

Seesaw mechanisms for Dirac and Majorana neutrino masses

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We investigate the seesaw mechanism for generally non-fine-tuned $n \times n$ mass matrices involving both Dirac and Majorana neutrinos. We specifically show that the number of naturally light neutrinos cannot exceed half of the dimension of the considered mass matrix. Furthermore, we determine a criterion for mass matrix textures leading to light Dirac neutrinos with the seesaw mechanism. In particular, we study 4×4 and 6×6 mass matrix textures and give some examples in order to highlight these types of texture. Next, we present a model scheme based on non-Abelian and discrete symmetries satisfying the above mentioned criterion for light Dirac neutrinos. Finally, we investigate the connection between symmetries and the invariants of a mass matrix on a formal level.

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I. INTRODUCTION

Neutrino mass squared differences have lately been more and more accurately measured by neutrino oscillation experiments. The latest values can be summarized as follows [1–4]:

$$\Delta m_{\odot}^2 \in (10^{-5}, 10^{-4}) \text{ eV}^2,$$

$$\Delta m_{\text{atm.}}^2 \approx 2.5 \times 10^{-3} \text{ eV}^2,$$

where Δm_{\odot}^2 is the solar mass squared difference of the preferred large mixing angle solution of the solar neutrino problem and $\Delta m_{\text{atm.}}^2$ is the atmospheric mass squared difference. These results were originally obtained in two flavor neutrino oscillation analyses, and are approximately valid in three flavor neutrino oscillation models at least as long as the vacuum mixing angle $\theta_2 \equiv \theta_{13}$ is small. This means that a three flavor neutrino oscillation model decouples into two two flavor neutrino scenarios. An upper bound of the vacuum mixing angle θ_2 has been found by the CHOOZ experiment [5], $\sin^2 2\theta_2 \leq 0.10$, indicating that it is indeed small.

Neutrino oscillations depend only on the mass squared differences and the absolute neutrino mass scale is bound from above only to about 3 eV [6–10]. It is also unknown if neutrinos are Dirac or Majorana particles [11,12]. Neutrino mass models, on the other hand, depend crucially on the absolute neutrino mass scale and on the question of whether the neutrinos are Majorana or Dirac particles. Small Majorana neutrino masses are, for example, naturally understood by the canonical seesaw mechanism [13–15], involving right-handed neutrinos with a Majorana mass matrix M_R with entries that are much heavier than the electroweak scale ϵ . After integrating out the superheavy right-handed neutrinos, the *effective neutrino mass matrix* M_ν is given in terms of the Dirac mass matrix M_D and M_R as $M_\nu =$

$-M_D M_R^{-1} M_D^T$. The Majorana mass matrix M_R is, however, in general unrelated to the Dirac mass matrix M_D , resulting in lack of predictivity [16]. This is different for models where certain symmetries enforce specific correlated textures for both M_D and M_R . This has, for example, been achieved by introducing a conserved U(1) charge, e.g., the lepton number $\tilde{L} = L_e - L_\tau$ in a minimal left-right symmetric model [17,18]. In general, Abelian horizontal U(1) symmetries have been widely used in string-inspired models [19] of Froggatt-Nielsen type [20] for hierarchical neutrino masses [21,22] in order to accommodate the observed large $\nu_\mu - \nu_\tau$ mixing [23]. Although Abelian flavor symmetries tend to exhibit mixings staying maximal under renormalization group running [24], maximal and bimaximal mixings appear more generically in models with non-Abelian flavor symmetries [25–27]. A drawback of non-Abelian symmetries is that the resulting neutrino mass matrices typically have entries of equal magnitude [16], which tends to result in degenerate neutrino masses [28]. Hence, a phenomenologically successful scenario requires that these degeneracies are broken to a certain extent. An interesting way to achieve this is if two degenerate neutrinos combine to one quasi Dirac neutrino. Thus, it is possible that seesaw mechanism schemes for Dirac neutrinos based on discrete or non-Abelian symmetries provide a natural link between (bi)maximal mixing and hierarchical neutrino masses.

In this paper, we will investigate the types of neutrino mass matrix textures allowing a seesaw mechanism for both Dirac and Majorana neutrinos when fine-tuning is absent. Unlike earlier approaches, we will not assume some conserved U(1) charge from the beginning, which is assigned in flavor basis [17,18]. Furthermore, we will not assume additional hierarchies between the entries of the Dirac mass matrix M_D or the Majorana mass matrix M_R [29], as they could, e.g., arise from a soft breaking of lepton numbers and permutation symmetries [30,31]. In particular we will not examine the singular seesaw mechanism for generating light sterile neutrinos [32–36] in connection with Dirac neutrinos. Instead, we will consider the most general description of textures yielding small neutrino masses solely provided by

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the seesaw mechanism (seesaw suppressed eigenvalues) and we will then discuss the connections to symmetries arising from such seesaw suppressions.

The paper is organized as follows. In Sec. II, we discuss some properties of the seesaw mechanism and the resulting mass spectrum. Furthermore, we will discuss the relations between fine-tuning and the principal invariants of a neutrino mass matrix. In Sec. III, the Zel'dovich-Konopinski-Mahmoud (ZKM) and pseudo Dirac neutrinos are briefly revisited before the concept of seesaw Dirac particles (e.g. neutrinos) is introduced. Then, it is shown that a 3×3 mass matrix cannot describe a seesaw Dirac particle, i.e., it cannot provide a seesaw mechanism for a Dirac particle. Next, 4×4 and 6×6 neutrino mass matrix textures for seesaw Dirac particles are discussed. At the end of Sec. III, we present a model scheme for seesaw Dirac neutrinos in the presence of non-Abelian and discrete symmetries as well as algebraic relations. In Sec. IV, we investigate the connection between symmetries and the principal invariants of a neutrino mass matrix on a formal mathematical level. At the end of this section, we examine this connection for the case of 4×4 neutrino mass matrices. Finally, in Sec. V, we summarize and give our conclusions.

II. THE SEESAW MECHANISM

A. Naturally small neutrino masses

The most widely accepted mechanism for the generation of small neutrino masses is the canonical seesaw mechanism [13–15]. It involves the only spontaneously generated mass scale of the standard model (SM), i.e., the electroweak scale, which is of the order $\epsilon \sim 10^2 \text{ GeV} - 10^3 \text{ GeV}$, and a large mass scale which is typically of the order $\Lambda \sim 10^{10} \text{ GeV} - 10^{16} \text{ GeV}$ or even as high as the Planck scale ($\sim 10^{19} \text{ GeV}$), i.e., we have the hierarchy

$$0 < \epsilon \ll \Lambda. \quad (1)$$

The complex symmetric neutrino mass matrix M takes in the flavor basis $\Psi = (\nu_{a,1} \dots \nu_{a,n_a} \nu_{s,1} \dots \nu_{s,n_s})^T$ the following form:

$$M = \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix}, \quad (2)$$

where n_a denotes the number of active neutrinos (in the SM, $n_a = 3$), which are elements of $SU(2)_L$ doublets, and n_s denotes the number of sterile (singlet) neutrinos. Thus, M is an $n \times n$ matrix with $n = n_a + n_s$. Furthermore, in Eq. (2), “0” denotes the $n_a \times n_a$ null matrix. The elements of the $n_a \times n_s$ Dirac mass matrix M_D arise from electroweak symmetry breaking and are thus of order ϵ . The elements of the “heavy” $n_s \times n_s$ Majorana mass matrix M_R are not forbidden by symmetry. These elements are therefore typically of order Λ , a scale provided by a grand unified theory (GUT) or some other embedding which is associated with the breaking of $B-L$ symmetry.

