

How the quark self-energy affects the color-superconducting gap

Qun Wang

*Institut für Theoretische Physik, J. W. Goethe-Universität, D-60054 Frankfurt am Main, Germany
and Physics Department, Shandong University, Jinan, Shandong 250100, People's Republic of China*

Dirk H. Rischke

Institut für Theoretische Physik, J. W. Goethe-Universität, D-60054 Frankfurt am Main, Germany

(Received 5 October 2001; published 22 January 2002)

We consider color superconductivity with two flavors of massless quarks which form Cooper pairs with total spin zero. We solve the gap equation for the color-superconducting gap parameter to subleading order in the QCD coupling constant g at zero temperature. At this order in g , there is also a previously neglected contribution from the real part of the quark self-energy to the gap equation. Including this contribution leads to a reduction of the color-superconducting gap parameter ϕ_0 by a factor $b'_0 = \exp[-(\pi^2+4)/8] \approx 0.177$. On the other hand, the BCS relation $T_c \approx 0.57\phi_0$ between ϕ_0 and the transition temperature T_c is shown to remain valid after taking into account corrections from the quark self-energy. The resulting value for T_c confirms a result obtained previously with a different method.

DOI: 10.1103/PhysRevD.65.054005

PACS number(s): 12.38.Mh, 24.85.+p

I. INTRODUCTION

Quantum chromodynamics (QCD) is the fundamental theory of the strong interaction. In strongly interacting matter at large density or, equivalently, large quark chemical potential μ , asymptotic freedom [1] implies that single-gluon exchange becomes the dominant interaction between quarks. Single-gluon exchange is attractive in the color-antitriplet channel [2]. By Cooper's theorem [3], any attractive interaction destabilizes the Fermi surface and, at sufficiently small temperature T , leads to the condensation of Cooper pairs. If the Cooper pair condensate carries charge quantum numbers of a local gauge symmetry, the Meissner effect leads to superconductivity. Strongly interacting matter, where quark Cooper pairs carry color charge, becomes a color superconductor. In a superconductor, exciting particle-hole pairs costs at least an amount of energy of $2\phi_0$, where ϕ_0 is the value of the superconducting gap parameter at the Fermi surface for $T=0$. Its value can be computed from a gap equation derived in the mean-field approximation, which, in QCD, involves single-gluon exchange [4,5].

Schematically, this gap equation can be written in the form [6]

$$\phi_0 = g^2 \left[\zeta \ln^2 \left(\frac{\mu}{\phi_0} \right) + \beta \ln \left(\frac{\mu}{\phi_0} \right) + \alpha \right] \phi_0. \quad (1)$$

For small values of the QCD coupling constant, $g \ll 1$, the solution is [4,5,7-9]

$$\phi_0 = 2b \mu \exp \left(-\frac{c}{g} \right) [1 + O(g)]. \quad (2)$$

The first term in Eq. (1) contains two powers of the logarithm $\ln(\mu/\phi_0)$. One logarithm is well known from the gap equation in standard BCS theory [3], where it arises from the integration over fermion momenta up to the Fermi surface. The other logarithm is special to theories with long-range

interactions, like the exchange of almost static magnetic gluons in QCD [5,7]. Its origin is a collinear singularity when integrating over angles between quark and gluon momenta in the gap equation. The weak-coupling solution (2) implies that the first term in brackets in Eq. (1) is $\sim 1/g^2$. It therefore dominates the right-hand side of Eq. (1). Together with the prefactor g^2 , it is of order $O(1)$ in the gap equation. The value of the coefficient ζ determines the constant c in Eq. (2).

The second term in Eq. (1) contains subleading contributions of order $O(g)$ to the gap equation, characterized by a single power of the logarithm $\ln(\mu/\phi_0) \sim 1/g$. A part of these contributions arises from the exchange of nonstatic magnetic and static electric gluons [5]. Both types of interactions are of short range: they are screened on a distance scale m_g^{-1} , where m_g is the gluon mass; $m_g^2 = N_f g^2 \mu^2 / (6\pi^2)$, N_f is the number of quark flavors. Consequently, the collinear logarithm characteristic for long-range interactions is absent, and one is left with the BCS logarithm. The coefficient β in Eq. (1) determines the constant b in Eq. (2).

The third term in Eq. (1) summarizes sub-subleading contributions of order $O(g^2)$ with neither a collinear nor a BCS logarithm. It was argued in Refs. [4,5,10] that at this order gauge-dependent terms enter the QCD gap equation. However, the gap parameter is in principle an observable quantity, and thus gauge independent. Therefore one concludes that the mean-field approach cannot be used to compute sub-subleading contributions to the gap parameter. It was also shown [11] that effects from the finite lifetime of quasiparticles in the Fermi sea influence the value of ϕ_0 at this order. In weak coupling, these contributions are suppressed by one power of g compared to the subleading terms and therefore constitute an order $O(g)$ correction to the prefactor b , as indicated in Eq. (2).

