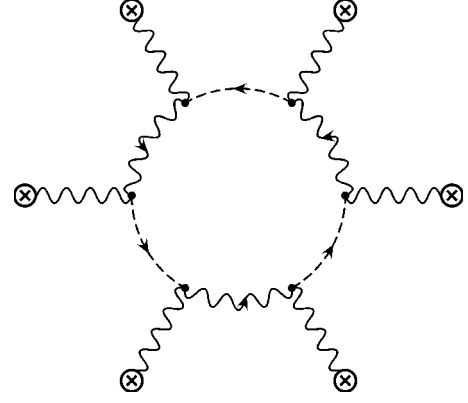The generic pseudoscalar–two-photon interaction (see Fig. 1) can be written as

FIG. 2. Loop diagram showing an infinite number of photon external legs.

$$\mathcal{L}_{\phi\gamma\gamma} = \frac{1}{8} g \phi \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \qquad (27)$$

We should mention that in the special case where the PS is an axion the coupling g is related to the mass m of the axion.

We need to evaluate the loop diagram of the type shown in Fig. 2 with an infinite number of zero-frequency photon external legs. The imaginary part of this diagram gives the probability for the decay of the external electromagnetic field.

To calculate this diagram, we first evaluate the process $\phi A \to \phi A$, where A is an external photon. We use $i\mathcal{L}_{\phi\gamma\gamma}$ from Eq. (27) in momentum space,

$$\frac{1}{4} g \tilde{\phi} \epsilon^{\mu\nu\rho\sigma} k_\mu \tilde{A}_\nu \tilde{F}_{\rho\sigma}. \qquad (28)$$

The two-photon two-PS interaction is then obtained contracting the internal photon legs,

$$4\left(\frac{1}{4} g \tilde{\phi}\right)^2 \epsilon^{\mu\nu\rho\sigma} k_\mu \tilde{F}_{\rho\sigma} \frac{-ig_{\nu\nu'}}{k^2} \epsilon^{\mu'\nu'\rho'\sigma'} (-k_{\mu'}) \tilde{F}_{\rho'\sigma'}. \qquad (29)$$

The factor of 4 in Eq. (29) is for the four possible ways of joining the photon legs. Due to the presence of the k^2 term in the denominator, the effective coupling (29) is non-local. However, when we calculate the effective action for the external EM field the momentum k is integrated over. One can therefore make use of the identity

$$\int d^4 k \, k_\mu k_{\mu'} g(k^2) = \int d^4 k \frac{g_{\mu\mu'} k^2}{4} g(k^2) \qquad (30)$$

to simplify Eq. (29). Thus, we can reduce the effective two PS-two photon interaction to a local interaction vertex. Back in configuration space, it is given by

$$\mathcal{L}_I = -\frac{1}{4} g^2 \phi^2 F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} g^2 \phi^2 (\mathbf{E}^2 - \mathbf{B}^2) \qquad (31)$$
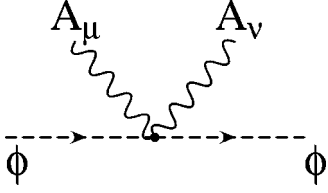
(see Fig. 3).

FIG. 3. Two-PS–two-photon interaction.

With the interaction Lagrangian (31) we can go back to the formalism of Sec. II and calculate the probability density. We can readily identify $f(x)$ in Eq. (7),

$$f(x) = g^2(\mathbf{E}^2 - \mathbf{B}^2). \tag{32}$$

In order to have a non-trivial $\mathcal{L}_{eff}$, one needs non-zero second derivatives of the EM fields as they appear in expression (32). As we said in Sec. I, depending on the sign of the corresponding γ^2 matrix we may have PS production. We illustrate it in some simple physical situations in the following sections.