For $n_a = n_s = 1$, i.e., $n = 2$, the diagonalization of the neutrino mass matrix in Eq. (2) yields a superlight Majorana

neutrino with mass of order ϵ^2/Λ and a superheavy Majorana neutrino with mass of order Λ . The smallness of the neutrino masses follows from the hierarchy in Eq. (1), which does not constitute a fine-tuning of the model parameters, since the presence of the large mass scale Λ is expected on the grounds of GUTs. This is the famous seesaw mechanism [13–15] in its simplest form, which can be generalized to $n > 2$. Note, however, that the results will depend crucially on the specific form of the Dirac and Majorana mass matrices M_D and M_R . Both these matrices are expected to emerge from scenarios involving flavor symmetries and their breakings, which lead, for example, to so-called “texture zeros.” A nontrivial flavor structure can have profound consequences and it is in general *not* true that the superlight neutrinos arising from the seesaw mechanism must be Majorana neutrinos. Instead, appropriate symmetries imposed on the fermions (and the Higgs fields) can, for example, enforce a texture of the mass matrix in Eq. (2), which allows the combination of two superlight Majorana neutrinos with opposite signs of the mass eigenvalues into one superlight Dirac neutrino.

B. Perturbation theory and the number of small neutrino masses

Diagonalization of the complex symmetric mass matrix M given in Eq. (2) yields the block-diagonal form

$$\mathcal{M} = U^T M U = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \quad (3)$$

where U is a unitary $n \times n$ matrix and M_1 and M_2 are $n_a \times n_a$ and $n_s \times n_s$ matrices, respectively. The hierarchy in Eq. (1) allows us to consider the Dirac mass matrix M_D in the neutrino mass matrix M in Eq. (2) as a small perturbation of the “unperturbed” matrix, where the Majorana mass matrix M_R is kept and $M_D = 0$. Therefore, we will choose for the unitary matrix U as an ansatz

$$U = \begin{pmatrix} C_1 & S_2^\dagger \\ -S_1 & C_2^\dagger \end{pmatrix}, \quad (4)$$

where C_1 is an $n_a \times n_a$ matrix, C_2 is an $n_s \times n_s$ matrix, S_1 and S_2 are $n_s \times n_a$ matrices, and the entries of the matrices S_i ($i = 1, 2$) are much smaller than those of the matrices C_i ($i = 1, 2$) [51]. Using the unitarity condition for the matrix U , $U^\dagger U = U U^\dagger = 1_n$, we find that the matrices C_i and S_i have to obey $C_1^\dagger C_1 + S_1^\dagger S_1 = 1_{n_a}$, $C_2 C_2^\dagger + S_2 S_2^\dagger = 1_{n_s}$, $C_1 C_1^\dagger + S_2^\dagger S_2 = 1_{n_a}$, $S_1 S_1^\dagger + C_2^\dagger C_2 = 1_{n_s}$, $S_2 C_1 - C_2 S_1 = 0$, and $C_1 S_1^\dagger - S_2^\dagger C_2 = 0$. Neglecting terms that are quadratic in the matrices S_i and do not appear in combination with the Majorana mass matrix M_R , we obtain, from Eq. (3)

$$M_1 = -(C_1^T M_D S_1 + S_1^T M_D^T C_1) + S_1^T M_R S_1, \quad (5)$$

$$M_2 = C_2^* M_R C_2^\dagger + (S_2^* M_D C_2^\dagger + C_2^* M_D^T S_2^\dagger), \quad (6)$$

$$S_1 \approx M_R^{-1} M_D^T C_1. \quad (7)$$

Note that we also obtain $S_1 \simeq (M_R^{-1})^T M_D^T C_1$ which together with Eq. (7) means that $M_R^T = M_R$. Furthermore, using Eq. (7) and the relation $S_2 C_1 - C_2 S_1 = 0$, we find that

$$S_2 \simeq C_2 M_R^{-1} M_D^T. \quad (8)$$

Inserting Eqs. (7) and (8) into Eqs. (5) and (6) and also using the fact that the Majorana mass matrix M_R is symmetric gives

$$M_1 \simeq -C_1^T M_D M_R^{-1} M_D^T C_1, \quad (9)$$

$$M_2 \simeq C_2^* M_R C_2^\dagger + C_2^* ((M_R^{-1})^* M_D^\dagger M_D + M_D^T M_D^* (M_R^{-1})^*) C_2^\dagger. \quad (10)$$

Since the entries of the matrix M_D are much smaller than those of the matrix M_R , which is consistent with Eqs. (7) and (8), we find that

$$M_1 \simeq -C_1^T M_D M_R^{-1} M_D^T C_1, \quad (11)$$

$$M_2 \simeq C_2^* M_R C_2^\dagger. \quad (12)$$

In the limit $M_D \rightarrow 0$, we can choose $C_1 = 1_{n_a}$ and $C_2 = 1_{n_s}$, i.e., after block diagonalization the mass matrices can to lowest order in the inverse seesaw scale Λ^{-1} be written as

$$M_1 \simeq -M_D M_R^{-1} M_D^T, \quad (13)$$

$$M_2 \simeq M_R. \quad (14)$$

The matrix $M_\nu \equiv -M_D M_R^{-1} M_D^T$ on the right-hand side of Eq. (13) is an *effective mass matrix* obtained from integrating out the heavy degrees of freedom represented by the heavy Majorana mass matrix $M_2 \simeq M_R$. However, the fact that the elements of the matrices M_D and M_R are of the orders ϵ and Λ , respectively, together with Eq. (13) does not imply n_a ‘‘seesaw mass eigenvalues’’ of superlight Majorana neutrinos with masses of order ϵ^2/Λ . Similarly, Eq. (14) does not imply n_s mass eigenvalues of order Λ for superheavy Majorana neutrinos. The diagonalization of the $n \times n$ mass matrix in Eq. (2) leads instead to the following pattern of eigenvalues. First, for a given Majorana mass matrix M_R with entries of order Λ there are $r \equiv \text{rank}(M_R) \leq n_s$ eigenvalues of order Λ . Then, block diagonalization of the $n_s \times n_s$ submatrix M_R leads to an r dimensional block of rank r with eigenvalues of order Λ , which is placed in the bottom-right corner of an $n_s \times n_s$ null matrix, whereas $n_s - r$ dimensions of M_R are not of order Λ . This can be used to divide the complete mass matrix M into blocks according to the magnitude of the entries. First, there is the r dimensional (diagonal) block of order Λ . Then, there is the complementary diagonal block with dimension $n_a + n_s - r$ and the off-diagonal blocks, all with elements that are maximally of order ϵ . The $n_a + n_s - r$ dimensional light block on the main diagonal is composed of the n_a dimensional null matrix of the original matrix in Eq. (2) and elements of order ϵ , arising from the reorganization into the light and heavy sectors. Thus, unless there exist specific structures in the mass matrix M that lead

to cancellations due to symmetries, the $n_a + n_s - r$ dimensional block on the main diagonal naturally yields $2(n_s - r)$ or $2n_a$ mass eigenvalues of order ϵ , depending on the sign of $n_a - n_s + r$. Written in a more compact form, there are, in total,

$$e \equiv n - r - |n_a - n_s + r|$$

mass eigenvalues of order ϵ in the $n_a + n_s - r$ dimensional light diagonal block. Including the remaining off-diagonal blocks with elements of order ϵ does not change this result, which can, for example, be seen by treating these blocks as perturbations to the stiff diagonal blocks. The remaining

$$z \equiv n - r - e = |n_a - n_s + r|$$

mass eigenvalues are not of order ϵ or Λ , i.e., they are seesaw mass eigenvalues of order ϵ^2/Λ , exact zeros, or further suppressed eigenvalues of order ϵ^{k+1}/Λ^k , where $k > 1$. With this we arrive at the important result: *The number of small mass eigenvalues naturally generated by the seesaw mechanism cannot exceed $|n_a - n_s + r|$.* For $n_a = n_s$ this implies, for example, that the number of seesaw mass eigenvalues is always equal to or smaller than half of the dimension of the mass matrix M . This means, for example, that it is impossible to obtain four or five seesaw mass eigenvalues of order $\epsilon^2/\Lambda \ll \epsilon$ from a 6×6 mass matrix M . Note, however, that the presence of symmetries may further reduce the order of magnitude of the eigenvalues, which will be discussed below.

