Comments on conformal stability of brane-world models

Mariam Bouhmadi-López*
Instituto de Matemáticas y Física Fundamental, CSIC, C/ Serrano 121, 28006 Madrid, Spain

Alexander Zhuk†
Instituto de Matemáticas y Física Fundamental, CSIC, C/ Serrano 121, 28006 Madrid, Spain
and Department of Physics, University of Odessa, 2 Dvoryanskaya St., Odessa 65100, Ukraine
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The stability of five-dimensional (5D) brane-world models under conformal perturbations is investigated. The analysis is carried out in the general case and then it is applied to particular solutions. It is shown that models with the Poincaré and de Sitter branes are unstable because they have a negative mass squared of gravexcitons whereas models with the anti–de Sitter branes have a positive gravexciton mass squared and are stable. It is also shown that 4D effective cosmological and gravitational constants on branes as well as gravexciton masses undergo a hierarchy: they have different values on different branes.

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I. INTRODUCTION

It is well known that part of any realistic multidimensional model should be a mechanism for extra dimension stabilization. This problem was the subject of numerous investigations. In the standard Kaluza-Klein approach, cosmological models are taken in the form of a warped product of Einstein spaces as internal spaces. The corresponding warp (scale) factors are assumed to be functions of external (our) space-time. If these scale factors are dynamical functions, then it results in a variation of the fundamental physical constants. To be in agreement with observations, internal spaces should be compact, static (or nearly static), and less or of the order of the electroweak scale (the Fermi length). The stability problem of these models with respect to conformal perturbations of the internal spaces was considered in detail in our paper [1]. It was shown that stability can be achieved with the help of an effective potential of a dimensionally reduced effective four-dimensional (4D) theory. Small conformal excitations of the internal spaces near the minima of the effective potential have the form of massive minimal scalar fields developing in the external space-time. These particles were called gravitational excitons (gravexcitons). Their physical meaning can be easily explained with the help of a simple 3D model where the 2D spatial part has the cylindrical topology: $S^1 \times R^1$. Here, S^1 plays the role of the compact internal space and R^1 describes 1D external space. Let us suppose that the size of S^1 is stabilized near some value by an effective potential. Then, conformal excitation of S^1 near its equilibrium position results in waves running along the cylinder (along R^1). Thus, any 1D observer living on the cylinder (on R^1) will detect these oscillations as massive scalar fields. Obviously, this effect takes place for any multidimensional cosmological model with compact internal spaces. In general, it does not depend on the presence or absence of branes in models. Masses of gravexcitons and equilibrium positions for the internal spaces depend on the form of the effective potential (on concrete topology and matter content of the model).

Recently [2–4], it was realized that it is not necessary for extra dimensions to be very small. They can be enlarged up to submillimeter scales in such a way that the standard model fields are localized on a 3-brane with thickness of the electroweak (or less) length in the extra dimensions, whereas the gravitational field can propagate in all multidimensional (bulk) space. This gives the possibility for lowering of the multidimensional fundamental gravitational constant down to the TeV scales (therefore this approach is often called the TeV gravity approach). Cosmological models in this approach are topologically equivalent to the standard Kaluza-Klein one. Problems of their stability against conformal perturbations of additional dimensions were considered in [4,5]. A comparison of old and new approaches from the point of view of conformal stability was given in [6]. In [4,5], conformal excitations of additional dimensions near the minimum position of the effective potential were called radions (to our knowledge, the first time that the term radion appeared was in [4]). However, we prefer to call such particles gravexcitons first from the point of priority, and second, and most importantly, the term radion is widely used now in the brane-world models in different contexts.

The brane-world models are motivated by the strongly coupled regime of $E_8 \times E_8$ heterotic string theory, which is interpreted as M theory on an orbifold $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$ with a set of E_8 gauge fields at each ten-dimensional orbifold fixed plane. After compactification on a Calabi-Yau threefold and dimensional reduction, one arrives at effective five-dimensional solutions which describe a pair of parallel 3-branes with opposite tension, and located at the orbifold planes [7]. For these models, the five-dimensional metric contains a four-dimensional metric component multiplied by a warp factor which is a function of the additional dimension. A cosmological solution of this type with flat 4D branes (which we shall refer to as Poincaré branes) was obtained in [8]. This model was generalized in numerous publications to the cases of bent branes in models with five or more dimen-

