Isospin breaking corrections to low-energy π-K scattering

A. Nehme* and P. Talavera
Centre de Physique Théorique, CNRS Luminy, Case 907, F-13288 Marseille Cedex 9, France
(Received 3 August 2001; published 12 February 2002)

We evaluate the matrix elements for the processes $\pi^0 K^0 \to \pi^0 K^0$ and $\pi^- K^+ \to \pi^0 K^0$ in the presence of isospin breaking terms at leading and next-to-leading order. As a direct application the relevant combination of the S-wave scattering lengths involved in the pion-kaon atom lifetime is determined. We discuss the sensitivity of the results with respect to the input parameters.

DOI: 10.1103/PhysRevD.65.054023 PACS number(s): 14.40.Aq, 12.39.Fe, 13.40.Ks, 13.75.Lb

I. INTRODUCTION

Chiral perturbation theory [1] has become one of the most used tools in exploring QCD low-energy dynamics. It applies in the nonperturbative regime of QCD. The initial QCD Lagrangian $\mathcal{L}_0$ is then replaced by an effective one which contains the same symmetries as the fundamental $\mathcal{L}_0$, composed of a string of higher and higher dimension operators involving derivatives and quark masses. Technically, the new Lagrangian is not renormalizable (in the Wilsonian sense) but fortunately at a given order in the momenta (quark mass) expansion the number of counterterms needed is finite. Thus, *assuming* that the chiral series converges [2], one can truncate it at a given order and deal with a finite number of unknown constants. Restricting the analysis to the next-to-leading order in the mesonic sector (see below), one can obtain the unknown constants from the existing data and large-N_c arguments. In this respect one makes use of the experimental knowledge of the pseudoscalar masses and decay constants, pion vector form factor, and K_{l4} form factors. Therefore none of these data can be used to claim any theoretical *predictability*. To exhibit the consistency of the theory one has to use other processes where the low-energy constants are given as mere inputs and confront the theoretical results with the experimental data. With this aim the π-π scattering lengths have deserved careful examination [3] but unfortunately they give information only about the SU(2) sector. In line with the previous general argument, π-K scattering is the simplest meson-meson scattering process that involves strangeness and can be used as an independent source of information on the validity of the extra assumptions that are commonly believed to hold in chiral perturbation theory, such as, for instance, large-N_c arguments. This will, hopefully, bring some insight into the role of the strange quark mass inside the chiral expansion. Recently, it has been noticed that the quark condensate depends strongly on the number of light sea quarks [4]. This fact might cast doubts on the validity of the chiral expansion in the SU(3) sector where m_s is treated as a small parameter. Before any judgment is made it is necessary to make more accurate experiments and precision calculations on testing processes. In that sense the next proposal for the measurement of the lifetime and the splitting between energy levels in π-K atoms ($A_{\pi K}$) at CERN constitutes a major step from the experimental side [5]. Experiments on this system will constitute one of the most stringent tests of chiral symmetry breaking existing up to now. The lifetime (τ) of the $A_{\pi K}$ atom is given in terms of [6]

$$\frac{1}{\tau} \propto (a_0^{1/2} - a_0^{3/2})^2, \tag{1.1}$$

where $a_0^{1/2}$ and $a_0^{3/2}$ denote the S-wave π-K scattering lengths. Thus any theoretical insight into the corrections to Eq. (1.1) due to the isospin breaking terms has a key role in the accurate determination of the scattering lengths.

In this paper our aim is to incorporate some of the theoretical effects that were not taken into account in previous analyses [7]. We shall deal first with the more academic $\pi^0 K^0 \to \pi^0 K^0$ process, where there is no presence of an explicit virtual soft photon, but electromagnetic effects will appear in the expressions for the scattering lengths as differences between charged and neutral pseudoscalar masses. We proceed with the analysis by considering the $\pi^- K^+ \to \pi^0 K^0$ transition, where in addition the explicit exchange of virtual photons should be considered.

The paper is organized as follows. In Sec. II we review briefly the inclusion of electromagnetic corrections inside the framework of effective Lagrangians, emphasizing the role of the low-energy constants. We continue with the isospin decomposition for the π-K scattering amplitudes, analyzing the scattering lengths at leading order in Sec. III. Next, in Sec. IV we proceed with the analysis of the $\pi^0 K^0 \to \pi^0 K^0$ process at next-to-leading order, first reviewing the kinematics and carefully explaining how to deal with isospin breaking effects, both strong and electromagnetic ones, in order to have an expression consistent with chiral power counting. In Sec. V we turn to the evaluation of the experimental mode $\pi^- K^+ \to \pi^0 K^0$ emphasizing the soft-photon contribution, the extraction of the Coulomb pole at threshold, and the proper definition of observables once isospin breaking corrections are taken into account. In a more technical section, Sec. VI, we explain how to perform the threshold expansion of the non-Coulomb part of the scattering amplitude. In Sec. VII we review the experimental and theoretical status of the π-K S-wave scattering lengths and we present our results, discussing them in terms of all input parameters. We make use of our findings to determine the lifetime of $A_{\pi K}$ in Sec. VIII. Section IX summarizes our results. Finally, in order not to interrupt the discussion, we have collected in the Appendixes all relevant expressions.

*Email address: nehme@cpt.univ-mrs.fr

II. EFFECTIVE LAGRANGIAN TO LOWEST ORDER

This section covers briefly the inclusion of electromagnetic corrections in a systematic way in the low-energy theory describing hadron interactions [8,9]. Because of the smallness of the electromagnetic constant $\alpha_{\rm e.m.}$, these effects have been theoretically neglected so far in the π-K scattering process, but it is well known that near threshold isospin breaking effects can considerably enhance some observables.

In the presence of electromagnetism it is convenient to split the lowest-order effective Lagrangian into three terms:

$$\mathcal{L}_2=\mathcal{L}^{\gamma}+\mathcal{L}_{\rm QCD}^{(2)}+\mathcal{L}^{C}. \tag{2.1}$$

The foregoing Lagrangian possesses the same symmetry restrictions as the one in the strong sector. Additionally, one has to impose an extra symmetry, charge conjugation, affecting only the photon fields and the spurious (see below). The first term in Eq. (2.1) corresponds to the usual Maxwell Lagrangian containing the classical photon field A_μ and the field strength tensor $F_{\mu\nu}$:

$$\mathcal{L}^{\gamma}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-\tfrac{1}{2}(\partial_\mu A^\mu)^2,\quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu. \tag{2.2}$$

The second term formally describes the dynamics of the strong interaction sector [10] and it is given at lowest order by

$$\mathcal{L}_{\rm QCD}^{(2)}=\frac{F_0^2}{4}\langle d^\mu U^\dagger d_\mu U+\chi^\dagger U+\chi U^\dagger\rangle. \tag{2.3}$$

As usual, the angular brackets $\langle\cdots\rangle$ stand for a trace over flavor. The field U parametrizes the dynamics of the low-energy modes in terms of elements of the Cartan subalgebra [11]. The covariant derivative is slightly modified with respect to the pure QCD interaction expression to accommodate the inclusion of the electromagnetic field:

$$d_\mu U=\partial_\mu U-i(v_\mu+Q_R A_\mu+a_\mu)U+iU(v_\mu+Q_L A_\mu-a_\mu). \tag{2.4}$$

Q_R and Q_L are the aforementioned spurion fields, containing the sources for the electromagnetic operators $A_\mu\overline{q}_L(\lambda_a/2)\gamma^\mu q_L$ and $A_\mu\overline{q}_R(\lambda_a/2)\gamma^\mu q_R$. Furthermore, from now on we set them to their constant value

$$Q_L(x)=Q_R(x)=Q,\quad Q=\frac{e}{3}\times{\rm diag}(2,-1,-1). \tag{2.5}$$

As usual, a_μ and v_μ stand for the axial and vector sources, respectively. The scalar and pseudoscalar sources are contained inside the SU(3) matrix χ as

$$\chi=s+ip=2B_0\mathcal{M}+\cdots,\quad \mathcal{M}={\rm diag}(m_u,m_d,m_s). \tag{2.6}$$

At lowest order the last term in Eq. (2.1), $\mathcal{L}^C$, determines the masses of the mesons in the chiral limit, which are of purely electromagnetic origin. This means that even at tree level the pole of the two-point Green function is shifted from its QCD value, modifying therefore the kinematics of the low-energy region,

$$\mathcal{L}^C=C\langle QUQU^\dagger\rangle=-2e^2\frac{C}{F_0^2}(\pi^-\pi^++K^-K^+)+\cdots. \tag{2.7}$$

Obviously, the coupling is taken to be universal; thus we expect the same contribution to the masses for charged pions and kaons. The inclusion of isospin breaking terms can be seen in a very naive way as coming through two different sources in Eq. (2.1): first, a pure *strong* isospin breaking, i.e., $m_u\neq m_d$, and second an *electromagnetic* interaction $e\neq 0$, Eq. (2.7). In view of the seemingly different roles of the two contributions, one has to relate them in a consistent way. For instance, the Lagrangian Eq. (2.1) involves an expansion in several parameters: p, m, e, and δ, where p refers to the external momenta, m to the quark masses, e to the electric charge, and finally $\delta=m_d-m_u$. To be consistent, Eq. (2.1) should contain operators with the same chiral order in the series expansion. To achieve this a possible choice can be $m\sim e^2\sim O(p^2)$. At the next-to-leading order $O(p^4)$, none of the following terms are prevented from appearing by chiral power counting: p^4, m^2, p^2m, p^2e^2, me^2, $p^2\delta$, $m\delta$, e^4, and δ^2, although there is quantitative support for the assumption, often used in phenomenological discussions, that the e^4 and δ^2 contributions are tiny and can therefore be safely disregarded compared with the rest.

Hitherto we have listed all possible electromagnetic lowest-order operators. Once quantum fluctuations are considered using vertices from the functional (2.1) the results are ultraviolet divergent. Those divergences depend on the regularization method employed in the loop diagrams. As is customary we shall adopt the modified minimal subtraction ($\overline{\rm MS}$) scheme. In order to remove these ultraviolet divergences, higher-order operators, modulated by simple constants, should be incorporated into the theory with the guidance of the previously mentioned symmetry requirements. This allows us to deal with a theory that is ultraviolet finite order by order in the parameter expansion and hopefully it adequately describes several observables.

These modulated constants are order parameters of the effective theory (low-energy constants) and in the case at hand are determined by the underlying low-energy dynamics of QCD and QED. For instance, at lowest order the order parameters are given by F_0 [Eq. (2.3)], B_0 [Eq. (2.6)], and C [Eq. (2.7)], describing the lowest pseudoscalar decay constant, the vacuum condensate parameter, and the electromagnetic pion mass in the chiral limit, respectively. For the strong SU(3) sector and at next-to-leading order, there are ten new low-energy constants [10] $L_1,\ldots,L_{10}$ and two high-energy constants. At this level of accuracy the ten low-energy constants may be extracted almost independently from one other by matching some observables with the corresponding experimental determination. In the electromagnetic SU(3) sector there is also need of higher-order operators, up to 16 modulated via $K_1,\ldots,K_{16}$, to render any observable free of ultraviolet divergences. To gain some information on these low-energy constants one has to resort to models [12], to sum rules [13], or to simple crude order estimates. All in all, one can see that the inclusion of electromagnetic corrections to hadronic processes increases the number of low-energy constants enormously and constitutes the major source of uncertainty when making use of the effective Lagrangian.

III. ISOSPIN BREAKING CORRECTIONS AT LEADING ORDER

Under strong interactions symmetry pions are assigned to a triplet of states, "isotriplet states," whereas kaons can be collected into doublets. This means that the pion and the kaon are isospin eigenstates with eigenvalues 1 and 1/2, respectively. Therefore, the amplitudes for the π-K scattering processes are solely described in terms of two independent isospin eigenstate amplitudes $T^{1/2}$ and $T^{3/2}$,

$$\begin{aligned}\mathcal{M}(\pi^0K^0\to\pi^0K^0)&=\tfrac{1}{3}T^{1/2}+\tfrac{2}{3}T^{3/2},\\ \mathcal{M}(\pi^-K^+\to\pi^0K^0)&=-\frac{\sqrt{2}}{3}T^{1/2}+\frac{\sqrt{2}}{3}T^{3/2},\\ \mathcal{M}(\pi^-K^+\to\pi^-K^+)&=\frac{2}{3}T^{1/2}+\frac{1}{3}T^{3/2},\\ \mathcal{M}(\pi^+K^+\to\pi^+K^+)&=T^{3/2}.\end{aligned}\tag{3.1}$$

It is obvious that under $s\leftrightarrow u$ crossing the last two matrix elements are related. In particular, one finds

$$T^{1/2}(s,t,u)=\tfrac{3}{2}T^{3/2}(u,t,s)-\tfrac{1}{2}T^{3/2}(s,t,u).\tag{3.2}$$

Thus in the isospin limit it is sufficient to compute one of the processes. It is convenient to use, instead of the invariant amplitudes T^I, the partial wave amplitudes t_l^I defined in the s channel by

$$T^I(s,\cos\theta)=32\pi\sum_l (2l+1)P_l(\cos\theta)t_l^I(s),\tag{3.3}$$

or by the inverse expression

$$t_l^I(s)=\frac{1}{64\pi}\int_{-1}^{+1}d(\cos\theta)P_l(\cos\theta)T^I(s,\cos\theta),\tag{3.4}$$

where l is the total angular momentum, θ the scattering angle in the center of mass frame, and P_l the Legendre polynomials with $P_0(\cos\theta)=1$. Near threshold the partial wave amplitudes can be parametrized in terms of the scattering lengths a_l^I and slope parameters b_l^I. Neglecting isospin breaking effects and in the normalization (3.3), the real part of the partial wave amplitude reads

$$\mathrm{Re}\, t_l^I(s)=q^{2l}\{a_l^I+b_l^Iq^2+O(q^4)\},\tag{3.5}$$

with q being the center of mass three-momentum.

We start by estimating the scattering lengths in the isospin limit at tree level. Using for instance the first two processes in Eq. (3.1), one can disentangle the values of each scattering length separately. In order to match the prescription of the scattering lengths given in [14] we define them in terms of the charged pion and kaon masses. They read[1]

$$\begin{aligned}a_0^{1/2}&=\frac{M_{\pi^\pm}M_{K^\pm}}{16\pi F_0^2}=0.160\ \ (0.131),\\ a_0^{3/2}&=-\frac{M_{\pi^\pm}M_{K^\pm}}{32\pi F_0^2}=-0.080\ \ (-0.065),\end{aligned}\tag{3.6}$$

where the first quoted number refers to the choice $F_0^2=F_\pi^2$ whereas the second is for[2] $F_0^2=F_\pi F_K$.

When switching on the isospin breaking effects, we are not longer allowed to refer to the scattering lengths in a given isospin state and new terms in addition to the previous ones arise. Furthermore, in principle relations such as Eq. (3.2) do not hold. This failure can be seen already at tree level in the isospin breaking contribution. The modified scattering lengths in the presence of isospin breaking can be split as

$$\begin{aligned}a_0(00;00)&=\tfrac{1}{3}a_0^{1/2}+\tfrac{2}{3}a_0^{3/2}+\Delta a_0(00;00),\\ a_0(+-;00)&=-\frac{\sqrt{2}}{3}a_0^{1/2}+\frac{\sqrt{2}}{3}a_0^{3/2}+\Delta a_0(+-;00),\\ a_0(+-;+-)&=\tfrac{2}{3}a_0^{1/2}+\tfrac{1}{3}a_0^{3/2}+\Delta a_0(+-;+-),\\ a_0(++;++)&=a_0^{3/2}+\Delta a_0(++;++),\end{aligned}$$

where $\Delta a_0(i,j,;l,m)$ represents the leading correction of scattering lengths due to the isospin breaking effects. As for the i, j and l, m in the arguments, they refer to the charges of the particles in the initial and final states, respectively. The evaluation of these corrections is straightforward and can be read from the scattering amplitude at leading order:

$$\begin{aligned}\Delta a_0(00;00)&=\frac{1}{32\pi F_0^2}\left[\frac{1}{4}(\Delta_\pi-\Delta_K)+\frac{\epsilon}{\sqrt{3}}(M_\pi^2+M_K^2)\right]\\ &=0.0032\ \ (0.0027),\\ \Delta a_0(+-;00)&=\frac{1}{32\pi\sqrt{2}F_0^2}\left[-\frac{\epsilon}{\sqrt{3}}(M_\pi^2+M_K^2)\right.\\ &\qquad\left.-\frac{\Delta_\pi}{4}\,\frac{3M_K+5M_\pi}{M_\pi+M_K}+\frac{\Delta_K}{4}\,\frac{M_\pi-M_K}{M_\pi+M_K}\right]\\ &=-0.0016\ \ (-0.0013),\\ \Delta a_0(+-;+-)&=\frac{\Delta_\pi}{32\pi F_0^2}=0.0014\ \ (0.0012),\\ \Delta a_0(++;++)&=\frac{\Delta_\pi}{32\pi F_0^2}=0.0014\ \ (0.0012),\end{aligned}$$

[1]See Sec. VII for the input values.
[2]In the following we shall use $F_K/F_\pi=1.22$.

