Physics of synchronized neutrino oscillations caused by self-interactions

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In the early universe or in some regions of supernovas, the neutrino refractive index is dominated by the neutrinos themselves. Several previous studies have found numerically that these self-interactions have the effect of coupling different neutrino modes in such a way as to synchronize the flavor oscillations which otherwise would depend on the energy of a given mode. We provide a simple explanation for this baffling phenomenon in analogy to a system of magnetic dipoles which are coupled by their self-interactions to form one large magnetic dipole which then precesses coherently in a weak external magnetic field. In this picture the synchronized neutrino oscillations are perfectly analogous to the weak-field Zeeman effect in atoms.

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I. INTRODUCTION

In a two-flavor system of mixed neutrinos, the flavor content of a given state oscillates with the frequency $\Delta m^2/2p$ where $\Delta m^2 = m_2^2 - m_1^2$ is the neutrino mass-squared difference and p is the momentum. Therefore, if a neutrino ensemble encompasses many modes with many different momenta, these modes develop growing relative phases so that the overall flavor content of the ensemble quickly decoheres. This trivial effect is illustrated in Fig. 1 for an ensemble of neutrinos (no antineutrinos) with a thermal momentum distribution at temperature T. The vacuum mixing angle was taken to be $\sin 2\theta = 0.8$ and all neutrinos were originally in a pure ν_e state. In our example the momentum distribution is very broad so that the flavor decoherence takes place within about one oscillation period (dotted line in Fig. 1).

This behavior changes dramatically when the neutrinos feel a significant weak-interaction potential caused by the presence of the other neutrinos. We express the strength of the neutrino-neutrino potential in terms of the parameter

$$\kappa \equiv \frac{2\sqrt{2} G_{\rm F} n_\nu p_0}{\Delta m^2} \tag{1}$$

where $p_0 \equiv \langle p^{-1} \rangle^{-1}$. When the neutrino-neutrino potential is comparable or much larger than a typical $\Delta m^2/2p$, corresponding to $\kappa = \mathcal{O}(1)$ or larger, the modes get locked to each other—the entire ensemble oscillates with a common frequency $\omega_{\rm synch}$ which corresponds to a certain average of $\Delta m^2/2p$ (Fig. 1). This stunning effect was first discovered in numerical studies of early-universe neutrino oscillations [1] and then elaborated and applied in a large series of papers [2–9].

We note that the mode synchronization effect discussed in [10] is unrelated to our present case. When frequent flavor-blind collisions occur on a time scale much faster than the oscillation period, then the energy of a given neutrino is averaged over an oscillation period, leading to a common oscillation frequency.

Our present mode synchronization effect illustrated in Fig. 1 is a strictly nonlinear effect caused by the neutrino-neutrino self-interactions and as such seems difficult to understand. We presently develop a very simple and physically transparent theory of this effect, taking full advantage of the equivalence of our problem with the spin precession of a magnetic dipole in magnetic fields. It will become clear that the essential effect of the neutrino self-coupling is to lock the individual neutrino modes to form one large "magnetic moment" which then spin-precesses in a weak external field. Therefore, the equation of motion returns to a simple linear form.

Our approach provides a transparent and intuitive analytic framework which nicely accommodates and illuminates the results of the previous literature which were largely based on numerical studies. Once the framework is established, it is easy to study various generalizations and special cases that

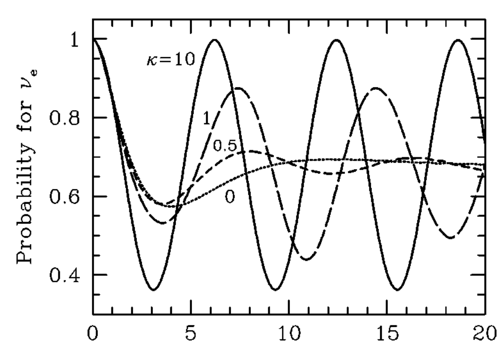

FIG. 1. Total ν_e survival probability as a function of time, where $\tau \equiv (\Delta m^2/2p_0)t$ and $p_0 = \langle p^{-1} \rangle^{-1} \simeq 2.2T$. The curves are for different values κ of the neutrino self-coupling as indicated where $\kappa = 0$ corresponds to vacuum oscillations.

otherwise would be difficult to predict or understand.

