Muonium hyperfine structure and hadronic effects

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A new result for the hadronic vacuum polarization correction to the muonium hyperfine splitting (HFS) is presented: $\Delta\nu(\mathrm{had-vp})=(0.233\pm0.003)$ kHz. Compared with previous calculations, the accuracy is improved by using the latest data on $e^+e^-\to$ hadrons. The status of the QED prediction for HFS is discussed.

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I. INTRODUCTION

The hyperfine structure of two-body atomic systems is of interest for a variety of reasons. In particular, it offers opportunities for precise tests of bound-state QED and an accurate determination of fundamental constants such as the fine structure constant α, or electron-to-muon mass ratio. The hyperfine interval in the hydrogen ground state used to be the most precisely measured quantity. The fractional uncertainty was about 10^{-12} (see, e.g. [1]), although the theory cannot make a prediction with an accuracy better than 10^{-5}. The theoretical uncertainty comes from the proton magnetic form factor unknown at low momentum transfers [2] and from the proton polarizability contribution.

Studying purely leptonic systems such as muonium (the bound state of a positive muon and an electron), one can avoid the problems of the proton structure. Despite the short muon lifetime (about 2.2 μsec), the hyperfine splitting (HFS) of the muonium ground state has been measured very precisely [3,4]:

$$\nu_{\mathrm{HFS}}(\mathrm{exp})=4\,463\,302.776(51)\ \ \mathrm{kHz}, \tag{1}$$

with a fractional uncertainty of 0.011 ppm. (An overview of the present status and perspectives of the precision experiments and theory of simple atoms can be found in [5].)

The theoretical prediction for the HFS is usually written in terms of the Fermi energy, resulting from a nonrelativistic interaction of electron and muon magnetic moments. It can be expressed as a combination of fundamental constants:

$$\nu_F=\frac{16}{3}(Z\alpha)^2cZ^2R_\infty\frac{m}{M}\left[\frac{m_R}{m}\right]^3(1+a_\mu). \tag{2}$$

Another convenient formula which will be used to determine the Fermi energy is

$$\nu_F=\frac{16}{3}(Z\alpha)^2cZ^2R_\infty\frac{\mu_\mu}{\mu_B^e}\left[\frac{m_R}{m}\right]^3. \tag{3}$$

The accuracy of these expressions is limited by the electron-to-muon mass ratio m/M or the ratio of the muon magnetic moment to the electron Bohr magneton μ_μ/μ_B^e. Since the muon anomalous magnetic moment a_μ is known with high accuracy [6–8], these two formulas are essentially equivalent. Given that the three most precise experiments [3,4,9] determined the magnetic moment ratio, we will use Eq. (3). Using the values $1/\alpha=137.035\,999\,58(52)$ [8], $cR_\infty=3\,289\,841\,960\,368(25)$ kHz [10], $\mu_\mu/\mu_p=3.183\,345\,24(37)$ [3,4], and $\mu_p/\mu_B^e=1.521\,032\,203(15)\times10^{-3}$ [10], we find

$$\nu_F=4\,459\,031.920(511)(34)\ \ \mathrm{kHz}, \tag{4}$$

where the first (larger) uncertainty is due to the magnetic moment ratio and the second to the fine structure constant. The value $M/m=206.768\,276(24)$ used in this work was obtained by us from the values of the magnetic moments above as well as the experimental value $a_\mu=1.165\,920\,3(15)\times10^{-3}$ [6]. Here Z is the nuclear charge and in the case of muonium $Z=1$; however, it is convenient to keep that for the classification of different contributions (see Sec. II). In practice, precise measurements of the muonium HFS [3,4] serve as the most precise determination of the muon-to-electron mass ratio [10]. In the future, muon mass might be determined more precisely from other experiments [11], thus enabling a new test of bound-state QED.

Further experimental progress in the muonium HFS is also expected [11] if high-intensity muon sources become available. In view of those advances, it is conceivable that the accuracy of bound-state QED tests may become limited by interactions beyond QED, in particular by the strong interaction effects. In this paper we study hadronic vacuum polarization, the leading contribution of the strong interac-

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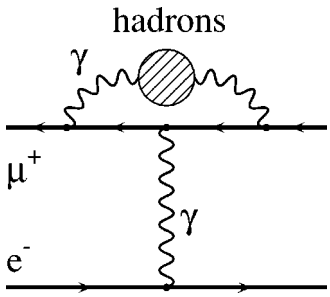

FIG. 1. Hadronic contribution to muonium HFS via the muon anomalous magnetic moment.

tions. These effects influence the HFS in two ways: through effects included in the Fermi energy (contributions to the muon a_μ shown in Fig. 1) and bound-state effects depicted in Fig. 2. Hadronic effects in a_μ are relatively large ($\sim\alpha^2$) and have been studied by a number of authors over the past 3 decades (see references in [7,12]). In the present case, we are using an experimental determination of the muon magnetic moment (μ_μ) and do not address this theoretical issue. We focus on the bound-state effects of the hadronic vacuum polarization [$\sim\alpha^2(m/M)$].