The value of the coefficient c was first computed by Son [7]:

$$\frac{c}{g} = \frac{\pi}{2g}, \quad \bar{g} \equiv \frac{g}{3\sqrt{2}}. \quad (3)$$

Son also gave an estimate for the constant b ,

$$b = \frac{b_0}{g^5}, \quad (4)$$

with a constant b_0 of order $O(1)$, which could not be determined in the approach of Ref. [7]. In Refs. [4,5] the constant b_0 was computed by solving the QCD gap equation including nonstatic magnetic and static electric gluon exchange. The result is

$$b_0 = 256 \pi^4 \left(\frac{2}{N_f} \right)^{5/2} b'_0, \quad (5)$$

with an undetermined constant b'_0 of order $O(1)$. In Refs. [4,5], where the only subleading contributions to the gap equation arise from nonstatic magnetic and static electric gluon exchange, $b'_0 = 1$. In principle, however, there could be other subleading contributions, which would change b'_0 to a value $b'_0 \neq 1$.

At sufficiently large temperature, thermal random motion breaks up Cooper pairs and the superconducting condensate melts. In Ref. [5] it was shown that the temperature T_c for the transition between the normal and the superconducting phase is related to the zero-temperature gap at the Fermi surface in the same way as in BCS theory,

$$T_c = \frac{e^\gamma}{\pi} \phi_0 \approx 0.57 \phi_0, \quad (6)$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant.

In Ref. [12], Brown, Liu, and Ren calculated T_c in a different approach with the result

$$T_c = 2 \frac{e^\gamma}{\pi} 256 \pi^4 \left(\frac{2}{N_f g^2} \right)^{5/2} c'_1 \mu \exp\left(-\frac{\pi}{2g}\right), \quad (7)$$

where

$$c'_1 = \exp\left(-\frac{\pi^2 + 4}{8}\right) \approx 0.177. \quad (8)$$

Furthermore, the authors of Ref. [12] assumed the validity of Eq. (6), and concluded that

$$b'_0 = c'_1, \quad (9)$$

as one readily checks with Eqs. (2)–(5), (7), and (8). Physically, the difference between the approach of Refs. [4,5] and that of Ref. [12] is that contributions from the quark self-energy were neglected in the former, but taken into account in the latter. If the above arguments are correct, one may therefore conclude that the quark self-energy constitutes a subleading correction to the gap equation and thus is responsible for the change of the value of b'_0 from 1 to c'_1 given by Eq. (8). The authors of Ref. [12] also assert that there are no further subleading contributions that could alter the value of c'_1 .

While manifestly gauge invariant, a disadvantage of the approach of Ref. [12] is that its range of applicability is restricted to the normal-conducting phase and thus can only determine the value of T_c , but not the value of the zero-temperature gap ϕ_0 . Therefore a relation between T_c and ϕ_0 , like Eq. (6), can in principle not be established within this approach. It is possible to derive such a relation with the help of the gap equation, as demonstrated in Ref. [5], but contributions from the quark self-energy were neglected in obtaining the result Eq. (6). It is conceivable that Eq. (6) changes, once these contributions are taken into account. Consequently, the validity of Eq. (9) is not obvious.

The aim of the present paper is twofold. On the one hand, we want to compute the contribution of the quark self-energy to the value of the constant b'_0 in the zero-temperature gap. On the other hand, we want to confirm the result (7) for T_c . To this end, it is necessary to first compute the value of the zero-temperature gap by directly solving the gap equation including the quark self-energy. Second, one has to prove that Eq. (6) remains valid in order to determine T_c , which then can be compared to the value (7) obtained in Ref. [12]. Our paper is organized as follows. In Sec. II we first clarify how the quark self-energy enters the gap equation. In Sec. III, the resulting gap equation is solved at zero temperature. In Sec. IV we determine T_c . We conclude in Sec. V with a summary of our results.

Our convention for the metric tensor is $g^{\mu\nu} = \text{diag}\{1, -1, -1, -1\}$. Our units are $\hbar = c = k_B = 1$. Four-vectors are denoted by capital letters, $K \equiv K^\mu = (k_0, \mathbf{k})$, and $k \equiv |\mathbf{k}|$, while $\hat{\mathbf{k}} \equiv \mathbf{k}/k$.