In Sec. II we begin by setting up the equations of motion for neutrino oscillations with the inclusion of a neutrino-neutrino potential. In Sec. III we develop the picture of coupled magnetic moments to explain the synchronization effect. In Sec. IV we include antineutrinos, a situation where the system can behave qualitatively different from the neutrino-only case. Finally we summarize our findings in Sec. V.

II. EQUATIONS OF MOTION

Our starting point is the well-known spin-precession picture for neutrino oscillations in vacuum [11–14]

$$\dot{\mathbf{P}}=\frac{\Delta m^2}{2p}\mathbf{B}\times\mathbf{P}. \tag{2}$$

Here, $\mathbf{P}$ is the polarization vector in flavor space of a neutrino mode with momentum p. In the usual way, the z component of $\mathbf{P}$ gives us the probability for finding the neutrino, say, in the electron flavor state by virtue of $\text{prob}(\nu_e)=\frac{1}{2}(1+P_z)$. The vector $\mathbf{B}=(\sin 2\theta,0,-\cos 2\theta)$ with the mixing angle θ gives us an effective ‘‘magnetic field’’ around which $\mathbf{P}$ precesses. Therefore, $\mathbf{P}$ plays the role of an angular momentum vector while $\mathbf{M}=(\Delta m^2/2p)\mathbf{P}$ plays the role of a magnetic dipole moment associated with $\mathbf{P}$. The quantity $\Delta m^2/2p=|\mathbf{M}|/|\mathbf{P}|$, gives us the proportionality between $\mathbf{M}$ and $\mathbf{P}$ and thus plays the role of the ‘‘gyromagnetic ratio’’ for a given mode, determining the rate of precession.

In much of the literature, the equation of motion is written in the form $\dot{\mathbf{P}}=\mathbf{V}\times\mathbf{P}$ without distinguishing clearly between the effective angular momentum $\mathbf{P}$ and its associated magnetic moment. For our present discussion this distinction is crucial. Still, we could split $\mathbf{V}$ in different ways between the gyromagnetic ratio, the unit of magnetic moment μ, and the $\mathbf{B}$ field which really stands for $\mu\mathbf{B}$. For example, we might have used p_0/p as the gyromagnetic ratio with p_0 some typical or average momentum, and defined $\mathbf{B}=\mu\mathbf{B}=(\Delta m^2/2p_0)(\sin 2\theta,0,-\cos 2\theta)$. However, we have preferred to avoid introducing an additional quantity p_0, and it is convenient to define $\mathbf{B}$ as a unit vector.

We will frequently consider the polarization vector for an entire ensemble of neutrinos:

$$\mathbf{J}\equiv\sum_{j=1}^{N_\nu}\mathbf{P}_j; \tag{3}$$

i.e. we consider a large volume $\mathcal{V}$ filled homogeneously with N_ν neutrinos. We also assume an isotropic distribution of the momenta so that it suffices to specify the modulus of the momentum of a given mode, $p_j=|\mathbf{p}_j|$.

There is no closed equation of motion for $\mathbf{J}$ because the individual modes oscillate with different frequencies. Evidently, however, the projection of $\mathbf{J}$ on $\mathbf{B}$ is conserved. On the other hand, the fast precession of the individual mode polarizations around $\mathbf{J}$ average the transverse components of the individual modes to zero so that the asymptotic value $\mathbf{J}_\infty=(\mathbf{B}\cdot\mathbf{J})\mathbf{B}$ obtains. (Recall that in our definition $\mathbf{B}$ is a unit vector.) For maximum mixing, and if initially all neutrinos were in a flavor eigenstate, $\mathbf{J}_\infty=0$, corresponding to an incoherent mixture of both flavors.

A background medium consisting, say, of protons, neutrons and electrons modifies the ‘‘magnetic field.’’ Assuming our two-flavor system involves electron neutrinos the substitution is

$$\frac{\Delta m^2}{2p}\mathbf{B}\to\frac{\Delta m^2}{2p}\mathbf{B}+\sqrt{2}G_\mathrm{F}n_e\hat{\mathbf{z}} \tag{4}$$

with n_e the electron number density. Therefore, the precession is no longer around a common direction for all modes. If we started with a situation of maximum mixing, then the medium reduces the effective mixing angle for all modes. For a very dense medium, the effective magnetic field will be almost perfectly along the z direction, suppressing flavor oscillations entirely.