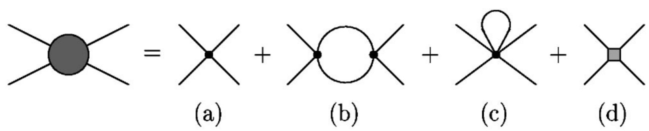

FIG. 1. Irreducible topologies. Vertices in (a), (b), and (c) come from either Eq. (2.3) or Eq. (2.7). The vertex in diagram (d) renders the amplitude ultraviolet finite.

where $\Delta_m = M^2_{m^\pm} - M^2_{m^0}$ and ϵ refers to the lowest-order π-η mixing angle. Notice that, whereas in the isospin limit the combination for the scattering lengths cancels in the $\pi^0 K^0 \to \pi^0 K^0$ process, this is no longer true when isospin breaking terms are considered. In the $\pi^- K^+ \to \pi^0 K^0$ process the isospin breaking in the combination of the scattering lengths is roughly two orders of magnitude smaller than the leading isospin limit quantity. Even though the future experimental bound on the combination of scattering lengths for this process is quite restrictive it is worthwhile to control higher-order corrections. For the rest of the processes the isospin effects are also rather small, roughly two orders of magnitude less than their isospin limit counterparts.

IV. $\pi^0 K^0 \to \pi^0 K^0$ PROCESS

Let us start by considering the following process:

$$\pi^0(p_{\pi_1}) K^0(p_{K_1}) \to \pi^0(p_{\pi_2}) K^0(p_{K_2}). \tag{4.1}$$

Our aim in this section will be to compute its amplitude taking into account all possible isospin breaking effects. Even if the process has not the same experimental interest as the $\pi^- K^+ \to \pi^0 K^0$ reaction, it is worth considering because both processes share almost the same features and complications, with the exception of the one-photon exchange contribution (see Sec. V A).

A. Kinematics

The amplitude for the process Eq. (4.1) can be studied on general grounds in terms of the Mandelstam variables

$$s = (p_{\pi_1} + p_{K_1})^2, \quad t = (p_{\pi_1} - p_{\pi_2})^2, \quad u = (p_{\pi_1} - p_{K_2})^2. \tag{4.2}$$

In the isospin limit and at lowest order of perturbation theory (corresponding to the partially conserved axial-vector current results) [see diagram (a) in Fig. 1], the off-shell amplitude is given by [14,15]

$$\mathcal{M}(s,t,u) = \frac{1}{6F_0^2}\left\{ M_K^2 + M_\pi^2 - \frac{u+s}{2} + t \right\}. \tag{4.3}$$

It is worth reviewing briefly the kinematics of the process that will be needed subsequently. In the center of mass frame the Mandelstam variables are defined in terms of q and θ by

$$s = \left(\sqrt{M^2_{\pi^0} + q^2} + \sqrt{M^2_{K^0} + q^2}\right)^2,$$

$$t = -2q^2(1 - \cos\theta),$$

$$u = \left(\sqrt{M^2_{\pi^0} + q^2} - \sqrt{M^2_{K^0} + q^2}\right)^2 - 2q^2(1 + \cos\theta).$$

B. General framework

Hitherto we have considered the process Eq. (4.1) at leading order. In this section we shall sketch the role of isospin breaking at next-to-leading order. Indeed, as mentioned in the Introduction, the isospin violating terms that are retained in π-K scattering differ from the ones in the π-π reaction due to the inclusion of the s quark. The latter process can be described fully in terms of SU(2) quantities where, for instance, the difference between the charged and neutral pion masses is of order δ^2 and thus can be disregarded, whereas in the former there exist intermediate strangeness states like π-K that give e and δ contributions.

In order to construct a matrix element consistent with the chiral power counting one has to take into account all possible scenarios for the construction of the graphs. For instance, given a generic $2 \to 2$ reaction mediated via the diagram (b) in Fig. 1 there are two different possibilities for incorporating isospin violating terms: (i) consider that one of the vertices breaks isospin through the e^2 terms or via the quark-mass difference, so that at the order we shall work at the other vertex and the two propagators are taken in the isospin limit; or (ii) if both vertices are taken in the isospin limit that forces us to consider the splitting between charged and neutral masses (in a given channel) in the same triplet for the pions or in the doublet for the kaons in the propagators. Thus within this prescription we shall consider in the chiral series terms up to and including δ and e^2 corrections as well as the usual p^4 at next-to-leading order.

The previous distinction, disentangling strong and electromagnetic contributions to the isospin breaking terms, is quite artificial as one realizes when the pseudoscalar masses are rewritten in terms of bare quantities. However, it constitutes a great conceptual help because ultraviolet divergences involving e^2 and δ terms do not mix at this order, allowing us to keep track of each term independently.

For the case we are interested in, involving only neutral particles, there is no direct exchange of virtual photons, and therefore the amplitude is safe from infrared singularities and all the e^2 dependence is due to the e.m. mass difference of the mesons or the integration of hard-photon loops. Hence, to obtain the amplitude including all $O(\alpha_{\mathrm{e.m.}})$ corrections, one needs to restrict the evaluation to the one-particle-irreducible diagrams depicted in Fig. 1 corresponding to the Born amplitude (a), unitary contribution (b), tadpole (c), and finally the counterterm piece (d). For the explicit expressions of this last contribution we refer the reader to the original literature [8,10].

To ascertain the correctness of our expression we look at the scale independence of the result once all contributions of the one-particle-irreducible diagrams are added, the wave function renormalization for the external field is taken into account, and the π-η mixing is treated correctly (see below). Furthermore, when we restrict the expressions to the isospin limit we recover the results given in [7].

Once the amplitude is finite in terms of bare quantities we have to renormalize the coupling constant F_0 and the masses appearing at lowest order. For the latter contribution we obtain agreement with the results quoted in [10] for the terms up to and including ϵ corrections and with [8] for the electromagnetic ones. For the former we shall use two choices: the first one is to fully renormalize F_0^2 as F_π^2 and for comparison purposes as the combination $F_\pi F_K$. To this end we use the isospin limit quantities [10]

$$F_\pi = F_0\left\{1+\frac{4}{F_0^2}(M_\pi^2+2M_K^2)L_4^r+\frac{4}{F_0^2}M_\pi^2 L_5^r-2\mu_\pi-\mu_K\right\} \tag{4.4}$$

and

$$F_K = F_0\left\{1+\frac{4}{F_0^2}(M_\pi^2+2M_K^2)L_4^r+\frac{4}{F_0^2}M_K^2 L_5^r-\frac{3}{4}\mu_\pi-\frac{3}{2}\mu_K-\frac{3}{4}\mu_\eta\right\}, \tag{4.5}$$

where μ_P is the finite part of the well-known tadpole integral. We do not use F_{π^0} in the numerical estimates of F_π because experimentally it is quite poorly known; instead we shall make use of the charged decay constant value.

The final contribution enters through the π-η mixing. At lowest order it is given by

$$\epsilon = \epsilon^{(2)} \equiv \frac{\sqrt{3}}{4}\frac{m_d-m_u}{m_s-\hat{m}}, \tag{4.6}$$

where $\hat{m}$ is the mean value of the light quark masses,

$$\hat{m} = \tfrac{1}{2}(m_u+m_d).$$

Notice that, given the order of accuracy we are considering, $\epsilon^{(2)}$ does not suffice and the next-to-leading order contribution $\epsilon^{(4)}$ to the mixing angle needs to be considered. We shall use the same approach as in [16], to which we refer for a more detailed explanation. It consists essentially in diagonalizing the mixing matrix at the lowest order, redefining in that way the π^0 and η fields, while higher-order terms in the mixing are treated by direct computation of the S-matrix off-diagonal elements. We have cross-checked our results explicitly using the procedure outlined in [17].

Taking into account all the mentioned contributions we obtain the renormalized S-matrix element $\mathcal{M}^{00;00}$ for the transition $\pi^0 K^0 \to \pi^0 K^0$ that is given in Appendix A, where we refer the reader for a detailed exposition.

V. $\pi^- K^+ \to \pi^0 K^0$ PROCESS

In this section we shall consider the more relevant process

$$\pi^-(p_{\pi^-})K^+(p_{K^+}) \to \pi^0(p_{\pi^0})K^0(p_{K^0}), \tag{5.1}$$

with the following Mandelstam variables:

$$s=(p_{\pi^-}+p_{K^+})^2, \quad t=(p_{\pi^-}-p_{\pi^0})^2, \quad u=(p_{\pi^-}-p_{K^0})^2. \tag{5.2}$$

In the center of mass frame these variables read

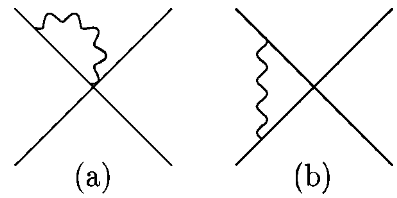

FIG. 2. Soft-photon contributions to the process $\pi^- K^+ \to \pi^0 K^0$. Diagram (a) has a crossed term. The photon interchanged in diagram (b) is possible only between the initial states.

$$s=(E_1+E_2)^2,$$

$$t=M_{\pi^\pm}^2+M_{\pi^0}^2-2E_1\sqrt{M_{\pi^0}^2+q'^2}+2qq'\cos\theta,$$

$$u=M_{\pi^\pm}^2+M_{K^0}^2-2E_1\sqrt{M_{K^0}^2+q'^2}-2qq'\cos\theta, \tag{5.3}$$

with

$$E_1^2=M_{\pi^\pm}^2+q^2,$$

$$E_2^2=M_{K^\pm}^2+q^2,$$

$$q'^2=\frac{1}{4}\left[E_1+E_2+\frac{M_{\pi^0}^2-M_{K^0}^2}{M_{\pi^\pm}^2-M_{K^\pm}^2}(E_1-E_2)\right]^2-M_{\pi^0}^2, \tag{5.4}$$

and q (q') the three-momentum of the charged (neutral) particles.

As has been pointed out earlier, the relevance of this process is intimately related to the lifetime of the π-K system, even though we want to stress that our formalism allows us to deal only with free, on-shell external particles, in clear contrast with the π-K atom where the states are bounded and off shell [18,19]. We shall not pursue the more complete approach here.

For the construction of most of the graphs (those equivalent to Fig. 1) we shall use the same arguments presented in the preceding section. For the remaining ones we shall sketch their treatment in the next section.

A. Soft-photon contribution

In the case of the $\pi^- K^+ \to \pi^0 K^0$ process, as well as the corrections due to the mass difference of the up and down quarks and those generated by the integration of hard photons, one has to consider corrections due to virtual photons. At order $e^2 p^2$ these corrections arise from the wave function renormalization of the charged particles and from the one-photon exchange diagrams depicted in Fig. 2. The result of diagram (a) in this figure reduces at threshold to combinations of polynomials and logarithms, whereas the second, (b), needs closer consideration. It develops a singular behavior at threshold. This singularity issues from the ultraviolet finite three-point function C defined by

$$C(M_P^2,M_Q^2,m_\gamma^2;p_1,p_2) \equiv \frac{1}{i}\int\frac{d^dl}{(2\pi)^d} \times\frac{1}{(l^2-m_\gamma^2)[(p_1-l)^2-M_P^2][(p_2-l)^2-M_Q^2]}, \quad (5.5)$$

with the on-shell conditions $p_1^2=M_P^2$, $p_2^2=M_Q^2$ and m_γ^2 acting as an infrared cutoff for the photon mass. Using standard techniques, the integral (5.5) can be expressed in terms of logarithms and dilogarithms. For $p^2\equiv(p_1-p_2)^2>(M_P+M_Q)^2$ and $m_\gamma^2\to 0$ it is given as follows:

$$\begin{aligned}
&32\pi^2\lambda_{PQ}^{1/2}(p^2)C(M_P^2,M_Q^2,m_\gamma^2;p_1,p_2)\\
&=-\left\{\log\left[\frac{\lambda_{PQ}(p^2)}{p^2m_\gamma^2}\right]-\frac{1}{2}\log\left[\frac{[\Delta_{PQ}-\lambda_{PQ}^{1/2}(p^2)]^2-p^4}{[\Delta_{PQ}+\lambda_{PQ}^{1/2}(p^2)]^2-p^4}\right]-i\pi\right\}\\
&\quad\times\left\{\log\left[\frac{[p^2-\lambda_{PQ}^{1/2}(p^2)]^2-\Delta_{PQ}^2}{[p^2+\lambda_{PQ}^{1/2}(p^2)]^2-\Delta_{PQ}^2}\right]+2i\pi\right\}\\
&\quad+2\,\mathrm{Li}_2\left[\frac{p^2+\Delta_{PQ}+\lambda_{PQ}^{1/2}(p^2)}{p^2+\Delta_{PQ}-\lambda_{PQ}^{1/2}(p^2)}\right]\\
&\quad-2\,\mathrm{Li}_2\left[\frac{p^2-\Delta_{PQ}-\lambda_{PQ}^{1/2}(p^2)}{p^2-\Delta_{PQ}+\lambda_{PQ}^{1/2}(p^2)}\right], \qquad (5.6)
\end{aligned}$$

where

$$\Delta_{PQ}\equiv M_P^2-M_Q^2,$$

$$\lambda_{PQ}(p^2)\equiv[p^2-(M_P+M_Q)^2][p^2-(M_P-M_Q)^2],$$

and finally the dilogarithm function is defined as

$$\mathrm{Li}_2(z)\equiv-\int_0^z dt\,\frac{\log(1-t)}{t}.$$

We have performed several checks on the validity of the expression (5.6): (i) when reduced to the equal mass case we recover the result as given in [20] and (ii) it numerically agrees with the results given in [21] for the general case.

The contribution of the C function via the diagram (b) in Fig. 2 to the amplitude is [cf. Eq. (B20)]

$$\frac{e^2}{\sqrt{2}F_0^2}(s-u)(s-M_\pi^2-M_K^2)C(M_\pi^2,M_K^2,m_\gamma^2;p_{\pi^-},-p_{K^+}).$$

Expanding the real part of the preceding function in the vicinity of the threshold by the use of Eqs. (5.6) and (5.4) one obtains a Coulomb type behavior, i.e., q^{-1}. Then schematically the threshold expansion of the real part of the amplitude takes the following form:

$$\mathrm{Re}\,\mathcal{M}^{-+;00}(s,t,u)=-\frac{M_{\pi^\pm}M_{K^\pm}}{\sqrt{2}F_0^2}\frac{e^2}{4}\frac{\mu_{\pi K}}{q}+\mathrm{Re}\,\mathcal{M}_{\mathrm{thr}}^{-+;00}+O(q), \qquad (5.7)$$

with

$$\mu_{\pi K}=\frac{M_{\pi^\pm}M_{K^\pm}}{M_{\pi^\pm}+M_{K^\pm}} \qquad (5.8)$$

the reduced mass of the π-K system.

A remarkable feature of the threshold expansion is that neither $\mathrm{Re}\,\mathcal{M}_{\mathrm{thr}}^{-+;00}$ nor the long-range force of the photon exchange is affected by the infrared singularity, which contributes only to $O(q^2)$ or to higher-order terms. These infrared terms arise from the wave function renormalization and from the three-point scalar integral function via the diagram (b) in Fig. 2. Adding the infrared contributions of both pieces one gets

$$\mathrm{Re}\,\mathcal{M}_{\mathrm{ir}}^{-+;00}=\left(\frac{s-u}{\sqrt{2}F_0^2}\right)\left[\frac{e^2}{16\pi^2}\log(m_\gamma^2)\right]\times\left\{1+\frac{1}{2}\frac{s-\Sigma_{\pi K}}{\lambda_{\pi K}^{1/2}(s)}\log\left(\frac{[s-\lambda_{\pi K}^{1/2}(s)]^2-\Delta_{\pi K}^2}{[s+\lambda_{\pi K}^{1/2}(s)]^2-\Delta_{\pi K}^2}\right)\right\}$$

with

$$\Sigma_{mn}=M_m^2+M_n^2.$$

As is expected, when evaluated at threshold, this expression vanishes, rendering $\mathrm{Re}\,\mathcal{M}_{\mathrm{thr}}^{-+;00}$ as an infrared finite quantity. Although the scattering lengths defined in this way are infrared finite the slope parameters are not. The proper definition of an infrared finite observable requires one in addition to take into account the real emission of a soft photon from the external particles. Notice that the experimental data will include this bremsstrahlung effect. The cancellation of infrared divergences takes place order by order in $\alpha_{\mathrm{e.m.}}$ and therefore it is just sufficient in our case to consider one single-photon emission, whose amplitude reads

$$\begin{aligned}
\mathcal{M}^{-+;00\gamma}=\frac{e}{2\sqrt{2}F_0^2}\epsilon_\mu(k_\gamma)\Bigg[&-2(p_{\pi^0}-p_{K^+})^\mu+(p_{K^+}+p_{K^0})\cdot(p_{\pi^+}+p_{\pi^0}-k_\gamma)\frac{(2p_{\pi^+}-k_\gamma)^\mu}{m_\gamma^2-2p_{\pi^+}\cdot k_\gamma}\\
&-(p_{\pi^+}+p_{\pi^0})\cdot(p_{K^-}+p_{K^0}-k_\gamma)\times\frac{(2p_{K^-}-k_\gamma)^\mu}{m_\gamma^2-2p_{K^-}\cdot k_\gamma}\Bigg],
\end{aligned}$$

with k_γ and $\epsilon_\mu(k_\gamma)$ being the momenta of the photon and its polarization vector, respectively. As a result, one can write the infrared finite cross section including all $O(\alpha_{\mathrm{e.m.}})$ but neglecting $O(\alpha_{\mathrm{e.m.}}^2)$ terms as

$$\sigma(s;\Delta E)=\sigma^{-+;00}(s)+\sigma^{-+;00\gamma}(s;\Delta E), \qquad (5.9)$$

where ΔE stands for the detector resolution. Once this is done, the corrected scattering length $\tilde{a}_0(+-;00)$ might be defined from the threshold expansion of the infrared finite cross section Eq. (5.9) by subtracting the Coulomb pole term and excluding the corrections due to the mass squared differences in the phase space in the following way [22]:

$$\sigma(s;\Delta E)=\frac{1}{32\pi s}\frac{\lambda^{1/2}_{\pi^0K^0}(s)}{\lambda^{1/2}_{\pi^-K^+}(s)}\left\{-\frac{M_{\pi^\pm}M_{K^\pm}}{\sqrt{2}F_0^2}\frac{e^2}{4}\frac{\mu_{\pi K}}{q}+32\pi\tilde{a}_0(+-;00)+O(q)\right\}^2. \quad (5.10)$$

We have checked that the corrections due to real soft-photon emission are negligible and thus we expect that the corrected scattering length $\tilde{a}_0(+-;00)$ will differ only beyond our accuracy from the one obtained using Eq. (5.7), that is, from the infrared finite real part of the scattering amplitude at threshold.