IV. PRODUCTION OF PSEUDOSCALARS IN DIPOLE MAGNETIC FIELDS

In a static dipole magnetic field the PS-pair–EM interaction is given by

$$\mathcal{L}_I = -\frac{1}{2} g^2 B^2(r) \phi^2$$

$$= -\frac{1}{2} g^2 \left(B_0^2 z_0^6 \frac{3z^2 + r^2}{4r^8} \right) \phi^2 \tag{33}$$

where B_0 is the field strength at a point $\vec{r}_0 = (0,0,z_0)$ on the z axis. We have now

$$f(\vec{r}) = -g^2 \left(B_0^2 z_0^6 \frac{3z^2 + r^2}{4r^8} \right). \tag{34}$$

Expanding $B^2(r)$ near the point $\vec{r}_0$

$$\mathcal{L} = \frac{1}{2} \phi \left[-\partial^2 - m^2 + f(\vec{r}_0) + \frac{\partial f}{\partial x_i} \bigg|_{\vec{r} = \vec{r}_0} (x_i - x_{i0}) \right. \tag{35}$$

$$\left. + \frac{1}{2} \frac{\partial^2 f}{\partial x_i \partial x_j} \bigg|_{\vec{r} = \vec{r}_0} (x_i - x_{i0})(x_j - x_{j0}) \right] \phi + \cdots \tag{36}$$

we find that the coefficients of the Taylor expansion are given by

$$\alpha = f(\vec{r}_0) = -g^2 B_0^2 \equiv \alpha_m \tag{37}$$

$$(\boldsymbol{\beta})_i = \frac{\partial f}{\partial x_i} \bigg|_{\vec{r} = \vec{r}_0} = (0,0,6g^2 B_0^2 z_0^{-1}) \tag{38}$$

and

$$(\gamma^2)_{ij} = \frac{1}{2} \frac{\partial^2 f}{\partial x_i \partial x_j} \bigg|_{\vec{r} = \vec{r}_0} = \frac{3g^2 B_0^2}{4z_0^2} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -28 \end{pmatrix}$$

$$\equiv \begin{pmatrix} a_m^2 & 0 & 0 \\ 0 & a_m^2 & 0 \\ 0 & 0 & -b_m^2 \end{pmatrix}. \tag{39}$$

Therefore we find using the notation and formalism of Sec. II that the effective action on integrating out the PS field is given by

$$\mathcal{L}_{eff} = -\frac{1}{32\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-is(m^2 - \alpha_m)} \frac{2a_m s}{\sinh 2a_m s}$$

$$\times \sqrt{\frac{2b_m s}{\sin 2b_m s}} e^{il_m(s)} \tag{40}$$

with

$$l_m(s) = \lambda_m(b_m s - \tan b_m s) \tag{41}$$

$$\lambda_m = \frac{9g^4 B_0^4}{z_0^2} \frac{1}{b_m^3} = \frac{3}{7\sqrt{21}} g B_0 z_0. \tag{42}$$

The imaginary part of the expression (40) can be performed by enclosing the simple poles at $s = -in\pi(2a_m)^{-1}$, $n = 0,1,\ldots$, with a contour from below. We get

$$\operatorname{Im} \mathcal{L}_{eff} = \frac{1}{8\pi^{5/2}} a_m^{3/2} b_m^{1/2} \sum_{n=1}^\infty (-1)^{n+1} C_n^{(m)} e^{-n\pi/2\eta_m} \tag{43}$$

$$C_n^{(m)} = n^{-3/2} \left[\sinh n \frac{b_m}{a_m} \pi \right]^{-1/2} e^{\tilde{l}_m} \tag{44}$$

with

$$\eta_m = \frac{a_m}{m^2 - \alpha_m} = \frac{\sqrt{15}}{2} \left(\frac{m^2 z_0}{g B_0} + z_0 g B_0 \right)^{-1} \tag{45}$$

and

$$\tilde{l}_m = \lambda_m \left[n \frac{b_m}{a_m} \frac{\pi}{2} - \tanh n \frac{b_m}{a_m} \frac{\pi}{2} \right]. \tag{46}$$