II. THE GAP EQUATION INCLUDING THE QUARK SELF-ENERGY

In fermionic systems at nonzero density, it is advantageous to treat fermions and charge-conjugate fermions as independent degrees of freedom and to work in the so-called Nambu-Gorkov basis. In this basis, the full inverse fermion propagator is defined as [11]

$$S^{-1} \equiv \begin{pmatrix} S_{11}^{-1} & S_{12}^{-1} \\ S_{21}^{-1} & S_{22}^{-1} \end{pmatrix} = \begin{pmatrix} S_{11}^{0-1} + \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & S_{22}^{0-1} + \Sigma_{22} \end{pmatrix}, \quad (10)$$

where S_{11}^0 is the propagator for free fermions, S_{22}^0 the propagator for free charge-conjugate fermions. In momentum space and, for $\mu \gg m$,

$$S_{11}^0(Q) = (\gamma^\mu Q_\mu + \mu \gamma_0)^{-1}, \quad S_{22}^0(Q) = (\gamma^\mu Q_\mu - \mu \gamma_0)^{-1}, \quad (11)$$

where γ^μ are Dirac matrices. The four components of the fermion self-energy are denoted as Σ_{ij} , $i, j = 1, 2$. The 11 component of the self-energy, Σ_{11} , is the standard one-loop self-energy for fermions; similarly, Σ_{22} is the self-energy for charge-conjugate fermions. The 21 component of the self-energy, Σ_{21} , which was denoted Φ^+ in Ref. [5], is the gap matrix in a superconductor, while $\Sigma_{12} = \gamma_0 \Sigma_{21}^\dagger \gamma_0$. In the fol-

lowing, we somewhat imprecisely use the term “self-energy” only for the diagonal components Σ_{11} and Σ_{22} .

Inverting Eq. (10) one obtains the full fermion propagator S , with the diagonal components

$$S_{11}=[S_{11}^{0-1}+\Sigma_{11}-\Sigma_{12}(S_{22}^{0-1}+\Sigma_{22})^{-1}\Sigma_{21}]^{-1}, \quad (12a)$$

describing the (normal) propagation of fermions, and

$$S_{22}=[S_{22}^{0-1}+\Sigma_{22}-\Sigma_{21}(S_{11}^{0-1}+\Sigma_{11})^{-1}\Sigma_{12}]^{-1}, \quad (12b)$$

describing the (normal) propagation of charge-conjugate fermions. In superconductors, due to the presence of a fermion-fermion condensate one can always convert an incoming fermion into an outgoing charge-conjugate fermion and vice versa. Therefore the full fermion propagator S also has off-diagonal components,

$$S_{12}=-\left(S_{11}^{0-1}+\Sigma_{11}\right)^{-1}\Sigma_{12}S_{22}, \quad (12c)$$

$$S_{21}=-\left(S_{22}^{0-1}+\Sigma_{22}\right)^{-1}\Sigma_{21}S_{11}, \quad (12d)$$

describing the *anomalous* propagation of fermions and charge-conjugate fermions. [Please note that our sign convention for the self-energy differs from that in Ref. [11], which leads to the difference between our Eqs. (12) and Eqs. (2.4) and (2.5) in Ref. [11].]

Let us now consider a system of quarks interacting via one-gluon exchange. In mean-field approximation [13], the four components of the fermion self-energy in momentum space are computed as

$$\Sigma_{ij}(K)=-g^2\frac{T}{V}\sum_Q\Delta_{\mu\nu}^{ab}(K-Q)[\hat{\Gamma}_a^\mu S(Q)\hat{\Gamma}_b^\nu]_{ij}, \quad i,j=1,2. \quad (13)$$

Here, $\Delta_{\mu\nu}^{ab}$ is the gluon propagator, and $\hat{\Gamma}_a^\mu$ is the diagonal Nambu-Gorkov matrix $\hat{\Gamma}_a^\mu=\text{diag}(\gamma^\mu T_a, -\gamma^\mu T_a^T)$, T_a are the Gell-Mann matrices. We compute the self-energy in the imaginary-time formalism, i.e., $T/V\Sigma_Q\equiv T\Sigma_n\int d^3\mathbf{q}/(2\pi)^3$, where n labels the fermionic Matsubara frequencies, $\omega_n=(2n+1)\pi T\equiv iq_0$. For $ij=11$, Eq. (13) becomes Eq. (2.7) of [11], for $ij=21$, we recover Eq. (2.6) of Ref. [11] (however, due to our different sign convention, only up to an overall sign).

