Conformal black hole solutions of axidilaton gravity in D dimensions

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Static, spherically symmetric solutions of axidilaton gravity in D dimensions are given in the Brans-Dicke frame for arbitrary values of the Brans-Dicke constant ω and an axion-dilaton coupling parameter k. The mass and the dilaton and axion charges are determined and a BPS bound is derived. There exists a one-parameter family of black hole solutions in the scale-invariant limit.

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I. INTRODUCTION

It is an exciting conjecture that all superstring models belong to a hypothetical eleven-dimensional M theory that would accommodate their apparent dualities. M theory as a classical theory may be considered in a low-energy limit where only the low-lying massless excitation modes contribute to an effective field theory. As such, it would be the same as simple eleven-dimensional supergravity theory. A subsequent Kaluza-Klein reduction would bring it to a ten-dimensional theory related with the type-IIA string model whose gravitational sector consists of the space-time metric tensor g, dilaton scalar ϕ, and the axion potential $(p+1)$ form A that would minimally couple to p branes. We call such an effective gravitational field theory an axidilaton gravity in D dimensions and consider in the following its static, spherically symmetric solutions for $p=D-4$.

The study of black-hole solutions of higher-dimensional gravity theories started in 1963 with the generalization of Schwarzschild and Reissner-Nordström solutions to $D>4$ dimensions by Tangherlini [1]. These solutions were later put in a wider context by Myers and Perry [2], while Gibbons and Maeda [3] emphasized the relevance of dilaton scalars for the interpretation of such solutions. They provided a wide range of static, spherically symmetric solutions of the coupled Einstein-antisymmetric tensor-massless scalar field equations (see also [4,5]). On the other hand, it is a well known fact that the scalar-tensor Brans-Dicke theory [6] may be rewritten in terms of a conformally rescaled metric as the coupled Einstein-massless scalar field theory [7–9]. For a particular value of the Brans-Dicke coupling parameter, namely for $\omega=-\frac{3}{2}$ in four dimensions, the theory becomes locally scale invariant and called the Einstein-conformal scalar field theory. We showed in a previous work that the conformal rescaling properties of the Brans-Dicke theory can be conveniently exhibited using the non-Riemannian reformulation involving space-time torsion expressed in terms of the gradient of the scalar field [10]. Brans-Dicke theory has also been generalized to D dimensions [11] and the black-hole solutions of the Brans-Dicke-Maxwell field equations were given [12,13].

In a remarkable paper, Bekenstein [14] found two classes of static, spherically symmetric solutions of the Einstein-conformal scalar field equations, and he argued [15] that one particular class describes black-hole solutions with scalar hair. His arguments were later repeated in $D>4$ dimensions [16]. It is essential here to note that such a subclass of conformal black-hole solutions cannot be reached by the assumptions of Ref. [3]. In this paper, we consider axidilaton gravity in D dimensions $(p=D-4)$ in the Brans-Dicke frame and give its static, spherically symmetric solutions for arbitrary values of two coupling parameters ω and k. A one-parameter family of conformal black-hole solutions is obtained for $\omega=(D-1)/(D-2)$ and $k=-(D-4)/(D-2)$.

II. AXIDILATON GRAVITY IN D DIMENSIONS

The dynamics of the axidilaton gravity will be determined by a variational principle from the action $I[e,\omega,\phi,A]=\int\mathcal{L}$, where the Lagrangian density D form is taken in the Brans-Dicke frame as

$$\mathcal{L}=\frac{\phi}{2}R_{ab}\wedge*(e^b\wedge e^b)-\frac{\omega}{2\phi}d\phi\wedge*d\phi-\frac{\phi^k}{2}H\wedge*H. \tag{2.1}$$

Here the basic gravitational field variables are the coframe 1-forms e^a, in terms of which the space-time metric $g=\eta_{ab}e^a\otimes e^b$, where $\eta_{ab}=\mathrm{diag}(-+++\cdots)$. The Hodge $*$ map is defined so that the oriented volume form $*1=e^0\wedge e^1\wedge\cdots e^n$. The metric compatible torsion-free connection 1-forms ω^a_b are obtained from the Cartan structure equations

$$de^a+\omega^a_b\wedge e^b=0 \tag{2.2}$$

and the corresponding curvature 2-forms

$$R^a_b=d\omega^a_b+\omega^a_c\wedge\omega^c_b. \tag{2.3}$$

ϕ is the dilaton 0-form and H is a $(p+2)$-form field that is derived from the $(p+1)$-form axion potential A such that $H=dA$. ω and k are real parameters.