In Eqs. (44) and (46) we can put $b_m/a_m = \sqrt{28/5}$. The main contribution to the above integral comes from the $n=1$ term and we find that w is given by the expression

$$w = 0.036 \frac{g^2 B_0^2}{z_0^2} e^{2.72\lambda_m} e^{-\pi/2 \eta_m}. \tag{47}$$

The probability w of field decay is extremely suppressed for realistic parameters. To illustrate this, let us choose a mass m and a coupling g consistent with the axion window:

$$m \sim 10^{-3} \text{ eV}$$

$$g \sim 10^{-13} \text{ GeV}^{-1}. \tag{48}$$

Also, let us choose

$$B_0 = 1 \text{ Tesla}$$

$$z_0 = 10 \text{ cm}. \tag{49}$$

With these values, we get

$$\eta_m \sim 10^{-19} \tag{50}$$

and since η_m appears in the exponential in Eq. (47), we see that the probability w is indeed extremely suppressed. For a dipole magnetic field to be unstable and decay into axions, one needs $\eta_m \gtrsim 1$, but this would correspond either to unrealistic values for the external field parameters (49) or to excluded values for the axion mass and coupling (48). For non-axion models, g and m are not related (still there are restrictions on these parameters, see Ref. [4]). One could have $\eta_m \sim 1$ by tuning g and m. Imposing that the field (49) does not decay into pseudoscalars leads to the constraint

$$\left(\frac{m}{10^{-12} \text{ eV}} \right)^2 \gtrsim \frac{g}{10^{-13} \text{ GeV}^{-1}}. \tag{51}$$

V. PRODUCTION OF PSEUDOSCALARS IN A CYLINDRICAL CAPACITOR

The modulus of the electric field inside a cylindrical capacitor whose axis lies along the z axis depends only on $\rho = (x^2+y^2)^{1/2}$,

$$E(\rho) = \frac{\lambda}{2\pi} \frac{1}{\rho} \tag{52}$$

with λ the linear electric charge density.

The bilinear interaction term (31) is

$$\mathcal{L}_I = \frac{1}{2} g^2 E^2(\rho) \phi^2(x)$$

$$= \frac{1}{2} g_c^2 \left(\frac{1}{\rho^2} \right) \phi^2(x) \tag{53}$$

where $g_c \equiv \lambda g/2\pi$. The corresponding function $f(x)$ is

$$f(\rho) = g_c^2 \left(\frac{1}{\rho^2} \right). \tag{54}$$

Expanding the fields near some reference point (x_0, y_0, z_0) with $\rho_0 = (x_0^2 + y_0^2)^{1/2}$

$$\mathcal{L} = \frac{1}{2} \phi \left[-\partial^2 - m^2 + f(\rho_0) + \frac{\partial f}{\partial \rho} \bigg|_{\rho=\rho_0} (\rho - \rho_0) \right.$$

$$\left. + \frac{1}{2} \frac{\partial^2 f}{\partial \rho^2} \bigg|_{\rho=\rho_0} (\rho - \rho_0)^2 \right] \phi + \cdots . \tag{55}$$

It can be written as in Eq. (8) with

$$\alpha = f(\rho_0) \equiv \alpha_c \tag{56}$$

$$(\boldsymbol{\beta})_i = \frac{f'}{\rho_0} (x_0, y_0) \tag{57}$$

and

$$(\boldsymbol{\gamma}^2)_{ij} = \frac{f'}{2\rho_0^3} \begin{pmatrix} y_0^2 & -x_0 y_0 \\ -x_0 y_0 & x_0^2 \end{pmatrix} + \frac{f''}{2\rho_0^2} \begin{pmatrix} x_0^2 & x_0 y_0 \\ x_0 y_0 & y_0^2 \end{pmatrix} \tag{58}$$

where primes denote derivatives with respect to ρ taken at $\rho = \rho_0$. In the above formulas, the spatial indices run over 1, 2.