*Electronic address: mbouhmadi@imaff.cfmac.csic.es
†Electronic address: ai_zhuk@imaff.cfmac.csic.es

sions and with single or many branes. In [9], the necessity of stabilization in the distance between branes to get conventional cosmology on branes was stressed. Here, the radius (scale factor) of the extra dimension was called a radion. But this definition is not very precise. We should note that there is a confusion in the literature concerning the term radion: quite different forms of the metric perturbations of the brane-world models were called radions. However, in [10] it was clearly shown and strictly emphasized that radions describe the relative distance between branes (see also [11,12]). It demonstrates the main difference between gravexcitons and radions: gravexcitons describe conformal excitations of geometry (in particular, conformal excitations of the additional dimensions), whereas radions describe relative motion of branes. Obviously, gravexcitons can exist in a model where branes are absent at all or in models with a single brane, and conversely, radions can exist in the absence of gravexcitons. The latter situation can be realized, for example, in the TeV scale approach where branes can move relatively with respect to each other due to interaction between them ''sliding'' on the background fixed geometry (gravexcitons are absent). Branes are considered here as ''probe bodies'' moving in the background geometry. Nevertheless, in the brane-world models, gravexcitons and radions are closely connected with each other (and this is the main reason for the confusion between them). Here, branes are 4D surfaces along which different 5D bulk solutions are gluing with each other. In this case, the positions of branes fix the shape and size of the geometry,[1] and the relative motion of branes results in conformal changes of the geometry. Thus, it is natural for such models to expect that solutions stable against radions are stable against gravexcitons and vice versa. Obviously, direct comparison of stability against gravexcitons and radions only makes sense in models with two (and more) branes where radions exist.

The radion stabilization problem was investigated in a number of papers devoted to the brane-world models. It was shown, in particular, that in the case of solutions with two de Sitter branes (bent branes with 4D effective positive cosmological constants) and two anti–de Sitter branes (bent branes with 4D effective negative cosmological constants), radions have negative and positive mass squared, respectively. Thus, the former solution is unstable but the latter one is stable against radions. In our paper, we find, first, that the model with one de Sitter brane is unstable against gravexcitons and, second, that the model with one or a number of parallel anti–de Sitter branes (connected with each other via wormhole throats) is stable under conformal excitations. In the case of the Randall-Sundrum solution [8] with two Poincaré branes, radions have zero mass [11,12]. From the particle physics point of view, such particles do not lead to instability. However, as is well known, such ultralight scalar fields, originating from the extra dimensions, produce a number of cosmological problems connected with the flatness of their effective potential. For example, in the homogeneous case, any small excitations near an equilibrium position (which can be chosen arbitrary for the flat potential) have a runaway behavior if we do not take into account the friction term due to the cosmological expansion. However, such dynamical stabilization is a very delicate problem (see, e.g., [13]) and needs a separate investigation for each case. Exactly this kind of instability for radions in the model with two Poincaré branes was found in [11]. It was shown here that a small departure from the equilibrium position results in either a colliding of the two branes or a runaway behavior. In our paper, we show that models with two as well as one Poincaré brane are also unstable under conformal perturbations. It is necessary to stress that in [11,12], the analysis was performed in the Brans-Dicke frame. In our paper, the problem of stability is investigated in the Einstein frame. Obviously, if models are stable in one frame, they are stable in another one because both frames coincide in the equilibrium position. However, the exact form of the dynamical behavior (time dependence) near the equilibrium position can depend on the frame. In paper [14], the role of conformal transformations is explicitly discussed, and it is shown that some solutions of the brane-world models exist in one frame but are absent in the other one. Therefore, the equivalence between these two frames depends on the concrete discussed problem and in some cases is a matter of delicate investigation.

In papers [11,12], mentioned above, the authors' conclusions concerning the radion stability or instability were obtained for the brane-world models where matter in bulk as well as on branes is absent (more precisely, it is considered there in its simplest form as a bulk cosmological constant and ''vacuum energies'' on branes). In this case, only the pair of anti–de Sitter branes are really stable. However, it was observed that inclusion of matter can stabilize radions for different types of branes. This can be done with the help of bulk scalar field [9,15–17], perfect fluid on branes [18], and the Casimir effect between branes [19,20].