Finally, in order to present our results we shall collect in a single, infrared finite expression (denoted by $\mathcal{M}_{\text{soft photon}}$ in the tables) the contributions at threshold of Eqs. (B6), (B7), (B20).

VI. THRESHOLD EXPANSION

Let us explain how we obtain the scattering lengths from the scale invariant amplitudes Eqs. (A1) and (B1). Since we are interested in only the S-wave threshold parameters it is just sufficient to expand the scattering amplitude around the threshold values. Even so, this step is not quite straightforward because in order to match the prescription given in [14] for the scattering lengths we need to shift the isospin limit and neutral masses to the corresponding charged ones. The procedure is rather cumbersome and we shall use the following substitutions for the masses:

$$M^2_{\pi^0}\to M^2_{\pi^\pm}-\Delta_\pi, \quad M^2_{K^0}\to M^2_{K^\pm}-\Delta_K, \quad (6.1)$$

where Δ_i are small quantities that in the case of pions contain at leading order e^2 pieces while for kaons they contain both e^2 and ϵ terms. It is therefore sufficient to expand all quantities up to first order in Δ_i. For instance, in the charged$\to$neutral transition we obtain, for the kinematical variables,

$$s=(M_{\pi^\pm}+M_{K^\pm})^2,$$

$$t=-\frac{M_{K^\pm}}{M_{\pi^\pm}+M_{K^\pm}}\Delta_\pi-\frac{M_{\pi^\pm}}{M_{\pi^\pm}+M_{K^\pm}}\Delta_K,$$

$$u=(M_{\pi^\pm}-M_{K^\pm})^2-\frac{M_{\pi^\pm}}{M_{\pi^\pm}+M_{K^\pm}}\Delta_\pi-\frac{M_{K^\pm}}{M_{\pi^\pm}+M_{K^\pm}}\Delta_K. \quad (6.2)$$

Once this step is performed, and in order to bookkeep the power counting properly, any charged mass multiplying Δ_i or ϵ pieces is settled at its isospin limit:

$$\frac{M_{\pi^\pm}}{M_{\pi^\pm}+M_{K^\pm}}\approx\frac{M_\pi}{M_\pi+M_K}[1+O(\epsilon)+O(e^2)]. \quad (6.3)$$

Only for estimating higher-order corrections shall we eventually keep the charged masses in the ratios introduced in Eq. (6.2).

Even though the outlined procedure for shifting the neutral masses is rather involved, it allows us to expand all one-loop integrals in an analytical form with quite compact expressions. For instance, in the s channel for the neutral $\to$neutral transition we obtain after performing the mentioned steps the following expansion:

$$\begin{aligned}\mathrm{Re}\,\bar{J}(M^2_{\pi^0},M^2_{K^0};(M_{\pi^0}+M_{K^0})^2)&=\frac{1}{16\pi^2}\left[1+\frac{M_{\pi^\pm}M_{K^\pm}}{M^2_{K^\pm}-M^2_{\pi^\pm}}\log\left(\frac{M^2_{K^\pm}}{M^2_{\pi^\pm}}\right)\right]\\&-\frac{1}{32\pi^2}\frac{1}{(M_K^2-M_\pi^2)^2}\left(\frac{M_K}{M_\pi}\Delta_\pi-\frac{M_\pi}{M_K}\Delta_K\right)\\&\times\left[-2(M_K^2-M_\pi^2)+(M_\pi^2+M_K^2)\log\left(\frac{M_K^2}{M_\pi^2}\right)\right].\end{aligned}$$

For loop functions involving the η mass the use of the Gell-Mann–Okubo relation reduces the expression considerably.

In the neutral$\to$neutral process, although the kinematics allows $t\propto q^2\approx 0$ at threshold, there is no need to consider the expansion of the $\bar{J}$ function in powers of q. This is the case because all channel contributions behave as polynomial$\times\bar{J}$ without any inverse power of the kinematical variables (t in this case). This does not turn out to be the case in the charged$\to$neutral transition. There, one deals with terms like $\bar{J}/t$, where in the isospin limit $t\propto q^2\to 0$. Expanding near threshold $\bar{J}[m_1^2,m_2^2;t(q^2)]\approx bq^2+\cdots$ we shall obtain contributions from terms linear in q^2. Apart from this, there are no more differences in the treatments of the two processes.

Because of the relevance of the process $\pi^-K^+\to\pi^0K^0$ we collect, as well as the expression for the scattering amplitude in Appendix B 1, the expression for the S-wave scattering lengths in Appendix B 2.

VII. RESULTS AND DISCUSSION

Let us first point out that, because of the high threshold of production $\sqrt{s_{\text{th}}}\sim 632$ MeV, it is not necessarily true that a single one-loop calculation is enough to approach the physical values for the scattering lengths. Because of the fuzzy existing π-K data, we consider that this question can only be answered once the size of the next-to-next-to-leading order term is computed. Also, there are openings for intermediate particle production, for instance $K\bar{K}$ in the t channel, that presumably strongly affect the chiral series convergence.

A. Input parameters

Before presenting our results we want to stress the relevance of the π-K scattering process. Its importance goes beyond the determination of some threshold quantities and can play a key role in our knowledge of spontaneous symmetry breaking. Up to now chiral perturbation theory has been used to *parametrize* the low-energy QCD phenomenology. Lacking enough processes to determine all low-energy constants, one has to resort to theoretical inputs (or prejudices). For instance, the low-energy constants in the electromagnetic sector have in general a quite mild impact on the results and therefore have been relegated to a secondary place and have only recently received some attention [12,13] due to increasing precision in the experiment. But very little is known about them with the exception of model estimates. In our treatment these constants can play an important role, and we include them as given in [12], where they are estimated by means of resonance saturation. (Hereafter all our results are given at the scale $\mu = M_\rho$.)

$$K_1^r = -6.4\times 10^{-3}, \quad K_2^r = -3.1\times 10^{-3},$$

$$K_3^r = 6.4\times 10^{-3}, \quad K_4^r = -6.2\times 10^{-3},$$

$$K_5^r = 19.9\times 10^{-3}, \quad K_6^r = 8.6\times 10^{-3},$$

$$K_7^r,\ldots,K_{10}^r = 0, \quad K_{11}^r = 0.6\times 10^{-3},$$

$$K_{12}^r = -9.2\times 10^{-3}, \quad K_{13}^r = 14.2\times 10^{-3},$$

$$K_{14}^r = 2.4\times 10^{-3}.$$

If instead we use a naive dimensional analysis the value assigned to each of them would be restricted to be inside the range

$$|K_i^r| \lesssim \frac{1}{16\pi^2},$$

which is taken as a crude indication of the error. Notice that the central values quoted in [23,13] lie inside this error band.

Contrary to the previous case the low-energy constants in the strong sector are better known. In a series of works [10,24] most of the next-to-leading low-energy constants were pinned down. In addition to the experimental datum large-N_c arguments were used to settle the marginal relevance of some operators (those entering together with L_4 and L_6). The use of π-K data in the $T^{3/2}$ channel can disentangle (in principle) the value of L_4^r, due to its product with M_K^2 which enhances its sensitivity to the role of m_s [25].

In order to have more complete control over our results we use two different sets of constants [16]. The first one was obtained by fitting simultaneously the next-to-leading expressions of the meson masses, decay constants, and threshold values of the K_{l4} form factors to their experimental values.[3] We shall refer to it as set I and it is given by[4]

$$10^3\times L_1^r = 0.46\pm 0.23, \quad 10^3\times L_2^r = 1.49\pm 0.23,$$

$$10^3\times L_3^r = -3.18\pm 0.85,$$

$$10^3\times L_4^r = 0\pm 0.5^*, \quad 10^3\times L_5^r = 1.46\pm 0.2,$$

$$10^3\times L_6^r = 0\pm 0.3^*,$$

$$10^3\times L_7^r = -0.49\pm 0.15, \quad 10^3\times L_8^r = 1.00\pm 0.20.$$

The second (set II) is obtained with the same inputs and under the same assumptions as the previous one, but this time the fitted expressions are next-to-next-to-leading quantities:

$$10^3\times L_1^r = 0.53\pm 0.25, \quad 10^3\times L_2^r = 0.71\pm 0.27,$$

$$10^3\times L_3^r = -2.72\pm 1.12,$$

$$10^3\times L_4^r = 0\pm 0.5^*, \quad 10^3\times L_5^r = 0.91\pm 0.15,$$

$$10^3\times L_6^r = 0\pm 0.3^*,$$

$$10^3\times L_7^r = -0.32\pm 0.15, \quad 10^3\times L_8^r = 0.62\pm 0.20.$$

As one can see by comparing both sets, the central value of some of the low-energy constants is sizably shifted from one to the other. We stress at this point that the error in the determination of the scattering lengths is mainly associated with the errors on the low-energy constants. The other quantities involved in the calculation are hadron masses, and for those there are rather accurate determinations. For the latter we use [26]

$$M_{\pi^\pm} = 139.570 \text{ MeV}, \quad M_{\pi^0} = 134.976 \text{ MeV},$$

$$M_{K^\pm} = 493.677 \text{ MeV}, \quad M_{K^0} = 497.672 \text{ MeV}.$$

Notice that in principle there is no need to consider M_η^2 as an additional input. It always appears through loop propagators and therefore it is sufficient to consider it via the Gell-Mann–Okubo relation in the presence of isospin breaking (including e.m. corrections),

$$\Delta_{\rm GMO} \equiv M_\eta^2 - \tfrac{2}{3}(M_{K^\pm}^2 + M_{K^0}^2) + \tfrac{1}{3}(2M_{\pi^\pm}^2 - M_{\pi^0}^2) = 0. \tag{7.1}$$

However, we shall also use the value $M_\eta = 547.30$ MeV and consider the difference as an indication of higher-order corrections.

Furthermore, we make use of the isospin limit quantities M_π and M_K, which can be defined through combinations of the physical masses

$$M_\pi^2 = M_{\pi^0}^2, \quad M_K^2 = \tfrac{1}{2}(M_{K^\pm}^2 + M_{K^0}^2 + \gamma[M_{\pi^0}^2 - M_{\pi^\pm}^2]).$$

[3]Notice that in the decay $K\to\pi\pi l v$ the s quark is involved. Thus although in principle it can be used to obtain the value of L_4^r the form factors F and G turn out to be rather insensitive to its actual value [16].

[4]Quantities with an asterisk are theoretical inputs.

TABLE I. Different chiral contributions to the combination of scattering lengths proportional to $a_0(00;00)$. We have chosen to renormalize the coupling constant as F_π^2 with the value $F_\pi = 92.4$ MeV. In parentheses we show the values obtained with set I.

	$a_0^{1/2}+2a_0 3/2$	ϵ	Δ_π	Δ_K	e^2
Tree	—	0.0053	0.0011	0.0034	—
Born (a)	−0.0038 (−0.0106)	−0.0021 (−0.0029)	−0.0007 (−0.0011)	−0.0020 (−0.0030)	0
Born (b)	0.0084 (0.0484)	0.0019 (0.0036)	0 (−0.0022)	−0.0001 (0.0002)	—
Born (c)	—	—	—	—	0
Mixing	—	0.0014	—	0	—
F_π^2	—	0.0014 (0.0014)	0.0003 (0.0002)	0.0009 (0.0009)	—
Tadpole	−0.0458	0.0021	0	−0.0007	—
s channel	0.0520	−0.0008	−0.0021	0.0036	—
t channel	0	−0.0008	0	0	—
u channel	0.0592	−0.0013	−0.0018	0.0001	—
	0.0700 (0.1032)	0.0070 (0.0080)	−0.0032 (−0.0059)	0.0052 (0.0045)	0

The factor γ will take into account any deviation from Dashen's theorem [27]. At lowest order ($\gamma=1$) the e.m. relation between the pseudoscalar masses reads

$$(M_{K^\pm}^2 - M_{K^0}^2)|_{\text{e.m.}} = M_{\pi^\pm}^2 - M_{\pi^0}^2.$$

Next-to-leading contributions to the previous relation can be quite sizable. As an indication we shall use [23]

$$(M_{K^\pm}^2 - M_{K^0}^2)|_{\text{e.m.}} = (1.84 \pm 0.25)(M_{\pi^\pm}^2 - M_{\pi^0}^2). \quad (7.2)$$

For the π-η mixing angle at lowest order we use the value [28]

$$\epsilon = (1.061 \pm 0.083) \times 10^{-2}.$$

The only remaining input is

$$F_\pi = 92.4 \text{ MeV},$$

corresponding to the charged pion decay constant with the electromagnetic effects removed [29].

B. Scattering lengths

The studies on the π-K scattering started quite early [14,15] within a current algebra approach and followed later by a series of works based on dispersion relations by means of unitarity and crossing symmetry [30] (see also [30–32] for recent work using the same technique). With the advent of chiral perturbation theory the process was analyzed once more [7]. Nowadays, its interest has been revived and new approaches like the inverse amplitude method [31–35], or like the treatment of kaons as heavy particles [36] have been considered. Broadly speaking, all mentioned techniques lead to a fairly constant prediction for the scattering lengths, inside the range [0.16, 0.24] for $a_0^{1/2}$ and [−0.05, −0.07] for $a_0^{3/2}$. The corrections to the current algebra values [cf. Eq. (3.6)] are roughly 20% for $a_0^{1/2}$ and 30% for $a_0^{3/2}$, thus being in the ballpark of the usual shifts between the next-to-leading and leading order quantities in processes where chiral perturbation applies.

Even if it seems that from the theoretical point of view there is some general consensus the experimental results on the scattering lengths are more widespread. In the earlier experiments most of the collected data on π-K were obtained via the scattering of kaons on proton or neutron targets. The data were then analyzed by determining the contribution of the one-pion exchange. This technique does not allow one to obtain the initial pion on the mass shell and hence some model dependent extrapolation is needed. The central values obtained are $a_0^{1/2} \in [0.168, 0.335]$ [37–40] and $a_0^{3/2} \in [-0.072, -0.14]$ [40–45]. Later experiments analyzed the reaction near the threshold using dispersive techniques. Even though the experimental results are slightly improved with respect to the oldest ones, the threshold parameters are known only within a factor of 2 [46–48]. A more interesting analysis was performed in [48]. There, using the forward sum rule and measured phase shifts a direct determination of the combination $|a^{1/2} - a^{3/2}|$ was obtained with the result

$$0.21 \leqslant a^{1/2} - a^{3/2} \leqslant 0.32. \quad (7.3)$$

This will be used as a reference point for us.

Let us turn now to discuss our findings. In Table I we have collected the partial contributions to each of the terms given in Appendix A with F_π^2 as the renormalization choice for the coupling constants. We show the isospin limit contribution (first column) and the different corrections to it. We shall bear in mind in the remainder the following set of conditions: (i) We have used the Gell-Mann–Okubo relation for the eta mass inside the loop functions; (ii) we have kept only the first term in the right-hand side (RHS) of Eq. (6.3); and (iii) we assume Dashen's theorem to hold at next-to-leading order. (We shall discuss below the uncertainties associated with these considerations.)

In parentheses are quoted the contributions corresponding to set I instead of set II. The first thing to remark is that isospin breaking corrections are roughly of the same order of magnitude as the isospin limit quantities. It is worth recalling that the isospin limit correction comes purely from the next-to-leading terms.

The final result is given by

TABLE II. Different chiral contributions to the combination of scattering lengths proportional to $\mathrm{Re}\,\mathcal{M}^{+-;00}$. We have choose to renormalize F_0^2 as F_π^2 with the value $F_\pi = 92.4$ MeV. We show the first four digits without any rounding. Quantities in parentheses correspond to set I.