Introducing the explicit form of f, we get

$$\alpha_c = \frac{g_c^2}{\rho_0^2} \tag{59}$$

$$(\boldsymbol{\beta})_i = -\frac{2g_c^2}{\rho_0^4} (x_0, y_0) \tag{60}$$

and the γ^2 matrix (58) reads

$$(\boldsymbol{\gamma}^2)_{ij} = \frac{g_c^2}{\rho_0^6} \begin{pmatrix} -y_0^2 + 3x_0^2 & 4x_0 y_0 \\ 4x_0 y_0 & -x_0^2 + 3y_0^2 \end{pmatrix}. \tag{61}$$

Next, we diagonalize γ^2 by rotating the coordinates with an orthogonal matrix. For example, we can use
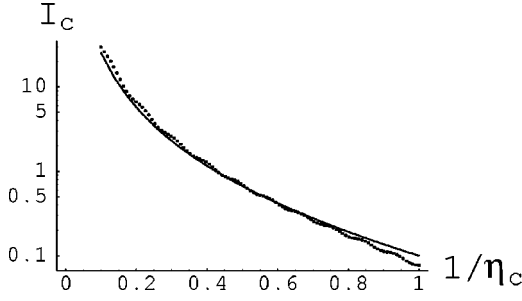
FIG. 4. I_c as a function of $1/\eta_c$ for the value $\lambda_c=0.1$. Dotted line: I_c obtained by numerical integration of Eq. (70). Full line: I_c as given by Eq. (76) with $A=0.28$ and $k=1.026$.

FIG. 5. Same as Fig. 4 for $\lambda_c=0.4$ and $A=0.43$ and $k=2.1$.

$$(O)_{ij}=\frac{1}{\rho_0}\begin{pmatrix} x_0 & y_0 \\ -y_0 & x_0 \end{pmatrix}. \tag{62}$$

In diagonal form we have

$$(\gamma_D^2)_{ij}=\frac{g_c^2}{\rho_0^4}\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\equiv\begin{pmatrix} a_c^2 & 0 \\ 0 & -b_c^2 \end{pmatrix}. \tag{63}$$

We need $\vec{\beta}$ in the diagonal basis given by $\vec{\beta}_D=O\cdot\vec{\beta}$. We get

$$(\boldsymbol{\beta}_D)_i=(-2g_c^2\rho_0^{-3},0). \tag{64}$$

The expression for $\mathcal{L}_{eff}$ (25) finally reads

$$\mathcal{L}_{eff}=-\frac{1}{32\pi^2}\int_0^\infty\frac{ds}{s^3}e^{-is(m^2-\alpha_c)}$$

$$\times\sqrt{\frac{2a_cs}{\sinh 2a_cs}}\sqrt{\frac{2b_cs}{\sin 2b_cs}}e^{il_c(s)} \tag{65}$$

where

$$l_c(s)=\lambda_c(a_cs-\tanh a_cs) \tag{66}$$

$$\lambda_c=g_c^4\rho_0^{-6}a_c^{-3}=\frac{g_c}{3\sqrt{3}}. \tag{67}$$

We are not able to perform the integration in Eq. (65) by the procedure of extending s to the complex plane, as we have done in Sec. IV. The reason is the presence of essential singularities contained in $l_c(s)$. (For a discussion of the implications of essential singularities in the context of QED pair production calculations at finite temperature, see Ref. [5].)

We shall calculate the integral (65) numerically. We make the change of variables

$$x=(m^2-\alpha_c-\lambda_ca_c)s \tag{68}$$

so that

$$w=2\,\mathrm{Im}\,\mathcal{L}_{eff}=\frac{1}{16\pi^2}\frac{1}{(m^2-\alpha_c-\lambda_ca_c)^2}I_c \tag{69}$$

where

$$I_c=\int_0^\infty\frac{dx}{x^3}\sin(\phi_c)\left(\sqrt{\frac{2\eta_cx}{\sinh 2\eta_cx}}\sqrt{\frac{2\eta_cx/\sqrt{3}}{\sin(\eta_cx/\sqrt{3})}}\right.$$

$$\left.+\frac{2}{9}x^2-1\right). \tag{70}$$

We have introduced the necessary subtractions and defined

$$\eta_c=\frac{a_c}{m^2-\alpha_c-\lambda_ca_c} \tag{71}$$

and

$$\phi_c=x+\lambda_c\tanh\eta_cx. \tag{72}$$

As it stands, I_c depends on the two adimensional parameters λ_c and η_c, which reflect its dependence on both the strength of the interaction and on the mass of the scalar particles. In order to explore numerically this two-dimensional space we should focus on those regions for which the results make physical sense. This is even more so because a blind computation of the integral leads easily to wild fluctuations due to the violent oscillations of the integrand for large domains of parameter space.