Some specific forms of instability in the brane-world solutions were observed in papers [21,22]. It was shown that a single Poincaré brane is unstable under small perturbations of the brane tension[2] [21] and a single de Sitter brane is unstable against thermal radiation [22].

The main goal of our present comments consists in the investigation of 5D brane-world stability against conformal perturbations. First, we elaborate a method to study the stability for a large class of solutions and obtain general expressions for 4D effective cosmological constants on branes and masses of gravitational excitons. Then we apply this method to a number of well-known solutions. In particular, we find that models with the Poincaré and the de Sitter branes are unstable because they have negative mass squared of gravexcitons, whereas models with the anti–de Sitter branes have positive gravexciton mass squared and are stable under con-

[1]In our paper, we shall consider the case of compact with respect to additional dimension brane-world models.

[2]Here it was mentioned about instability of the single brane Randall-Sundrum solution under homogeneous gravitational perturbations. In our paper, we show that this model is unstable also under conformal excitations.

formal perturbations. We show also that 4D effective cosmological and gravitational constants on branes as well as gravexciton masses undergo a hierarchy: they have different values on different branes (if different branes have different warp factors).

The paper is organized as follows. In Sec. II, we explain the general setup of our model, perform dimensional reduction of the brane-world models to an effective 4D theory in a general case, and apply this procedure to a number of well-known solutions. In Sec. III, we elaborate on a method of the investigation of the brane-world solution stability against conformal perturbations and apply it to particular solutions considered in Sec. II. Here we show also that physical masses of gravexcitons undergo hierarchy on different branes. The brief conclusions of the paper are followed by three Appendixes. In Appendix A, we present useful expressions for the Ricci tensor components and scalar curvature in the case of block-diagonal metrics. Some useful formulas of the conformal transformation are summarized in Appendix B. In Appendix C, we show that the results of the paper do not change if only an additional dimension undergoes conformal perturbations: we arrive here at the same 4D effective theory and the same gravexciton masses as in the case of total geometry conformal perturbations. This provides an interesting analogy between gravity and an elastic media where the eigenfrequencies of elastic body oscillations do not depend on the manner of excitation.

II. MODEL AND GENERAL SETUP: DIMENSIONAL REDUCTION OF BRANE-WORLD MODELS

We consider 5D cosmological models on a manifold $M^{(5)}$ which is divided on n pieces by $n-1$ branes: $M^{(5)} = \cup_{i=1}^{n} M_i^{(5)}$. Branes are 4D hypersurfaces $r = r_i = \text{const}$, $i = 1, \dots, n-1$, where r is an extra dimension. Each brane is characterized by its own tension $T_i(r_i)$, $i = 1, \dots, n-1$. We suppose that a boundary $\partial M^{(5)}$ also corresponds to two hypersurfaces $r = \text{const}$, $r = r_0$ and $r = r_n$, and either 4D geometry on $\partial M^{(5)}$ is closed (induced 4D metric vanishes there) or opposite points r_0 and r_n are identified with each other. In the first case, boundary terms corresponding to $\partial M^{(5)}$ are equal to zero. In the second case, the boundary $\partial M^{(5)}$ is absent, however if the geometry is not smoothly matched here, it results in the appearance of an additional brane with a tension $T_0(r_0)$. For simplicity, bulk matter is considered in the form of a cosmological constant, in general different for each of $M_i^{(5)}$. Thus, our model is described by the following action:

$$S^{(5)} = \frac{1}{2\kappa_5^2} \int_{M^{(5)}} d^5X \sqrt{|g^{(5)}|} (R[g^{(5)}] - 2\Lambda_5(r)) + S_{\text{YGH}} - \sum_{i=0}^{n-1} T_i(r_i) \int d^4x \sqrt{|g^{(4)}|}\bigg|_{r_i}, \tag{2.1}$$

where $S_{\text{YGH}} = -\kappa_5^{-1} \int_{\partial M^{(5)}} d^4x \sqrt{|g^{(4)}|}\, K$ is the standard York-Gibbons-Hawking (YGH) boundary term.[3] The Einstein equation corresponding to action (2.1) reads

$$R_{MN}[g^{(5)}] - \tfrac{1}{2} g_{MN}^{(5)} R[g^{(5)}] = -\Lambda_5(r) g_{MN}^{(5)} - \frac{\kappa_5^2}{\sqrt{|g^{(5)}|}} \times \sum_{i=0}^{n-1} T_i(r_i) \sqrt{|g^{(4)}(x,r_i)|}\, g_{\mu\nu}^{(4)}(x,r_i)\, \delta_M^{\mu} \delta_N^{\nu} \delta(r - r_i). \tag{2.2}$$