This is very different if we consider a neutrino ensemble so dense that the neutrinos themselves produce a significant refractive index. In that case the medium's contribution to the refractive index is not along the flavor direction (i.e. along the z axis), but rather along the direction of $\mathbf{J}$. Put another way, neutrinos produce an ‘‘off-diagonal refractive index’’ [15] because a given background neutrino may be a coherent superposition of flavor states. The equation of motion for a single mode j now reads [16]

$$\dot{\mathbf{P}}_j=\frac{\Delta m^2}{2p_j}\mathbf{B}\times\mathbf{P}_j+\frac{\sqrt{2}G_\mathrm{F}}{\mathcal{V}}\mathbf{J}\times\mathbf{P}_j, \tag{5}$$

where the second term represents the self-interactions. If we sum this equation over all modes, then the second term becomes proportional to $\mathbf{J}\times\mathbf{J}=0$ so that

$$\dot{\mathbf{J}}=\mathbf{B}\times\sum_{j=1}^{N_\nu}\frac{\Delta m^2}{2p_j}\mathbf{P}_j. \tag{6}$$

Therefore, the first derivative $\dot{\mathbf{J}}$ of the ensemble's polarization vector is not affected by the self-interactions. Still, the evolution of $\mathbf{J}$ is changed because the evolution of the individual modes is affected. However, the vacuum oscillations are not obviously suppressed even in a dense gas of neutrinos, in contrast to a standard background medium.

Our case of active-active neutrino oscillations is very different from the active-sterile case [16,17]. Sterile neutrinos do not produce a weak potential so that there is no off-diagonal refractive index. The neutrino contribution to the ‘‘effective magnetic field’’ is along the z direction, i.e. not proportional to $\mathbf{J}$. Therefore, the self-interaction term is somewhat more similar to the effect of an external background medium, although the oscillation equations, of course, remain non-linear.

III. EXPLANATION OF SYNCHRONIZED OSCILLATIONS

It is now easy to demonstrate that Eq. (5) implies a synchronized precession of all modes around $\mathbf{B}$ if the neutrino density is sufficiently large. To this end we first imagine the vacuum oscillation term to be absent, i.e. we consider the equation of motion

$$\dot{\mathbf{P}}_j=\frac{\sqrt{2}G_\mathrm{F}}{\mathcal{V}}\mathbf{J}\times\mathbf{P}_j\,. \tag{7}$$

Every individual mode precesses around the direction of **J**. Of course, if all neutrinos were initially prepared in a specific flavor state, then all $\mathbf{P}_j$ as well as **J** are aligned along the z direction, and no precession takes place (perfectly coherent state). Likewise, if the individual $\mathbf{P}_j$ point in random directions so that $\mathbf{J}=0$, again there are no precessions (perfectly incoherent flavor mixture). We consider the general case where the $\mathbf{P}_j$ initially point in many different directions, but do not add to zero.

Next we switch on the vacuum term from Eq. (2), i.e. a weak external ''magnetic field.'' With ''weak'' we mean that for a typical mode the precession frequency around **J** is much larger than the one around **B**. This implies that the evolution of a given mode remains dominated by **J**. The fast precession around **J** implies that the projection of $\mathbf{P}_j$ on **J** is conserved while the transverse component averages to zero on a fast time scale relative to the slow precession around **B**. If **J** moves slowly, then the individual modes will follow **J**. Put another way, the individual modes are coupled to each other by their strong ''internal magnetic fields,'' forming a compound system with one large magnetic moment. It is this compound object which precesses around **B**. Of course, if the external field is much larger than the internal ones (dilute neutrino gas), then the modes will decouple and precess individually around the external field with their separate vacuum oscillation frequencies.

Our picture is perfectly analogous to the Zeeman effect in atoms. An atomic state is characterized by its spin angular momentum **S**, its orbital angular momentum **L**, and the total angular momentum $\mathbf{J}=\mathbf{L}+\mathbf{S}$. In a weak external magnetic field the spin-orbit coupling caused by the internal magnetic fields (Russell-Saunders coupling) remains intact, the external field is only a perturbation. In this case it is the total angular momentum **J** which precesses, i.e. which determines the atomic level splittings caused by the external B field. On the other hand, if the external field is much stronger than the internal one, then **L** and **S** decouple and precess independently around the external field: the atomic levels are determined by the separate quantization of **L** and **S** along the external B field (the Paschen-Back effect).