	$a_0^{1/2} + 2a_0 3/2$	ϵ	Δ_π	Δ_K	e^2
Tree	0.2408	0.0026	0.0018	−0.0009	—
Born (a)	−0.1060 (−0.1389)	−0.0010 (−0.0014)	−0.0007 (−0.0008)	−0.0001 (0.0003)	—
Born (b)	0.0540 (0.0867)	0.0006 (0.0012)	−0.0006 (−0.0008)	0 (−0.0005)	—
Mixing	—	0.0010	—	0	0
F_π^2	0.0664 (0.0688)	0.0007 (0.0007)	−0.0008 (−0.0009)	−0.0003 (−0.0003)	—
Tadpole	0.0415	0.0011	0.0002	−0.0005	—
s channel	0.0283	0.0013	0.0005	0	—
t channel	−0.0312	−0.0004	0.0001	−0.0004	—
u channel	−0.0265	−0.0009	0	0.0001	—
Soft photon	—	—	—	—	−0.0004
	0.2674 (0.2695)	0.0047 (0.0053)	0.0005 (0)	−0.0011 (−0.0020)	−0.0004

$$3a_0(00;00) = a_0^{1/2} + 2a_0^{3/2} + 3\Delta_0(00;00) = 0.0700 + 0.0090 \pm 0.0511 \quad (0.1032 + 0.0066 \pm 0.0462), \tag{7.4}$$

renormalizing F_0^2 as F_π^2. The first quoted number corresponds to the isospin limit and the second to the isospin breaking corrections. Interesting enough are the assigned errors, which wash out any sensitivity with respect to the choice of the set or to any of the choices in the renormalization of the coupling constant (see below). They come about through the uncertainty in the low-energy constants and we have propagated them in quadrature. The dominant contributions by far come from the low-energy constants L_3^r and L_4^r of the pure strong sector, whereas the electromagnetic ones give an imperceptible contribution. Unfortunately, this is not an experimental mode because this strong sensitivity would constitute a cross-check on the consistency of the values for L_3^r and L_4^r. For the sake of completeness we also show the final result if instead we renormalize F_0^2 as $F_\pi F_K$ and use set II:

$$3a_0(00;00) = a_0^{1/2} + 2a_0^{3/2} + 3\Delta_0(00;00) = 0.0423 + 0.0172 \pm 0.0343. \tag{7.5}$$

Let us turn now to discuss the most interesting mode, the charged→neutral transition. The experimental proposal claims an accuracy of 20–30 % on the measurement of the lifetime; this roughly translates to 10–15 % accuracy for the determination of the scattering lengths. As before we have disentangled all contributions and explored all possible scenarios allowed by the input parameters. Even though the most natural choice for the coupling constants is $F_\pi F_K$ we shall proceed as in the previous case and also show the results using F_π^2 as an indication of the sensitivity to this parameter.

We have collected the results for F_π^2 in Table II. As one can see the difference between the two sets is at most 2% in the final result. Also, the isospin breaking effects are roughly two orders of magnitude smaller than the isospin limit quantity. In this case it seems that the experimental setup sensitivity is not enough to detect the new effect we have incorporated. Adding all partial contributions we obtain

$$-\frac{3}{\sqrt{2}} a_0(+-;00) = a_0^{1/2} - a_0^{3/2} - \frac{3}{\sqrt{2}}\Delta_0(+-;00) = 0.2674 + 0.0037 \pm 0.0022 \quad (0.2695 + 0.0033 \pm 0.0023). \tag{7.6}$$

We stress that the content of the previous expression is indicative only, because of the way we have chosen to renormalize the coupling constant. Notice that the assigned errors are on the same footing as the isospin breaking terms. A closer look at the errors reveals that they have mainly an electromagnetic origin. Furthermore, they are clearly dominated by the K_{10}^r and K_{11}^r low-energy constants. In the strong sector the dominant errors come (by order of dominant contribution) from L_5^r and L_8^r. Those low-energy constants are strongly correlated.

Our main results are collected in Table III. It corresponds to the choice of renormalization $F_\pi F_K$ for the decay constant. Once more isospin breaking effects turn out to be two orders of magnitude smaller than the isospin limit quantities. Furthermore, like Table II but in a more accentuated way, Table III shows a strong cancellation between the isospin breaking contributions (see, for instance, the contributions of Δ_π and Δ_K). Adding all contributions from the table, one obtains

$$-\frac{3}{\sqrt{2}} a_0(+-;00) = 0.2412 + 0.0037 \pm 0.0034 \quad (0.2520 + 0.0038 \pm 0.0043). \tag{7.7}$$

Notice that once more the theoretical errors are small but still competitive in size with the isospin breaking effects. It also seems that with this choice of renormalization of the cou-

TABLE III. Different chiral contributions to the combination of scattering lengths proportional to Re $\mathcal{M}^{+-;00}$. The entries in the table differ from Table II just in the renormalization of the coupling constant. Here we have used the combination $F_\pi F_K$ with the constraint $F_K/F_\pi=1.22$ and the value $F_\pi=92.4$ MeV. We shows the first four digits without any rounding. Quantities in parentheses correspond to set I.

	$a_0^{1/2}+2a_0 3/2$	ϵ	Δ_π	Δ_K	e^2
Tree	0.1974	0.0021	0.0015	−0.0008	—
Born (a)	−0.0712 (−0.0933)	−0.0007 (−0.0009)	−0.0004 (−0.0005)	0 (0.0002)	—
Born (b)	0.0363 (0.0582)	0.0004 (0.0008)	−0.0004 (−0.0005)	0 (0)	—
Mixing	—	0.0007	—	0	0
$F_\pi F_K$	0.0705 (0.0815)	0.0007 (0.0008)	0 (0)	−0.0001 (−0.0001)	—
Tadpole	0.0279	0.0007	0.0001	−0.0003	—
s channel	0.0192	0.0009	0.0003	0	—
t channel	−0.0209	−0.0003	0.0001	0.0002	—
u channel	−0.0179	−0.0006	0	0.0001	—
Soft photon	—	—	—	—	−0.0003
	0.2412 (0.2520)	0.0038 (0.0043)	0.0011 (0.0009)	−0.0009 (−0.0011)	−0.0003

pling constants we weight more the role of the low-energy constant L_5^r (this is reflected in the sizable uncertainty if we use set I). It is worth stressing at this point that the expression for the amplitude for the charged→neutral transition does not depend on the low-energy constant L_4^r whereas L_6^r only occurs inside isospin breaking terms. Because of the numerical irrelevance of these last terms and even lacking a complete knowledge of the role of large-N_c suppressed operators, our estimates (in that sense) are precise and unambiguous. The difference between the central values of the two sets is at most 5%. Hence, without taking into account the error bars, and in the most optimistic case, 10% of accuracy in the experimental result, any sensitivity to the set of low-energy constants is just borderline. Before concluding let us comment on the role of the physical eta mass, the full expression in the LHS in Eq. (6.3), and Dashen's theorem Eq. (7.2). If instead of using the Gell-Mann–Okubo relation (7.1) one uses the physical eta mass, the central value in Eq. (7.7) increases by $\sim$0.0003. Higher-order terms as introduced by Eq. (6.3) also increase Eq. (7.7) roughly by an amount $\sim$0.003. And finally the shift allowed by the violation of Dashen's theorem and by the error induced by the lowest π-η mixing goes beyond the accuracy we quote. A consistent way of incorporating these effects in our estimates is to consider them as a crude guess at higher-order corrections, and we shall treat them as theoretical uncertainties; therefore we add all three in quadrature and to the previous error bar in Eq. (7.7). This leads to our final estimate for the combination of scattering lengths:

$$-\frac{3}{\sqrt{2}}a_0(+-;00)=a_0^{1/2}-a_0^{3/2}-\frac{3}{\sqrt{2}}\Delta_0(+-;00)$$
$$=\begin{cases}(\text{set I}) & 0.2520+0.0038\pm0.0073,\\ (\text{set II}) & 0.2412+0.0037\pm0.0045.\end{cases} \tag{7.8}$$

VIII. PERSPECTIVES

Using the previous estimates for the combination of scattering lengths, Eq. (7.8), we can partially calculate the lifetime of the $A_{\pi K}$ atoms. The probability of the transition of an $A_{\pi K}$ atom

$$A_{\pi K}\to\pi^0+K^0$$

can be cast as

$$W_{n,0}(\pi^0K^0)\propto(\mathrm{Re}\,\mathcal{M}_{\mathrm{thr}}^{\pm;0})^2.$$

Focusing on the isospin limit, the previous relation is given by [6]

$$W_{n,0}(\pi^0K^0)\approx\frac{8\pi}{9}\left(\frac{2\Delta m}{\mu_{\pi K}}\right)^{1/2}\times\frac{(a_0^{1/2}-a_0^{3/2})^2|\Psi_{n,0}(0)|^2}{1+\frac{2}{9}\mu_{\pi K}\Delta m(a_0^{1/2}+2a_0^{3/2})^2}+\cdots\approx\frac{1}{\tau_{n,0}}, \tag{8.1}$$

where n is the principal quantum number,

$$\Delta m=(M_{K^\pm}+M_{\pi^\pm})-(M_{\pi^0}+M_{K^0}), \tag{8.2}$$

and $\mu_{\pi K}$ is given in Eq. (5.8). Finally, $\Psi_{n,0}(0)$ is the $A_{\pi K}$ Coulomb wave function at the origin and is given by

$$|\Psi_{n,0}(0)|^2=\frac{p_B^3}{\pi n^3},\quad p_B=\frac{e^2}{4\pi}\mu_{\pi K}. \tag{8.3}$$

Furthermore, the orbital angular momentum l can be safely taken equal to zero. If one allows $l>0$ the transition is suppressed by chiral power counting, i.e., it turns to be of $O(e^4)$. Notice that in the previous relation (8.1) enter precisely both combinations of scattering lengths we have found, $a_0^{1/2}-a_0^{3/2}$ and $a_0^{1/2}+2a_0^{3/2}$. Even so, the term added to unity in the denominator of Eq. (8.1) can be neglected, since it is of order 10^{-5}, and thus beyond the accuracy of our calculation or the experimental sensitivity; hence only the combination $a_0^{1/2}-a_0^{3/2}$ turns out to be of relevance. This simplify the expression slightly to

$$\tau_{1,0}\approx\frac{0.274\times10^{-15}}{M_{\pi^\pm}^2}\left(\frac{2}{9}\right)a_0(+-;00)^{-2}. \tag{8.4}$$

Inserting Eq. (7.8) in Eq. (8.4) one finds the following value for the $A_{\pi K}$ lifetime in the ground state:

$$\tau_{1,0}=4.58\times10^{-15}\text{ s}\quad(4.20\times10^{-15}\text{ s}), \tag{8.5}$$

where the quantity quoted in the parentheses corresponds to set I.

Let us stress once more that in writing Eq. (8.1) we have not taken into account any source of isospin breaking and consequently we are only halfway to a theoretical determination of the π-K lifetime. Hence the content of Eq. (8.5) is just indicative. In fact Eq. (8.1) is not suitable for handling bound state systems, the framework of a nonrelativistic Lagrangian [49] being by far the most efficient way of treating bound states. Even so, without any further control over the errors on the low-energy constants it seems not worthwhile to pursue this analysis.

IX. SUMMARY

In this work we have estimated the role of isospin breaking effects in the transitions $\pi^0K^0\to\pi^0K^0$ and $\pi^-K^+\to\pi^0K^0$. They turn out to be rather mild. Furthermore, the former reaction is quite interesting because of its sensitivity to the large-N_c suppressed operator involving L_4^r. From a more practical point of view (essentially because of the existence of experimental data) we have carefully evaluated the shift in the scattering lengths for the $\pi^-K^+\to\pi^0K^0$ reaction. While the error from the low-energy constants turns out to be compatible with future experimental sensitivity, estimates of higher-order corrections are on the same footing. Contrary to the previous case this reaction does not contain sizable contributions from large-N_c suppressed operators. Bearing in mind the results in Eq. (7.8) and the expected experimental sensitivity, we can conclude that in principle isospin breaking effects do not affect the determination of the lifetime. An accuracy in the determination of the π-K S-wave scattering lengths at the same level as in pionium experiments will distinguish between the two sets of low-energy constants and thus will constitute a major step in understanding the basic structure of the effective Lagrangian. As a direct application we have evaluated (partially) the expected lifetime of the $A_{\pi K}$ atom.

Note added. When we were completing this work, Ref. [50] appeared. It contains partially the same work. As has been shown the isospin breaking effects are quite mild and hidden by sizable error bars. This reference essentially leads to the same numerical conclusions. However, after some partial checks we noticed some differences: (i) our expressions contain some logarithmic dependence from the three-point function that is missing in Ref. [50]. As we mentioned, our results have been cross checked with [8,21]. (ii) The π-η mixing is not treated correctly in Ref. [50] where there is a double counting of some pieces. Our results are cross checked with [17]. (iii) Finally in [50] the pole appearing in the t channel is mistreated.

ACKNOWLEDGMENTS

We thank Ll. Ametller for collaborating in the earlier stages of this project, discussions, and a critical reading of the manuscript. We also express our gratitude to M. Knecht and H. Sadzjian for discussions and encouragement in pursuing the presented work. P.T. was supported by EC Contract No. ERBFMRX-CT980169.

APPENDIX A: $\pi^0K^0\to\pi^0K^0$ SCATTERING AMPLITUDE

In this appendix we collect all relevant formulas for the neutral→neutral amplitude. We are aware that the "digestion" of this kind of expression is always hard; thus for the sake of clarity we have not mixed the contributions very much. This has probably enlarged the expressions slightly but we find this worthwhile for any future comparison.

The amplitude is written as

$$\mathcal{M}(s,t,u)=\left\{\mathcal{M}|_{\text{tree}}+\sum_{i=a,b,c}\mathcal{M}_{(i)}|_{\text{Born}}+\mathcal{M}|_{\text{mixing}}+\mathcal{M}|_{\text{tadpole}}+\mathcal{M}|_{t\ \text{channel}}\right\}+\{\mathcal{M}|_{s\ \text{channel}}+s\leftrightarrow u\}, \tag{A1}$$

making explicit use of the $s\leftrightarrow u$ symmetry. In the following we briefly discuss the relevant pieces. Notice also that we explicitly use F_0 at all orders. This stands for the *nonrenormalized* decay constant.

The leading order contribution is given by

$$\mathcal{M}|_{\text{tree}}=\frac{1}{4F_0^2}(t+\Delta_\pi-\Delta_K)+\frac{1}{2F_0^2}\left(\frac{\epsilon}{\sqrt{3}}\right)(s+u-2t). \tag{A2}$$

The Born-type term containing the wave-function renormalization and bare mass renormalization contributions is given by

$$\mathcal{M}_{(a)}|_{\rm Born}=\frac{1}{4F_0^2}(t+\Delta_\pi-\Delta_K)\Bigg\{\frac{1}{6}(\mu_{\pi^0}+3\mu_\eta+6\mu_{K^0}+10\mu_{\pi^\pm}+4\mu_{K^\pm})-\frac{8}{F_0^2}[2(M_\pi^2+2M_K^2)L_4^r+\Sigma_{\pi K}L_5^r]+\frac{\epsilon}{\sqrt{3}}(\mu_\eta-\mu_\pi)$$
$$-\frac{16}{F_0^2}\left(\frac{\epsilon}{\sqrt{3}}\right)\Delta_{K\pi}L_5^r\Bigg\}+\frac{M_{K^0}^2}{6F_0^2}\Bigg\{-\frac{2}{3}\mu_\eta-\frac{8\epsilon}{\sqrt{3}}(\mu_\eta-\mu_\pi)-\frac{16}{F_0^2}\left(\frac{\epsilon}{\sqrt{3}}\right)\Delta_{K\pi}(2L_8^r-L_5^r)-\frac{8}{F_0^2}[(M_\pi^2+2M_K^2)(2L_6^r-L_4^r)$$
$$+M_K^2(2L_8^r-L_5^r)]\Bigg\}+\frac{M_{\pi^0}^2}{6F_0^2}\Bigg\{\mu_{\pi^0}+\frac{1}{3}\mu_\eta-2\mu_{\pi^\pm}-\frac{8}{F_0^2}[(M_\pi^2+2M_K^2)(2L_6^r-L_4^r)+M_\pi^2(2L_8^r-L_5^r)]\Bigg\}$$
$$+\frac{\Delta_K}{4F_0^2}\Bigg\{\frac{2}{3}\mu_\eta+\frac{8}{F_0^2}[(M_\pi^2+2M_K^2)(2L_6^r-L_4^r)+2M_K^2(2L_8^r-L_5^r)]\Bigg\}+\frac{\Delta_\pi}{4F_0^2}\Bigg\{\frac{M_\pi^2}{16\pi^2F_0^2}+4\mu_\pi-2\mu_K-\frac{2}{3}\mu_\eta$$
$$-\frac{8}{F_0^2}[(M_\pi^2+2M_K^2)(2L_6^r-L_4^r)+2M_K^2(2L_8^r-L_5^r)-\Delta_{\pi K}L_5^r]\Bigg\}+\frac{1}{2F_0^2}\left(\frac{\epsilon}{\sqrt{3}}\right)(s+u-2t)\Bigg\{\frac{1}{6}(11\mu_\pi+3\mu_\eta+10\mu_K)$$
$$-\frac{8}{F_0^2}[2(M_\pi^2+2M_K^2)L_4^r+\Sigma_{\pi K}L_5^r]\Bigg\}-\frac{e^2t}{18F_0^2}[24(K_1^r+K_2^r)-18K_3^r+9K_4^r+14(K_5^r+K_6^r)]+\frac{e^2M_K^2}{6F_0^2}\Bigg[\frac{3}{8\pi^2}-9F_0^2\tilde{\mu}_K$$
$$+\frac{2}{9}[12(K_1^r+K_2^r-K_7-K_8^r)-5K_5^r-5K_6^r-4K_9^r+50K_{10}^r+54K_{11}^r]\Bigg]+\frac{e^2M_\pi^2}{6F_0^2}\Bigg[-\frac{3}{8\pi^2}+9F_0^2\tilde{\mu}_\pi+\frac{1}{9}[24(K_1^r+K_2^r$$
$$-K_7-K_8^r)+18K_3^r-9K_4^r+20(K_5^r+K_6^r)-2K_9^r-110K_{10}^r-108K_{11}^r]\Bigg], \tag{A3}$$

where as is customary $\Sigma_m=M_{m^\pm}^2+M_{m^0}^2$.