The theoretical expression for the hyperfine splitting can be written in the form

$$\nu_{\rm HFS}({\rm theor}) = \nu_F + \Delta\nu({\rm QED}) + \Delta\nu({\rm weak}) + \Delta\nu({\rm had}). \tag{5}$$

The biggest correction to the Fermi splitting (ν_F) comes from the QED effects [$\Delta\nu$(QED)] and it is under consideration in Sec. II. The two terms beyond QED are due to weak [$\Delta\nu$(weak)] and strong [$\Delta\nu$(had)] interactions.

Influence of the weak interaction effects on the muonium HFS was studied in the leading order [13,14]. The leading correction is induced by the neutral currents, given by a Z-boson exchange diagram. Recently it was found that the sign of the corrections had not been well understood and the absolute value was verified. In a recent paper [15] that problem was solved. The contribution is

$$\Delta\nu({\rm weak}) = -\frac{G_F m M}{\sqrt{2}}\frac{3}{4\pi Z\alpha}\nu_F \simeq -0.065 \ \ {\rm kHz}, \tag{6}$$

where G_F is the Fermi constant of the weak interaction. Next-to-leading-order contributions, studied in Ref. [16], are $\mathcal{O}(1\%)$ of the leading contribution (6) and thus negligible.

The strong interaction effects (Fig. 2) were studied in [17–19]. We examine the hadronic contribution in Sec. III

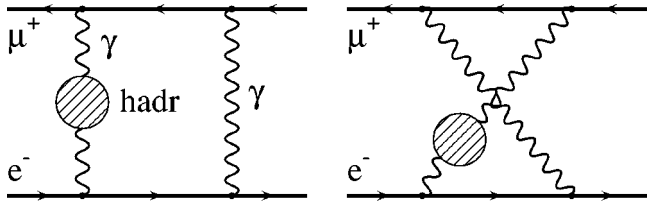

FIG. 2. Bound-state hadronic effects to muonium HFS considered in this paper.

following the approach of Ref. [20], where the QED and hadronic parts are separated and the final result in the leading order is presented in the form

$$\Delta\nu({\rm had}--{\rm vp}) = -\frac{Z}{2\pi^3}\frac{m}{M}\nu_F\int_{4m_\pi^2}^{\infty} ds\,\sigma(s)H(s), \tag{7}$$

where s is the center-of-mass energy squared, $\sigma(s)$ is the total cross section of electron-positron annihilation into hadrons and $H(s)$ is the QED kernel calculated in [20]. Our calculation gives

$$\Delta\nu({\rm had}--{\rm vp}) = (0.233\pm 0.003) \ \ {\rm kHz}. \tag{8}$$

The uncertainty of this result, based on recent electron-positron annihilation data, is less than half of that of the most accurate previous calculation [19]. The higher-order hadronic corrections are considered in part in [20].

In Sec. IV we discuss our results.

II. QED CONTRIBUTIONS

The problem of the theoretical accuracy of QED effects in the muonium HFS was considered in [16], where the importance of higher-order logarithmic corrections was pointed out. The complete QED theory includes a variety of contributions: electron and muon magnetic moments, external field terms, recoil and radiative-recoil corrections. The theoretical expression can be presented as an expansion in small parameters: α, $Z\alpha$ and m/M. In the case of muonium $Z=1$, however, it is customary to keep Z in order to distinguish between QED effects (which are counted by powers of α) and the binding Coulomb effect giving rise to $Z\alpha$. In some papers the muon charge is denoted by Ze and so the QED effects due to the radiative corrections on the muon line involve $Z^2\alpha$. Since that is still a QED effect, in our paper this correction is just α.