Since the electric field should be inhomogeneous on scales of the order of a particle's Compton wavelength, we have

$$\left|E^{-1}\left(\frac{dE}{d\rho}\right)_0m^{-1}\right|>1. \tag{73}$$

In our case, this means

$$\rho_0m<1. \tag{74}$$

Moreover, we should restrict our survey to subcritical conditions, i.e. conditions such that we find ourselves below the point where catastrophical pair production starts in

earnest and vacuum breakdown occurs. The system will be subcritical whenever the field energy stored in a volume of size m^{-3}, and which is converted into a PS pair, is at most of the order of twice the scalar particle rest mass. A crude back of an envelope estimate gives

$$\lambda_c^2/m^2\rho_0^2 \lesssim \mathcal{O}(1). \tag{75}$$

Using both restrictions as a rough guide, we perform the numerical integration of I_c as a function of η_c^{-1} for η_c and λ_c in the ballpark of the values required by Eqs. (74) and (75). The results for two values of λ_c are displayed in Figs. 4 and 5. The curves are accurately fit by an analytic expression of the form,

$$I_c = A\,\eta_c^2 e^{-k/\eta_c} \tag{76}$$

with A and k depending on λ_c. (A and k are positive.) Equation (76) has a form that closely resembles the classical Schwinger result and characterizes a typical non-perturbative process.

The electric field would break down into pseudoscalars if the exponent in Eq. (76) becomes small. Again we choose the values consistent with the axion window (48) and for the field parameters we take

$$E_0 \sim 10^4 \text{ V/m}$$

$$\rho_0 \sim 0.1 \text{ cm}. \tag{77}$$

This leads to

$$\eta_c \sim 10^{-22} \tag{78}$$

which implies a large negative value of the exponent in Eq. (76) that suppresses field decay.

VI. PRODUCTION OF PSEUDOSCALARS IN A SPHERICAL CAPACITOR

The modulus of the electric field inside a spherical capacitor depends only on $r=|\vec{r}|$,

$$E(r)=\frac{Q}{4\pi}\frac{1}{r^2} \tag{79}$$

where Q is the electric charge.

The bilinear interaction term is then

$$\mathcal{L}_I=\frac{1}{2}g^2E^2(r)\phi^2(x)$$

$$\frac{1}{2}g_s^2\left(\frac{1}{r^4}\right)\phi^2(x) \tag{80}$$

where $g_s=Qg/4\pi$.

The corresponding function $f(x)$ depends only on r,

$$f(r)=g_s^2\left(\frac{1}{r^4}\right). \tag{81}$$

Expanding the fields near some reference point with (x_0,y_0,z_0) with modulus r_0

$$\mathcal{L}=\frac{1}{2}\phi\left[-\partial^2-m^2+f(r_0)+\frac{\partial f}{\partial r}\bigg|_{r=r_0}(r-r_0)+\frac{1}{2}\frac{\partial^2 f}{\partial r^2}\bigg|_{r=r_0}\right.$$

$$\left.\times(r-r_0)^2\right]\phi+\cdots. \tag{82}$$

In Cartesian coordinates it can be written as in Eq. (8) with

$$\alpha=f(r_0)\equiv\alpha_s \tag{83}$$

$$(\boldsymbol{\beta})_i=\frac{f'}{r_0}(x_0,y_0,z_0) \tag{84}$$

and

$$(\boldsymbol{\gamma}^2)_{ij}=\frac{f'}{2r_0^3}\begin{pmatrix} y_0^2+z_0^2 & -x_0y_0 & -x_0z_0 \\ -x_0y_0 & x_0^2+z_0^2 & -y_0z_0 \\ -x_0z_0 & -y_0z_0 & x_0^2+y_0^2 \end{pmatrix}$$