Granting that in the neutrino case the polarization vectors of the individual modes are locked in the sense that **J** indeed forms one large ''angular momentum,'' the evolution of the compound system is governed by the equation

$$\dot{\mathbf{J}}=\omega_\mathrm{synch}\mathbf{B}\times\mathbf{J}. \tag{8}$$

It remains to determine the value of ω_synch which plays the role of the gyromagnetic ratio for the compound system. Of the individual modes, the external field ''sees'' only the projection along **J** because the transverse components average to zero. Therefore, the contribution of mode j to the total magnetic moment **M** is $(\Delta m^2/2p_j)\hat{\mathbf{J}}\cdot\mathbf{P}_j$ so that

$$\mathbf{M}=\hat{\mathbf{J}}\sum_{j=1}^{N_\nu}\frac{\Delta m^2}{2p_j}\hat{\mathbf{J}}\cdot\mathbf{P}_j\,, \tag{9}$$

where $\hat{\mathbf{J}}$ is a unit vector in the direction of **J**. Therefore, the gyromagnetic ratio is

$$\omega_\mathrm{synch}=\frac{|\mathbf{M}|}{|\mathbf{J}|}=\frac{1}{|\mathbf{J}|}\sum_{j=1}^{N_\nu}\frac{\Delta m^2}{2p_j}\hat{\mathbf{J}}\cdot\mathbf{P}_j\,. \tag{10}$$

In particular, if all modes started aligned (coherent flavor state) then $|\mathbf{J}|=N_\nu$ and $\hat{\mathbf{J}}\cdot\mathbf{P}_j=1$ so that

$$\omega_\mathrm{synch}=\left\langle\frac{\Delta m^2}{2p}\right\rangle=\frac{1}{N_\nu}\sum_{j=1}^{N_\nu}\frac{\Delta m^2}{2p_j} \tag{11}$$

in agreement with [5].

If an external medium is present, the uncoupled modes precess around different **B**-field vectors rather than a common direction. When the modes are coupled by self-interactions, it is still the one **J** that precesses around one common **B** field which is a suitable average of the individual $\mathbf{B}_j$ which easily can be worked out.

Returning to the simpler case of a common **B** for all modes, the calculation of ω_synch amounts to determining the Landé factor for **J** in the atomic analogy. This is the problem of calculating the magnetic moment of a system if the angular momentum is the sum of individual components which have magnetic moments with different gyromagnetic ratios. In this case the vector sums of the angular momenta and of the magnetic moments are not colinear. In atomic physics, the spin angular momentum produces twice the magnetic moment of the orbital angular momentum, hence the complication.

We stress that, contrary to the previous literature, our analysis shows that there is nothing special about the initially aligned state, even though this state may be mostly motivated by the neutrino applications when all of them start in one flavor state. Any initial configuration of **J** precesses as one vector. Again, in atomic physics **L** and **S** can be combined in different ways to form one **J**. For example, a p state with $L=1$ and $S=1/2$ can combine to a $J=3/2$ or a $J=1/2$ state; there is nothing special about the ''aligned'' state ($J=3/2$).

Moreover, even though the initial **J** precesses as a compound system, the individual modes do not oscillate in unison, except for the special case of perfect initial alignment and infinitely strong self-coupling. In the general case the motion of every $\mathbf{P}_j$ is a fast precession around **J**, superimposed on a slow precession of **J** around **B**. The compound motion of $\mathbf{P}_j$ is generally rather complicated and different for every mode. This is illustrated in Fig. 2, where the synchronized oscillations of three different modes are shown for $\kappa=10$, starting with a perfectly aligned state.

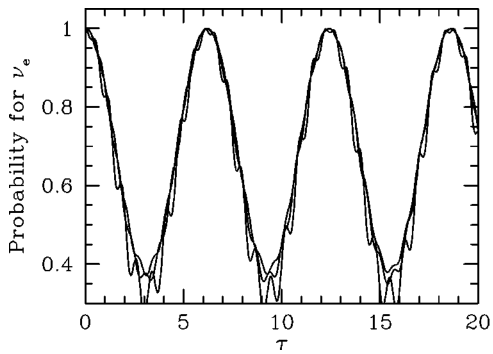

FIG. 2. Evolution of the ν_e survival probability for three values of the neutrino momenta in the presence of a strong neutrino self-potential term ($\kappa=10$).