Here we discuss briefly the progress made since the publication of Ref. [16]. We divide the QED effects in three parts,

$$\Delta\nu({\rm QED}) = \Delta\nu(a_e) + \Delta\nu({\rm QED3}) + \Delta\nu({\rm QED4}), \tag{9}$$

because of their different status. The first term

$$\Delta\nu(a_e) = a_e\,\nu_F = 5\,170.926(1) \ \ {\rm kHz} \tag{10}$$

arises due to the electron anomalous magnetic moment $a_e = 1\,159\,652\,187(4)\times 10^{-3}$ for which we use the experimental value [21]. The next term, ν(QED3) includes the bound-state QED effects up to the 3rd order in the relevant small parameters, to be discussed in Sec. II A. These effects are known with good accuracy. The last term, ν(QED4), is related to the 4th order contributions and is subject to ongoing investigations. Its present status will be reviewed in Sec. II B.

A. QED up to third order

The contributions up to the third order in those parameters give

$$\Delta\nu(\mathrm{QED3}) = \nu_F\left\{\frac{3}{2}(Z\alpha)^2 + \alpha(Z\alpha)\left(\ln 2 - \frac{5}{2}\right) + \frac{\alpha(Z\alpha)^2}{\pi}\left[-\frac{2}{3}\ln\frac{1}{(Z\alpha)^2}\left(\ln\frac{1}{(Z\alpha)^2} + 4\ln 2 - \frac{281}{240}\right) + 17.122\,339\ldots\right.\right.$$
$$\left.\left. - \frac{8}{15}\ln 2 + \frac{34}{225}\right] + 0.7718(4)\,\frac{\alpha^2(Z\alpha)}{\pi}\right\} + \frac{\nu_F}{1+a_\mu}\,\frac{(Z\alpha)m}{M}\left\{\left[-\frac{3}{\pi}\ln\frac{M}{m} + (Z\alpha)\left(\ln\frac{1}{(Z\alpha)^2} - 8\ln 2 + \frac{65}{18}\right)\right]\right.$$
$$\left. + \frac{\alpha}{\pi^2}\left[-2\ln^2\frac{M}{m} + \frac{13}{12}\ln\frac{M}{m} + \frac{21}{2}\zeta(3) + \frac{\pi^2}{6} + \frac{35}{9}\right]\right\}. \tag{11}$$

Origins of individual terms in Eq. (11) are discussed in detail in the review [22]. The numerical value resulting from Eq. (11) is

$$\Delta\nu(\mathrm{QED3}) = -899.557 \ \ \mathrm{kHz}, \tag{12}$$

with an error beyond the displayed digits.

B. Fourth order

The essential part of the fourth order corrections has been evaluated (see e.g. [23]). The known terms are summarized in Table I. Some of them were calculated in the leading logarithmic approximation (using $\ln 1/\alpha \sim \ln M/m \sim 5$) and in those cases we follow [16] and estimate the uncertainty due to the non-leading terms as half of the leading ones. Table I contains a number of recent results, published after the review [16]. Some of the contributions will be reviewed in more detail below. We also present in Table I a very small contribution of the tau lepton loops replacing the hadronic loop in Fig. 2. This effect is $\mathcal{O}(\alpha(Z\alpha)mM/m_\tau^2)$ and amounts to about 2×10^{-3} kHz.

C. One-loop corrections

The electron one-loop self-energy has been studied in some detail. After direct calculations of the $\alpha(Z\alpha)^2$ term [36,37], two independent exact [without an expansion in $(Z\alpha)$] numerical evaluations were performed [38,39]. Unfortunately, the accuracy for $Z=1$ was not high in both cases and the final results for muonium were obtained by extrapolating the data from higher values of Z to $Z=1$, using previously found coefficients of $\alpha(Z\alpha)^2$ [36,37] and $\alpha(Z\alpha)^3\ln Z\alpha$ [16]. The result of the later calculation [39] is presented in Table II as $\alpha(Z\alpha)^{3+}$. It contains the nonlogarithmic $\alpha(Z\alpha)^3$ term and higher-order corrections. The result of [38] is slightly higher than that of [39]. The calculation gives for this term $-14.3(1.1)\,\alpha(Z\alpha)^3\nu_F/\pi$ [39] and $-12(2)\,\alpha(Z\alpha)^3\nu_F/\pi$ [38]. Recently the nonlogarithmic term was calculated directly by expansion in $Z\alpha$ to be $-15.9(1.6)\alpha(Z\alpha)^3\nu_F/\pi$ [41].