A fully self-consistent treatment of the mean-field approximation requires us to solve the coupled system of Eqs. (12) and (13). The mean-field solution obtained in this way resums terms of infinite order in the coupling constant. However, because only a particular class of diagrams is taken into account (the so-called “rainbow” diagrams), such a solution is in general not gauge invariant. On the other hand, the quasiparticle properties encoded in the propagator, like their excitation spectrum, are physical observables and thus in principle gauge invariant. Indeed, a complete solution of the Schwinger-Dyson equations, as well as a perturbative expansion in powers of g , preserve gauge invariance. Nevertheless, as was discussed in the introduction, an expansion of the mean-field equation (13) for the color-superconducting gap matrix $\Phi^+\equiv\Sigma_{21}$ in powers of g is believed to be gauge

invariant up to terms of subleading order [the first two terms in Eq. (1)], and the gauge dependence only surfaces at sub-subleading order [the third term in Eq. (1), or the terms $\sim O(g)$ in Eq. (2)]. To preserve gauge invariance beyond subleading order, other diagrams than those of rainbow topology have to be added to Eq. (13), or in other words, one has to go beyond the mean-field approximation to solve for the gap matrix Σ_{21} .

If one restricts the computation of the gap to subleading accuracy, however, the mean-field equation (13) should be sufficient to obtain a gauge-invariant result. It is then mandatory to identify *all* terms that can contribute to subleading order. In Eq. (13) for Σ_{21} , the term in square brackets becomes $-\gamma^\mu T_a^T S_{21}(Q)\gamma^\nu T_b$. With the exception of Ref. [11], previous calculations of the QCD gap parameter neglected the terms Σ_{11} and Σ_{22} in S_{21} , see Eq. (12). A perturbative calculation of these self-energies, i.e., approximating $[\hat{\Gamma}_a^\mu S(Q)\hat{\Gamma}_b^\nu]_{11}\approx\gamma^\mu T_a S_{11}^0(Q)\gamma^\nu T_b$, and analogously for $ij=22$, and analytical continuation to real energies q_0 gives the result [14]

$$\Sigma^0(Q)\equiv\Sigma_{11}^0(Q)=\Sigma_{22}^0(Q)\approx\gamma_0 g^2\left(q_0\ln\frac{M^2}{q_0^2}+i\pi|q_0|\right), \quad (14)$$

where $M^2=(3\pi/4)m_g^2$. On the quasiparticle mass shell, $q_0=\epsilon_q$, and near the Fermi surface, $\epsilon_q\approx\phi_0$, the real part of the self-energy is of order $g^2\phi_0\ln(\mu/\phi_0)\sim g\phi_0$, while the imaginary part is $\sim g^2\phi_0$ and thus down by a factor of g compared to the real part.

In Ref. [11], the real part of Σ^0 was neglected and the effect of the imaginary part on the magnitude of the color-superconducting gap was studied. It was found that a nonvanishing imaginary part leads to sub-subleading corrections [terms included in the third term $\sim\alpha$ in Eq. (1)] and to corrections of order $O(g)$ to the prefactor of the gap, cf. Eq. (2). Therefore they are of the same order as terms that violate gauge invariance in the mean-field approximation [4,5,10].

Since the real part of the self-energy is parametrically larger than the imaginary part by one power of g , we expect the former to contribute to subleading order, $O(g)$, to the gap equation, and therefore lead to a correction of order $O(1)$ to the prefactor. As discussed in the introduction, this is precisely what the authors of Ref. [12] found, assuming the validity of Eq. (6). In the next section, we solve the gap equation including the quark self-energy and compute the value of b'_0 at zero temperature. In Sec. IV we then check the validity of Eq. (6).

First note that, since the real part of the quark self-energy is expected to influence the value of ϕ_0 only at subleading order in the gap equation, it is sufficient to approximate the value of Σ_{11} or Σ_{22} in the propagator in Eq. (13) [cf. Eq. (12)] by the perturbative expression Σ^0 , Eq. (14), the difference contributing at sub-subleading order to the gap equation. In order to solve the gap equation, let us revert the analytic continuation to real energies in Eq. (14), i.e., q_0 is purely imaginary in the following. From Eqs. (12) and (14), the effect of including Σ^0 is to replace

$$q_0 \rightarrow \frac{q_0}{Z(q_0)} \quad (15)$$

in the quark propagator, where

$$Z(q_0) \equiv \left(1 + \bar{g}^2 \ln \frac{M^2}{q_0^2} \right)^{-1} \quad (16)$$

is the quark wave-function renormalization factor. Since we only want to consider the real part of the quark self-energy, we shall ignore the cut of the logarithm in Eq. (16) when performing the Matsubara sum in Eq. (13) by contour integration. In other words, we assume that the quark propagator has only simple poles in the complex q_0 plane, corresponding to the excitation energies of quasiparticles with infinite lifetime. This approximation is valid up to subleading order in the gap equation, because, as explained above, effects from a finite quasiparticle lifetime enter only at sub-subleading order.