$$+\frac{f''}{2r_0^2}\begin{pmatrix} x_0^2 & x_0y_0 & x_0z_0 \\ x_0y_0 & y_0^2 & y_0z_0 \\ x_0z_0 & y_0z_0 & z_0^2 \end{pmatrix} \tag{85}$$

where primes denote derivatives with respect to r taken at $r=r_0$. Introducing the form of $f(x)$, we get

$$\alpha_s=\left(\frac{g_s^2}{r_0^4}\right) \tag{86}$$

and

$$(\boldsymbol{\beta})_i=-\frac{4g_s^2}{r_0^6}(x_0,y_0,z_0) \tag{87}$$

and

$$(\gamma^2)_{ij} = \frac{2g_s^2}{r_0^8} \begin{pmatrix} 5x_0^2 - y_0^2 - z_0^2 & 6x_0y_0 & 6x_0z_0 \\ 6x_0y_0 & -x_0^2 + 5y_0^2 - z_0^2 & 6y_0z_0 \\ 6x_0z_0 & 6y_0z_0 & -x_0^2 - y_0^2 + 5z_0^2 \end{pmatrix}. \tag{88}$$

The γ^2 matrix (88) can be diagonalized by rotating the coordinates with an orthogonal matrix. For example, we can use

$$(O)_{ij} = \frac{1}{r_0 d_0} \begin{pmatrix} -z_0 r_0 & 0 & x_0 r_0 \\ -x_0 y_0 & d_0^2 & -y_0 z_0 \\ x_0 d_0 & y_0 d_0 & z_0 d_0 \end{pmatrix} \tag{89}$$

where $d_0 = \sqrt{x_0^2 + z_0^2}$.

In diagonal form we have

$$(\gamma_D^2)_{ij} = \frac{2g_s^2}{r_0^6} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \equiv \begin{pmatrix} -b_s^2 & 0 & 0 \\ 0 & -b_s^2 & 0 \\ 0 & 0 & a_s^2 \end{pmatrix}. \tag{90}$$

One must also use $\vec{\beta}$ in the diagonal basis given by $\vec{\beta}_D = O \cdot \vec{\beta}$. We get

$$(\boldsymbol{\beta}_D)_i = (0, 0, -4g_c^2 r_0^{-5}). \tag{91}$$

The expression for $\mathcal{L}_{eff}$ (25) is given in this case by

$$\mathcal{L}_{eff} = -\frac{1}{32\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-is(m^2 - \alpha_s)} \sqrt{\frac{2a_s s}{\sinh 2a_s s} \frac{2b_s s}{\sin 2b_s s}} e^{il_s(s)} \tag{92}$$

where

$$l_s(s) = \lambda_s(a_s s - \tanh a_s s) \tag{93}$$

$$\lambda_s = \frac{4g_s^4}{r_0^{10}} \frac{1}{a_s^3} = \frac{2}{5\sqrt{10}} \frac{g_s}{r_0}. \tag{94}$$

As in the precedent section, we cannot perform the integration in Eq. (92) by extending s to the complex plane since

again there are essential singularities and we do a numerical integration. The procedure is very similar. It is convenient to change variables,

$$x = (m^2 - \alpha_s - \lambda_s a_s)s \tag{95}$$

so that

$$w = 2 \ \text{Im}\,\mathcal{L}_{eff} = \frac{1}{16\pi^2} \frac{1}{(m^2 - \alpha_s - \lambda_s a_s)^2} I_s \tag{96}$$

where

$$I_s = \int_0^\infty \frac{dx}{x^3} \sin(\phi_s) \left(\sqrt{\frac{2\eta_s x}{\sinh 2\eta_s x} \frac{2\eta_s x/\sqrt{5}}{\sin(\eta_s x/\sqrt{5})}} + \frac{1}{5}x^2 - 1 \right). \tag{97}$$