C. Fine-tuning, principal invariants, and generic mass scales

We have so far discussed the natural eigenvalue spectrum of a mass matrix with the structure in Eq. (2) without specifying any structural details of the M_D and M_R matrices, which can arise from flavor symmetries and their breakings. Such symmetries are expected to exist and they lead, for example, to so-called ‘‘texture zeros’’ or other exact algebraic relations between different matrix elements. It is important to observe that flavor symmetries can (but need not) change the discussed generic mass eigenvalue spectrum such that one or more of the eigenvalues do not assume their natural order of magnitude. This means that an eigenvalue may turn out, for example, to be of order ϵ instead of order Λ , of order ϵ^2/Λ instead of order ϵ , or 0 instead of order ϵ^2/Λ . Since $\epsilon \ll \Lambda$, this leads to a drastic change in the order of magnitude of the corresponding eigenvalue. An eigenvalue that is many orders of magnitude smaller than its natural order of magnitude may thus be understood in terms of some symmetry in the given mass matrix. Without such a symmetry, such a drastic deviation from the natural order of magnitude in the mass eigenvalue spectrum requires a fine-tuning of parameters. This relation between deviations from the natural mass eigenvalue spectrum and flavor symmetries will be further discussed in Sec. IV.

Other (in some sense also more natural) quantities for the discussion of the properties of the mass matrices are their

TABLE I. The generic order of magnitude of the principal invariants T_i ($i=1,2,\dots,6$) of a neutrino mass matrix M of arbitrary dimension and rank r ($r=0,1,\dots,6$) for large n_a and n_s .

Principal invariant	Rank r						
	0	1	2	3	4	5	6
T_1	0	Λ	Λ	Λ	Λ	Λ	Λ
T_2	ϵ^2	ϵ^2	Λ^2	Λ^2	Λ^2	Λ^2	Λ^2
T_3	ϵ^3	$\epsilon^2\Lambda$	$\epsilon^2\Lambda$	Λ^3	Λ^3	Λ^3	Λ^3
T_4	ϵ^4	ϵ^4	$\epsilon^2\Lambda^2$	$\epsilon^2\Lambda^2$	Λ^4	Λ^4	Λ^4
T_5	ϵ^5	$\epsilon^4\Lambda$	$\epsilon^4\Lambda$	$\epsilon^2\Lambda^3$	$\epsilon^2\Lambda^3$	Λ^5	Λ^5
T_6	ϵ^6	ϵ^6	$\epsilon^4\Lambda^2$	$\epsilon^4\Lambda^2$	$\epsilon^2\Lambda^4$	$\epsilon^2\Lambda^4$	Λ^6

invariants. The basis independent principal invariants T_i of the mass matrix M , where $i=1,2,\dots,n$, are defined by the characteristic equation

$$\det(M - \lambda 1_n) = \lambda^n + \sum_{i=1}^n T_i \lambda^{n-i} \quad (15)$$

and can be entirely determined from the mass eigenvalues alone. In the same way as the block structure of the neutrino mass matrix M given in Eq. (2) naturally leads to a generic neutrino mass eigenvalue spectrum, each of the invariants is characterized by generic powers of the mass scales ϵ and Λ . Without the block structure in Eq. (2), which is related to the representations of the fields under $SU(2)_L \times U(1)_Y$, one would expect all entries of the matrix M to be of order Λ and the natural scale of all invariants would therefore be $T_i = \mathcal{O}(\Lambda^i)$. The presence of gauge symmetries imposes, however, the block structure in Eq. (2) and we have a first example where symmetries change the natural order of magnitude of the invariants T_i .

The discussion of the seesaw mechanism above provided the natural neutrino mass scales of orders ϵ^2/Λ , ϵ , and Λ . We can now discuss, in a similar way, the generic mass scales of the invariants T_i . If we denote by $T_{i,r}$ the invariant T_i of a mass matrix with r mass eigenvalues of order Λ and further elements of the mass matrix of order ϵ , then it is easy to see that one obtains for the generic scales

$$T_{i,r} = \epsilon^{i-r} \Lambda^r \quad \text{for } r < i, \quad (16)$$

$$T_{i,r} = \Lambda^i \quad \text{for } r \geq i. \quad (17)$$

For large n_a and n_s the specific structure of the mass matrix M in Eq. (2) reduces, however, the power of Λ by one unit for $r < i$ (balanced by an extra factor of ϵ) whenever one of i or r is even, while the other one is odd. In Table I, the resulting generic orders of the $T_{i,r}$'s are given for large n_a and n_s . For small n_a and n_s these generic powers are even further reduced. For a 4×4 mass matrix M with a structure as in Eq. (2) and $r=2$ one obtains

$$T_{1,2} = \mathcal{O}(\Lambda), \quad T_{2,2} = \mathcal{O}(\Lambda^2),$$

$$T_{3,2} = \mathcal{O}(\epsilon^2 \Lambda), \quad T_{4,2} = \mathcal{O}(\epsilon^4).$$

Note that $T_{1,2}$, $T_{2,2}$, and $T_{3,2}$ are as in Table I, whereas $T_{4,2}$ is further reduced due to the small values of n_a and n_s .

In analogy with the discussion of the eigenvalues above, a deviation of the invariants from this generic mass scale dependence is either a fine-tuning of parameters or a consequence of a symmetry. Note, however, that not all symmetries lead to a change in the generic mass scales of the invariants. The presence or absence of mass scales which actually contribute to the invariants of the neutrino mass matrix M nevertheless sheds light in an interesting way on the symmetries constraining the neutrino mass matrix texture.

III. TEXTURES FOR SEESAW-DIRAC PARTICLES

A. Seesaw-Dirac particles

Consider in the Majorana basis a real symmetric $n \times n$ neutrino mass matrix M . Let us assume that it can be brought via some orthogonal transformation to the block-diagonal form

$$M \mapsto \tilde{M} = \mathcal{O}^T M \mathcal{O} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad (18)$$

where A denotes a real symmetric 2×2 matrix given by

$$A = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \quad (19)$$

and B refers to some real symmetric $(n-2) \times (n-2)$ matrix. The fields that span the matrix A will be denoted by $(\nu_{1L} \nu_{2L})$. Diagonalization of the mass matrix \tilde{M} then leads to the mass matrix

$$m = \text{diag}(\alpha, -\alpha, \beta, \gamma, \dots), \quad (20)$$

where the two degenerate Majorana neutrinos with opposite signs of the mass eigenvalues can be combined to one Dirac neutrino. Therefore, we will speak of the real symmetric neutrino mass matrix M as containing a *Dirac particle* if diagonalization finally leads to a mass matrix of the type given in Eq. (20). Note that the mass matrix \tilde{M} respects a conserved $U(1)$ symmetry acting on the fermionic fields with ν_{1L} carrying a charge of $+1$ and ν_{2L} carrying a charge of -1 .