The effect of the π-η mixing is taken into account by

$$\mathcal{M}|_{\rm mixing}=\frac{1}{144F_0^2}[2(s+u-2t)-\Delta_{\pi^0\eta}]\Bigg\{-3(\mu_{K^\pm}-\mu_{K^0})-\frac{36\epsilon}{\sqrt{3}}(\mu_\pi-\mu_\eta)-\frac{24\epsilon}{\sqrt{3}}(\mu_\pi-\mu_K)+\frac{864\epsilon}{\sqrt{3}F_0^2}\Delta_{\pi\eta}(3L_7+L_8^r)\Bigg\}$$
$$+\frac{1}{144F_0^2}\left[2\left(\frac{s+u-2t}{M_{\pi^0}^2-M_\eta^2}\right)-1\right]\Bigg\{12\Sigma_{\pi^0K^0}(\mu_{K^\pm}-\mu_{K^0})-\frac{96\epsilon}{\sqrt{3}}M_\pi^2(\mu_\pi-\mu_K)+8e^2M_\pi^2[3(2K_3^r-K_4^r)-2(K_5^r+K_6^r)$$
$$+2(K_9^r+K_{10}^r)]\Bigg\}. \tag{A4}$$

The Born-type contribution containing one insertion of the strong $O(p^4)$ counterterms is written as

$$\mathcal{M}_{(b)}|_{\rm Born}=\frac{1}{F_0^4}\sum_{i=1}^{8}\mathcal{P}_iL_i^r, \tag{A5}$$

with

$$\mathcal{P}_1=8(2M_{\pi^0}^2-t)(2M_{K^0}^2-t),$$

$$\mathcal{P}_2=4[(\Sigma_{\pi^0K^0}-s)^2+(\Sigma_{\pi^0K^0}-u)^2],$$

$$\mathcal{P}_3=2\left(1-\frac{2\epsilon}{\sqrt{3}}\right)(2M_{\pi^0}^2-t)(2M_{K^0}^2-t)+\left(1+\frac{2\epsilon}{\sqrt{3}}\right)[(\Sigma_{\pi^0K^0}-s)^2+(\Sigma_{\pi^0K^0}-u)^2],$$

$$\mathcal{P}_4=-\frac{2}{3}\left[M_\pi^2+14M_K^2-\frac{6\epsilon}{\sqrt{3}}(5M_\pi^2-2M_K^2)\right]\times(2M_{\pi^0}^2-t)-\frac{2}{3}\left[13M_\pi^2+2M_K^2-\frac{6\epsilon}{\sqrt{3}}(M_\pi^2+2M_K^2)\right](2M_{K^0}^2-t)+\frac{2}{3}t\left[M_\pi^2+2M_K^2-\frac{6\epsilon}{\sqrt{3}}(M_\pi^2+2M_K^2)\right],$$

$$\mathcal{P}_5=-\frac{2}{3}\left(M_\pi^2+3M_K^2-\frac{12\epsilon}{\sqrt{3}}M_\pi^2\right)(2M_{\pi^0}^2-t)-\frac{2}{3}\left[3M_\pi^2+M_K^2-\frac{4\epsilon}{\sqrt{3}}(5M_\pi^2-2M_K^2)\right](2M_{K^0}^2-t)+\frac{2}{3}t\left[-\Sigma_{\pi K}+\frac{4\epsilon}{\sqrt{3}}(2M_\pi^2-5M_K^2)\right],$$

$$\mathcal{P}_6=\frac{8}{3}\left[2M_K^4+15M_\pi^2M_K^2+M_\pi^4+\frac{16\epsilon}{\sqrt{3}}(M_K^4+M_\pi^2M_K^2-2M_\pi^4)\right],$$

$$\mathcal{P}_7=\frac{64\epsilon}{\sqrt{3}}(2M_K^4-M_\pi^2M_K^2-M_\pi^4),$$

$$\mathcal{P}_8=\frac{8}{3}\left[M_K^4+6M_\pi^2M_K^2+M_\pi^4+\frac{2\epsilon}{\sqrt{3}}(19M_K^4-6M_\pi^2M_K^2-13M_\pi^4)\right].$$

The same kind of diagram but with the e.m. counterterm is given by

$$\mathcal{M}_{(c)}|_{\text{Born}}=-\frac{2e^2}{27F_0^2}(3K_1^r+3K_2^r+K_5^r+K_6^r)(s+u-2t)+\frac{4e^2}{27F_0^2}(3K_7+3K_8^r+K_9^r+K_{10}^r)\Sigma_{\pi K}. \quad \text{(A6)}$$

If mesons were really massless, any tadpole type contribution would vanish, as SU(3) symmetry is broken precisely by quark masses this does not turn out to be the case in nature. One thus obtains

$$\mathcal{M}|_{\text{tadpole}}=-\frac{\mu_{\pi^0}}{18F_0^2}\left[t+2M_{\pi^0}^2-\frac{12\epsilon}{\sqrt{3}}(t+M_\pi^2-2M_K^2)\right]-\frac{\mu_\eta}{18F_0^2}\left[3t-4M_{K^0}^2-\frac{12\epsilon}{\sqrt{3}}(t-2M_K^2)\right]-\frac{\mu_{\pi^\pm}}{9F_0^2}\left[2(t-3M_{\pi^0}^2)-\frac{\epsilon}{\sqrt{3}}(9t-2M_\pi^2-4M_K^2)\right]-\frac{\mu_{K^0}}{9F_0^2}\left[3t-2M_{\pi^0}^2-\frac{6\epsilon}{\sqrt{3}}(3t-2M_\pi^2-4M_K^2)\right]-\frac{\mu_{K^\pm}}{18F_0^2}\left[t+2M_{\pi^0}^2-8M_{K^0}^2-\frac{4\epsilon}{\sqrt{3}}(3t+4M_\pi^2-4M_K^2)\right]. \quad \text{(A7)}$$

Hitherto we have shown terms that are polynomials. When performing the one-loop corrections one obtains a unitary piece. This is given in terms of nonanalytical functions, i.e., essentially functions with a momentum dependence in the argument. In order to present them we have kept track of the internal propagators; thus the identification of each diagram is straightforward. They are given in the t channel by

$$\mathcal{M}_{t\ \text{channel}}=\mathcal{M}_{\pi^0\pi^0}+\mathcal{M}_{\eta\eta}+\mathcal{M}_{\pi^0\eta}+\mathcal{M}_{\pi^+\pi^-}+\mathcal{M}_{K^0\bar{K}^0}+\mathcal{M}_{K^+K^-},$$

where

$$\mathcal{M}_{\pi^0\pi^0}=\frac{M_{\pi^0}^2}{24F_0^4}\left\{4F_0^2\left(1-\frac{6\epsilon}{\sqrt{3}}\right)\mu_{\pi^0}+3\left[t+\Delta_\pi-\Delta_K-\frac{2\epsilon}{\sqrt{3}}(3t-2\Sigma_{\pi K})\right]\bar{B}(M_{\pi^0}^2,M_{\pi^0}^2;t)\right\}, \quad \text{(A8)}$$

$$\mathcal{M}_{\eta\eta}=\frac{M_{\pi^0}^2}{72F_0^4}\left\{12F_0^2\left(1+\frac{2\epsilon}{\sqrt{3}}\right)\mu_\eta+\left[9t-6M_\eta^2-2M_{\pi^0}^2+3(\Delta_K-\Delta_\pi)+\frac{6\epsilon}{\sqrt{3}}(3t-2\Sigma_{\eta K})\right]\bar{B}(M_\eta^2,M_\eta^2;t)\right\}, \quad \text{(A9)}$$

$$\mathcal{M}_{\pi^0\eta}=\frac{1}{3F_0^4}\left(\frac{\epsilon}{\sqrt{3}}\right)\Delta_{K\pi}[2F_0^2\mu_\pi+2F_0^2\mu_\eta+(3t-4M_K^2)\overline{B}(M_\pi^2,M_\eta^2;t)], \quad \text{(A10)}$$

$$\mathcal{M}_{\pi^+\pi^-}=\frac{1}{36F_0^4}[4F_0^2\mu_{\pi^\pm}(5t-3M_{\pi^0}^2)+9t(t-M_{\pi^0}^2)\overline{B}(M_{\pi^\pm}^2,M_{\pi^\pm}^2;t)], \quad \text{(A11)}$$

$$\mathcal{M}_{K^0\bar{K}^0}=\frac{1}{72F_0^4}\left\{-4F_0^2\mu_{K^0}\left[-5t+3(\Delta_K-\Delta_\pi)+\frac{6\epsilon}{\sqrt{3}}(5t-2\Sigma_{\pi K})\right]+9t\left[t+\Delta_\pi-\Delta_K-\frac{2\epsilon}{\sqrt{3}}(3t-2\Sigma_{\pi K})\right]\overline{B}(M_{K^0}^2,M_{K^0}^2;t)\right\}, \quad \text{(A12)}$$

$$\mathcal{M}_{K^+K^-}=\frac{1}{144F_0^4}\left\{4F_0^2\mu_{K^\pm}\left[5t+3(\Delta_K-\Delta_\pi)+\frac{6\epsilon}{\sqrt{3}}(5t-2\Sigma_{\pi K})\right]+9t\left[t+\Delta_K-\Delta_\pi+\frac{2\epsilon}{\sqrt{3}}(3t-2\Sigma_{\pi K})\right]\overline{B}(M_{K^\pm}^2,M_{K^\pm}^2;t)\right\}. \quad \text{(A13)}$$

For the s (u) channel the intermediate particles contribution is

$$\mathcal{M}|_{s\ \text{channel}}=\mathcal{M}_{\pi^+\pi^-}+\mathcal{M}_{\pi^0K^0}+\mathcal{M}_{\eta K^0}.$$

The above terms are given by

$$\mathcal{M}_{\pi^+K^-}=\frac{1}{8F_0^4}\left\{2F_0^2\mu_{K^\pm}\left[3s-3M_{\pi^\pm}^2-M_{K^\pm}^2+4\Delta_\pi+\frac{2\epsilon}{\sqrt{3}}(3s-3M_\pi^2-M_K^2)\right]+\left[s-\Sigma_{\pi^\pm K^\pm}+2\Delta_\pi+\frac{\epsilon}{\sqrt{3}}(s-\Sigma_{\pi K})\right]^2\overline{B}(M_{\pi^\pm}^2,M_{K^\pm}^2;s)+2\left[s-\Sigma_{\pi^\pm K^\pm}+2\Delta_\pi+\frac{\epsilon}{\sqrt{3}}(s-\Sigma_{\pi K})\right]\left[s-\Delta_{\pi^0K^0}+\frac{\epsilon}{\sqrt{3}}(s+3\Delta_{\pi K})\right]\overline{B}_1(M_{\pi^\pm}^2,M_{K^\pm}^2;s)+\left[s-\Delta_{\pi^0K^0}+\frac{\epsilon}{\sqrt{3}}(s+3\Delta_{\pi K})\right]^2\overline{B}_{21}(M_{\pi^\pm}^2,M_{K^\pm}^2;s)+2s\left[2M_{K^0}^2-t-\frac{2\epsilon}{\sqrt{3}}(2u-t-2M_\pi^2)\right]\overline{B}_{22}(M_{\pi^\pm}^2,M_{K^\pm}^2;s)\right\}, \quad \text{(A14)}$$

$$\mathcal{M}_{\pi^0K^0}=\frac{1}{144F_0^4}\left\{6F_0^2\mu_{K^0}\left[s-M_{\pi^0}^2-3M_{K^0}^2-\frac{4\epsilon}{\sqrt{3}}(3s-3M_\pi^2-5M_K^2)\right]+\left[5M_{\pi^0}^2+M_{K^0}^2-s-\frac{6\epsilon}{\sqrt{3}}(5M_\pi^2-3M_K^2-s)\right]^2\overline{B}(M_{\pi^0}^2,M_{K^0}^2;s)-2(s+3\Delta_{\pi^0K^0})\left[5M_{\pi^0}^2+M_{K^0}^2-s+\frac{12\epsilon}{\sqrt{3}}(s-5M_\pi^2+M_K^2)\right]\overline{B}_1(M_{\pi^0}^2,M_{K^0}^2;s)+\left(1-\frac{12\epsilon}{\sqrt{3}}\right)(s+3\Delta_{\pi^0K^0})^2\overline{B}_{21}(M_{\pi^0}^2,M_{K^0}^2;s)+2s\left(1-\frac{12\epsilon}{\sqrt{3}}\right)[4(u-t+\Delta_{\pi^0K^0})+2M_{K^0}^2-t]\overline{B}_{22}(M_{\pi^0}^2,M_{K^0}^2;s)\right\}, \quad \text{(A15)}$$

$$M_{\eta K^0}=\frac{1}{432F_0^4}\left\{18F_0^2\mu_{K^0}\left[3s+M_{\pi^0}^2-5M_{K^0}^2-\frac{2\epsilon}{\sqrt{3}}(6s+6M_\pi^2-14M_K^2)\right]+\left[3s-7M_{\pi^0}^2+M_{K^0}^2-\frac{2\epsilon}{\sqrt{3}}(3s+M_\pi^2-7M_K^2)\right]^2\overline{B}(M_\eta^2,M_{K^0}^2;s)+6(s+3\Delta_{\pi^0K^0})\left[3s-7M_{\pi^0}^2+M_{K^0}^2-\frac{12\epsilon}{\sqrt{3}}(s-M_\pi^2-M_K^2)\right]\overline{B}_1(M_\eta^2,M_{K^0}^2;s)+9\times\left(1-\frac{4\epsilon}{\sqrt{3}}\right)(s+3\Delta_{\pi^0K^0})^2\overline{B}_{21}(M_\eta^2,M_{K^0}^2;s)+18s\left(1-\frac{4\epsilon}{\sqrt{3}}\right)[4(u-t+\Delta_{\pi^0K^0})+2M_{K^0}^2-t]\overline{B}_{22}(M_\eta^2,M_{K^0}^2;s)\right\}. \quad \text{(A16)}$$

The $\overline{B}_{ij}$ functions are defined in Appendix C.

APPENDIX B: $\pi^- K^+ \to \pi^0 K^0$

In this appendix we display the relevant formulas concerning the more interesting process. As before our philosophy has been not to mix the terms too much. One can verify that the following results are scale invariant.

1. Scattering amplitude

The amplitude is written as

$$\mathcal{M}(s,t,u)=\left\{\mathcal{M}|_{\text{tree}}+\sum_{i=a,b,c,d}\mathcal{M}_{(i)}|_{\text{Born}}+\mathcal{M}|_{\text{mixing}}+\mathcal{M}|_{\text{tadpole}}+\mathcal{M}|_{s\ \text{channel}}+\mathcal{M}|_{t\ \text{channel}}+\mathcal{M}|_{u\ \text{channel}}+\mathcal{M}|_{\text{one photon}}\right\}. \tag{B1}$$

In the above expression the splitting between the terms is essentially the same as in the pure neutral case, but in addition we have the last term, which concerns the explicit photon exchange.