Our treatment predicts that any set of initial $\mathbf{P}_j$ leads to synchronized oscillations in the sense of a precession of the initial $\mathbf{J}$ with a single frequency $\omega_{\rm synch}$. The length of the initial state $\mathbf{J}$ is conserved while the oscillation frequency depends on details of the initial distribution of polarization vectors. The system does not prefer one particular synchronized state over another. For example, the completely incoherent state will stay that way, and will not spontaneously align the $\mathbf{P}_j$ to form a coherent state, in agreement with the stability analysis of [9]. Any initially prepared $\mathbf{J}$ will do nothing but precess about $\mathbf{B}$. Put another way, the nonlinear aspect of the neutrino system manifests itself in the coupling of the individual $\mathbf{P}_j$ to each other to form a compound $\mathbf{J}$ which acts as one large angular momentum. Beyond this, the system is easily understood in terms of a linear equation of motion.

Our analysis is also entirely independent of the number of neutrinos or modes. For example, if there are only two modes nothing in our analysis changes so that the synchronization effect should not be viewed as a collective phenomenon. In fact, the atomic example was one consisting of two coupled magnetic moments, the orbital and spin terms. We have checked with a simple numerical code that a system of two or three modes indeed behaves as expected according to our treatment. Actually, a two-mode system can be fully solved analytically so that in principle the physical essence of the synchronization effect can be calculated without recourse to numerical methods.

There are exceptions to our general statements which will be of interest in the next section. Evidently we can construct ''pathological'' cases where several angular momenta $\mathbf{P}_j$ add up to a vanishing or arbitrarily small $\mathbf{J}$ while the magnetic moments do not, leading to a gyromagnetic ratio which can be constructed to become arbitrarily large. In this case $\omega_{\rm synch}$ will no longer represent some typical $\Delta m^2/2p$, but can be constructed to become arbitrarily large. In this case the motion of $\mathbf{J}$ is not necessarily slow compared to the internal precessions of the individual $\mathbf{P}_j$ so that our treatment is no longer adequate. However, the perfectly incoherent state of an ensemble of many modes will not be pathological in this sense because the random distribution of $\mathbf{P}_j$ will ensure that both $|\mathbf{J}|\sim|\mathbf{M}|\sim 1/\sqrt{N_\nu}$, where N_ν is the number of modes. In the limit $N_\nu\to\infty$ both $\mathbf{J}$ and $\mathbf{M}$ vanish simultaneously.

IV. NEUTRINOS PLUS ANTINEUTRINOS

As a next step we may extend our analysis to the case where neutrinos and antineutrinos are simultaneously present, an inevitable situation in a realistic system such as the early universe unless the neutrino chemical potentials are extremely large. To first order in $G_{\rm F}$ the equations of motion are [16]

$$\dot{\mathbf{P}}_j=+\frac{\Delta m^2}{2p_j}\mathbf{B}\times\mathbf{P}_j+\frac{\sqrt{2}G_{\rm F}}{\mathcal{V}}(\mathbf{J}-\overline{\mathbf{J}})\times\mathbf{P}_j$$
$$\dot{\overline{\mathbf{P}}}_k=-\frac{\Delta m^2}{2p_k}\mathbf{B}\times\overline{\mathbf{P}}_k+\frac{\sqrt{2}G_{\rm F}}{\mathcal{V}}(\mathbf{J}-\overline{\mathbf{J}})\times\overline{\mathbf{P}}_k, \tag{12}$$

where overbarred quantities refer to antineutrinos. In particular, the total polarization vectors are

$$\mathbf{J}=\sum_{j=1}^{N_\nu}\mathbf{P}_j \quad \text{and} \quad \overline{\mathbf{J}}=\sum_{k=1}^{N_{\bar\nu}}\overline{\mathbf{P}}_k. \tag{13}$$

As explained in [16], the definition of the polarization vector for antineutrinos is ''reversed'' in the sense that in vacuum it precesses in the opposite direction of neutrinos.[1] This corresponds to fermions with a true magnetic moment which have opposite gyromagnetic ratios for particles and antiparticles. For example, a neutron and an antineutron spin precess in opposite directions in the same external magnetic field.