If the mass matrix \tilde{M} is identical to the mass matrix M formulated in flavor basis, then the neutrinos ν_{1L} and ν_{2L} could represent two different flavor fields. In this case, two Majorana neutrinos of different flavors combine to one Dirac particle and the conserved $U(1)$ charge is called a lepton number of the ZKM type [37,38]. If, for example, the identifications $\nu_{1L} = \nu_{\mu L}$ and $\nu_{2L} = \nu_{\tau L}$ hold, then the mass matrix $\tilde{M} = M$ exhibits a $U(1)_{\text{ZKM}}$ symmetry characterized by a ZKM lepton number $\hat{L} = L_\mu - L_\tau$.

If instead $\tilde{M} \neq M$, then the fields ν_{1L} and ν_{2L} of the matrix A are not flavor fields and the basis, where the assignment of the conserved $U(1)$ charges (the ν_{1L} number minus the ν_{2L} number) takes place, is different from the flavor basis. In this case, the emerging Dirac particle is called a *pseudo-Dirac particle* [39]. To zeroth order in the gauge in-

interactions a pseudo-Dirac particle cannot be distinguished from a genuine Dirac particle characterized by a conserved $U(1)_{\text{ZKM}}$ lepton number. However, higher order gauge interactions induce a splitting of the Dirac particle into two Majorana particles with *nearly* degenerate masses [39]. Since we will be concerned only with zeroth order mass matrices, we will, if nothing else is mentioned, refer to both the ZKM neutrino and the pseudo Dirac neutrino briefly as Dirac neutrinos. A Dirac neutrino will be called a *seesaw Dirac particle* if its mass is superlight due to the seesaw mechanism without invoking any fine-tuning in the sense of Sec. II C.

As a quick application of the results given above, let us consider a 3×3 mass matrix M , which is assumed to describe a Dirac neutrino. If fine-tuning is absent, then this Dirac neutrino cannot have a small mass suppressed by a seesaw mechanism, since the necessary number of two light mass eigenvalues would exceed half of the dimension of the mass matrix M , which is forbidden by the results of Sec. II C.

In the next section, we will show that seesaw Dirac neutrinos in the absence of any fine-tuning are first possible in the case of 4×4 mass matrices.

B. 4×4 textures

1. Criterion for seesaw Dirac particles

According to Sec. II, we can write any real symmetric 4×4 mass matrix M , providing an effective seesaw mechanism in block form as in Eq. (2), where the Dirac mass matrix M_D is given by the real 2×2 matrix

$$M_D \equiv \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \quad (21)$$

with $D_i = \mathcal{O}(\epsilon)$ or $D_i = 0$, where $i = 1, 2, 3, 4$, and the heavy Majorana mass matrix M_R is given by the real symmetric 2×2 matrix

$$M_R \equiv \begin{pmatrix} R_1 & R_2 \\ R_2 & R_3 \end{pmatrix} \quad (22)$$

with $R_i = \mathcal{O}(\Lambda)$ or $R_i = 0$, where $i = 1, 2, 3$. It is assumed that the two submatrices satisfy $\det M_D \neq 0$ and $\det M_R \neq 0$, i.e., $r = 2$.

If we assume that the mass matrix M describes a seesaw Dirac particle in the sense of Sec. III A, then the real symmetric mass matrix M can be brought by some orthogonal transformation $M \mapsto m = \mathcal{O}^T M \mathcal{O}$ to the form

$$m = \text{diag}(\alpha, -\alpha, \beta, \gamma). \quad (23)$$

The invariants T_1 and T_3 defined by the characteristic equation

$$\begin{aligned} \det(M - \lambda 1_4) &= \det(m - \lambda 1_4) \\ &= \lambda^4 + T_1 \lambda^3 + T_2 \lambda^2 + T_3 \lambda + T_4 = 0 \end{aligned} \quad (24)$$

are given as functions of the matrix elements of the matrix M by

$$T_1(D_1, D_2, \dots, R_3) = -R_1 - R_3, \quad (25a)$$

$$\begin{aligned} T_3(D_1, D_2, \dots, R_3) &= D_1^2 R_3 + D_2^2 R_1 + D_3^2 R_3 + D_4^2 R_1 \\ &\quad - 2(D_1 D_2 + D_3 D_4) R_2, \end{aligned} \quad (25b)$$

and as functions of the matrix elements of the diagonal matrix m by

$$T_1(\alpha, \beta, \gamma) = -\beta - \gamma, \quad (26a)$$

$$T_3(\alpha, \beta, \gamma) = \alpha^2(\beta + \gamma). \quad (26b)$$

Equations (26a) and (26b) imply that

$$T_3 = -\alpha^2 T_1. \quad (27)$$

Assume that $T_1 \neq 0$. Then, it follows from Eq. (27) that

$$\alpha^2 = -\frac{T_3(D_1, D_2, \dots, R_3)}{T_1(D_1, D_2, \dots, R_3)} \ll \epsilon^2 \quad (28)$$

and from Eqs. (25a) and (25b) we find that the numerator is of order $\epsilon^2 \Lambda$ and the denominator is of order Λ . Hence, it is clear that the relation (28) can only be satisfied in the presence of some fine-tuned cancellations between the matrix elements of M . If we reject fine-tuning, then it follows that $T_1 = 0$, i.e., *the mass matrix must be traceless*. The tracelessness of the mass matrix implies that $m = \text{diag}(\alpha, -\alpha, \beta, -\beta)$, i.e., the four Majorana neutrinos combine to two Dirac neutrinos: one seesaw Dirac neutrino and one heavy Dirac neutrino. Thus, Eq. (27) reads

$$T_1 = T_3 = 0. \quad (29)$$

The above considerations tell us that Eq. (29) together with the auxiliary relations $\det M_D \neq 0$ and $\det M_R \neq 0$ represent necessary and sufficient conditions for superlight Dirac neutrinos in the absence of fine-tuning. We will therefore refer to Eq. (29) as the *criterion for seesaw Dirac particles*. Since the mass matrix $M = (M_{ij})$ is originally defined in flavor basis, there is a fundamental difference between matrices whose matrix elements can be considered as independent parameters and those matrices where the matrix elements are related by some specific exact algebraic relations.

2. Textures with independent entries

Under the assumption that the matrix elements M_{ij} are either exact texture zeros or independent parameters, we will now determine all possible textures of the mass matrix M in Eq. (2), that describe a seesaw Dirac particle. Since we will treat the matrix elements, which are not texture zeros, as independent parameters, the tracelessness of the mass matrix M can only be satisfied if $R_1 = R_3 = 0$. Thus, we obtain from Eq. (25b) that the criterion for seesaw Dirac particles (29) reduces to

$$D_1 D_2 + D_3 D_4 = 0. \quad (30)$$

Since the matrix elements can be varied independently, the condition (30) together with $\det M_D \neq 0$ can be satisfied only if $D_1 = D_4 = 0$ or $D_2 = D_3 = 0$. Thus, we arrive at the conclusion that up to trivial permutations all 4×4 textures with independent nonvanishing entries, describing a superlight Dirac particle, are of the canonical form

$$M \equiv \begin{pmatrix} 0 & 0 & 0 & \epsilon_1 \\ 0 & 0 & \epsilon_2 & 0 \\ 0 & \epsilon_2 & 0 & \Lambda \\ \epsilon_1 & 0 & \Lambda & 0 \end{pmatrix} \quad (31)$$

with $\epsilon_1, \epsilon_2 = \mathcal{O}(\epsilon)$. Textures equivalent to the one displayed in Eq. (31) have been obtained within left-right-symmetric and horizontal models implementing a conserved horizontal U(1) charge of the ZKM type [40].