The uncertainty of the extrapolation [39] is determined as an estimate of the unknown higher-order terms rather than the accuracy of numerical data. However, the final uncertainty is not clear, because of the value of the nonlogarithmic $\alpha(Z\alpha)^2$ term [36,37] (cf. [42,16]):

$$(17.122\,339\ldots)\,\frac{\alpha(Z\alpha)^2}{\pi}\,\nu_F \quad [36],$$

$$(17.122\,7\pm0.001\,1)\,\frac{\alpha(Z\alpha)^2}{\pi}\,\nu_F \quad [37]. \tag{13}$$

These results appear to be in fair agreement with each other. However, it has been pointed out [37] that the term-by-term comparison reveals inconsistencies. The low-energy contribution in Ref. [37] is by 0.0257(2) higher than that in Ref. [36], while the medium- and high-energy parts are by 0.0246(14) lower. It should also be mentioned that different fits for extrapolation to the zero photon mass λ in Ref. [37] are not consistent. Such extrapolation was necessary to calculate the high momentum part. These problems remain to be clarified. Since the same method was used in [41], it is the result of [39] that we include to Table II.

TABLE I. The fourth order corrections. The first column gives the order of corrections relative to ν_F. The uncertainty due to the unknown term $\alpha^3(Z\alpha)$ is estimated as $(\alpha/\pi)^2$ times the known $\alpha(Z\alpha)\nu_F$ term in Eq. (11).

Correction	Contribution to the HFS	Reference
$(Z\alpha)^4$	0.03 kHz	[24]
$(Z\alpha)^3\frac{m}{M}$	−0.29(13) kHz	[25,26,14,16,27,28]
$(Z\alpha)^2\left(\frac{m}{M}\right)^2$	−0.02(1) kHz	[29,30]
$(Z\alpha)\left(\frac{m}{M}\right)^3$	−0.02 kHz	[31]
$\alpha(Z\alpha)^3$	−0.52(3) kHz	see Table II
$\alpha(Z\alpha)^2\frac{m}{M}$	0.39(17) kHz	[25,27,28]
$\alpha(Z\alpha)\left(\frac{m}{M}\right)^2$	−0.04 kHz	[32,33]
$\alpha^2(Z\alpha)^2$	−0.04(2) kHz	[25]
$\alpha^2(Z\alpha)\frac{m}{M}$	−0.04(3) kHz	[34,35]
$\alpha^3(Z\alpha)$	±0.01 kHz	see caption
τ lepton	0.002 kHz	[17]
Total	−0.55(22) kHz	

TABLE II. Higher-order one-loop contributions.

Order	Contribution to the HFS	Reference	Comments
$\alpha(Z\alpha)^3 \ln\frac{1}{Z\alpha}$	-0.53 kHz	[16]	self-energy
$\alpha(Z\alpha)^{3+}$	-0.06 kHz	[39]	self-energy
$\alpha(Z\alpha)^3 \ln\frac{1}{Z\alpha}$	0.03 kHz	[16]	vacuum polarization
$\alpha(Z\alpha)^3$	0.03 kHz	[40]	vacuum polarization
$\alpha(Z\alpha)^3$	± 0.03 kHz	see Sec. II C	Wichmann-Kroll
one-loop	$-0.52(3)$ kHz		sum of all terms

Recently, the one-loop vacuum polarization was calculated exactly (to all orders in $Z\alpha$) [43,40]. For our purpose it is enough to know the $\alpha(Z\alpha)^3$ terms only

$$\Delta\nu = \alpha(Z\alpha)^3\left(\frac{13}{24}\ln\frac{2}{Z\alpha} + \frac{539}{288}\right)\nu_F, \tag{14}$$

the logarithmic part of which was obtained in Ref. [16], while the constant was found in Ref. [40].

The only unknown term in the order $\alpha(Z\alpha)^3$ is now the so-called Wichmann-Kroll contribution. We estimate it by the value of the nonlogarithmic part of the VP term in Eq. (14).

D. Recoil effects

A number of results on various recoil effects were obtained after the publication of [16].

Pure recoil effects of the order $(Z\alpha)^2$ were studied without an m/M expansion in Ref. [44] for arbitrary mass and numerical results for several values of m/M were obtained. Recently the same correction was found for muonium [30] (see Table I).

For radiative recoil corrections $(\alpha(Z\alpha)m/M)$, there are minor discrepancies between the published numerical and analytical results, summarized in Table III. All analytical results were obtained by two groups which are in agreement. It is likely that the uncertainty of the numerical calculations was underestimated [45] and the discrepancies have no connection with higher-order corrections. This explanation is supported by the similar situation with the radiative recoil corrections in positronium, shown in the same table.