For the tree-level amplitude we have

$$\mathcal{M}|_{\text{tree}}=\frac{u-s}{2\sqrt{2}F_0^2}-\frac{\Delta_\pi}{2\sqrt{2}F_0^2}+\frac{\Delta_K-\Delta_\pi}{4\sqrt{2}F_0^2}-\frac{1}{2\sqrt{2}F_0^2}\left(\frac{\epsilon}{\sqrt{3}}\right)(s+u-2t). \tag{B2}$$

The Born amplitude is given by

$$\begin{aligned}\mathcal{M}_{(a)}|_{\text{Born}}=&\frac{u-s}{2\sqrt{2}F_0^2}\left\{\frac{1}{6}(3\mu_{\pi^0}+3\mu_\eta+5\mu_{K^0}+8\mu_{\pi^\pm}+5\mu_{K^\pm})-\frac{8}{F_0^2}[2(M_\pi^2+2M_K^2)L_4^r+\Sigma_{\pi K}L_5^r]\right\}+\frac{M_K^2}{\sqrt{2}F_0^2}\left(\frac{\epsilon}{\sqrt{3}}\right)(\mu_\eta-\mu_\pi)\\&+\frac{\Delta_K}{4\sqrt{2}F_0^2}\left\{\frac{1}{6}(11\mu_\pi-\mu_\eta+10\mu_K)-\frac{8}{F_0^2}[(M_\pi^2+2M_K^2)(L_4^r+2L_6^r)-\Delta_{K\pi}L_5^r+4M_K^2L_8^r]\right\}-\frac{\Delta_\pi}{4\sqrt{2}F_0^2}\left\{\frac{3M_\pi^2}{16\pi^2F_0^2}\right.\\&\left.+\frac{1}{6}(129\mu_\pi+5\mu_\eta+42\mu_K)-\frac{8}{F_0^2}[(M_\pi^2+2M_K^2)(3L_4^r+2L_6^r)+2M_K^2(L_5^r+2L_8^r)]\right\}-\frac{1}{2\sqrt{2}F_0^2}\left(\frac{\epsilon}{\sqrt{3}}\right)(s+u-2t)\\&\times\left\{\frac{1}{6}(11\mu_\pi+3\mu_\eta+10\mu_K)-\frac{8}{F_0^2}[2(M_\pi^2+2M_K^2)L_4^r+\Sigma_{\pi K}L_5^r]\right\}.\end{aligned} \tag{B3}$$

As in the neutral to neutral transition, there is a term from the treatment of the π-η mixing:

$$\begin{aligned}\mathcal{M}|_{\text{mixing}}=&-\frac{1}{144\sqrt{2}F_0^2}[2(s+u-2t)-\Delta_{\pi^0\eta}+2\Delta_\pi]\left\{-3(\mu_{K^\pm}-\mu_{K^0})-\frac{36\epsilon}{\sqrt{3}}(\mu_\pi-\mu_\eta)-\frac{24\epsilon}{\sqrt{3}}(\mu_\pi-\mu_K)+\frac{864\epsilon}{\sqrt{3}F_0^2}\Delta_{\pi\eta}(3L_7\right.\\&\left.+L_8^r)\right\}-\frac{1}{144\sqrt{2}F_0^2}\left[2\left(\frac{s+u-2t+\Delta_\pi}{M_{\pi^0}^2-M_\eta^2}\right)-1\right]\left\{12\Sigma_{\pi^0K^0}(\mu_{K^\pm}-\mu_{K^0})-\frac{96\epsilon}{\sqrt{3}}M_\pi^2(\mu_\pi-\mu_K)+8e^2M_\pi^2[3(2K_3^r-K_4^r)\right.\\&\left.-2(K_5^r+K_6^r)+2(K_9^r+K_{10}^r)]\right\}.\end{aligned} \tag{B4}$$

The strong leading Lagrangian at $O(p^4)$ contributes as

$$\mathcal{M}_{(b)}|_{\text{Born}}=\frac{2}{\sqrt{2}F_0^4}\sum_{i=3}^{8}\mathcal{P}_iL_i^r, \tag{B5}$$

where

$$\mathcal{P}_3=\frac{2\epsilon}{\sqrt{3}}(2M_\pi^2-t)(2M_K^2-t)+\left(1-\frac{\epsilon}{\sqrt{3}}\right)(\Sigma_{\pi^0K^+}-u)(\Sigma_{\pi^+K^0}-u)-\left(1+\frac{\epsilon}{\sqrt{3}}\right)(\Sigma_{\pi^0K^0}-s)(\Sigma_{\pi^+K^-}-s),$$

$$\mathcal{P}_4=-2(M_\pi^2+2M_K^2)(s-u)-\frac{2\epsilon}{\sqrt{3}}(M_\pi^2+2M_K^2)(2\Sigma_{\pi K}-3t),$$

$$\mathcal{P}_5=-2\Sigma_{\pi K}(s-u)+\frac{6\epsilon}{\sqrt{3}}\Sigma_{\pi K}t+\frac{4\epsilon}{\sqrt{3}}(M_K^4-M_\pi^4-4M_\pi^2M_K^2),$$

$$\mathcal{P}_6=-\frac{8\epsilon}{\sqrt{3}}(2M_K^4-M_\pi^4-M_\pi^2M_K^2),$$

$$\mathcal{P}_7=-\frac{32\epsilon}{\sqrt{3}}(2M_K^4-M_\pi^4-M_\pi^2M_K^2),$$

$$\mathcal{P}_8=-\frac{8}{3}\left(\frac{\epsilon}{\sqrt{3}}\right)(17M_K^4-7M_\pi^4-10M_\pi^2M_K^2).$$

The polynomials not displayed explicitly do not contribute to the process.

The equivalent contribution but in the e.m. sector is cast as

$$\mathcal{M}_{(c)}|_{\text{Born}}=\frac{2e^2}{3\sqrt{2}F_0^2}(2K_1^r+2K_2^r-4K_3^r+2K_4^r+K_5^r+4K_6^r)(\Sigma_{\pi K}-s)-\frac{2e^2}{9\sqrt{2}F_0^2}(6K_1^r+6K_2^r-6K_3^r+3K_4^r+4K_5^r+4K_6^r)(\Sigma_{\pi K}-u)$$
$$+\frac{e^2}{9\sqrt{2}F_0^2}(12K_3^r-6K_4^r-K_5^r-K_6^r)(2M_\pi^2-t)+\frac{e^2}{3\sqrt{2}F_0^2}(K_5^r+K_6^r)(2M_K^2-t)-\frac{2e^2}{9\sqrt{2}F_0^2}[9(M_\pi^2+2M_K^2)K_8^r-M_\pi^2K_9^r$$
$$+(17M_\pi^2+18M_K^2)K_{10}^r+18\Sigma_{\pi K}K_{11}^r]. \tag{B6}$$

The e^2 contribution from the wave function renormalization reads

$$\mathcal{M}_{(d)}|_{\text{Born}}=\frac{e^2(u-s)}{2\sqrt{2}F_0^2}\left[-2F_0^2\tilde{\mu}_\pi-2F_0^2\tilde{\mu}_K-\frac{1}{16\pi^2}\left(2+\log\frac{m_\gamma^2}{M_\pi^2}+\log\frac{m_\gamma^2}{M_K^2}\right)-\frac{1}{9}(48K_1^r+48K_2^r-18K_3^r+9K_4^r+34K_5^r+34K_6^r)\right]$$
$$-\frac{e^2M_K^2}{4\sqrt{2}F_0^2}\left[\frac{1}{4\pi^2}-6F_0^2\tilde{\mu}_K-\frac{4}{3}(K_5^r+K_6^r+12K_8^r-6K_{10}^r-6K_{11}^r)\right]+\frac{e^2M_\pi^2}{4\sqrt{2}F_0^2}\left[\frac{3}{4\pi^2}-18F_0^2\tilde{\mu}_\pi-\frac{2}{3}(18K_3^r-9K_4^r-12K_8^r\right.$$
$$\left.+2K_9^r-34K_{10}^r-36K_{11}^r)\right], \tag{B7}$$

with

$$\tilde{\mu}_P=\frac{1}{32\pi^2F^2}\log\left(\frac{M_P^2}{\mu^2}\right).$$

The tadpole diagram contributes as

$$\mathcal{M}|_{\rm tadpole}=\frac{\mu_{\pi^0}}{3\sqrt{2}F_0^2}\left[s-u+\Delta_\pi-\frac{\epsilon}{\sqrt{3}}(3t+3M_\pi^2-4M_K^2)\right]+\frac{\mu_\eta}{6\sqrt{2}F_0^2}\left[s-u+\Delta_\pi-\frac{\epsilon}{\sqrt{3}}(3t+2M_\pi^2-12M_K^2)\right]+\frac{\mu_{\pi^\pm}}{18\sqrt{2}F_0^2}\left[9(s-u)+3(\Delta_K+4\Delta_\pi)-\frac{\epsilon}{\sqrt{3}}(15t+4M_\pi^2-16M_K^2)\right]+\frac{\mu_{K^0}}{18\sqrt{2}F_0^2}\left[8s+5t-4u-4\Sigma_\pi+3\Delta_\pi+\frac{\epsilon}{\sqrt{3}}(21s+27u-40M_\pi^2-20M_K^2)\right]+\frac{\mu_{K^\pm}}{18\sqrt{2}F_0^2}\left[4s-5t-8u+4\Sigma_\pi+27\Delta_\pi+\frac{\epsilon}{\sqrt{3}}(27s+21u-40M_\pi^2-20M_K^2)\right]. \tag{B8}$$

As in the neutral case, we display the unitary contribution disentangling each term separately. We recall that the subscripts refer to the internal particles running in the propagators. For the s channel one gets

$$\mathcal{M}|_{s\ \ \rm channel}=-\frac{1}{12\sqrt{2}F_0^4}\left\{\mathcal{M}_{\pi^-K^+}+\frac{1}{2}\mathcal{M}_{\pi^0K^0}+\frac{1}{18}\mathcal{M}_{\eta K^0}\right\},$$

where

$$\mathcal{M}_{\pi^-K^+}=2F_0^2\left(1+\frac{\epsilon}{\sqrt{3}}\right)[2(s-M_{\pi^0}^2)+s-\Delta_{\pi^-K^+}+6\Delta_\pi]\mu_{K^\pm}+(s+\Delta_{\pi^-K^+}+6\Delta_\pi)\left[s-\Sigma_{\pi^0K^0}+\frac{\epsilon}{\sqrt{3}}(s-5M_\pi^2+3M_K^2)\right]\overline{B}(M_{\pi^\pm}^2,M_{K^\pm}^2;s)+\left\{(s-3\Delta_{\pi^-K^+})\left[s-\Sigma_{\pi^0K^0}+\frac{\epsilon}{\sqrt{3}}(s-5M_\pi^2+3M_K^2)\right]+(s+\Delta_{\pi^-K^+}+6\Delta_\pi)\left[s-\Delta_{\pi^0K^0}+\frac{\epsilon}{\sqrt{3}}(s+3\Delta_{\pi K})\right]\right\}\overline{B}_1(M_{\pi^\pm}^2,M_{K^\pm}^2;s)+(s-3\Delta_{\pi^-K^+})\left[s-\Delta_{\pi^0K^0}+\frac{\epsilon}{\sqrt{3}}(s+3\Delta_{\pi K})\right]\overline{B}_{21}(M_{\pi^\pm}^2,M_{K^\pm}^2;s)-2s\left[\Sigma_{\pi^-K^0}-u+2(t-\Sigma_{K^+K^0})+\frac{4\epsilon}{\sqrt{3}}(\Sigma_{\pi K}-t)-\frac{5\epsilon}{\sqrt{3}}(\Sigma_{\pi K}-u)\right]\overline{B}_{22}(M_{\pi^\pm}^2,M_{K^\pm}^2;s), \tag{B9}$$

$$\mathcal{M}_{\pi^0K^0}=-2F_0^2\left(1+\frac{\epsilon}{\sqrt{3}}\right)\left[3s-3M_{\pi^0}^2-5M_{K^0}^2-\frac{6\epsilon}{\sqrt{3}}(3s-3M_\pi^2-M_K^2)\right]\mu_{K^0}+\left[s-\Sigma_{\pi^0K^0}+\frac{\epsilon}{\sqrt{3}}(s-9M_\pi^2+7M_K^2)\right]\times\left[5M_{\pi^0}^2+M_{K^0}^2-s-\frac{6\epsilon}{\sqrt{3}}(5M_\pi^2-3M_K^2-s)\right]\overline{B}(M_{\pi^0}^2,M_{K^0}^2;s)+\left\{\left[s-\Delta_{\pi^-K^+}+\frac{\epsilon}{\sqrt{3}}(s+3\Delta_{\pi K})\right]\left[5M_{\pi^0}^2+M_{K^0}^2-s-\frac{6\epsilon}{\sqrt{3}}(5M_\pi^2-3M_K^2-s)\right]-\left(1-\frac{6\epsilon}{\sqrt{3}}\right)(s+3\Delta_{\pi^0K^0})\left[s-\Sigma_{\pi^0K^0}+\frac{\epsilon}{\sqrt{3}}(s-9M_\pi^2+7M_K^2)\right]\right\}\overline{B}_1(M_{\pi^0}^2,M_{K^0}^2;s)-\left(1-\frac{6\epsilon}{\sqrt{3}}\right)(s+3\Delta_{\pi^0K^0})\left[s-\Delta_{\pi^-K^+}+\frac{\epsilon}{\sqrt{3}}(s+3\Delta_{\pi K})\right]\overline{B}_{21}(M_{\pi^0}^2,M_{K^0}^2;s)-2s\left(1-\frac{6\epsilon}{\sqrt{3}}\right)\left[2(\Sigma_{\pi^0K^+}-u)+t-\Sigma_{K^+K^0}+\frac{4\epsilon}{\sqrt{3}}(2M_\pi^2-t)+\frac{\epsilon}{\sqrt{3}}(2M_K^2-t)-\frac{4\epsilon}{\sqrt{3}}(\Sigma_{\pi K}-u)\right]\overline{B}_{22}(M_{\pi^0}^2,M_{K^0}^2;s), \tag{B10}$$

$$\mathcal{M}_{\eta K^0}=18F_0^2\left[3s+M_{\pi^0}^2-5M_{K^0}^2-\frac{\epsilon}{\sqrt{3}}(15s+13M_\pi^2-33M_K^2)\right]\mu_{K^0}+\left[3s-7M_{\pi^0}^2+M_{K^0}^2-\frac{\epsilon}{\sqrt{3}}(9s-5M_\pi^2-13M_K^2)\right]\times\left[3s-7M_{\pi^0}^2+M_{K^0}^2-\frac{2\epsilon}{\sqrt{3}}(3s+M_\pi^2-7M_K^2)\right]\overline{B}(M_\eta^2,M_{K^0}^2;s)+\left\{3\left(1-\frac{2\epsilon}{\sqrt{3}}\right)(s+3\Delta_{\pi^0K^0})\left[3s-7M_{\pi^0}^2+M_{K^0}^2-\frac{\epsilon}{\sqrt{3}}(9s-5M_\pi^2-13M_K^2)\right]+3\left[3s-7M_{\pi^0}^2+M_{K^0}^2-\frac{2\epsilon}{\sqrt{3}}(3s+M_\pi^2-7M_K^2)\right]\left[s+3\Delta_{\pi^+K^-}-\frac{3\epsilon}{\sqrt{3}}(s-\Delta_{\pi K})\right]\right\}$$

$$\times\overline{B}_1(M_\eta^2,M_{K^0}^2;s)+9\left(1-\frac{2\epsilon}{\sqrt{3}}\right)(s+3\Delta_{\pi^0K^0})\left[s+3\Delta_{\pi^-K^+}-\frac{3\epsilon}{\sqrt{3}}(s-\Delta_{\pi K})\right]\overline{B}_{21}(M_\eta^2,M_{K^0}^2;s)+18s$$
$$\times\left(1-\frac{2\epsilon}{\sqrt{3}}\right)\left[2(u-s-t)+4(\Sigma_\pi-t)+\Sigma_K-t+\frac{3\epsilon}{\sqrt{3}}(2M_K^2-t)-\frac{6\epsilon}{\sqrt{3}}(\Sigma_{\pi K}-u)\right]\overline{B}_{22}(M_\eta^2,M_{K^0}^2;s). \quad \text{(B11)}$$

In the t channel

$$\mathcal{M}|_{t\ \text{channel}}=\mathcal{M}_{\pi^-\pi^0}+\mathcal{M}_{\pi^-\eta}+\mathcal{M}_{K^-\overline{K}^0}, \quad \text{(B12)}$$

with

$$\mathcal{M}_{\pi^-\pi^0}=-\frac{1}{6\sqrt{2}F_0^4}\Bigg\{\frac{4\epsilon}{\sqrt{3}}(3t-M_\pi^2-4M_K^2)F_0^2\mu_\pi+2(3M_\pi^2-t)\left[\Delta_\pi+\frac{\epsilon}{\sqrt{3}}(2M_\pi^2-t)\right]\overline{B}(M_\pi,M_\pi;t)-2\Bigg[3M_\pi^2\Delta_\pi+\frac{\epsilon}{\sqrt{3}}t(2M_\pi^2$$
$$-t)+\frac{\epsilon}{\sqrt{3}}(3M_\pi^2-t)(t+4\Delta_{\pi K})\Bigg]\overline{B}_1(M_\pi^2,M_\pi^2;t)+2t\left[\Delta_\pi+\frac{\epsilon}{\sqrt{3}}(t-\Delta_{K\pi})\right]\overline{B}_{21}(M_\pi^2,M_\pi^2;t)-2t\Bigg[3(s-u)-\Delta_\pi$$
$$-\frac{\epsilon}{\sqrt{3}}(t-4\Delta_{K\pi})\Bigg]\overline{B}_{22}(M_{\pi^\pm}^2,M_{\pi^0}^2;t)\Bigg\}, \quad \text{(B13)}$$

$$\mathcal{M}_{\pi^-\eta}=\frac{1}{3\sqrt{2}F_0^4}\left(\frac{\epsilon}{\sqrt{3}}\right)\{-2(4M_K^2-3t)F_0^2\mu_\eta+(\Sigma_{\pi\eta}-t)^2\overline{B}(M_\pi^2,M_\eta^2;t)-2t(\Sigma_{\pi\eta}-t)\overline{B}_1(M_\pi^2,M_\eta^2;t)+t^2\overline{B}_{21}(M_\pi^2,M_\eta^2;t)$$
$$+t^2\overline{B}_{22}(M_\pi^2,M_\eta^2;t)\}, \quad \text{(B14)}$$

$$\mathcal{M}_{K^-\overline{K}^0}=-\frac{1}{12\sqrt{2}F_0^4}\Bigg\{-\frac{4\epsilon}{\sqrt{3}}(3t-2M_K^2)F_0^2\mu_K+2t\left[\Delta_\pi+\frac{\epsilon}{\sqrt{3}}(2M_K^2-t)\right]\overline{B}(M_K^2,M_K^2;t)+\frac{4\epsilon}{\sqrt{3}}t(M_K^2-t)\overline{B}_1(M_K^2,M_K^2;t)$$
$$-2t\left(\Delta_\pi+\frac{\epsilon}{\sqrt{3}}t\right)\overline{B}_{21}(M_K^2,M_K^2;t)-2t\left[3(s-u)+\Delta_\pi+\frac{\epsilon}{\sqrt{3}}t\right]\overline{B}_{22}(M_{K^\pm}^2,M_{K^0}^2;t)\Bigg\}. \quad \text{(B15)}$$