3. Textures in presence of algebraic relations

We will now investigate the 4×4 textures leading to seesaw Dirac particles if algebraic relations between the matrix elements M_{ij} are allowed. If we assume $R_1 = R_3 = 0$, but allow for algebraic relations within the Dirac matrix M_D , then the criterion for seesaw Dirac particles is still given by Eq. (30) and the corresponding texture is equal to

$$M = \begin{pmatrix} 0 & 0 & \epsilon_1 & \epsilon_2 \\ 0 & 0 & -\frac{\epsilon_1 \epsilon_2}{\epsilon_3} & \epsilon_3 \\ \epsilon_1 & -\frac{\epsilon_1 \epsilon_2}{\epsilon_3} & 0 & \Lambda \\ \epsilon_2 & \epsilon_3 & \Lambda & 0 \end{pmatrix} \quad (32)$$

with $\epsilon_i = \mathcal{O}(\epsilon) \ll \mathcal{O}(\Lambda)$, where $i = 1, 2, 3$. Let us now assume that the relation $R_1 = -R_3 \neq 0$ holds. Then, the criterion for seesaw Dirac particles (29) reads

$$T_3 = R_1(-D_1^2 + D_2^2 - D_3^2 + D_4^2) - 2R_2(D_1 D_2 + D_3 D_4) = 0. \quad (33)$$

If we assume that R_1 and R_2 can be varied independently, then both parentheses in Eq. (33) are necessarily equal to zero. The second parenthesis vanishes for the choice $D_3 = -D_1 D_2 / D_4$, which means that the first parenthesis vanishes when $D_1^2 = D_4^2$. Thus, we can in this case determine the texture to be

$$M = \begin{pmatrix} 0 & 0 & \epsilon_1 & \epsilon_2 \\ 0 & 0 & \mp \epsilon_2 & \pm \epsilon_1 \\ \epsilon_1 & \mp \epsilon_2 & \Lambda_1 & \Lambda_2 \\ \epsilon_2 & \pm \epsilon_1 & \Lambda_2 & -\Lambda_1 \end{pmatrix} \quad (34)$$

with $\epsilon_1, \epsilon_2 = \mathcal{O}(\epsilon)$ and $\Lambda_1, \Lambda_2 = \mathcal{O}(\Lambda)$. Let the mass matrix given in Eq. (34) be spanned by the fields $(\nu_{1L} \nu_{2L} N_1 N_2)$, where ν_{1L} and ν_{2L} denote two active neutrinos and N_1 and N_2 denote two SM singlets. Then, the texture in Eq. (34) can be naturally obtained by imposing the discrete symmetry

$$\mathcal{D}: \begin{cases} \nu_{1L} \rightarrow \mp i \nu_{2L}, & \nu_{2L} \rightarrow \pm i \nu_{1L}, \\ N_1 \rightarrow i N_2, & N_2 \rightarrow -i N_1, \end{cases} \quad (35)$$

in the Majorana basis.

C. 6×6 textures

1. Criterion for seesaw Dirac particles

In order to maximize the number of superlight particles emerging from the seesaw mechanism, we can again according to Sec. II write a real symmetric 6×6 mass matrix in block form as in Eq. (2), where the real Dirac mass matrix M_D is given by the 3×3 matrix

$$M_D \equiv \begin{pmatrix} D_1 & D_2 & D_3 \\ D_4 & D_5 & D_6 \\ D_7 & D_8 & D_9 \end{pmatrix} \quad (36)$$

with $D_i = \mathcal{O}(\epsilon)$ or $D_i = 0$, where $i = 1, 2, \dots, 9$, and the real symmetric heavy Majorana mass matrix M_R is given by the 3×3 matrix

$$M_R \equiv \begin{pmatrix} R_1 & R_2 & R_3 \\ R_2 & R_4 & R_5 \\ R_3 & R_5 & R_6 \end{pmatrix} \quad (37)$$

with $R_i = \mathcal{O}(\Lambda)$ or $R_i = 0$, where $i = 1, 2, \dots, 6$. To guarantee an effective seesaw mechanism, it is additionally assumed that $\det M_D \neq 0$ and $\det M_R \neq 0$. Let us assume that the mass matrix M describes a seesaw Dirac particle. This means that the superlight mass eigenvalue spectrum is of the type $\alpha, -\alpha, \beta = \epsilon^2 / \Lambda$, where α can be regarded as the mass of the Dirac particle. Consider the characteristic equation of the mass matrix M written in block form

$$\det(M - \lambda 1_6) = \begin{vmatrix} -\lambda 1_3 & M_D \\ M_D^T & M_R - \lambda 1_3 \end{vmatrix} = 0. \quad (38)$$

Since $\det M_R \neq 0$, the matrix $M_R - \lambda 1_3$ can be inverted for $\lambda \ll \Lambda$. One can therefore apply the Gauss elimination algorithm to the block matrix $M - \lambda 1_6$, since it leaves the determinant invariant. From Eq. (38), we therefore obtain that the superlight mass eigenvalues of the mass matrix M are also solutions of

$$|-\lambda 1_3 - M_D (M_R - \lambda 1_3)^{-1} M_D^T| = 0. \quad (39)$$

Expanding this determinant for a small parameter λ , Eq. (39) can be rewritten as

$$\begin{aligned} & \left| -\lambda 1_3 - M_D M_R^{-1} M_D^T + \mathcal{O}\left(\frac{\epsilon^4}{\Lambda^3}\right) \right| \\ &= \lambda^3 + \left[T_1 + \mathcal{O}\left(\frac{\epsilon^4}{\Lambda^3}\right) \right] \lambda^2 + \left[T_2 + \mathcal{O}\left(\frac{\epsilon^8}{\Lambda^6}\right) \right] \lambda \\ & \quad + \left[T_3 + \mathcal{O}\left(\frac{\epsilon^{12}}{\Lambda^9}\right) \right] = 0, \end{aligned} \quad (40)$$

where the invariants T_1 , T_2 , and T_3 are defined by the characteristic equation $\det(M_\nu - \lambda 1_3) = \lambda^3 + T_1 \lambda^2 + T_2 \lambda + T_3 = 0$ of the effective mass matrix $M_\nu = -M_D M_R^{-1} M_D^T$. Using $\lambda = \mathcal{O}(\epsilon^2/\Lambda)$, we find that

$$\lambda^3 + T_1 \lambda^2 + T_2 \lambda + T_3 = \mathcal{O}\left(\frac{\epsilon^8}{\Lambda^5}\right). \quad (41)$$

Taking the previously assumed light mass eigenvalue spectrum into account, we obtain from Eq. (41) the following equations for the mass α of the seesaw Dirac particle

$$T_1 \alpha^2 + T_3 = \mathcal{O}\left(\frac{\epsilon^8}{\Lambda^5}\right), \quad \alpha^3 + T_2 \alpha = \mathcal{O}\left(\frac{\epsilon^8}{\Lambda^5}\right). \quad (42)$$

Since $\alpha \neq 0$, this system of equations can have a solution only if

$$T_3 - T_1 T_2 = \mathcal{O}\left(\frac{\epsilon^8}{\Lambda^5}\right). \quad (43)$$

Note that the invariant T_3 and the product $T_1 T_2$ are both of the order ϵ^6/Λ^3 . Hence, Eq. (43) expresses a fine-tuning of the mass matrix M unless the right-hand side vanishes. Since we reject fine-tuning, the invariants must therefore *exactly* satisfy

$$T_3 - T_1 T_2 = 0. \quad (44)$$

In fact, it is easy to show that Eq. (44) is also a sufficient condition for a superlight Dirac particle. Hence, we will call it the *criterion for seesaw Dirac particles*. It has thus been shown that the first order in the inverse seesaw scale given by the effective mass matrix M_ν is already sufficient to determine non-fine-tuned neutrino mass textures leading to seesaw Dirac particles.