The uncertainty of the QED calculations of the muonium hyperfine structure arises mainly from two sources: $\alpha(Z\alpha)^2 m/M$ and $(Z\alpha)^3 m/M$. Only double logarithmic corrections were calculated until recently, when some single-logarithmic terms $[\sim\ln(Z\alpha)]$ were found. The single-logarithmic terms with a recoil logarithm $[\ln(M/m)]$ are still unknown. We discuss a possible consequence for the accuracy of the theoretical calculations in the next section.

E. Nonleading logarithmic terms and uncertainty of the calculations

In this paper we follow [16] and estimate any unknown nonleading terms by a half-value of the corresponding leading logarithmic terms and sum them as independent. There is no proof that the nonleading terms can be estimated in this way. However, all experience with different contributions to the Lamb shift and the hyperfine structure (see e.g. the review [22] with a collection of various terms) demonstrates that this assumption is reasonable.

The fourth order leading corrections contain cubic and quadratic logarithmic terms and in the case of some contributions, known in the logarithmic approximation, more than the leading term has been evaluated (see Table I for references):

$$\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\nu_F\left\{-\frac{4}{3}\ln^3\frac{M}{m} + \frac{4}{3}\ln^2\frac{M}{m}\right\},$$

$$\frac{\alpha(Z\alpha)^2}{\pi}\frac{m}{M}\nu_F\left\{\frac{16}{3}\ln^2\frac{1}{Z\alpha} + \left(1 + \frac{32}{3}\ln 2 - \frac{431}{90}\right)\ln\frac{1}{Z\alpha}\right\},$$

and

TABLE III. Radiative-recoil corrections in units of $\alpha^2(m/M)\nu_F$. The numerical results include all orders in m/M. The analytical results refer to sums of the third and fourth order, $\alpha^2(m/M)$ and $\alpha^2(m/M)^2$ for muonium. For positronium, $M=m$ and ν_F is defined in Eq. (2).

Contribution	Numerical result	Analytical result	Discrepancy
e-line	3.335(58) [46,17]	3.499 [47,48,32,33]	0.164(58)
μ-line	$-1.0372(91)$ [46,17]	-1.0442 [49,32]	$-0.0070(91)$
Positronium	$-1.787(4)$ [46,17]	-1.805 [50–52]	$-0.018(4)$

$$\frac{(Z\alpha)^3}{\pi}\frac{m}{M}\nu_F\left\{-3\ln\frac{M}{m}\ln\frac{1}{Z\alpha}-\frac{2}{3}\ln^2\frac{1}{Z\alpha}+\left(\frac{101}{9}-20\ln 2\right)\ln\frac{1}{Z\alpha}\right\}.$$

Even in this case we estimate the unknown terms as the half-value of the *leading* term. From comparison with, e.g., known contributions of the third order $\alpha(Z\alpha)^2\nu_F$ [see Eq. (11)] one sees that the constant term can be even bigger than the nonleading logarithm. Error estimates based on a nonleading logarithmic term may be misleading. The reason is that the leading logarithm in most cases originates from relatively simple diagrams under well understood conditions. The cancelations between different contributions occur very seldom. The leading logarithmic term has a *natural* value and is useful for estimates. In case of a cancelation which, although rare, is possible, an estimate should be based on the contributions *before* the cancelation. In case of the nonleading logarithmic terms, there are usually a few sources of different nature and some cancelations take place very often.

The total magnitude of the muonium HFS interval calculated within QED,

$$\nu_{\rm QED}=4\,463\,302.738(511)(34)(220)\ \ {\rm kHz}, \qquad (15)$$

is found by adding the values given by Eqs. (10),(12), and the sum of the 4th order contributions listed in Table I to the Fermi splitting (4). The third uncertainty is due to the 4th order QED effects in Table I.