Finally, for the u channel we get the following set of results:

$$\mathcal{M}|_{u\ \text{channel}}=\mathcal{M}_{K^-\pi^0}+\mathcal{M}_{K^-\eta}+\mathcal{M}_{\pi^-K^0}. \quad \text{(B16)}$$

For each term separately,

$$\mathcal{M}_{K^-\pi^0}=-\frac{1}{24\sqrt{2}F_0^4}\Bigg(-2F_0^2\left[5M_{\pi^0}^2+3M_{K^0}^2-3u+3\Delta_\pi-\frac{\epsilon}{\sqrt{3}}(15u-29M_{\pi^0}^2+13M_{K^0}^2)\right]\mu_{\pi^0}+\left(1-\frac{\epsilon}{\sqrt{3}}\right)(\Sigma_{K^-\pi^+}-u+\Delta_\pi)$$
$$\times\left[5M_{K^\pm}^2+M_{\pi^0}^2-u+\frac{6\epsilon}{\sqrt{3}}(\Sigma_{\pi K}-u)\right]\overline{B}(M_{K^\pm}^2,M_{\pi^0}^2;u)+\Bigg\{-\left[5M_{K^\pm}^2+M_{\pi^0}^2-u+\frac{6\epsilon}{\sqrt{3}}(\Sigma_{\pi K}-u)\right]\Bigg[\Delta_{\pi^+K^0}+u$$
$$+\frac{\epsilon}{\sqrt{3}}(3\Delta_{\pi K}-u)\Bigg]+\left(1-\frac{\epsilon}{\sqrt{3}}\right)\left(1+\frac{6\epsilon}{\sqrt{3}}\right)(\Sigma_{K^-\pi^+}-u+\Delta_\pi)(3\Delta_{\pi^0K^-}-u)\Bigg\}\overline{B}_1(M_{K^\pm}^2,M_{\pi^0}^2;u)-\left(1+\frac{6\epsilon}{\sqrt{3}}\right)(3\Delta_{\pi^0K^-}$$
$$-u)\left[\Delta_{\pi^+K^0}+u+\frac{\epsilon}{\sqrt{3}}(3\Delta_{\pi K}-u)\right]\overline{B}_{21}(M_{K^\pm}^2,M_{\pi^0}^2;u)-2u\left(1+\frac{6\epsilon}{\sqrt{3}}\right)\Bigg[\Sigma_\pi-t+2(s-\Sigma_{\pi^+K^-})+\frac{\epsilon}{\sqrt{3}}(2M_\pi^2-t)$$
$$+\frac{4\epsilon}{\sqrt{3}}(2M_K^2-t)-\frac{4\epsilon}{\sqrt{3}}(\Sigma_{\pi K}-s)\Bigg]\overline{B}_{22}(M_{K^\pm}^2,M_{\pi^0}^2;u)\Bigg), \quad \text{(B17)}$$

$$\mathcal{M}_{K^-\eta}=\frac{1}{24\sqrt{2}F_0^4}\left(1+\frac{2\epsilon}{\sqrt{3}}\right)\Bigg(-2F_0^2\left[5M_{\pi^0}^2-M_{K^0}^2-3u+3\Delta_\pi-\frac{3\epsilon}{\sqrt{3}}(3u-M_\pi^2-3M_K^2)\right]\mu_\eta+\left(1+\frac{3\epsilon}{\sqrt{3}}\right)(\Sigma_{K^-\eta}-u-2\Delta_\pi)$$
$$\times(\Sigma_{K^-\eta}-u+4\Delta_\pi)\overline{B}(M_{K^\pm}^2,M_\eta^2;u)+\left\{(\Sigma_{K^-\eta}-u+4\Delta_\pi)\left[3\Delta_{\pi^+K^0}-u-\frac{3\epsilon}{\sqrt{3}}(\Delta_{\pi K}+u)\right]+\left(1+\frac{3\epsilon}{\sqrt{3}}\right)(\Sigma_{K^-\eta}-u\right.$$
$$\left.-2\Delta_\pi)(3\Delta_{\pi^0K^-}-u)\right\}\overline{B}_1(M_{K^\pm}^2,M_\eta^2;u)+(3\Delta_{\pi^0K^-}-u)\left[3\Delta_{\pi^+K^0}-u-\frac{3\epsilon}{\sqrt{3}}(\Delta_{\pi K}+u)\right]\overline{B}_{21}(M_{K^\pm}^2,M_\eta^2;u)+2u\Big[2(s$$
$$-t-u)+\Sigma_\pi-t+4(\Sigma_K-t)+\frac{6\epsilon}{\sqrt{3}}(\Sigma_{\pi K}-s)-\frac{3\epsilon}{\sqrt{3}}(2M_\pi^2-t)\Big]\overline{B}_{22}(M_{K^\pm}^2,M_\eta^2;u)\Bigg),\tag{B18}$$

$$\mathcal{M}_{\pi^-K^0}=-\frac{1}{12\sqrt{2}F_0^4}\Bigg(-2F_0^2\left(1-\frac{\epsilon}{\sqrt{3}}\right)(3u-3M_{\pi^0}^2+M_{K^0}^2-3\Delta_\pi)\mu_{K^0}+\left(1-\frac{\epsilon}{\sqrt{3}}\right)(u+\Delta_{\pi^-K^0})(\Sigma_{\pi^-K^0}-u$$
$$+2\Delta_\pi)\overline{B}(M_{\pi^\pm}^2,M_{K^0}^2;u)+\left\{-\left(1-\frac{\epsilon}{\sqrt{3}}\right)(\Sigma_{\pi^-K^0}-u+2\Delta_\pi)(3\Delta_{\pi^-K^0}-u)+(u+\Delta_{\pi^-K^0})\left[\Delta_{\pi^0K^-}-u+\frac{\epsilon}{\sqrt{3}}(u\right.\right.$$
$$\left.\left.+3\Delta_{\pi K})\right]\right\}\overline{B}_1(M_{\pi^\pm}^2,M_{K^0}^2;u)-(3\Delta_{\pi^-K^0}-u)\left[\Delta_{\pi^0K^-}-u+\frac{\epsilon}{\sqrt{3}}(u+3\Delta_{\pi K})\right]\overline{B}_{21}(M_{\pi^\pm}^2,M_{K^0}^2;u)-2u\Big[s-\Sigma_{\pi^+K^-}$$
$$+2(\Sigma_K-t)+\frac{4\epsilon}{\sqrt{3}}(\Sigma_{\pi K}-t)-\frac{5\epsilon}{\sqrt{3}}(\Sigma_{\pi K}-s)\Big]\overline{B}_{22}(M_{\pi^\pm}^2,M_{K^0}^2;u)\Bigg).\tag{B19}$$

The only remaining piece is the soft-photon exchange diagram. It contains an infrared divergence (included in the C function). Its result is given by

$$\mathcal{M}|_{\text{one photon}}=-\frac{e^2}{4\sqrt{2}F_0^2}(u+\Delta_{\pi K})\left(-6F_0^2\tilde{\mu}_\pi+\frac{1}{4\pi^2}\right)-\frac{e^2}{4\sqrt{2}F_0^2}(u-\Delta_{\pi K})\left(-6F_0^2\tilde{\mu}_K+\frac{1}{4\pi^2}\right)+\frac{e^2}{2\sqrt{2}F_0^2}\Big\{-F_0^2\tilde{\mu}_\pi(\Sigma_{\pi K}+u$$
$$-2s)-F_0^2\tilde{\mu}_K(\Delta_{\pi K}+u-2s)+2(u-s)\left(\frac{1}{16\pi^2}\right)+(\Sigma_{\pi K}-s)\overline{B}(M_\pi^2,M_K^2;s)+(s-\Delta_{\pi K})\overline{B}_1(M_\pi^2,M_K^2;s)+2(s-u)$$
$$\times(s-\Sigma_{\pi K})C_{\pi K}(s)+4(s-\Sigma_{\pi K})[uG_{\pi K}^-(s)+\Delta_{\pi K}G_{\pi K}^+(s)]\Big\},\tag{B20}$$

where the functions $G_{\pi K}^-$ and $G_{\pi K}^+$ are defined in Appendix C.

2. Scattering lengths

Because of the relevance of the process $\pi^-K^+\to\pi^0K^0$ we display the combination for the S-wave scattering lengths. For the sake of simplification, the results will be presented as functions of the constant Z defined at leading order by

$$M_{\pi^\pm}^2-M_{\pi^0}^2=2Ze^2F_0^2+\cdots.\tag{B21}$$

After performing all the steps mentioned in Sec. VI we make the following substitutions:

$$\Delta_K\to\Delta_\pi-\frac{4\epsilon}{\sqrt{3}}(M_K^2-M_\pi^2),\quad \Delta_\pi\to2e^2ZF_0^2.$$

This leads to the following expression:

$$-\frac{3}{\sqrt{2}}a_0(+-;00)=a_0^{1/2}-a_0^{3/2}-\frac{3}{\sqrt{2}}\Delta_0(+-;00)$$

$$=\frac{3}{32\pi}\frac{M_{\pi^\pm}M_{K^\pm}}{F_0^2}+\frac{3}{256\pi F_0^2}\frac{M_K-M_\pi}{M_K+M_\pi}\Delta_K+\frac{3}{256\pi F_0^2}\frac{3M_K+5M_\pi}{M_K+M_\pi}\Delta_\pi+\frac{\sqrt{3}}{64\pi F_0^2}(M_K^2+M_\pi^2)\epsilon$$

$$+\frac{3}{32\pi}\frac{M_{\pi^\pm}M_{K^\pm}}{F_0^4}\Big\{-8(M_{\pi^\pm}^2+2M_{K^\pm}^2)L_4^r-\frac{1}{576\pi^2}\left(\frac{1}{M_{K^\pm}^2-M_{\pi^\pm}^2}\right)\Big[27M_{\pi^\pm}^4\log\left(\frac{M_{\pi^\pm}^2}{\mu^2}\right)-2M_{K^\pm}^2(18M_{K^\pm}^2$$

$$+5M_{\pi^\pm}^2)\log\left(\frac{M_{K^\pm}^2}{\mu^2}\right)+M_{\pi^\pm}^2(28M_{K^\pm}^2-9M_{\pi^\pm}^2)\log\left(\frac{M_\eta^2}{\mu^2}\right)\Big]\Big\}+\frac{M_{\pi^\pm}^2M_{K^\pm}^2}{32\pi F_0^4}\mathcal{B}_--\frac{\sqrt{3}\epsilon}{32\pi F_0^4}$$

$$\Big\{-\frac{M_\pi}{432\pi^2(M_K-M_\pi)(4M_K^2-M_\pi^2)}(216M_K^6-604M_K^5M_\pi+862M_K^4M_\pi^2+1331M_K^3M_\pi^3-245M_K^2M_\pi^4$$

$$-283M_KM_\pi^5+M_\pi^6)+4\{(M_K^4-M_\pi^4)(L_5^r-2L_8^r)+2M_KM_\pi[2L_3^r(M_K-M_\pi)M_\pi+L_4^r(6M_K^2-3M_\pi^2)]\}$$

$$+\frac{M_KM_\pi^2}{36(M_K^2-M_\pi^2)(4M_K^2-M_\pi^2)}[(304M_K^5-60M_K^4M_\pi-472M_K^3M_\pi^2+10M_K^2M_\pi^3+99M_KM_\pi^4+8M_\pi^5)\mathcal{B}_+$$

$$+(-160M_K^5+24M_K^4M_\pi+310M_K^3M_\pi^2+26M_K^2M_\pi^3-81M_KM_\pi^4-8M_\pi^5)\mathcal{B}_-]$$

$$+\frac{1}{\pi^2(M_K-M_\pi)^2(M_K+M_\pi)}\Big[\frac{M_\pi^2}{192}(71M_K^5-79M_K^4M_\pi-174M_K^3M_\pi^2+43M_K^2M_\pi^3-44M_KM_\pi^4$$

$$+3M_\pi^5)\log\left(\frac{M_\pi^2}{\mu^2}\right)+\frac{M_KM_\pi}{864}(-162M_K^5+68M_K^4M_\pi+455M_K^3M_\pi^2+645M_K^2M_\pi^3-154M_KM_\pi^4$$

$$+128M_\pi^5)\log\left(\frac{M_K^2}{\mu^2}\right)+\frac{1}{1728}(-36M_K^7+36M_K^6M_\pi-415M_K^5M_\pi^2+251M_K^4M_\pi^3-174M_K^3M_\pi^4-277M_K^2M_\pi^5$$

$$+266M_KM_\pi^6+9M_\pi^7)\log\left(\frac{M_\eta^2}{\mu^2}\right)\Big]\Big\}-\frac{3Ze^2}{32\pi F_0^2}\Big\{\frac{1}{1728\pi^2(M_K-M_\pi)^2(4M_K^2-M_\pi^2)(M_K+M_\pi)}(4M_K^7$$

$$-124M_K^6M_\pi+671M_K^5M_\pi^2+3967M_K^4M_\pi^3+2276M_K^3M_\pi^4-1136M_K^2M_\pi^5-575M_KM_\pi^6+29M_\pi^7)+2[(8L_4^r$$

$$+3L_5^r)M_K^2-4(L_3^r+6L_4^r)M_KM_\pi+(4L_4^r-L_5^r)M_\pi^2]+\frac{M_KM_\pi}{72(M_K-M_\pi)^2(M_K+M_\pi)(4M_K^2-M_\pi^2)}[(-160M_K^5$$

$$+48M_K^4M_\pi+274M_K^3M_\pi^2+14M_K^2M_\pi^3-45M_KM_\pi^4-20M_\pi^5)\mathcal{B}_++(-32M_K^5+252M_K^4M_\pi-88M_K^3M_\pi^2$$

$$-362M_K^2M_\pi^3+51M_KM_\pi^4+68M_\pi^5)\mathcal{B}_-]+\frac{1}{\pi^2(M_K-M_\pi)^2(M_K^2-M_\pi^2)}\Big[\frac{M_\pi}{192}(8M_K^5-4M_K^4M_\pi+30M_K^3M_\pi^2$$

$$+70M_K^2M_\pi^3-29M_KM_\pi^4+15M_\pi^5)\log\left(\frac{M_\pi^2}{\mu^2}\right)-\frac{M_K}{1728}(27M_K^5-220M_K^4M_\pi+854M_K^3M_\pi^2-54M_K^2M_\pi^3$$

$$+269M_KM_\pi^4+104M_\pi^5)\log\left(\frac{M_K^2}{\mu^2}\right)+\frac{M_\pi}{1728}(32M_K^5+296M_K^4M_\pi-270M_K^3M_\pi^2+260M_K^2M_\pi^3-121M_KM_\pi^4$$

$$-27M_\pi^5)\log\left(\frac{M_\eta^2}{\mu^2}\right)\Big]\Big\}-\frac{3e^2}{64\pi^3F_0^2}\Big(\frac{M_\pi^2\pi^2}{9}\left(\frac{2M_K^2+M_\pi^2}{M_K^2-M_\pi^2}\right)[6K_3^r-3K_4^r-2(K_5^r+K_6^r-K_9^r-K_{10}^r)]+\frac{M_K^2}{64}\Big[-12$$

$$+\left(\frac{9M_K+11M_\pi}{M_K+M_\pi}\right)\log\left(\frac{M_K^2}{\mu^2}\right)+64\pi^2[K_5^r+K_6^r-6(K_{10}^r+K_{11}^r)]\Big]+\frac{M_\pi^2}{144}\Big[9-\frac{9}{4}\left(\frac{M_K+3M_\pi}{M_K+M_\pi}\right)\log\left(\frac{M_\pi^2}{\mu^2}\right)$$

$$-8\pi^2(6K_3^r-3K_4^r+4K_5^r+4K_6^r+2K_9^r-34K_{10}^r-36K_{11}^r)\Big]+\frac{M_\pi M_K}{36}\Big\{8\pi^2(24K_1^r+24K_2^r+18K_3^r-9K_4^r$$

$$+20K_5^r+2K_6^r)+\frac{9M_K^2}{2(M_K+M_\pi)^2}\left[2+\log\left(\frac{M_K^2}{\mu^2}\right)+\log\left(\frac{M_\pi^2}{\mu^2}\right)\right]+\frac{9M_\pi^2}{2(M_K+M_\pi)^2}\left[10+3\log\left(\frac{M_K^2}{\mu^2}\right)\right.$$
$$\left.-\log\left(\frac{M_\pi^2}{\mu^2}\right)\right]+\frac{18M_KM_\pi}{(M_K+M_\pi)^2}\left[3+\log\left(\frac{M_\pi^2}{\mu^2}\right)\right]\Bigg\}\Bigg), \tag{B22}$$

where for the sake of clarity we have expressed the combination $a_0(+-;00)$ in terms of the bare coupling constant F_0. If one chooses to renormalize the decay constants as $F_\pi F_K$ the following term should be added to the previous expression:

$$\delta_{F_\pi F_K}=-\frac{3M_{\pi^\pm}M_{K^\pm}}{256\pi}\left(\frac{1}{16\pi^2F_0^4}\right)\Bigg\{-512\pi^2[2(M_{\pi^\pm}^2+2M_{K^\pm}^2)L_4^r+(M_{\pi^\pm}^2+M_{K^\pm}^2)L_5^r]+11M_{\pi^\pm}^2\log\left(\frac{M_{\pi^\pm}^2}{\mu^2}\right)$$
$$+10M_{K^\pm}^2\log\left(\frac{M_{K^\pm}^2}{\mu^2}\right)+(4M_{K^\pm}^2-M_{\pi^\pm}^2)\log\left(\frac{M_\eta^2}{\mu^2}\right)\Bigg\}-\frac{3M_\pi M_K}{256\pi}\left(\frac{1}{16\pi^2F_0^4}\right)\left(\frac{\epsilon}{\sqrt{3}}\right)\Bigg\{20(M_K^2-M_\pi^2)-512\pi^2[3M_K^2(4L_4^r$$
$$+L_5^r)-M_\pi^2(6L_4^r+L_5^r)]+11M_\pi^2\log\left(\frac{M_\pi^2}{\mu^2}\right)+10(3M_K^2-2M_\pi^2)\log\left(\frac{M_K^2}{\mu^2}\right)+3(4M_K^2-3M_\pi^2)\log\left(\frac{M_\eta^2}{\mu^2}\right)\Bigg\}$$
$$+\frac{3Ze^2}{256\pi}\left(\frac{1}{16\pi^2F_0^2}\right)\Bigg\{42M_\pi M_K+512\pi^2[M_K^2(4L_4^r+L_5^r)-4M_\pi M_K(3L_4^r+L_5^r)+M_\pi^2(2L_4^r+L_5^r)]+11M_\pi(2M_K$$
$$-M_\pi)\log\left(\frac{M_\pi^2}{\mu^2}\right)-10M_K(M_K-2M_\pi)\log\left(\frac{M_K^2}{\mu^2}\right)+(-4M_K^2+6M_\pi M_K+M_\pi^2)\log\left(\frac{M_\eta^2}{\mu^2}\right)\Bigg\}. \tag{B23}$$