2. Textures not yielding seesaw Dirac neutrinos

Consider an ansatz where the Dirac mass matrix M_D is equivalent to the matrix

$$M_D \equiv \text{diag}(D_1, D_5, D_9), \quad (45)$$

the simplest form consistent with $\det M_D \neq 0$. For this Dirac mass matrix M_D there exist *exactly* two inequivalent Majorana mass matrices, that never lead to a superlight Dirac particle, even if arbitrary algebraic relations between the matrix elements are allowed. These matrices are

$$M_R = \begin{pmatrix} R_1 & R_2 & 0 \\ R_2 & 0 & R_5 \\ 0 & R_5 & 0 \end{pmatrix}, \quad M'_R = \begin{pmatrix} 0 & R_2 & R_3 \\ R_2 & 0 & R_5 \\ R_3 & R_5 & 0 \end{pmatrix}.$$

The criterion for seesaw Dirac particles given in Eq. (44), corresponding to the Majorana mass matrix M_R , reads

$$(T_3 - T_1 T_2) \det M_R = D_9^4 D_5^2 R_1^2 R_2^2 R_5^2 = 0, \quad (46)$$

which can be satisfied only if an additional texture zero were be introduced in the mass matrix M , leading to a different class of textures, or letting $\det M$ vanish. Similar considerations hold for the Majorana mass matrix M'_R .

3. Textures with independent entries

Assume now that all nonvanishing matrix elements of the mass matrix M are independent parameters. Then, all possible textures consistent with the criterion for seesaw Dirac particles given in Eq. (44) are equivalent to the unique form

$$M = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 & 0 & 0 \\ \epsilon_1 & \Lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \epsilon_2 \\ 0 & 0 & 0 & 0 & \epsilon_3 & 0 \\ 0 & 0 & 0 & \epsilon_3 & 0 & \Lambda_2 \\ 0 & 0 & \epsilon_2 & 0 & \Lambda_2 & 0 \end{pmatrix} \quad (47)$$

with $\epsilon_1, \epsilon_2, \epsilon_3 = \mathcal{O}(\epsilon)$ and $\Lambda_1, \Lambda_2 = \mathcal{O}(\Lambda)$. From Eq. (47) we observe that the superlight particles obtain their masses via two decoupled mechanisms:

(1) The Majorana particle obtains its mass β from the ordinary 2×2 seesaw mechanism.

(2) The Dirac particle obtains its mass α from the canonical 4×4 texture given in Eq. (31).

Hence, it is shown for 6×6 textures with independent entries that the model-independent method presented here to determine all possible textures consistent with the criterion for seesaw Dirac particles is equivalent to the introduction of a conserved lepton number of the ZKM type [17,18].

4. Model scheme for textures in presence of non-Abelian and discrete symmetries

Non-Abelian as well as discrete symmetries between leptons mostly predict lepton mass matrix textures where some of the matrix elements are equal in magnitude, which can result in maximal or bimaximal mixing and degenerate neutrino masses [16,28]. In the case of non-Abelian symmetries, models have been proposed where only one mixing angle that is *nearly* maximal is generic [41]. However, most of these models have problems in naturally accommodating large $\nu_\mu - \nu_\tau$ mixing and hierarchical mass squared differences [16]. In particular the presence of a ZKM lepton number in the neutrino sector as a source of (nearly) maximal atmospheric mixing usually implies the reverse order of the hierarchy between Δm_\odot^2 and Δm_{atm}^2 [42]. On the other hand, a combination of neutrinos to a pseudo Dirac particle, in the

sense of Sec. III A, does not establish a specific mixing between the corresponding generations, and hence allows more freedom in accommodating mixing angles and mass squared differences.

Contrary to the usual approach, we will therefore concentrate on the natural generation of hierarchical mass squared differences by imposing a conserved U(1) charge and assume that the U(1) generator is not diagonal in flavor basis, i.e., some of the active neutrinos combine to a pseudo Dirac particle instead of a genuine Dirac particle. The pseudo Dirac character of some of the neutrinos requires the presence of exact algebraic relations between the matrix elements.

Let ν_e , ν_μ , and ν_τ denote the SM neutrino flavor fields and N_1 , N_2 , and N_3 denote the charge conjugates of three right-handed neutrinos. To be specific, we will assume in the basis $(N_1^c N_1 N_2^c N_2 N_3^c N_3)$ that the corresponding 3×3 Majorana mass matrix M_R in Eq. (2) is of the form [43,44]

$$M_R = \Lambda \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (48)$$

This kind of Majorana matrix has received interest in the framework of SO(3) flavor symmetries, where the three SM singlets N_1 , N_2 , and N_3 are combined to an SO(3) triplet field [41,43–48]. Invariance of the neutrino mass terms under the discrete Z_2 symmetry

$$\mathcal{D}: \begin{cases} \nu_e \rightarrow -i\nu_\mu, & \nu_\mu \rightarrow i\nu_e, & \nu_\tau \rightarrow -\nu_\tau, \\ N_1 \rightarrow iN_3, & N_2 \rightarrow -N_2, & N_3 \rightarrow -iN_1 \end{cases} \quad (49)$$

enforces the 6×6 neutrino mass matrix M to be of the form

$$M = \begin{pmatrix} 0 & 0 & 0 & \epsilon_1 & 0 & \epsilon_2 \\ 0 & 0 & 0 & -\epsilon_2 & 0 & \epsilon_1 \\ 0 & 0 & 0 & 0 & \epsilon_3 & 0 \\ \epsilon_1 & -\epsilon_2 & 0 & 0 & 0 & \Lambda \\ 0 & 0 & \epsilon_3 & 0 & -\Lambda & 0 \\ \epsilon_2 & \epsilon_1 & 0 & \Lambda & 0 & 0 \end{pmatrix}, \quad (50)$$

where ϵ_1, ϵ_2 , and ϵ_3 denote three different real matrix elements of the order of the electroweak scale ϵ . The algebraic relations between some of the entries of the mass matrix M in Eq. (50), which are established by the discrete symmetry \mathcal{D} , allow the existence of a seesaw Dirac neutrino in the absence of a conserved ZKM charge. Because of fact that there is no conserved ZKM charge present, radiative corrections will induce a small splitting of the degenerate masses and could thereby, in principle, establish the hierarchy between Δm_\odot^2 and $\Delta m_{\text{atm.}}^2$.

IV. CONNECTION BETWEEN SYMMETRIES AND INVARIANTS

The smallness of the neutrino masses is well understood in terms of the seesaw mechanism, which is a consequence

of the general block form of the neutrino mass matrix M given in Eq. (2). The sub-blocks of M , however, leave the mixing angles and mass spectrum of the light neutrinos unspecified. Predictions for the mixing angles and mass spectra of neutrinos require further horizontal or flavor symmetries enforcing specific textures of the Dirac and Majorana mass matrices. Therefore, it is important to relate the general properties of the mass matrix as discussed above to the symmetries acting on the different neutrino flavors.