In Eqs. (2.1) and (2.2),

$$\Lambda_5(r) := \sum_{i=1}^{n} \Lambda_i \theta_i(r), \quad \Lambda_i = \text{const}, \tag{2.3}$$

with piecewise discontinuous functions

$$\theta_i(r) = \eta(r - r_{i-1}) - \eta(r - r_i) = \begin{cases} 0, & r < r_{i-1}, \\ 1, & r_{i-1} \le r < r_i, \\ 0, & r \ge r_i, \end{cases} \tag{2.4}$$

where step functions $\eta(r - r_i)$ are equal to zero for $r < r_i$ and unity for $r \ge r_i$.

Now, we suppose that a metric[4]

$$g^{(5)}(X) = g_{MN}^{(5)} dX^M \otimes dX^N = dr \otimes dr + a^2(r) \gamma_{\mu\nu}^{(4)}(x) dx^{\mu} \otimes dx^{\nu}, \tag{2.5}$$

$$a(r) = \sum_{i=1}^{n} a_i(r) \theta_i(r)$$

is the solution of the Einstein equation (2.2) and has the following matching conditions: $a_i(r_i) = a_{i+1}(r_i)$, $i = 1, \dots, n-1$ and $a_1(r_0) = a_n(r_n)$. Scale factors $a_i(r)$ are supposed to be non-negative smooth functions in intervals $[r_{i-1}, r_i]$. Boundary points r_0 and r_n are either identified with each other, $r_0 \leftrightarrow r_n$, or they are not identified and the geometry in the latter case is closed, $a_1(r_0) = a_n(r_n) = 0$, i.e., the induced metric $g_{\mu\nu}^{(4)}(x,r) = a^2(r) \gamma_{\mu\nu}^{(4)}(x)$ vanishes in these points.

Having at hand solution (2.5), we can perform a dimensional reduction of action (2.1). Here, the dimensional reduction means an integration over extra dimensions in the 5D

[3]In compact brane-world models, it is worth while to include this term even if the boundary $\partial M^{(5)}$ is absent because it is convenient here (as well as for all models with branes) to split manifold $M^{(5)}$ by branes into n submanifolds: $M^{(5)} = \cup_{i=1}^{n} M_i^{(5)}$. Each of them has boundaries $\partial M_i^{(5)}$ defined by positions of the branes. Such boundary terms at $\partial M_i^{(5)}$ take into account the presence of the branes and are needed in order to satisfy the variational principle and the junction conditions on the branes [23]. These junction conditions coincide with the ones following directly from the Einstein equation (2.2).

[4]Different parts of the manifold $M^{(5)}$ can be covered by different coordinates charts. We show an explicit example below.

part of action (2.1) to get 4D effective action. To do that, let us perform first some preliminary calculations.

Applying Eq. (A3) to our case, we obtain

$$R[g^{(5)}]=a^{-2}(r)R[\gamma^{(4)}]-f_1(r), \tag{2.6}$$

where

$$f_1(r):=8\frac{a''}{a}+12\left(\frac{a'}{a}\right)^2. \tag{2.7}$$

Using properties of the θ function—$\theta_i^p=\theta_i$, $p>0$; $\theta_i\theta_j=0$, $i\neq j \Rightarrow a^p=\sum_{i=1}^n a_i^p\theta_i$, $\forall p$, and $\theta_i'=\delta(r-r_{i-1})-\delta(r-r_i)$—the function $f_1(r)$ can be written in the following form:

$$f_1(r)=12\sum_{i=1}^n\frac{(a_i')^2}{a_i^2}\theta_i+8\sum_{i=1}^n\frac{a_i''}{a_i}\theta_i-2\Bigg[K(r_0^+)\delta(r-r_0)-K(r_n^-)\delta(r-r_n)+\sum_{i=1}^{n-1}\hat{K}(r_i)\delta(r-r_i)\Bigg], \tag{2.8}$$