We have introduced the necessary substractions and defined

$$\eta_s = \frac{a_s}{m^2 - \alpha_s - \lambda_s a_s} \tag{98}$$

and

$$\phi_s = x + \lambda_s \tanh \eta_s x. \tag{99}$$

Here we follow a similar strategy as before to pinpoint the relevant parameter space. We get similar restrictions:

$$r_0 m < 1 \quad \text{and} \quad \lambda_s^2 / r_0^2 m^2 \lesssim \mathcal{O}(1). \tag{100}$$

In Figs. 6 and 7 we display I_s as a function of η_s^{-1} for a couple of values of λ_s. Again, we see that the approximate behavior is that of a decreasing exponential,

$$I_s = A \eta_s^2 e^{-k/\eta_s} \tag{101}$$

with positive constants A and k (that depend on λ_s).

When one considers realistic values for the field and PS parameters it turns out that the probability of field break-
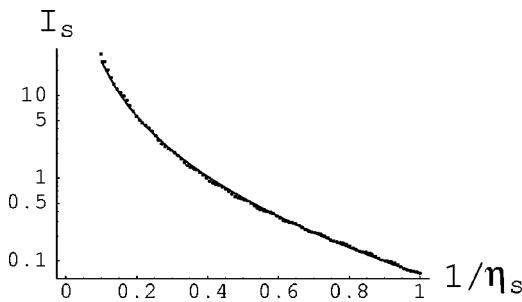
FIG. 6. I_s as a function of $1/\eta_s$ for the value $\lambda_s = 0.2$. Dotted line: I_s obtained by numerical integration of Eq. (97). Full line: I_s given by Eq. (101) with $A = 0.30$ and $k = 1.46$.
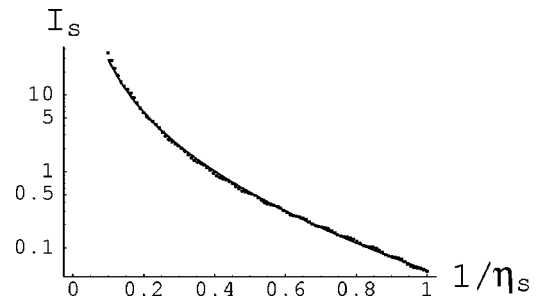
FIG. 7. Same as Fig. 6 for $\lambda_s = 0.5$ and $A = 0.35$ and $k = 1.92$.

down is extremely suppressed as it happens in the cases that have been analyzed in the precedent sections, namely the dipole magnetic field and the cylindrical capacitor.

VII. CONCLUSIONS AND FINAL REMARKS

In the presence of strong external fields, the physical vacuum breaks down because particle-antiparticle pairs are being pumped out of it at the expense of field energy. The case of a strong uniform electric field spontaneously creating electron-positron pairs is the best known (QED) example for this phenomenon. Such a process is of a nonperturbative nature and the QED case has been solved exactly by Schwinger and others [1]. Their solution, however, does not include the back reaction on the external field exerted by the presence of the produced e^+e^- pairs. Clearly, creation of pairs requires the supply of mass energy and kinetic energy which must be furnished by the external field. A balanced energy budget is therefore only possible through a corresponding reduction of the energy stored in the field. Because electrons and positrons carry charge they will fly to the external sources of the field and thus the field (and hence its energy) will diminish. So, unless from the outside the field is restored, the pair production process cannot be indefinitely sustained. If nothing is done from the outside a catastrophic breakdown of the initially strong (critical) electric field will inevitably follow.