Assume that there exists a (flavor symmetry) group G , which is mapped into a reducible unitary representation [52] D acting on the flavor space:

$$g \mapsto D(g), \quad g \in G. \quad (51)$$

The matrix representation $D(g)$ acts on the state vectors $\Psi = (\nu_{a,1} \dots \nu_{a,n_a} \nu_{s,1} \dots \nu_{s,n_s})^T$ (here given in flavor basis), where n_a denotes again the number of active neutrinos, which are elements of $SU(2)_L$ doublets, and n_s denotes the number of sterile neutrinos (SM singlets):

$$\Psi \mapsto D(g)\Psi, \quad g \in G. \quad (52)$$

Next, consider two irreducible subrepresentations $D^{(\alpha)}(g)$ and $D^{(\beta)}(g)$ of the representation $D(g)$ and some matrix X which satisfies

$$(D^{(\beta)}(g))^T X D^{(\alpha)}(g) = X \quad (53)$$

for all $g \in G$. Using Eq. (53) and the unitarity of the representations, we obtain that

$$X^\dagger X D^{(\alpha)}(g) = D^{(\alpha)}(g) X^\dagger X. \quad (54)$$

Since the representation $D^{(\alpha)}(g)$ is irreducible, it follows from Schur's lemma [49,50] that $X^\dagger X$ is proportional to the unit matrix, i.e., the matrix X is a *unitary* matrix times some arbitrary mass scale, which is given as a physical input parameter. Furthermore, it follows that the matrix X is, up to this overall factor, uniquely determined by the choice of the matrix representations $D^{(\alpha)}(g)$ and $D^{(\beta)}(g)$. Moreover, Schur's lemma tells us that in the case that as well as Eq. (53) for some irreducible representation $D^{(\gamma)}(g)$ there holds also a relation

$$(D^{(\beta)}(g))^T X D^{(\gamma)}(g) = X \quad (55)$$

for all $g \in G$, the representation $D^{(\gamma)}(g)$ is equivalent to the representation $D^{(\alpha)}(g)$.

We will now consider the case when the symmetry group G is unbroken at low energies. The origin of the mass terms in M_R of the heavy sterile neutrinos generated at the GUT or embedding scale is in general quite different from the origin of the Dirac mass terms in M_D , which emerge from Yukawa couplings and the electroweak vacuum expectation value. It can therefore be assumed that in flavor basis the representation $D(g)$ takes on the block-diagonal form

$$D(g) = D_a(g) \oplus D_s(g), \quad (56)$$

where $D_a(g)$ is an n_a dimensional unitary representation acting on the subspace of the active neutrinos only and $D_s(g)$ is an n_s dimensional representation that acts only on the sterile neutrinos.

From the unitarity of the representation $D(g)$ it follows that an appropriate change of basis allows us to write the representations $D_a(g)$ and $D_s(g)$ in reduced form as

$$D_a(g) = \bigoplus_{\alpha} k_{\alpha}^a D_a^{(\alpha)}(g), \quad D_s(g) = \bigoplus_{\alpha} k_{\alpha}^s D_s^{(\alpha)}(g), \quad (57)$$

where $\alpha=1,2,\dots$ runs only over the inequivalent irreducible representations and the integer k_{α}^x , $x=a,s$, specifies how often the irreducible representation $D_x^{(\alpha)}(g)$ occurs in the reduction. This means that the basis has been chosen such that the matrices of equivalent representations are identical and

$$k_{\alpha}^x D_x^{(\alpha)}(g) \equiv \underbrace{D_x^{(\alpha)}(g) \oplus \dots \oplus D_x^{(\alpha)}(g)}_{k_{\alpha}^x}, \quad x=a,s. \quad (58)$$

Specifically, if two representations $D_a^{(\alpha)}(g)$ and $D_s^{(\beta)}(g)$ are equivalent, they are also understood to be identical and we can therefore choose the labeling such that $\beta \equiv \alpha$. The dimension of the irreducible representation $D_x^{(\alpha)}(g)$ will be denoted by d_{α}^x , i.e., the representation $D_x^{(\alpha)}(g)$ is a $d_{\alpha}^x \times d_{\alpha}^x$ matrix.

Consider now the neutrino mass matrix M in Eq. (2) and assume for simplicity that the Dirac mass matrix M_D as well as the Majorana mass matrix M_R are nonsingular. From Eq. (56) the unitary transformation V that brings the representation $D(g)$ to the completely reduced form in Eq. (57) decomposes in flavor basis into $V = V_a \oplus V_s$, where V_a and V_s are unitary matrices of dimensions n_a and n_s , respectively. Therefore, in the basis where the representation $D(g)$ is in the completely reduced form in Eq. (57), the neutrino mass matrix $M' \equiv V^T M V$ reads

$$M' = \begin{pmatrix} 0 & M'_D \\ M_D'^T & M'_R \end{pmatrix} \equiv \begin{pmatrix} 0 & V_a^T M_D V_s \\ V_s^T M_D^T V_a & V_s^T M_R V_s \end{pmatrix} \quad (59)$$

and the effective neutrino mass matrix is simply given by $M'_\nu \equiv -M'_D M_R'^{-1} M_D'^T$. In this basis, we can now apply the above stated implications of Schur's lemma in order to determine the neutrino mass matrix M' , following from the invariance of the neutrino mass term $\overline{\Psi^c} M' \psi \sim \Psi^T M' \Psi$ [53] under the transformations

$$\Psi^T M' \Psi \mapsto \Psi^T D^T(g) M' D(g) \Psi, \quad g \in G. \quad (60)$$

As a result, we obtain that the Majorana mass matrix M'_R is, up to trivial permutations, in block-diagonal form

$$M'_R = \text{diag}(A_1, A_2, \dots), \quad (61)$$

where each submatrix A_{α} , $\alpha=1,2,\dots$, defines a bilinear form which is invariant under the symmetry G in the representation $k_{\alpha}^s D_s^{(\alpha)}(g)$.

If for some $\alpha=1,2,\dots$ the representation $D_s^{(\alpha)}(g)$ is equivalent to its complex conjugate representation $D_s^{(\alpha)*}(g)$, then the matrix A_{α} can be written as

$$A_{\alpha} = A'_{\alpha} \otimes U_{\alpha}, \quad (62a)$$

where A'_{α} denotes some arbitrary invertible symmetric $k_{\alpha}^s \times k_{\alpha}^s$ matrix with undetermined entries of the generic order Λ , whereas the unitary $d_{\alpha}^s \times d_{\alpha}^s$ matrix U_{α} is determined by the choice of the representation $D_s^{(\alpha)}(g)$ [54]. If instead the representation $D_s^{(\alpha)}(g)$ is *complex*, i.e., not equivalent to its complex conjugate, then the condition $\det M_R \neq 0$ requires the complete reduction of the representation $D_s(g)$ in Eq. (57) also to contain the complex conjugate representation $D_s^{(\alpha)*}(g)$ exactly k_{α}^s times. Thus, the matrix A_{α} is of the form

$$A_{\alpha} = \begin{pmatrix} 0 & A'_{\alpha} \\ A'^T_{\alpha} & 0 \end{pmatrix} \otimes U_{\alpha}, \quad (62b)$$

where “0” denotes the $k_{\alpha}^s \times k_{\alpha}^s$ null matrix, A'_{α} denotes some arbitrary invertible $k_{\alpha}^s \times k_{\alpha}^s$ matrix with undetermined entries of the order Λ , and the unitary $d_{\alpha}^s \times d_{\alpha}^s$ matrix U_{α} is again determined by the choice of the representation $D_s^{(\alpha)}(g)$.

As a simple example, consider the Majorana mass matrix M_R in Eq. (48). Before symmetry breaking, the representation $D_s(g) = D_s^{(\alpha)}(g)$, where $\alpha \equiv 1$, is the (irreducible) 3-dimensional representation of $\text{SO}(3)$, which is equivalent to its complex conjugate representation. Therefore, Eq. (62a) applies when Λ is the arbitrary “matrix” A'_{α} and M_R/Λ is the uniquely determined unitary (even orthogonal) matrix U_{α} .