The field equations obtained from this action are

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$$-\frac{\phi}{2}R^{bc}\wedge*(e_a\wedge e_b\wedge e_c)=\frac{\omega}{\phi}\tau_a[\phi]+\phi^k\tau_a[H]+D(\iota_a*d\phi), \tag{2.4}$$

$$\tilde{k}d*d\phi=\frac{\alpha}{2}\phi^k H\wedge*H, \tag{2.5}$$

$$d(\phi^k*H)=0,\quad dH=0, \tag{2.6}$$

where the dilaton and axion stress-energy $(D-1)$ forms are given by

$$\tau_a[\phi]=\tfrac{1}{2}(\iota_a d\phi\wedge*d\phi+d\phi\wedge\iota_a*d\phi), \tag{2.7}$$

$$\tau_a[H]=\tfrac{1}{2}[\iota_a H\wedge*H+(-)^{p-1}H\wedge\iota_a*H], \tag{2.8}$$

respectively. We set $\alpha=k+\{[2p-(n-3)]/(n-1)\}$ and $\tilde{k}=\omega+[n/(n-1)]$.

The same action may be rewritten in terms of the $(D-p-2)$-form field

$$G\equiv\phi^k*H \tag{2.9}$$

that is dual to the axion $(p+2)$-form field H. We have, in terms of G,

$$\mathcal{L}=\frac{\phi}{2}R_{ab}\wedge*(e^a\wedge e^b)-\frac{\omega}{2\phi}d\phi\wedge*d\phi+\frac{\phi^{-k}}{2}G\wedge*G. \tag{2.10}$$

Hence given any solution $\{g,\phi,H\}$ of the field equations derived from Eq. (2.1), we may write down a dual solution $\{g,\phi,G\}$ to the field equations derived from Eq. (2.10). This notion of duality generalizes the usual electric-magnetic duality in $D=4$ source-free electromagnetism.

Finally, we wish to point out that the passage to the Einstein frame is achieved by the following conformal rescaling of the field variables:

$$\tilde{g}=\phi^{2/(n-1)}g,\quad \tilde{\phi}=\tilde{k}^{1/2}\ln\phi,\quad \tilde{H}=H. \tag{2.11}$$

The resulting Lagrangian density D form will be

$$\mathcal{L}=\tfrac{1}{2}\tilde{R}_{ab}\wedge\tilde{*}(\tilde{e}^a\wedge\tilde{e}^b)-\tfrac{1}{2}d\tilde{\phi}\wedge\tilde{*}d\tilde{\phi}-\tfrac{1}{2}\exp\left(\frac{\alpha}{\tilde{k}^{1/2}}\tilde{\phi}\right)\tilde{H}\wedge\tilde{*}\tilde{H}. \tag{2.12}$$

Given the above information, it is not difficult to compare solutions obtained in the Brans-Dicke frame with those given in the Einstein frame.

III. STATIC, SPHERICALLY SYMMETRIC SOLUTIONS

We will be giving below the most general static, spherically symmetric $p=(D-4)$ brane solution to the field equations (2.4)–(2.6). This family of solutions generalizes the usual magnetically charged Reissner-Nordström black-hole solution in $D=4$ to higher dimensions in a natural way. To this end, we start with the ansatz

$$g=-f^2(r)dt\otimes dt+h^2(r)dr\otimes dr+R^2(r)d\Omega_{n-1} \tag{3.1}$$

for the metric tensor $(D=n+1)$, $\phi=\phi(r)$ for the dilaton 0-form, and $H=g(r)e^1\wedge e^2\wedge e^3\cdots\wedge e^{n-1}$ for the axion field $(D-2)$ form. We set $e^0=f(r)dt$ and $e^n=h(r)dr$. Then the Einstein field equations reduce to the following set of ordinary coupled differential equations (a prime denotes a partial derivative with respect to r):

$$\phi\left[\frac{(n-2)(n-1)h}{2R^2}\left[1-\left(\frac{R'}{h}\right)^2\right]-\frac{(n-1)}{R}\left(\frac{R'}{h}\right)'\right]=\frac{\omega}{2\phi}\left(\frac{\phi'^2}{h}\right)+\frac{\phi^k}{2}g^2h+\left(\frac{\phi'}{h}\right)'+(n-1)\frac{\phi'R'}{hR}, \tag{3.2}$$

$$\phi\left\{\frac{(n-2)f'R'}{hR}+\left(\frac{f'}{h}\right)'+\frac{(n-2)f}{R}\left(\frac{R'}{h}\right)'-\frac{(n-3)(n-2)fh}{2R^2}\left[1-\left(\frac{R'}{h}\right)^2\right]\right\}=-\frac{\omega f}{2\phi h}\phi'^2+\frac{\phi^k g^2 fh}{2}-\left(\frac{\phi' f}{h}\right)'-(n-2)\frac{\phi' f}{hR}, \tag{3.3}$$