It is obvious that a system consisting of neutrinos and antineutrinos behaves the same way as one consisting of neutrinos only, except that the role of the total angular momentum is now played by $\mathbf{I}=\mathbf{J}-\overline{\mathbf{J}}$. The anti-particles appear as normal modes of the system, except that they sport negative gyromagnetic ratios. It is $\mathbf{I}=\mathbf{J}-\overline{\mathbf{J}}$ which precesses slowly around the external field, while all $\mathbf{P}_j$ and $\overline{\mathbf{P}}_k$ remain pinned to $\mathbf{I}$. The corresponding evolution equation for the compound system is

[1] In the flavor oscillation case one can avoid this minus sign by defining both polarization vectors in the same way so that the vacuum oscillation equations look the same. In this case the sign change has to be introduced by hand in the non-linear potential terms by flipping the sign of one component of the potential. This can be achieved by defining $\mathbf{P}^*$ as the vector with the second component reversed, leading to equations of motion of the form

$$\dot{\mathbf{P}}_j=+\frac{\Delta m^2}{2p_j}\mathbf{B}\times\mathbf{P}_j+\frac{\sqrt{2}G_{\rm F}}{\mathcal{V}}(\mathbf{J}-\overline{\mathbf{J}}^*)\times\mathbf{P}_j,$$
$$\dot{\overline{\mathbf{P}}}_k=+\frac{\Delta m^2}{2p_k}\mathbf{B}\times\overline{\mathbf{P}}_k-\frac{\sqrt{2}G_{\rm F}}{\mathcal{V}}(\mathbf{J}^*-\overline{\mathbf{J}})\times\overline{\mathbf{P}}_k.$$

This approach was used in much of the literature. While it is mathematically equivalent to our treatment, we think that it obscures the simplicity of the equations because their vector form is destroyed.

$$\dot{\mathbf{I}}=\omega_{\rm synch}\mathbf{B}\times\mathbf{I} \tag{14}$$

where the total gyromagnetic ratio is

$$\omega_{\rm synch}=\frac{1}{|\mathbf{I}|}\left(\sum_{j=1}^{N_\nu}\frac{\Delta m^2}{2p_j}\hat{\mathbf{I}}\cdot\mathbf{P}_j+\sum_{k=1}^{N_{\bar\nu}}\frac{\Delta m^2}{2p_k}\hat{\mathbf{I}}\cdot\overline{\mathbf{P}}_k\right). \tag{15}$$

We have checked with a numerical code several situations with a thermal population of neutrinos and antineutrinos, and the results were always as expected.

Therefore, a system consisting of neutrinos and antineutrinos typically behaves qualitatively similar to the neutrino-only case. However, the negative gyromagnetic ratios of antineutrinos relative to neutrinos allow for "pathological" situations where the system behaves qualitatively differently than above.

One such case is a thermal ensemble without chemical potential, i.e. a situation where $N_\nu=N_{\bar\nu}$. If all neutrinos start in a given flavor state we have $\mathbf{I}=\mathbf{J}-\overline{\mathbf{J}}=0$. This situation is "pathological" in the sense that a vanishing or very small $\mathbf{I}$ is associated with a large magnetic moment because particles and anti-particles enter with exactly opposite magnetic moments. To illustrate this case we consider only one mode of particles and anti-particles so that the equations of motion are

$$\begin{aligned}\dot{\mathbf{P}}&=+\omega\mathbf{B}\times\mathbf{P}+(\mathbf{P}-\overline{\mathbf{P}})\times\mathbf{P},\\ \dot{\overline{\mathbf{P}}}&=-\omega\mathbf{B}\times\overline{\mathbf{P}}+(\mathbf{P}-\overline{\mathbf{P}})\times\overline{\mathbf{P}},\end{aligned} \tag{16}$$

with a suitable ω. This implies

$$\dot{\mathbf{P}}-\dot{\overline{\mathbf{P}}}=\omega\mathbf{B}\times(\mathbf{P}+\overline{\mathbf{P}}). \tag{17}$$

Evidently $\mathbf{I}=\mathbf{P}-\overline{\mathbf{P}}$ is not conserved if the two polarization vectors start aligned so that at first $\mathbf{I}=0$. The effect of the external field is to drive $\mathbf{P}$ and $\overline{\mathbf{P}}$ apart, creating a net $\mathbf{I}$ which is orthogonal to $\mathbf{B}$. In the case of large mixing both $\mathbf{P}$ and $\overline{\mathbf{P}}$ will oscillate much faster than they would in vacuum, yet convert to the other flavor.