Similarly to the treatment of the Majorana mass matrix M'_R in Eq. (61), one also verifies that the Dirac mass matrix M'_D decomposes into the unitary submatrices U_{α} known from Eqs. (62a) and (62b) times a mass scale of the order ϵ , in such a way that the effective neutrino mass matrix M'_ν can be determined by consistently carrying out block multiplications. After an appropriate relabeling of the representations $D_a^{(\alpha)}(g)$ and $D_s^{(\alpha)}(g)$, we obtain for the effective neutrino mass matrix the block-diagonal form

$$M'_\nu = \text{diag}(B_1, B_2, \dots), \quad (63)$$

where each matrix B_{α} , $\alpha=1,2,\dots$, defines a bilinear form, which is invariant under the symmetry G in the representation $k_{\alpha}^a D_a^{(\alpha)}(g)$. If for some $\alpha=1,2,\dots$ the representation $D_a^{(\alpha)}(g)$ is equivalent to its complex conjugate representation $D_a^{(\alpha)*}(g)$, then the matrix B_{α} reads

$$B_{\alpha} = B'_{\alpha} \otimes U_{\alpha}, \quad (64a)$$

where B'_{α} denotes some arbitrary invertible symmetric $k_{\alpha}^a \times k_{\alpha}^a$ matrix with entries of the order ϵ^2/Λ and the unitary $d_{\alpha}^a \times d_{\alpha}^a$ matrix U_{α} is fixed by the representation $D_a^{(\alpha)}(g)$.

For a complex representation $D_a^{(\alpha)}(g)$ the invertibility of the Dirac matrix M_D requires the matrix B_{α} to be of the form

$$B_\alpha = \begin{pmatrix} 0 & B'_\alpha \\ B'^T_\alpha & 0 \end{pmatrix} \otimes U_\alpha, \quad (64b)$$

where “0” denotes the $k_\alpha^a \times k_\alpha^a$ null matrix, B'_α denotes some invertible $k_\alpha^a \times k_\alpha^a$ matrix with entries of order ϵ^2/Λ , and the unitary $d_\alpha^a \times d_\alpha^a$ matrix U_α again depends on the choice of the representation $D_a^{(\alpha)}(g)$. The important thing here is to note that each matrix element of the neutrino mass matrix M' in Eq. (59) can serve as a parameter of one and only one of the matrices B_α , i.e., for $\alpha \neq \beta$ the matrices B_α and B_β are described by decoupled mass parameters and are therefore independent in a parametrical sense.

Taking $B_\alpha^\dagger B_\alpha$, it is readily seen that block diagonalization of B_α yields k_α^a different sets of neutrino mass eigenvalues, which are, up to relative phases, d_α^a -fold [or, if the representation $D_a^{(\alpha)}(g)$ is complex, $2d_\alpha^a$ -fold] *degenerate*. The neutrino masses, each of which is d_α^a -fold ($2d_\alpha^a$ -fold) degenerate, will be denoted by $m_{\alpha l}$, where $l=1,2,\dots,k_\alpha^a$. In this case, the k_α^a different neutrino masses $m_{\alpha l}$ are actually correlated, i.e., they are in a parametrical sense dependent. However, it is crucial to note that this correlation is due only to the diagonalization of the matrix B'_α , whose entries are not constrained by the symmetry G , which has already been fully taken into account when introducing the unitary matrices U_α in Eqs. (62) and (64). Except of their common mass scale, which is ϵ^2/Λ , the k_α^a different masses exhibit no further relations that are protected by the symmetry G . Hence, we can regard them as independent in a *generic* sense.

Let us now specialize to the case when the neutrino mass matrix M is real. The principal invariants T_i ($i=1,2,\dots,n$) of the neutrino mass matrix M can be expanded as finite sums of powers of the electroweak scale ϵ and the GUT or embedding scale Λ as follows:

$$T_i = \sum_j \sum_{k=j} a_{jk}^i \left(\frac{\epsilon^2}{\Lambda}\right)^j \Lambda^k, \quad (65)$$

where the non-negative integers j and k have to obey $j+k=i$. In Eq. (65), the term $a_{jk}^i \epsilon^{2j} \Lambda^{k-j}$ is the sum over all products of mass eigenvalues, where j eigenvalues are of the order ϵ^2/Λ and k eigenvalues are of the order Λ . Since the symmetry G implies that some of the eigenvalues are degenerate up to a sign it can happen that some (or all) of the coefficients a_{jk}^i vanish exactly. However, since the absolute value of the neutrino masses $m_{\alpha l}$ is undetermined by the symmetry G , any nonvanishing coefficient $a_{jk}^i \neq 0$ must generically be of order unity. In other words, a situation where $0 < |a_{jk}^i| \ll 1$ cannot be understood in terms of the symmetry G and must therefore be the result of some fine-tuning of the model parameters.

Within the framework presented above one can now even test the validity of the criteria for seesaw Dirac particles given in Secs. III B and III C. First, we note that following Eq. (65), the principal invariants T_1 and T_3 of the 4×4 mass matrix M in Sec. III B can be written as

$$T_1 = a_{01}^1 \Lambda, \quad (66a)$$

$$T_3 = a_{12}^3 \epsilon^2 \Lambda. \quad (66b)$$

Next, using the fact that the generic mass scale of the seesaw Dirac particle is of the order ϵ^2/Λ , Eq. (27) implies that

$$a_{12}^3 \simeq - \left(\frac{\epsilon}{\Lambda}\right)^2 a_{01}^1. \quad (67)$$

Since the coefficients a_{01}^1 and a_{12}^3 can generically either be of order unity or vanish exactly, it indeed follows from Eq. (67) that $T_1 = T_3 = 0$, which is the criterion for seesaw Dirac particles in the case of 4×4 matrices. Similarly, in Sec. III C, one can confirm the validity of the criterion for seesaw Dirac particles in the case of 6×6 matrices.

V. SUMMARY AND CONCLUSIONS

In conclusion, we have discussed general properties of neutrino mass matrices involving a fine-tuning condition and the connection to flavor symmetries. We pointed out in particular that the number of light neutrino masses generated by the seesaw mechanism cannot exceed half of the dimension of the considered mass matrix if fine-tuning is absent. Furthermore, we have introduced the concept of seesaw Dirac particles. In the light of this, we formulated for the examples of real symmetric 4×4 and 6×6 neutrino mass matrices a necessary and sufficient criterion in order to obtain seesaw Dirac particles. For these cases it was shown that the imposition of a conserved ZKM charge is equivalent to the assumption that the mass terms in flavor basis represent independent parameters. As an application of our methods, we demonstrated that small pseudo Dirac neutrino masses can be generated in a natural way by the seesaw mechanism if discrete or non-Abelian symmetries are taken into account. Then, we presented a model scheme based on one continuous non-Abelian symmetry and one discrete Abelian symmetry generically leading to a 6×6 mass matrix, which satisfies the criterion for seesaw Dirac particles in the absence of a conserved ZKM charge. Furthermore, we have found that a considerably wide class of reducible representations of unbroken unitary flavor symmetries accounts only for the degeneracy of some neutrino masses, but does not establish any relations between the nondegenerate neutrino masses. Finally, we confirmed the above formulated fine-tuning condition, i.e., the criterion for seesaw Dirac particles coming from our symmetry considerations.

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- [51] The entries of the matrices S_i are of the order ϵ/Λ , whereas the entries of the matrices C_i are of the order 1.
- [52] Note that all representations of finite groups are equivalent to unitary representations.
- [53] Here we have omitted the gamma matrices, since they act only on the Lorentz indices, but not on the flavor indices.
- [54] In fact, the unitary matrix is in this case either symmetric or antisymmetric, depending on the question whether $D_s^{(a)}$ is an integer or a half-integer representation.