where $\hat{K}(r_i)=K(r_i^+)-K(r_i^-)$, $K(r_i^+)=-4a_{i+1}'/a_{i+1}|_{r_i^+}$, and $K(r_i^-)=-4a_i'/a_i|_{r_i^-}$ in accordance with Eq. (B6). As we can see, the function f_1 contains all the information about boundary terms, and for correct dimensional reduction of action (2.1), it is not necessary to include additional boundary term S_{YGH} because it will lead in this case to double counting.[5] It can be easily seen also that the integral

$$\int_{r_0}^{r_n}dr\,a^4(r)f_1(r)=-12\sum_{i=1}^n\int_{r_{i-1}}^{r_i}dr\,a_i^2(a_i')^2, \tag{2.9}$$

where we used integration by parts. Thus, dimensional reduction of action (2.1) will result in the following effective 4D action:

$$S_{\mathrm{eff}}^{(4)}=\frac{1}{2\kappa_4^2}\int_{M^{(4)}}d^4x\sqrt{|\gamma^{(4)}|}\{R[\gamma^{(4)}]-2\Lambda_{\mathrm{eff}}^{(4)}\}, \tag{2.10}$$

where the effective 4D cosmological constant is

$$\Lambda_{\mathrm{eff}}^{(4)}=\frac{1}{B_0}[B_1+B_2+B_3] \tag{2.11}$$

and

$$B_0=\sum_{i=1}^n\int_{r_{i-1}}^{r_i}dr\,a_i^2, \tag{2.12}$$

$$B_1=-6\sum_{i=1}^n\int_{r_{i-1}}^{r_i}dr\,a_i^2(a_i')^2, \tag{2.13}$$

$$B_2=\sum_{i=1}^n\Lambda_i\int_{r_{i-1}}^{r_i}dr\,a_i^4, \tag{2.14}$$

$$B_3=\kappa_5^2\sum_{i=0}^{n-1}a_i^4(r_i)T_i(r_i). \tag{2.15}$$

An effective 4D gravitational constant is defined as follows:[6] $\kappa_4^2=\kappa_5^2/B_0$. Equation (2.10) shows that solution (2.5) of Eq. (2.2) takes place only if the 4D metric $\gamma^{(4)}$ is the Einstein space metric.

A. Examples

In this subsection, we apply the above-considered procedure of the dimension reduction to some well known solutions (see, e.g., [8,12,17,24,25]).

1. Poincaré branes

In this model,[7] $r_0=-L$, $r_1=0$, $r_2=L$,

$$\begin{aligned} a_1(r)&=\exp(r/l), \quad -L\leq r\leq 0,\\ a_2(r)&=\exp(-r/l), \quad 0\leq r\leq L, \end{aligned} \tag{2.16}$$

and bulk cosmological constants $\Lambda_1=\Lambda_2=-6/l^2$, where l is the AdS radius. The points r_0 and r_2 are identified with each other. A free parameter L defines the size of the models in the additional dimension. The geometry is not smooth at points $r=0$ and $r=r_0\equiv r_2$, thus we have two branes with tensions: $-T_0(r_0)=T_1(r_1)=6/(\kappa_5^2 l)$. Substituting concrete expressions into formulas (2.12)–(2.15), we obtain, respectively,

[5] There are two equivalent ways of the dimensional reduction. First, we can divide action integral (2.1) into n integrals in accordance with the splitting procedure described in footnote 3 and take into account the boundary terms at $\partial M_i^{(5)}$ arising due to the presence of branes. In this case, the scale factors $a_i(r)$ for each of the submanifolds $M_i^{(5)}$ are smooth functions and their derivatives do not result in δ functions. Here, the brane boundary terms are taken into account directly in the action functional. In the second approach, we consider full nonsplit action (2.1) without the brane boundary terms but take into account that the scale factor $a(r)$ is not a smooth function in points corresponding to the brane location. Thus, its second derivative has δ-function terms which are completely equivalent to the brane boundary terms [see Eq. (2.8)]. It can be easily checked that integration over extra dimensions in both of these approaches results in the same 4D effective action. In the present paper, we applied the second approach.