In the present paper we dealt with pseudoscalar particles. Pseudoscalars are fundamental ingredients of many completions of particle physics models. Examples run from axions to superlight partners of gravitinos. In the previous sections we have derived the probability for pair production of PS in electric and magnetic fields. Contrary to the QED case mentioned above, constant fields do not cause the disruption of the vacuum. Field gradients are necessary for the phenomenon to occur. Hence, we studied PS pair production in inhomogeneous fields. We have calculated the probability in a general case and based our computation on an effective action formalism formulated by Brown and Duff. We then have applied the general formulas to a few specific cases: PS production in a magnetic dipole field and between the plates of a charged capacitor (either cylindrical or spherical). Again, back reaction was ignored and therefore adequate boundary conditions were implicitly assumed that take into account the fact that pairwise creation of PS requires field energy to be depleted.

In the three cases studied, we found that our probability shows the non-perturbative behavior $w \sim \exp(-\text{const} \times m^2/g)$ expected for subcritical fields. Finally, we should point out that in a previous paper [3] we erroneously estimated axion emission in the Coulomb field of an atomic nucleus. This result is incorrect because we overlooked the question of appropriate boundary conditions that guarantee energy conservation and which are clearly not met in this microscopic system.

ACKNOWLEDGMENTS

APPENDIX: DECAY OF A CONSTANT ELECTRIC FIELD INTO CHARGED SCALARS

We start with the equation for the Green's function $G(p)$:

$$[-(p-eA)^2+m^2]G(p)=1$$

$$[-p^2+m^2+e(A^\mu p_\mu + p_\mu A^\mu)-e^2 A^2]G(p)=1 . \quad \text{(A1)}$$

We assume constant **E** and **B** fields. The vector potential can be choosen as

$$A_\mu = -\frac{1}{2}F_{\mu\nu}x^\nu \rightarrow \frac{i}{2}F_{\mu\nu}\frac{\partial}{\partial p_\nu}. \quad \text{(A2)}$$

When inserted in Eq. (A1), one gets an equation of the form given in Eq. (11), except for a term

$$F^\mu{}_\nu p_\mu \frac{\partial}{\partial p_\nu}G(p) \quad \text{(A3)}$$

that leads to an expression containing

$$F^\mu{}_\nu p_\mu p_\alpha A^{\alpha\mu} . \quad \text{(A4)}$$

The antisymmetry of F and the fact that F and A commute makes this term vanish. Then our equation is

$$\left[-p^2+m^2+\frac{e^2}{4}F_{\mu\nu}F^\mu{}_\rho \frac{\partial^2}{\partial p_\nu \partial p_\rho}\right]G(p)=1 \quad \text{(A5)}$$

and with our definitions in Eq. (8), $\beta=0$ and

$$\gamma_{\nu\rho}^2 = -\frac{e^2}{4}F_{\mu\nu}F^\mu{}_\rho$$

$$\equiv -\frac{e^2}{4}F_{\nu\rho}^2 . \quad \text{(A6)}$$

Let us work out the special case of a constant **E** field. We have

$$(\gamma^2)_{\nu\rho} = -\frac{e^2}{4}\begin{pmatrix} E^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -E^2 \end{pmatrix}. \quad \text{(A7)}$$

(We have chosen the z direction as the direction of **E**.) The eigenvalues of $\gamma^2{}_\rho^\nu$ are negative so

$$\left[\det\left(\frac{2\gamma s}{\sin 2\gamma s}\right)\right]^{1/2} = \frac{eEs}{\sinh eEs} \quad \text{(A8)}$$

and

$$\mathcal{L}_{eff} = -\frac{1}{32\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-ism^2} \frac{eEs}{\sinh eEs}.$$ (A9)

To this expression for $\mathcal{L}_{eff}$ one should add a subtraction to make it finite at $s=0$. When this is done, $\mathcal{L}_{eff} \to 0$ when $eE \to 0$.

The probability of scalar production can now be calculated using Eq. (5). The integral can be calculated by contour integration by closing the real axis with a contour on the negative imaginary plane. This contour encloses poles at $s = -in\pi/eE$ which contribute to the integral. The final result for the constant electric field decay probability density is

$$w = \frac{\alpha E^2}{2\pi^2} \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2} \exp\left(-\frac{n\pi m^2}{eE}\right)$$ (A10)

which coincides with the well-known formula found in textbooks [6].

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