A wave-function renormalization of the form (16) is known from nonrelativistic systems [15], where it leads to non-Fermi-liquid behavior. In relativistic systems, non-Fermi-liquid behavior has been recently studied in great detail by Boyanovsky and de Vega [16].

After these introductory remarks, we may immediately proceed to Eq. (3.3) of Ref. [11] or Eq. (32) of Ref. [5]. This equation determines the spin-zero gap in a two-flavor color superconductor in pure Coulomb gauge. With the replacement (15) it reads

$$\begin{aligned} \phi(K) = & \frac{2}{3} g^2 \frac{T}{V} \sum_{\mathcal{Q}} Z^2(q_0) \frac{\phi(\mathcal{Q})}{q_0^2 - [Z(q_0)\epsilon_q]^2} \\ & \times \left[\Delta_l(K - \mathcal{Q}) \frac{1 + \hat{\mathbf{k}} \cdot \hat{\mathbf{q}}}{2} + \Delta_l(K - \mathcal{Q}) \right. \\ & \left. \times \left(-\frac{3 - \hat{\mathbf{k}} \cdot \hat{\mathbf{q}}}{2} + \frac{1 + \hat{\mathbf{k}} \cdot \hat{\mathbf{q}}}{2} \frac{(k - q)^2}{(\mathbf{k} - \mathbf{q})^2} \right) \right], \quad (17) \end{aligned}$$

where we neglected the contribution of antiparticles. The next step is to perform the Matsubara sum over q_0 . We use spectral representations for the propagators, as in Ref. [5]. The only difference to the calculation of Ref. [5] is that the poles of the fermion propagator are shifted. To leading order, they are now given by

$$q_0 \approx \pm Z(\epsilon_q) \epsilon_q \equiv \pm \tilde{\epsilon}_q. \quad (18)$$

The rest of the calculation is straightforward. We also take the external quark energy k_0 to be on the new quasiparticle mass shell, $k_0 = Z(\epsilon_k) \epsilon_k = \tilde{\epsilon}_k$. Then, in analogy to Eq. (3.4) of Ref. [11] and Eq. (72) of Ref. [5], the final result for the gap equation, including the quark self-energy, reads

$$\phi_k \approx \bar{g}^2 \int_0^{\delta d(q - \mu)} \frac{\delta d(q - \mu)}{\tilde{\epsilon}_q} Z^2(\tilde{\epsilon}_q) \tanh\left(\frac{\tilde{\epsilon}_q}{2T}\right) \frac{1}{2} \ln\left(\frac{\tilde{b}^2 \mu^2}{|\tilde{\epsilon}_q^2 - \tilde{\epsilon}_k^2|}\right) \phi_q, \quad (19)$$

where $\tilde{b} \equiv 256\pi^4 [2/(N_f g^2)]^{5/2}$. Note that we have replaced the symbol b in Eq. (72) of Ref. [5] by \tilde{b} , because the definition of b , cf. Eqs. (4),(5), includes b'_0 , the value of which has yet to be determined. In Ref. [5], this distinction was not necessary, because there $b'_0 \equiv 1$. We abbreviated $\phi_k \equiv \phi(\tilde{\epsilon}_k, \mathbf{k})$; ϕ_q is defined similarly.

III. SOLVING THE GAP EQUATION

Let us now solve the gap equation (19) at zero temperature. In this case, the factor $\tanh[\tilde{\epsilon}_q/(2T)] = 1$. Moreover, to leading order we can make the replacements $\tilde{\epsilon}_q \rightarrow \epsilon_q$ and $\tilde{\epsilon}_k \rightarrow \epsilon_k$ in the logarithm $\ln(\tilde{b}^2 \mu^2 / |\tilde{\epsilon}_q^2 - \tilde{\epsilon}_k^2|)$. For similar reasons, $Z(\tilde{\epsilon}_q) \approx Z(\epsilon_q)$. Following Ref. [7], we approximate

$$\frac{1}{2} \ln\left(\frac{\tilde{b}^2 \mu^2}{|\tilde{\epsilon}_q^2 - \tilde{\epsilon}_k^2|}\right) \rightarrow \ln\left(\frac{\tilde{b} \mu}{\epsilon_q}\right) \theta(q - k) + \ln\left(\frac{\tilde{b} \mu}{\epsilon_k}\right) \theta(k - q), \quad (20)$$