[6] In this paper, we focus on the problem of the stability of the considered models and we do not discuss cosmology on branes. It is clear that from the point of an observer on a brane, the physical metric is the induced metric on this brane (let it be the ith brane): $g_{(ph)\mu\nu}^{(4)}=a_i^2(r_i)\gamma_{\mu\nu}^{(4)}$. It means we should perform evident substitutions $a_j(r)\rightarrow a_j(r)/a_i(r_i)$ in corresponding formulas. For example, for this observer, physically effective 4D cosmological and gravitational constants read as follows: $\Lambda_{\mathrm{eff}}^{(4)}\rightarrow\Lambda_{(ph)\mathrm{eff}}^{(4)}=\Lambda_{\mathrm{eff}}^{(4)}/a_i^2(r_i)$ and $\kappa_4^2\rightarrow\kappa_{(ph)4}^2=\kappa_4^2a_i^2(r_i)$. The proportionality of the effective 4D Newton's constant on the brane to $a^2(r)|_{\mathrm{brane}}$ was pointed out, e.g., in [10].

[7] Here, we follow the original solution [8], where scale factors are dimensionless.

$$B_0 = l(1-e^{-2L/l})>0, \tag{2.17}$$

$$B_1 = \frac{3}{l}(e^{-4L/l}-1), \tag{2.18}$$

$$B_2 = \frac{3}{l}(e^{-4L/l}-1), \tag{2.19}$$

$$B_3 = \frac{6}{l}(1-e^{-4L/l}). \tag{2.20}$$

Thus, in this model

$$\Lambda_{\rm eff}^{(4)} \equiv 0 \tag{2.21}$$

and $\gamma_{\mu\nu}^{(4)}$ is a flat space-time metric. The Randall-Sundrum one-brane solution [24] corresponds to the trivial limit $L\to+\infty$ and also results in Eq. (2.21). In this case, the extra coordinate r runs over $\mathbb{R}$, but all integrals of the type (2.12)–(2.15) are convergent due to an exponential decrease of the warp factors $a_{1,2}$ when $|r|\to\infty$. Effectively, this model is compact with respect to the extra dimension.

2. de Sitter brane (symmetric solution)

In this model, $r_0=0,\ r_1=L,\ r_2=2L$,

$$\begin{aligned} a_1(r) &= l \sinh\frac{r}{l}, \quad 0\le r\le L,\\ a_2(r) &= l\sinh\frac{2L-r}{l}, \quad L\le r\le 2L, \end{aligned} \tag{2.22}$$

and bulk cosmological constants $\Lambda_1=\Lambda_2=-6/l^2$. In the points r_0 and r_2, the geometry is closed: $a_1(r_0)=a_2(r_2)=0$ (r_0 and r_2 are horizons of AdS_5). The geometry is not smooth in r_1. Therefore, in this model we have one brane with tensions: $T_1(r_1)=[6/(\kappa_5^2 l)]\coth(L/l)$. Substituting these expressions into formulas (2.12)–(2.15), we obtain, respectively,

$$B_0 = l^3\left(\frac{1}{2}\sinh\frac{2L}{l}-\frac{L}{l}\right)>0, \tag{2.23}$$

$$B_1 = -3l^3\left(\sinh^3\frac{L}{l}\cosh\frac{L}{l}+\frac{1}{4}\sinh 2\,\frac{L}{l}-\frac{1}{2}\,\frac{L}{l}\right), \tag{2.24}$$

$$B_2 = -3l^3\left(\sinh^3\frac{L}{l}\cosh\frac{L}{l}-\frac{3}{4}\sinh 2\,\frac{L}{l}+\frac{3}{2}\,\frac{L}{l}\right), \tag{2.25}$$

$$B_3 = 6l^3\sinh^3\frac{L}{l}\cosh\frac{L}{l}. \tag{2.26}$$

So, in this model,

$$\Lambda_{\rm eff}^{(4)} = 3 \tag{2.27}$$

and $\gamma_{\mu\nu}^{(4)}$ describes either a Riemannian 4-sphere with scalar curvature $R[\gamma^{(4)}]=D_0(D_0-1)=12$ or 4D de Sitter space-time with cosmological constant $\Lambda=3$ and scalar curvature $R[\gamma^{(4)}]=[2D_0/(D_0-2)]\Lambda=12$.