III. HADRONIC CONTRIBUTIONS

A. Calculation of the hadronic contributions to muonium HFS

The lowest order hadronic contribution to the muonium hyperfine splitting is given by the following expression:

$$\Delta\nu({\rm had}-{\rm vp})=-\frac{1}{2\pi^3}\frac{m}{M}\nu_F\int_{4m_\pi^2}^{\infty}{\rm d}s\,\sigma(s)H(s), \qquad (16)$$

where the kernel $H(s)$ calculated in [20] is

$$H(s)=\left(\frac{s}{4M^2}+2\right)r\ln\frac{1+r}{1-r}-\left(\frac{s}{4M^2}+\frac{3}{2}\right)\ln\frac{s}{M^2}+\frac{1}{2},$$

$$r\equiv\sqrt{1-\frac{4M^2}{s}}. \qquad (17)$$

The bulk of those effects is computed using the experimental data on the cross sections of $e^+e^-\to$hadrons in the energy range $\sqrt{s}<\sqrt{s_0}$ where s_0 is a scale above which perturbative formulas can be used. We choose $\sqrt{s_0}=12$ GeV. The result is insensitive to this choice, because the experimental data are in agreement with perturbative QCD in this region and in any case such high-energy region contributes very insignificantly to the muonium HFS [see Eq. (23)].

Up to s_0, we perform a direct numerical integration of the experimental points, similar to the approach of [53] where hadronic corrections to a_μ were calculated. In contrast to the methods in which some approximation of the data is used for the integration, in our approach model dependence is avoided as much as possible and, moreover, the calculation of the uncertainties is straightforward. In addition to the data set used in [53], one can take into account significant progress in the measurement of the hadronic cross sections in the energy range below 1.4 GeV with two detectors at VEPP-2M in Novosibirsk [54,55] and the R determination in the energy range 2 to 5 GeV by the BES detector in Beijing [56].

The integration procedure gives for the contribution of this part

$$\Delta\nu({\rm bulk})=(0.2031\pm0.0031)\ \ {\rm kHz}. \qquad (18)$$

The narrow resonances (ω, ϕ, J/ψ and Υ families) are evaluated separately. In a zero width approximation the contribution of a resonance B with mass m_B and electronic width Γ_{ee} is

$$\Delta\nu({\rm resonance\ B})=-\frac{6}{\pi}\frac{m}{M}\nu_F\frac{\Gamma(B\to e^+e^-)}{m_B}H(m_B^2)\times[1-\Delta\alpha(m_B^2)]^2,$$

$$\Delta\alpha(s)=\frac{\alpha}{3\pi}\sum_f Q_f^2N_{cf}\left(\ln\frac{s}{m_f^2}-\frac{5}{3}\right). \qquad (19)$$

The leptonic widths relevant for our study should correspond to the lowest order (Born) graphs. However, experimentally measured leptonic widths listed and averaged by the Particle Data Group [57], contain an additional contribution of the vacuum polarization by leptons and hadrons. Therefore, for the transition to the lowest order widths one should multiply the experimental values by $[1-\Delta\alpha(m_B^2)]^2$ (see the discussion of this issue in [53]). Q_f and m_f are the charge and mass of the fermion f, and N_{cf} is the number of colors for the corresponding fermion ($N_{cf}=1$ for leptons). The above approximate formula for $\Delta\alpha(s)$ is valid for $s\ll M_W$ and describes contributions of fermions much lighter than $\sqrt{s/4}$. In the present calculation we consider only effects of the lightest leptons, e and μ, and neglect the hadronic contributions to $\Delta\alpha(s)$. Hadronic loops are relevant only for heavy flavors (the J/ψ and Υ family of resonances), which however give a smaller contribution and have a larger relative error. The hadronic effects, had they been included, would have shifted the J/ψ family contribution by a few percent. The Υ-family contributes about 2×10^{-3} of the total resonance contribution.

The sum of the individual resonance contributions gives

$$\Delta\nu({\rm res})=(0.0290\pm0.0006)\ \ {\rm kHz}. \qquad (20)$$

In the region above s_0 one can use a perturbative formula for the hadronic cross section:

TABLE IV. Contributions to muonium HFS.

Final state	Energy range (GeV)	$\Delta\nu$ (Hz)
2π	0.28–1.4	158.8±1.9
ω		12.4±0.4
ϕ		13.2±0.4
Hadrons	0.6–1.4	10.7±0.8
Hadrons	1.4–3.1	23.8±2.2
J/ψ		3.4±0.2
Hadrons	3.1–12.0	9.8±0.5
Hadrons	>12.0	1.2
Total		233.3±3.1

TABLE V. Contributions to muonium HFS and a_μ.