In the case of small mixing the result strongly depends on the sign of ω in Eqs. (16), i.e. on the sign of Δm^2. For $\omega>0$, the directions of the vectors $\mathbf{P}$, $\overline{\mathbf{P}}$ and $\mathbf{B}$ almost coincide. The role of the strong neutrino self-potential term is just to increase the oscillation frequency, while the amplitude of P_z is the same as in the vacuum case.

For an inverted mass hierarchy ($\Delta m^2<0$) and small mixing angle, $\mathbf{B}$ is close to the z axis, but it is almost opposite to the initial directions of $\mathbf{P}$ and $\overline{\mathbf{P}}$. A small seed of $|\mathbf{I}|\neq 0$ is enough to drive $\mathbf{P}$ and $\overline{\mathbf{P}}$ to the opposite direction from their initial orientation, i.e. one can achieve complete flavor conversion. We illustrate this situation in Fig. 3, where the z component of the polarization vector and the modulus of $\mathbf{I}$ are plotted for $\omega=-0.01$ and $\sin 2\theta=0.01$. One can see that P_z evolves from 1 to -1 and back. This resonance has a different origin from the Mikheev-Smirnov-Wolfenstein mechanism because it comes from the neutrino self-potential term.

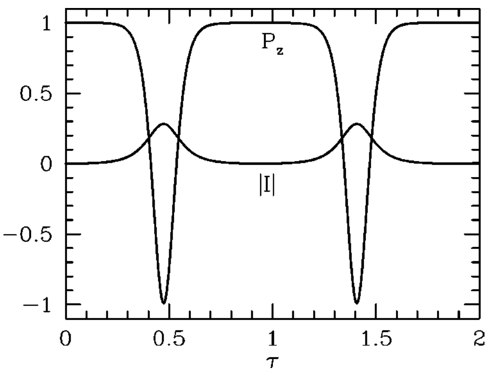

FIG. 3. Evolution of P_z and $|\mathbf{I}|$ for the system of Eqs. (16), when $\sin 2\theta=0.01$ and in the presence of a strong neutrino self-potential with $\omega=-0.01$, corresponding to an inverted mass hierarchy $\Delta m^2<0$.

These phenomena were discovered and discussed in [4] and subsequent papers of that series. We merely stress that our way of writing the equations allows for a straightforward visualization of the motion of the polarization vectors.

Another physically motivated "pathological" case is when two flavors of neutrinos are thermally populated with large but opposite chemical potentials. In the early universe this corresponds to a hypothetical initial condition of a vanishing or small lepton-number density, yet an anomalously large density of radiation in the form of neutrinos. Since the chemical potentials are assumed to be opposite we have $\mathbf{J}=-\overline{\mathbf{J}}$ so that $\mathbf{I}=\mathbf{J}-\overline{\mathbf{J}}=2\mathbf{J}$ is now "large." On the other hand, $\omega_{\rm synch}=0$ so that flavor oscillations take place with a vanishing frequency, i.e. the initial condition is frozen without further evolution. This appears to be the only case where the large neutrino-neutrino self-potential acts to prevent flavor oscillations.

These "pathological" cases have an atomic counterpart in the form of positronium where the two spins are associated with equal but opposite magnetic moments. Therefore, the $I=1$ state (ortho-positronium) has a vanishing magnetic moment and thus shows no Zeeman splittings in a weak external field. Likewise, the $I=0$ state (para-positronium) consists of only one level and thus cannot split. Therefore, even though we have a system of two spins associated with two magnetic moments, there is no weak-field Zeeman effect. In a strong field the electron and positron spins and magnetic moments are separately quantized along the external B field, giving rise to a nontrivial level structure.

V. CONCLUSIONS

In summary, we have provided a simple and physical explanation of the synchronized oscillations observed in the numerical treatment of dense neutrino ensembles. The effect is perfectly analogous to the coupling of several angular momenta, for example spin and orbital angular momentum in an atom, to form one large compound angular momentum with one large associated magnetic dipole moment which precesses as one object in a weak external field. Antineutrinos are naturally included in our picture if one observes that they carry ‘‘negative gyromagnetic ratios’’ in flavor space. The nonlinear nature of neutrinos oscillating in a background of neutrinos is thus reduced to a very simple and well-known coupling effect of magnetic moments to each other. Our picture provides a transparent framework that accounts perfectly for all of the previously discussed synchronization phenomena in the literature, and that allows one to derive both general properties and special cases in a unified framework. The practical impact of synchronized neutrino oscillations in the early universe will be studied in a forthcoming paper [18].

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