$$\phi\left[\frac{(n-1)(n-2)f}{2R^2}\left[1-\left(\frac{R'}{h}\right)^2\right]-(n-1)\frac{f'R'}{h^2R}\right]=-\frac{\omega f\phi'^2}{2\phi h^2}+\frac{g^2f\phi^k}{2}+\frac{f'\phi'}{h^2}+(n-1)\frac{fR'\phi'}{h^2R}, \tag{3.4}$$

while the dilaton field equation becomes

$$\tilde{k}\left(\phi'\frac{f}{h}R^{n-1}\right)'=\frac{\alpha}{2}\phi^k g^2 fhR^{n-1} \tag{3.5}$$

and the axion field equation reads

$$(gR^{n-1})'=0. \tag{3.6}$$

Solutions to the above field equations can be written as

$$\begin{aligned}
R(r)&=r\left[1-\left(\frac{C_1}{r}\right)^{n-2}\right]^{\alpha_3},\\
f(r)&=\left[1-\left(\frac{C_2}{r}\right)^{n-2}\right]^{\alpha_4}\left[1-\left(\frac{C_1}{r}\right)^{n-2}\right]^{\alpha_5},\\
h(r)&=\left[1-\left(\frac{C_2}{r}\right)^{n-2}\right]^{\alpha_2}\left[1-\left(\frac{C_1}{r}\right)^{n-2}\right]^{\alpha_1},\\
\phi&=\left[1-\left(\frac{C_1}{r}\right)^{n-2}\right]^{2\gamma/\alpha},\\
g(r)&=\frac{Q}{R^{n-1}},
\end{aligned} \tag{3.7}$$

where C_1 and C_2 are two independent integration constants and the third integration constant

$$Q^2=\frac{(n-1)(n-2)}{1+\frac{\alpha^2}{4k}\left(\frac{n-1}{n-2}\right)}(C_1C_2)^{n-2}.$$

The exponents are

$$\alpha_1=\gamma\left(\frac{1}{(n-2)}-\frac{2}{(n-1)\alpha}\right)-\frac{1}{2},\quad \alpha_2=-\frac{1}{2},$$

$$\alpha_3=\gamma\left(\frac{1}{(n-2)}-\frac{2}{(n-1)\alpha}\right),\quad \alpha_4=\tfrac{1}{2},$$

$$\alpha_5=-\gamma\left(1+\frac{2}{(n-1)\alpha}\right)+\frac{1}{2},$$

with

$$\gamma=\frac{1}{1+\frac{4\tilde{k}}{\alpha^2}\left(\frac{n-2}{n-1}\right)}.$$

Some special cases deserve attention.

(i) For $Q=0$ and $\phi=\text{const}$, we obtain the Tangherlini solution [1], which is a generalization of the Schwarzschild solution in $D=n+1$ dimensions,

$$g=-\left(1-\frac{2M}{r^{n-2}}\right)dt^2+\left(1-\frac{2M}{r^{n-2}}\right)^{-1}dr^2+r^2d\Omega_{n-1}. \tag{3.8}$$

(ii) For $k=0$ and $\phi=\text{const}$, we obtain the $(D=n+1)$-dimensional generalization of the Reissner-Nordström metric

$$g=-\left(1+\frac{Q^2}{(n-1)(n-2)r^{2(n-2)}}-\frac{2M}{r^{n-2}}\right)dt^2+\left(1+\frac{Q^2}{(n-1)(n-2)r^{2(n-2)}}-\frac{2M}{r^{n-2}}\right)^{-1}\times dr^2+r^2d\Omega_{n-1}. \tag{3.9}$$

The electric dual of this solution was also given by Tangherlini.

(iii) For $Q=0$, we obtain solutions that generalize the Janis-Newman-Winicour solutions of the Einstein-massless scalar field equations to D dimensions [4]:

$$R(r)=rh(r),$$

$$f(r)=\left(\frac{r^{n-2}-r_0^{n-2}}{r^{n-2}+r_0^{n-2}}\right)^{\beta_1-\beta_2},$$

$$h(r)=\left[1-\left(\frac{r_0}{r}\right)^{2(n-2)}\right]^{1/(n-2)}\left(\frac{r^{n-2}-r_0^{n-2}}{r^{n-2}+r_0^{n-2}}\right)^{-\beta_1/(n-2)-\beta_2},$$

$$\phi(r)=\left(\frac{r^{n-2}-r_0^{n-2}}{r^{n-2}+r_0^{n-2}}\right)^{\beta_2}, \tag{3.10}$$

where in order to ease comparison, we use the parametrization

$$\beta_2=\sqrt{\frac{4(n-1)}{(n-2)\tilde{k}}(4-\beta_1^2)}$$

and β_1 satisfies $4(n-2)r_0^{n-2}\beta_1=C$, where r_0 and C are integration constants.