Otherwise, if the desired renormalization is as F_π^2 the term to be added is

$$\delta_{F_\pi^2}=\frac{3M_{\pi^\pm}M_{K^\pm}}{32\pi}\left(\frac{1}{16\pi^2F_0^4}\right)\Bigg\{128\pi^2[(M_{\pi^\pm}^2+2M_{K^\pm}^2)L_4^r+M_{\pi^\pm}^2L_5^r]-2M_{\pi^\pm}^2\log\left(\frac{M_{\pi^\pm}^2}{\mu^2}\right)-M_{K^\pm}^2\log\left(\frac{M_{K^\pm}^2}{\mu^2}\right)\Bigg\}$$
$$-\frac{3M_\pi M_K}{32\pi}\left(\frac{1}{16\pi^2F_0^4}\right)\left(\frac{\epsilon}{\sqrt{3}}\right)\Bigg\{2(M_K^2-M_\pi^2)-128\pi^2[3(2M_K^2-M_\pi^2)L_4^r+M_\pi^2L_5^r]+2M_\pi^2\log\left(\frac{M_\pi^2}{\mu^2}\right)$$
$$+(3M_K^2-2M_\pi^2)\log\left(\frac{M_K^2}{\mu^2}\right)\Bigg\}$$
$$-\frac{3Ze^2}{32\pi}\left(\frac{1}{16\pi^2F_0^2}\right)\Bigg\{-6M_\pi M_K-128\pi^2[2M_K^2L_4^r-2M_\pi M_K(3L_4^r+L_5^r)+M_\pi^2(L_4^r+L_5^r)]$$
$$+2M_\pi(-2M_K+M_\pi)\log\left(\frac{M_\pi^2}{\mu^2}\right)+M_K(M_K-2M_\pi)\log\left(\frac{M_K^2}{\mu^2}\right)\Bigg\}. \tag{B24}$$

Obviously, once the renormalization of the coupling constant is taken into account there are some cancellations that slightly simplify the expression for the combination of scattering lengths. In writing this expression we have applied the Gell-Mann–Okubo relation extensively to the polynomial terms and to the $\bar{J}$ functions but otherwise kept the eta mass inside the logarithmic functions. The functions $\mathcal{B}_+$ and $\mathcal{B}_-$ are defined in Appendix C. Notice that this combination of scattering lengths is independent at leading order of the ratio $m_s/\hat{m}$ this implies that the uncertainty from this quantity is very small [51].

APPENDIX C: LOOP INTEGRALS

In this appendix we collect for completeness some familiar formulas. The $\bar{B}_{ij}$ functions are the finite components of those defined in [52–54]. Using Lorenz decomposition and some simpler algebraic manipulation they can be reduced to the tadpole integral μ and the one-loop function $\overline{J}$, through the following set of finite relations:

$$\overline{B}(m_1^2,m_2^2;p^2)=\frac{\mu_{m_1}-\mu_{m_2}}{\Delta_{m_1m_2}}+\overline{J}(m_1^2,m_2^2;p^2),$$

$$\overline{B}_1(m_1^2,m_2^2;p^2)=\frac{1}{2}\left[\left(1+\frac{\Delta_{m_1m_2}}{p^2}\right)\overline{J}(m_1^2,m_2^2;p^2)+\frac{\mu_{m_1}-\mu_{m_2}}{\Delta_{m_1m_2}}\right],$$

$$\overline{B}_{21}(m_1^2,m_2^2;p^2)=\frac{1}{3p^2}\left\{\frac{1}{p^2}[\lambda_{m_1m_2}(p^2)+3p^2m_1^2]\overline{J}(m_1^2,m_2^2;p^2)+\left(\frac{p^2-m_2^2}{\Delta_{m_1m_2}}\right)\mu_{m_1}-\left(\frac{p^2-m_1^2}{\Delta_{m_1m_2}}\right)\mu_{m_2}+\frac{p^2}{6}\left(\frac{1}{16\pi^2}\right)\left(1-\frac{3\Sigma_{m_1m_2}}{p^2}\right)\right\},$$

$$\overline{B}_{22}(m_1^2,m_2^2;p^2)=-\frac{1}{12p^4}\left\{\lambda_{m_1m_2}(p^2)\overline{J}(m_1^2,m_2^2;p^2)+\frac{2p^4}{3}\left(\frac{1}{16\pi^2}\right)\left(1-\frac{3\Sigma_{m_1m_2}}{p^2}\right)-(p^2+\Delta_{m_1m_2})\mu_{m_1}-(p^2-\Delta_{m_1m_2})\mu_{m_2}+\lambda_{m_1m_2}(p^2)\frac{\mu_{m_1}-\mu_{m_2}}{\Delta_{m_1m_2}}\right\}. \tag{C1}$$

In the $\overline{J}$ function we have to consider at least two branches:

$$32\pi^2\overline{J}(m_1^2,m_2^2;p^2)=\begin{cases}2+\left(-\frac{\Delta_{m_1m_2}}{p^2}+\frac{\Sigma_{m_1m_2}}{\Delta_{m_1m_2}}\right)\log\left(\frac{m_1^2}{m_2^2}\right)-\frac{\lambda_{m_1m_2}}{p^2}\log\frac{(p^2+\lambda_{m_1m_2})^2-\Delta_{m_1m_2}^2}{(p^2-\lambda_{m_1m_2})^2-\Delta_{m_1m_2}^2} & \text{if } p^2\geqslant(m_1+m_2)^2,\\ 2+\left(-\frac{\Delta_{m_1m_2}}{p^2}+\frac{\Sigma_{m_1m_2}}{\Delta_{m_1m_2}}\right)\log\left(\frac{m_1^2}{m_2^2}\right)-2\frac{\sqrt{-\lambda_{m_1m_2}}}{p^2}\left[\arctan\left(\frac{p^2+\Delta_{m_1m_2}}{\sqrt{-\lambda_{m_1m_2}}}\right)-\arctan\left(\frac{-p^2+\Delta_{m_1m_2}}{\sqrt{-\lambda_{m_1m_2}}}\right)\right] & \text{if } (m_1-m_2)^2\leqslant p^2\leqslant(m_1+m_2)^2.\end{cases} \tag{C2}$$

With the use of the Gell-Mann–Okubo relation the $\overline{J}$ function can be reduced to simpler combinations of logarithmic and arctangent functions. In the latter case we define the functions

$$\mathcal{B}_\pm\equiv\mathcal{B}(M_K,M_\pi)\pm\mathcal{B}(M_K,-M_\pi),$$

where

$$\mathcal{B}(x,y)=-\frac{\sqrt{(x-y)(2x+y)}}{12\pi^2(x+y)}\left[\arctan\left(\frac{\sqrt{(x-y)(2x+y)}}{2(x-y)}\right)+\arctan\left(\frac{x+2y}{2\sqrt{(x-y)(2x+y)}}\right)\right]. \tag{C3}$$

In addition to the previous integrals the three-point function integrals are needed in the calculation. These arise through the photon exchange diagram. The scalar function, the only one that is actually IR divergent, is given in Eq. (5.5), while the remaining one is

$$C^\mu(m_P^2,m_Q^2,m_\gamma^2;p,k)\equiv-i\mu^{4-D}\int\frac{d^Dl}{(2\pi)^D}\frac{l^\mu}{(l^2-m_\gamma^2)[(p-l)^2-M_P^2][(k-l)^2-M_Q^2]}$$
$$=(p-k)^\mu G_{PQ}^-(p,k)+(p+k)^\mu G_{PQ}^+(p,k). \tag{C4}$$

In the convention of [52] the previous decomposition reads

$$G^{+}_{PQ}(p,k)\propto -\frac{C_{11}(p,k)}{2},\quad G^{-}_{PQ}(p,k)\propto -\frac{C_{11}(p,k)}{2}+C_{12}(p,k).$$

In terms of the basic functions we obtain

$$G^{-}_{PQ}(q^2,p\cdot k)=\frac{1}{\mathcal{G}}\left\{-\Delta_{PQ}\left[\overline{J}_{PQ}(q^2)-\frac{1}{16\pi^2}\right]-\frac{1}{32\pi^2}[(p+k)^2-\Sigma_{PQ}]\log\left(\frac{M_P^2}{M_Q^2}\right)\right\},$$

$$G^{+}_{PQ}(q^2,p\cdot k)=\frac{1}{\mathcal{G}}\left\{q^2\left[\overline{J}_{PQ}(q^2)-\frac{1}{16\pi^2}\right]-\frac{1}{32\pi^2}\left(q^2\frac{\Sigma_{PQ}}{\Delta_{PQ}}-\Delta_{PQ}\right)\log\left(\frac{M_P^2}{M_Q^2}\right)\right\}, \tag{C5}$$

where

$$\mathcal{G}=[q^2(p+k)^2-\Delta_{PQ}^2]\quad\text{and}\quad q^2=(p-k)^2.$$

[1] S. Weinberg, Physica A **96**, 327 (1979).
[2] F. Dyson, Phys. Rev. **85**, 631 (1952); A. R. Zhitnitsky, Phys. Rev. D **54**, 5148 (1996).
[3] G. Colangelo, J. Gasser, and H. Leutwyler, Phys. Lett. B **488**, 261 (2000).
[4] B. Moussallam, Eur. Phys. J. C **14**, 111 (2000).
[5] B. Adeva *et al.*, ''First Observation of πK Atom and Its Lifetime Measurement,'' proposal to the SPSLC, Report No. CERN/SPSLC 2000-032.
[6] S. M. Bilen'kiĭ, Nguyen van Hieu, L. L. Nemenov, and F. G. Tkebuchava, Yad. Fiz. **10**, 812 (1969); L. L. Nemenov, *ibid.* **41**, 629 (1984); A. A. Bel'kov, V. N. Pervushin, and F. G. Tkebuchava, *ibid.* **44**, 300 (1985).
[7] V. Bernard, N. Kaiser, and U. G. Meissner, Phys. Rev. D **43**, 2757 (1991); Nucl. Phys. **B357**, 129 (1991).
[8] R. Urech, Nucl. Phys. **B433**, 234 (1995).
[9] H. Neufeld and H. Rupertsberger, Z. Phys. C **68**, 91 (1995); **71**, 131 (1996).
[10] J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985).
[11] C. G. Callan, S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2247 (1969).
[12] R. Baur and R. Urech, Nucl. Phys. **B499**, 319 (1997).
[13] B. Moussallam, Nucl. Phys. **B504**, 381 (1997).
[14] S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966).
[15] J. Cronin, Phys. Rev. **161**, 1483 (1967); R. W. Griffith, *ibid.* **176**, 1705 (1968).
[16] G. Amorós, J. Bijnens, and P. Talavera, Nucl. Phys. **B602**, 87 (2001).
[17] G. Ecker, G. Muller, H. Neufeld, and A. Pich, Phys. Lett. B **477**, 88 (2000).
[18] H. Jallouli and H. Sazdjian, Phys. Rev. D **58**, 014011 (1998); **58**, 099901(E) (1998).
[19] A. Gall, J. Gasser, V. E. Lyubovitskij, and A. Rusetsky, Phys. Lett. B **462**, 335 (1999); J. Gasser, V. E. Lyubovitskij, and A. Rusetsky, *ibid.* **471**, 244 (1999).
[20] M. Knecht and R. Urech, Nucl. Phys. **B519**, 329 (1998).
[21] A. Denner, Fortschr. Phys. **41**, 307 (1993).
[22] F. S. Roig and A. R. Swift, Nucl. Phys. **B104**, 533 (1976).
[23] J. Bijnens and J. Prades, Nucl. Phys. **B490**, 239 (1997).
[24] C. Riggenbach, J. Gasser, J. F. Donoghue, and B. R. Holstein, Phys. Rev. D **43**, 127 (1991); J. Bijnens, Nucl. Phys. **B337**, 635 (1990).
[25] B. Ananthanarayan, P. Buttiker, and B. Moussallam, Eur. Phys. J. C **22**, 133 (2001).
[26] Particle Data Group, D. E. Groom *et al.*, Eur. Phys. J. C **15**, 1 (2000).
[27] R. Dashen, Phys. Rev. **183**, 1245 (1969).
[28] H. Leutwyler, Phys. Lett. B **378**, 313 (1996).
[29] B. R. Holstein, Phys. Lett. B **244**, 83 (1990).
[30] J. Sa Borges, Nucl. Phys. **B109**, 357 (1976).
[31] C. B. Lang, Fortschr. Phys. **26**, 509 (1978).
[32] B. Ananthanarayan and P. Buttiker, Eur. Phys. J. C **19**, 517 (2001).
[33] T. Hannah, Phys. Rev. D **52**, 4971 (1995).
[34] A. Dobado and J. R. Pelaez, Phys. Rev. D **56**, 3057 (1997).
[35] M. Jamin, J. A. Oller, and A. Pich, Nucl. Phys. **B587**, 331 (2000).
[36] A. Roessl, Nucl. Phys. **B555**, 507 (1999).
[37] M. J. Matison *et al.*, Phys. Rev. D **9**, 1872 (1974).
[38] H. H. Bingham *et al.*, Nucl. Phys. **B41**, 1 (1972).
[39] G. C. Fox and M. L. Griss, Nucl. Phys. **B80**, 403 (1974).
[40] P. Estabrooks *et al.*, Nucl. Phys. **B133**, 490 (1978).
[41] A. M. Bakker *et al.*, Nucl. Phys. **B24**, 211 (1970).
[42] Y. Cho *et al.*, Phys. Lett. **32B**, 409 (1970).
[43] P. Antich *et al.*, Nucl. Phys. **B29**, 305 (1971).
[44] A. R. Kirschbaum *et al.*, Phys. Rev. D **4**, 3254 (1971).
[45] B. Jongejans *et al.*, Nucl. Phys. **B67**, 381 (1973).
[46] C. B. Lang, Nuovo Cimento A **41**, 73 (1977).
[47] N. Johannesson and G. Nilsson, Nuovo Cimento A **43**, 376 (1978); **43**, 389 (1978).
[48] A. Karabarbouris and G. Shaw, J. Phys. G **6**, 583 (1980).
[49] W. E. Caswell and G. P. Lepage, Phys. Lett. **167B**, 437 (1986).
[50] B. Kubis and U. G. Meissner, hep-ph/0107199.
[51] M. Knecht, H. Sazdjian, J. Stern, and N. H. Fuchs, Phys. Lett. B **313**, 229 (1993).
[52] G. Passarino and M. Veltman, Nucl. Phys. **B160**, 151 (1979).
[53] B. Adeva *et al.*, ''Lifetime measurement of $\pi^+\pi^-$ atoms to test low-energy QCD predictions,'' proposal to the SPSLC, Report No. CERN/SPSLC/P 284, 1994.
[54] U.-G. Meißner, G. Müller, and S. Steininger, Phys. Lett. B **406**, 154 (1997); **407**, 454(E) (1997).