Final state	Energy range (GeV)	$\Delta\nu$ (%)	a_μ (%)
2π	0.28–1.4	68.0	71.8
ω		5.3	5.7
ϕ		5.7	5.8
Hadrons	0.6–1.4	4.6	4.2
Hadrons	1.4–3.1	10.2	8.2
J/ψ		1.5	1.3
Hadrons	3.1–12.0	4.2	2.8
Hadrons	>12.0	0.5	0.2
Total		100.0	100.0

$$\sigma(s)=\frac{4\pi\alpha^2}{3s}R(s),$$

$$R(s)=R^{(0)}(s)\left[1+\frac{\alpha_s}{\pi}+C_2\left(\frac{\alpha_s}{\pi}\right)^2+C_3\left(\frac{\alpha_s}{\pi}\right)^3\right],$$

$$R^{(0)}(s)=N_c\left(\frac{1}{9}N_{1/3}+\frac{4}{9}N_{2/3}\right)=\frac{11}{3}\quad(s>s_0),\tag{21}$$

where $N_{1/3(2/3)}$ is the number of ''active'' quarks with a charge 1/3 (2/3) at given s, $C_2=1.411$ and $C_3=-12.8$ [58].

At $s\gg M^2$ one can use the asymptotic formula for $H(s)$ [20]:

$$H(s)\to-\frac{M^2}{s}\left(\frac{9}{2}\ln\frac{s}{M^2}+\frac{15}{4}\right)\quad(\text{for } s\to\infty),\tag{22}$$

so that after the numerical integration one finally obtains

$$\Delta\nu(\text{cont})\simeq 0.0012\ \text{kHz}.\tag{23}$$

The final result for the lowest order hadronic contribution is found as a sum of Eqs. (18), (20), (23):

$$\Delta\nu(\text{had}-\text{vp})=\Delta\nu(\text{bulk})+\Delta\nu(\text{res})+\Delta\nu(\text{cont})$$
$$=(0.233\pm 0.003)\ \text{kHz}.\tag{24}$$

The individual contributions are presented in Table IV. It is clear that the dominant contribution (about 84%) comes from the very low energy range below 1.4 GeV studied in Novosibirsk. The regions below 1.4 GeV and from 1.4 to 3.1 GeV give about the same contribution to the total uncertainty. For most of the hadronic channels below 1.4 GeV analysis is still in progress in Novosibirsk. However, one can hardly expect significant improvement in this region since the dominant fraction of the error comes from the 2π channel whose cross section is already known with a very high accuracy of about 0.6% around the ρ meson [59]. Very promising for decreasing the uncertainty of the 2π channel could be use of the τ lepton data [60], but the real accuracy of this approach has been recently questioned [61,62]. The improvement of the uncertainty for $\sqrt{s}$ from 1.4 to 3.1 GeV will require new e^+e^- colliders covering this energy range [63]. Some progress in this region is also possible at the B factories and Cornell Electron Storage Ring (CESR) which can measure hadronic events below their nominal center-of-mass energy by tagging photons from the initial state radiation [64,65].

B. Comparison with other calculations

It is instructive to compare the relative contributions of different energy ranges for $\nu_{\rm HFS}$ and a_μ. From Table V one can see that they are very close to each other indicating the importance of the low energy regions. This is a natural consequence of the similar kernel structure, so that the main contribution in both cases comes from the low range of $\sqrt{s}$. One can also note some enhancement of the high energy contribution to the HFS compared to that to the anomalous magnetic moment. That is due to the different asymptotic behavior of the QED kernels for these two problems. For a_μ the asymptotics of the kernel is proportional to M^2/s, whereas for the muonium HFS it contains an additional logarithmic enhancement.

In Table VI we compare results of various calculations of the leading order hadronic contribution to the muonium HFS. The first estimate of this effect was performed in [17]. The authors took into account the contributions of the ρ, ω, and ϕ mesons and parametrized the hadron continuum above 1 GeV under the assumption that R is constant and equals 2. This simplified approach gave nevertheless a result fairly close to those of the later, more sophisticated analyses. This is not surprising since the dominant part of the hadronic contribution comes just from the lowest vector mesons

TABLE VI. Comparison of various calculations.

Source	Reference	$\Delta\nu$(had −vp) (kHz)
Sapirstein *et al.* (1984)	[17]	0.22±0.03
Karimkhodzaev and Faustov (1991)	[18]	0.250±0.016
Faustov *et al.* (1999)	[19]	0.240±0.007
Czarnecki *et al.* (2001)	This work	0.233±0.003

whose properties were known quite accurately at that time. In [18] additional experimental data were taken into account above 1 GeV, including effects of heavy quarkonia. The hadron continuum was parametrized with a function $R(s) = As^B$ in five energy ranges below 47 GeV and above this energy the asymptotic QCD formula with six quarks was used. Finally, in the recent work [19] the authors used a similar approach with a slightly more sophisticated parameterization of the hadron continuum in seven energy ranges below 60 GeV. Unfortunately, the authors of the above mentioned papers ignore the model dependence of their results which can be fairly strong and the error of their calculations only reflects the quality of the fit in their rather artificial models describing the data. Even larger is the effect of the systematic uncertainties completely ignored in most of the calculations.