3. de Sitter brane (nonsymmetric solution)

We obtain this solution gluing together two submanifolds covered by different charts. The first submanifold describes the truncated Garriga-Sasaki instanton [25] and the second one describes flat 5D space:

$$\begin{aligned} a_1(r) &= l\sinh\frac{r}{l}, \quad 0\le r\le L,\\ a_2(R) &= R_0-R, \quad 0\le R\le R_0, \end{aligned} \tag{2.28}$$

where $R_0=l\sinh(L/l)$. Bulk cosmological constants $\Lambda_1=-6/l^2$ and $\Lambda_2\equiv 0$. In the points $r=0$ and $R=R_0$, the geometry is closed: $a_1(0)=a_2(R_0)=0$. The geometry is not smooth on the hypersurfaces of gluing $r=L$ and $R=0$. That is why we have one brane with tensions: $T_1|_{r=L,R=0}=3/\kappa_5^2[1/R_0+(1/l)\coth(L/l)]$. Substituting these expressions into formulas (2.12)–(2.15), we obtain, respectively,

$$B_0 = \frac{1}{2}l^3\left(\frac{1}{2}\sinh\frac{2L}{l}-\frac{L}{l}\right)+\frac{1}{3}R_0^3>0, \tag{2.29}$$

$$B_1 = -\frac{3}{2}l^3\left(\sinh^3\frac{L}{l}\cosh\frac{L}{l}+\frac{1}{4}\sinh 2\,\frac{L}{l}-\frac{1}{2}\,\frac{L}{l}\right)-2R_0^3, \tag{2.30}$$

$$B_2 = -\frac{3}{2}l^3\left(\sinh^3\frac{L}{l}\cosh\frac{L}{l}-\frac{3}{4}\sinh 2\,\frac{L}{l}+\frac{3}{2}\,\frac{L}{l}\right), \tag{2.31}$$

$$B_3 = 3l^3\sinh^3\frac{L}{l}\cosh\frac{L}{l}+3R_0^3. \tag{2.32}$$

Therefore, as in the symmetric case, in this model

$$\Lambda_{\rm eff}^{(4)} = 3 \tag{2.33}$$

and $\gamma_{\mu\nu}^{(4)}$ describes either 4-sphere or the de Sitter space with cosmological constant $\Lambda=3$.

4. Anti–de Sitter brane

In this model, $r_0=-L,\ r_1=0,\ r_2=L$,

$$\begin{aligned} a_1(r) &= l\cosh\frac{L+r}{l}, \quad -L\le r\le 0,\\ a_2(r) &= l\cosh\frac{L-r}{l}, \quad 0\le r\le L, \end{aligned} \tag{2.34}$$

and bulk cosmological constants $\Lambda_1=\Lambda_2=-6/l^2$. The points r_0 and r_2 are identified with each other. The geometry is not closed here and can be smoothly glued in these points. The points $r_{0,2}$ correspond to wormhole throats in the Riemannien space. The geometry is not smooth in r_1. Therefore, in this model we have only one brane[8] with tension: $T_1(r_1)=[6/(\kappa_5^2 l)]\tanh(L/l)$. Substituting these expressions into formulas (2.12)–(2.15), we obtain, respectively,

$$B_0=l^3\left(\frac{1}{2}\sinh\frac{2L}{l}+\frac{L}{l}\right)>0, \tag{2.35}$$

$$B_1=-3l^3\left(\sinh^3\frac{L}{l}\cosh\frac{L}{l}+\frac{1}{4}\sinh 2\frac{L}{l}-\frac{1}{2}\frac{L}{l}\right), \tag{2.36}$$

$$B_2=-3l^3\left(\sinh^3\frac{L}{l}\cosh\frac{L}{l}+\frac{5}{4}\sinh 2\frac{L}{l}+\frac{3}{2}\frac{L}{l}\right), \tag{2.37}$$

$$B_3=6l^3\left(\sinh^3\frac{L}{l}\cosh\frac{L}{l}+\frac{1}{2}\sinh 2\frac{L}{l}\right). \tag{2.38}$$

Thus, in this model,

$$\Lambda_{\rm eff}^{(4)}=-3 \tag{2.39}$$

and $\gamma_{\mu\nu}^{(4)}$ describes either a Riemannian 4-hyperboloid with scalar curvature $R[\gamma^{(4)}]=-D_0(D_0-1)=-12$ or 4D Anti de Sitter space-time with cosmological constant $\Lambda=-3$ and scalar curvature $R[\gamma^{(4)}]=[2D_0/(D_0-2)]\Lambda=-12$.