A consideration of the asymptotic behavior of the fields in the Brans-Dicke frame will allow us to determine a relationship satisfied by the mass, dilaton charge, and magnetic charge Q. The mass of the black hole is defined to be

$$2M\equiv\lim_{r\to\infty}r^{n-2}(1-f^2)=(C_2)^{n-2}+(\tilde{\gamma}-2\gamma)(C_1)^{n-2}, \tag{3.11}$$

where $\tilde{\gamma}=1-[4\gamma/(n-1)\alpha]$. The scalar charge

$$\Sigma\equiv\lim_{r\to\infty}r^{n-1}\frac{\phi'}{\phi}=2(n-2)\frac{\gamma}{\alpha}(C_1)^{n-2}. \tag{3.12}$$

Finally, the magnetic charge can be found from

$$Q\equiv\lim_{r\to\infty}r^{n-1}g=Q. \tag{3.13}$$

Therefore, by eliminating the integration constants C_1 and C_2 above, we can find the following relationship between these three physical parameters:

$$Q^2=\frac{2(n-2)\Sigma}{\alpha}\tilde{k}\left[(2\gamma-\tilde{\gamma})\frac{\alpha\Sigma}{2(n-2)\gamma}+2M\right]. \tag{3.14}$$

From this relationship, since Σ is a real parameter, the BPS bound respected by the mass and charge of a black hole follows after some algebra:

$$(n-1)(n-2)M\geqslant\sqrt{\frac{1+\frac{n-2}{n-1}\omega-\left(\frac{k-1}{2}\right)^2}{\omega+\frac{n}{n-1}}}|Q| \tag{3.15}$$

provided

$$\left(\frac{k-1}{2}\right)^2\leqslant\frac{n-2}{n-1}\omega+1. \tag{3.16}$$

IV. CONCLUSION

A conformally scale-invariant theory (2.1) is obtained for the parameter values $\omega=-[n/(n-1)]$ and $k=-[(n-3)/(n-1)]$. A class of static, spherically symmetric solutions to the conformally scale invariant theory may be reached from the solutions (3.7) above by taking the limit $\alpha\to 0$ and $\tilde{k}\to 0$ with the ratio $\tilde{k}/\alpha$ kept fixed:

$$R(r)=r\left[1-\left(\frac{C_1}{r}\right)^{n-2}\right]^{-\beta/(n-1)},$$

$$f(r)=\left[1-\left(\frac{C_2}{r}\right)^{n-2}\right]^{1/2}\left[1-\left(\frac{C_1}{r}\right)^{n-2}\right]^{1/2-[\beta/(n-1)]},$$

$$h(r)=\left[1-\left(\frac{C_2}{r}\right)^{n-2}\right]^{-1/2}\left[1-\left(\frac{C_1}{r}\right)^{n-2}\right]^{-1/2-[\beta/(n-1)]}, \tag{4.1}$$

$$\phi=\left[1-\left(\frac{C_1}{r}\right)^{n-2}\right]^{\beta},$$

$$g(r)=\frac{Q}{R^{n-1}},$$

where C_1 and C_2 are constants and β and Q should satisfy

$$2\beta(n-2)^2(C_1C_2)^{n-2}=Q^2.$$

We also verified this solution directly by substituting into the scale-invariant field equations. The special case of parameter values $Q=0$ and $C_2=0$ in $D=4$ dimensions brings Eqs. (4.1) to Bekenstein's Einstein-conformal scalar solution [14]. The fact that this solution describes black holes was later clarified by Bekenstein [15]. His argument is based on the observation that the scalar particles being postulated to follow geodesic world lines in Brans-Dicke theory [17] presupposes that the scalar field does not couple directly to matter. On the other hand, by assuming a different type of scalar field coupling to matter, one can show that neutral test particles follow conformal world lines as argued by Gürsey [18] and Dirac [19]. With this assumption, Bekenstein was able to verify that solution (4.1) describes a black hole with finite scalar charge. It is now known that the conformal world lines are merely autoparallel curves in a non-Riemannian reformulation of the Brans-Dicke theory [20]. A further reevaluation of the locally scale-invariant solutions above from the non-Riemannian point of view will be taken up in a separate study.

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