and then introduce the variables [5]

$$x = \bar{g} \ln\left(\frac{2\tilde{b} \mu}{k - \mu + \epsilon_k}\right), \quad (21a)$$

$$y = \bar{g} \ln\left(\frac{2\tilde{b} \mu}{q - \mu + \epsilon_q}\right), \quad (21b)$$

$$x^* = \bar{g} \ln\left(\frac{2\tilde{b} \mu}{\phi_0}\right), \quad (21c)$$

$$x_0 = \bar{g} \ln\left(\frac{\tilde{b} \mu}{\delta}\right). \quad (21d)$$

Note that in contrast to Ref. [5] we choose to include a factor \bar{g} in the definition of these variables. Consequently, since $\phi_0 \sim \mu \exp(-1/\bar{g})$, $x^* \sim O(1)$ and $x_0 \sim O(\bar{g})$. Furthermore, x and y are of order $O(1)$ near and of order $O(\bar{g})$ away from the Fermi surface.

In analogy to Eqs. (84) and (85) of Ref. [5], the gap equation and its derivatives read in these new variables

$$\phi(x) \approx x \int_x^{x^*} dy (1 - 2\bar{g}y) \phi(y) + \int_{x_0}^x dy y (1 - 2\bar{g}y) \phi(y), \quad (22a)$$

$$\frac{d\phi(x)}{dx} \approx \int_x^{x^*} dy (1 - 2\bar{g}y) \phi(y), \quad (22b)$$

$$\frac{d^2\phi(x)}{dx^2} \approx -(1 - 2\bar{g}x) \phi(x). \quad (22c)$$

In these equations, we neglected contributions of order $O(\bar{g}^2)$, for instance, a term $\bar{g}^2 \ln(\tilde{b} \mu / M)$ in the wave-function renormalization factor $Z(\epsilon_q)$.

In order to solve Eq. (22c), we replace x with a new variable z ,

$$z \equiv -(2\bar{g})^{-2/3}(1-2\bar{g}x), \quad (23)$$

and obtain Airy's differential equation [17],

$$\frac{d^2\phi(z)}{dz^2} = z\phi(z). \quad (24)$$

The solution $\phi(z)$ of Eq. (24) is a linear combination of the Airy functions $\text{Ai}(z)$ and $\text{Bi}(z)$,

$$\phi(z) = C_1 \text{Ai}(z) + C_2 \text{Bi}(z). \quad (25)$$

In weak coupling, z is always negative, and the Airy functions and their first derivatives can be expressed in terms of modulus and phase, defined as

$$\text{Ai}(z) = M(|z|)\cos\theta(|z|), \quad \text{Bi}(z) = M(|z|)\sin\theta(|z|),$$

$$M(|z|) = \sqrt{\text{Ai}'^2(z) + \text{Bi}'^2(z)}, \quad \theta(|z|) = \arctan\left[\frac{\text{Bi}(z)}{\text{Ai}(z)}\right],$$

$$\text{Ai}'(z) = N(|z|)\cos\varphi(|z|), \quad \text{Bi}'(z) = N(|z|)\sin\varphi(|z|),$$

$$N(|z|) = \sqrt{\text{Ai}'^2(z) + \text{Bi}'^2(z)}, \quad \varphi(|z|) = \arctan\left[\frac{\text{Bi}'(z)}{\text{Ai}'(z)}\right]. \quad (26)$$

At the Fermi surface, the value of the zero-temperature gap function is $\phi(z^*) = \phi_0$ and its derivative vanishes, $d\phi(z^*)/dz = 0$. Consequently, we obtain for the gap function

$$\phi(z) = \phi_0 \frac{M(|z|) \sin[\varphi(|z^*|) - \theta(|z|)]}{M(|z^*|) \sin[\varphi(|z^*|) - \theta(|z^*|)]}. \quad (27)$$

In order to determine ϕ_0 , we use Eq. (22a) at the Fermi surface, $z = z^*$, and substitute the integration variable y by $u \equiv -(2\bar{g})^{-2/3}(1-2\bar{g}y)$ to obtain

$$\phi(z^*) = \int_{z^*}^{z_0} du [u + (2\bar{g})^{-2/3}] u \phi(u), \quad (28)$$

where $z_0 = -(2\bar{g})^{-2/3}[1-2\bar{g}x_0]$. According to Eq. (24), we can replace $u\phi(u)$ with $d^2\phi(u)/du^2$. Integrating by parts, this leads to the condition

$$[z_0 + (2\bar{g})^{-2/3}]\phi'(z_0) = \phi(z_0). \quad (29)$$

Note that the above equation depends on z^* through Eq. (27). It seems that Eq. (29) also depends on z_0 which is arbitrary and far from the Fermi surface. In weak coupling, however, the dependence on z_0 disappears, as we shall show in the following. We first rewrite the condition (29) as

$$\begin{aligned} & [z_0 + (2\bar{g})^{-2/3}]\sin[\varphi(|z^*|) - \varphi(|z_0|)] \\ &= \frac{M(|z_0|)}{N(|z_0|)}\sin[\varphi(|z^*|) - \theta(|z_0|)]. \end{aligned} \quad (30)$$