This is particularly true for the dominant contribution to the muonium HFS coming from the 2π channel. For example, in [19] its accuracy calculated within the model of [66] is unrealistically high and reaches 0.65% although the model itself is based on the data of [67] where the systematic uncertainty in the dominant ρ meson region varies from 2 to 4.4%. Direct integration of the experimental points in our work also assumes some model for the energy dependence of the data. We estimated the model dependence by comparing the trapezoidal integration in which experimental points are connected with straight lines with even simpler rectangular integration which assumes a constant cross section within a small energy range. We found that both methods led to the same result, the difference being much smaller than the error. Both models allow a relatively simple estimation of the uncertainty arising because of the systematic errors. As already noted above, the effect is dominated by the contributions of the low lying vector mesons. Therefore, in all described calculations the central values of $\Delta\nu(\text{had}-\text{vp})$ are close to each other. However, in our opinion their uncertainty might have been underestimated. The decrease of the uncertainty in the result presented in this paper became possible thanks to the utilization of the most recent data set coming from the high precision e^+e^- experiments in the low energy region.

IV. CONCLUSIONS

The calculation of the hyperfine splitting in the ground state of muonium is summarized in Table VII. The biggest uncertainty of the theoretical expression comes from the inaccuracy of experimental determination of the muon mass and its magnetic moment. This inaccuracy is essentially related to the statistical uncertainty of the present experiments [3,4,9]. Improvement of the statistical accuracy in muonium spectroscopy studies will be possible with future intense muon sources. Construction of such facilities is being considered in connection with muon storage rings which would serve as neutrino factories [68,69]. There are plans to build a high-intensity muon source at the Japanese Hadron Facility [70]. The muon production rates at those future sources are expected to be around $10^{11}-10^{12}$ μ/s, higher than the most intense present beams (at the Paul Scherrer Institute) by a factor of 300–3000 or more. If muonium spectroscopy becomes part of the research program at these facilities, the statistical errors, which limit the present accuracy, will be decreased by one–two orders of magnitude [11]. In the more remote future, if a muon collider is constructed, further increase of beam intensity by a factor of about 100 will be possible.

TABLE VII. The muonium hyperfine structure: various theoretical contributions discussed in this paper.

Correction	Contribution to the HFS
ν_F	4 459 031.920(511)(34) kHz
$\Delta\nu(a_e)$	5 170.926(1) kHz
$\Delta\nu$(QED3)	−899.557 kHz
$\Delta\nu$(QED4)	−0.55(22) kHz
$\Delta\nu$(weak)	−0.065 kHz
$\Delta\nu$(had −vp)	0.233(3) kHz
$\Delta\nu$(had −h.o.)	0.007(2) kHz
ν_{HFS}(theor)	4 463 302.913(511)(34)(220) kHz
ν_{HFS}(exp)	4 463 302.776(51) kHz

In the present paper we have studied the limits of possible bound-state QED tests with muonium hyperfine structure. Given the recent progress in multiloop QED calculations [22,5], one can expect that higher-order radiative effects will be evaluated when warranted by the experimental precision. However, the ultimate accuracy of every theoretical prediction will be limited by the knowledge of the hadronic effects. Here, we have studied the leading order vacuum polarization effects and found that they can be calculated with 1.3% (or 3.1 Hz) accuracy. Higher order corrections can modify this contribution by a few percent. In the leading logarithmic approximation, they were found to increase the lowest order hadronic contribution by about 3% [20]. In the case of the muon anomalous magnetic moment, this logarithmic correction is not dominant [71]. At present, it is not clear how good this analogy is and how reliable the logarithmic approximation is in the case of the muonium hyperfine splitting. When the experimental accuracy is improved, the higher order hadronic effects will have to be scrutinized. The evaluation of the diagrams with the vacuum polarization is straightforward, since only moderate relative accuracy is sufficient. The theoretical accuracy will be limited by the knowledge of the hadronic vacuum polarization in the leading order diagram, and by the ability to compute the hadronic light-by-light scattering diagrams which at present is possible only within models of low-energy hadronic interactions.

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