In weak coupling, $|z| \sim (2\bar{g})^{-2/3} \gg 1$, and we may use the asymptotic formulas [17]

$$\varphi(|z|) \approx \frac{3\pi}{4} - \frac{2}{3}|z|^{3/2} - \frac{7}{48}|z|^{-3/2} + O(|z|^{-9/2})$$

$$\approx -\frac{1}{3\bar{g}} + \frac{3\pi}{4} + x - \bar{g}\left(\frac{x^2}{2} + \frac{7}{24}\right) + O(\bar{g}^2),$$

$$\theta(|z|) \approx \frac{\pi}{4} - \frac{2}{3}|z|^{3/2} + \frac{5}{48}|z|^{-3/2} + O(|z|^{-9/2})$$

$$\approx -\frac{1}{3\bar{g}} + \frac{\pi}{4} + x - \bar{g}\left(\frac{x^2}{2} - \frac{5}{24}\right) + O(\bar{g}^2),$$

$$\frac{M(|z|)}{N(|z|)} \approx |z|^{-1/2}[1 + O(|z|^{-3})], \quad (31)$$

where we employed $|z| \approx (2\bar{g})^{-2/3}(1-2\bar{g}x)$, cf. Eq. (23). We now expand Eq. (30) to order $O(\bar{g})$ and obtain

$$x^* = \arctan\left[-\frac{2}{\bar{g}(1+x^{*2})}\right]. \quad (32)$$

In weak coupling, the argument of the arctan is large, and we can expand the right-hand side to order $O(\bar{g})$ around $\pi/2$. The result is

$$x^* \approx \frac{\pi}{2} + \bar{g}\frac{1+x^{*2}}{2}. \quad (33)$$

To order $O(\bar{g})$, we can approximate $x^{*2} \approx \pi^2/4$ on the right-hand side of Eq. (33), and using the definition of x^* , Eq. (21c), we obtain the zero-temperature gap value at the Fermi surface,

$$\phi_0 = 2\bar{b}b'_0\mu \exp\left(-\frac{\pi}{2\bar{g}}\right), \quad (34)$$

where b'_0 is given by Eq. (9), with c'_1 of Eq. (8). In conclusion, the effect of including the quark self-energy in the gap equation changes the value of b'_0 from one, as in Refs. [4,5], to the value c'_1 given in Eq. (8).

IV. DETERMINING THE TRANSITION TEMPERATURE

Within the gap equation approach, we can also determine the temperature T_c for the transition between the normal and the superconducting phase. We follow Ref. [5] and consider the gap equation (22a) at the Fermi surface, restoring the factor $\tanh[\epsilon(y)/(2T)]$ present at nonzero temperature. As in

Ref. [5], we assume that to leading order the shape of the gap function at nonzero temperature is the same as at zero temperature, and that only the overall magnitude changes with temperature, $\phi(x, T) = \phi(T)\phi(x, 0)/\phi_0$. Then the gap equation reads

$$1 = \int_{x_0}^{x^*} dy y (1 - 2\bar{g}y) \tanh \frac{\epsilon(y)}{2T} \frac{\phi(y, 0)}{\phi_0}. \quad (35)$$

We now separate the range of integration into two pieces, $[x_0, x^*] \rightarrow [x_0, x_\kappa] + [x_\kappa, x^*]$, where $x_\kappa = x^* - \bar{g} \ln(2\kappa) = \bar{g} \ln[\bar{b}\mu/(\kappa\phi_0)]$. The main contribution to the integral in Eq. (35) comes from the region of momenta away from the Fermi surface, $[x_0, x_\kappa]$. In this region, $\epsilon(y) \gg \phi_0 \sim T$, such that we can approximate the factor $\tanh[\epsilon(y)/(2T)] \approx 1$. By making use of Eqs. (24) and (29), the integral over the region $[x_0, x_\kappa]$ is evaluated as

$$\begin{aligned} \mathcal{I} &= \int_{x_0}^{x_\kappa} dy y (1 - 2\bar{g}y) \frac{\phi(y, 0)}{\phi_0} \\ &= \frac{1}{\phi_0} \{ \phi(z_\kappa) - [z_\kappa + (2\bar{g})^{-2/3}] \phi'(z_\kappa) \} \\ &= 1 - \frac{\pi}{2} \bar{g} \ln 2\kappa + O(\bar{g}^2), \end{aligned} \quad (36)$$

where $z_\kappa = z^* - (2\bar{g})^{1/3} \bar{g} \ln 2\kappa$. The last line is obtained by expanding the right-hand side of the second equality to order $O(\bar{g})$ around z^* . Equation (35) becomes

$$\int_{x_\kappa}^{x^*} dy y (1 - 2\bar{g}y) \tanh \frac{\epsilon(y)}{2T} \frac{\phi(y, 0)}{\phi_0} = \frac{\pi}{2} \bar{g} \ln 2\kappa + O(\bar{g}^2). \quad (37)$$

The integral on the left-hand side may now be computed to order $O(\bar{g})$. As in Ref. [5], this amounts to approximating $y \approx x^*$ and $\phi(y, 0)/\phi_0 \approx 1$. Furthermore, the correction from the quark self-energy can be neglected, $1 - 2\bar{g}y \approx 1$. In this way, we obtain Eq. (104) of Ref. [5]; consequently the BCS result (6) remains valid to leading order in g , even when the quark self-energy is taken into account in the gap equation. With Eqs. (6) and (34), we thus conclude that our result for T_c is the same as that obtained in Ref. [12].

V. CONCLUSIONS

In this paper, we have computed the spin-zero gap in a two-flavor color superconductor at zero temperature from a mean-field gap equation. In contrast to earlier studies [4,5,8,9,11], we have included subleading contributions from the real part of the quark self-energy. We found that these contributions reduce the gap parameter at the Fermi surface by a factor $b'_0 = \exp[-(\pi^2 + 4)/8] \approx 0.177$. We then computed the transition temperature T_c between the normal and superconducting phase and found that the BCS relation $T_c \approx 0.57\phi_0$ remains valid to leading order in g after including the corrections from the quark self-energy. Therefore we obtain the same value for T_c as in Ref. [12].

ACKNOWLEDGMENTS

The authors thank R. Pisarski and I. Shovkovy for discussions and a critical reading of the manuscript. Q.W. acknowledges financial support from the Alexander von Humboldt-Foundation. He also appreciates help and support from the Institut für Theoretische Physik of the J. W. Goethe-Universität, and especially from Professor W. Greiner and Professor H. Stöcker.

-
- [1] D.J. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973); H.D. Politzer, *ibid.* **30**, 1346 (1973).
[2] D. Bailin and A. Love, Phys. Rep. **107**, 325 (1984).
[3] J.R. Schrieffer, *Theory of Superconductivity* (Benjamin, New York, 1964).
[4] T. Schäfer and F. Wilczek, Phys. Rev. D **60**, 114033 (1999).
[5] R.D. Pisarski and D.H. Rischke, Phys. Rev. D **61**, 051501 (2000); **61**, 074017 (2000).
[6] D.H. Rischke, Phys. Rev. D **64**, 094003 (2001).
[7] D.T. Son, Phys. Rev. D **59**, 094019 (1999).
[8] D.K. Hong, Phys. Lett. B **473**, 118 (2000); Nucl. Phys. **B582**, 451 (2000).
[9] I.A. Shovkovy and L.C.R. Wijewardhana, Phys. Lett. B **470**, 189 (1999); D.K. Hong, V.A. Miransky, I.A. Shovkovy, and L.C.R. Wijewardhana, Phys. Rev. D **61**, 056001 (2000); **62**, 059903(E) (2000).
[10] K. Rajagopal and E. Shuster, Phys. Rev. D **62**, 085007 (2000).
[11] C. Manuel, Phys. Rev. D **62**, 114008 (2000).
[12] W.E. Brown, J.T. Liu, and H.-C. Ren, Phys. Rev. D **61**, 114012 (2000); **62**, 054013 (2000); **62**, 054016 (2000).
[13] R.D. Pisarski and D.H. Rischke, Phys. Rev. D **60**, 094013 (1999).
[14] C. Manuel, Phys. Rev. D **62**, 076009 (2000).
[15] T. Holstein, R.E. Norton, and P. Pincus, Phys. Rev. B **8**, 2649 (1973).
[16] D. Boyanovsky and H.J. de Vega, Phys. Rev. D **63**, 034016 (2001).
[17] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I.A. Stegun (Dover, New York, 1965).