To conclude this section, we consider in more detail the Einstein equation (2.2) with the help of formulas (A1)–(A3). In addition to Eq. (2.6), we obtain

$$\begin{aligned} &R_{rr}[g^{(5)}]=-4\frac{a''}{a},\\ &R_{\mu r}[g^{(5)}]=R_{r\mu}[g^{(5)}]=0,\\ &R_{\mu\nu}[g^{(5)}]=R_{\mu\nu}[\gamma^{(4)}]-a^2\gamma_{\mu\nu}^{(4)}\left[\frac{a''}{a}+3\left(\frac{a'}{a}\right)^2\right]. \end{aligned} \tag{2.40}$$

Then, rr and $\mu\nu$ components of Eq. (2.2) are reduced correspondingly to

$$R[\gamma^{(4)}]=2\sum_{i=1}^{n} a_i^2(r)\Lambda_i\theta_i(r)+12\sum_{i=1}^{n}(a_i')^2\theta_i(r)\equiv f_2(r) \tag{2.41}$$

and

$$\begin{aligned} R_{\mu\nu}[\gamma^{(4)}]-\tfrac{1}{2}\gamma_{\mu\nu}^{(4)}R[\gamma^{(4)}]=&-\gamma_{\mu\nu}^{(4)}\left\{3\sum_{i=1}^{n}(a_i')^2\theta_i\right.\\ &\left.+\sum_{i=1}^{n}\Lambda_i a_i^2\theta_i+3\sum_{i=1}^{n}a_i a_i''\theta_i\right\}\\ \equiv&-\gamma_{\mu\nu}^{(4)}f_3(r). \end{aligned} \tag{2.42}$$

In the latter equation, the δ-function terms originated from a'' and tension terms cancel each other. It can be easily seen that for the Poincaré brane model, we obtain $f_2(r)\equiv f_3(r)\equiv 0$ in accordance with Eq. (2.21). For the symmetric de Sitter brane model, $f_2(r)\equiv 12$ and $f_3(r)\equiv 3$, which corresponds to Eq. (2.27). In the nonsymmetric de Sitter brane model, $f_2(r)\equiv f_2(R)\equiv 12$ and $f_3(r)\equiv f_3(R)\equiv 3$, in accordance with Eq. (2.33). For the anti–de Sitter brane model, $f_2(r)\equiv -12$ and $f_3(r)\equiv -3$, which corresponds to Eq. (2.39).

III. STABILITY UNDER CONFORMAL EXCITATIONS

Let us investigate now the stability of metric $g^{(5)}(X)$ defined in Eq. (2.5) with respect to conformal excitations. In other words, we want to investigate the dynamical behavior of the conformal metric excitations developing on the fixed background $g^{(5)}(X)$. To do this, we consider a perturbed metric of the form of Eq. (B1), $\bar{g}^{(5)}=\Omega^2 g^{(5)}\equiv e^{2\beta}g^{(5)}$, where $\Omega=1$ corresponds to the background solution and $\beta\ll 1$ describes the small perturbation limit. Obviously, the background solution is stable against such perturbations if Ω oscillates with time near the value $\Omega=1$ and is unstable if Ω has a runaway behavior from this value. According to the perturbation theory, full analysis should consist of two steps. The first one is the investigation of the dynamical behavior of perturbations on the fixed background, and the second one is the study of the backreaction of perturbations on the background solution. In the present paper, we are concentrating on the first problem, namely to find which brane-world solutions are stable against the conformal perturbations, and we postpone the second problem to our future investigation.

According to the standard approach, the equation of motion for perturbations (in our case for Ω or β), developing on the fixed background $g^{(5)}$, can be obtained substituting $\bar{g}^{(5)}$ in Eq. (2.2) and taking into account the background solution $g^{(5)}$ (e.g., solutions from Sec. II A). However, it is possible to investigate this problem in a different way, namely starting from action (2.1), putting in the perturbed metric $\bar{g}^{(5)}$, and, after that, taking into account the background solutions $g^{(5)}$. Then, the resulting effective action will describe the dynamical behavior of the perturbations on the fixed background. From this action, we can obtain the energy momentum tensor of the perturbations to study the backreaction of them on the background metric.

In our paper, we follow the second approach and put the perturbed metric $\bar{g}^{(5)}$ into action (2.1), which yields[9]

$$\begin{aligned} \bar{S}^{(5)}=&\frac{1}{2\kappa_5^2}\int_{M^{(5)}}d^5X\sqrt{|\bar{g}^{(5)}|}(R[\bar{g}^{(5)}]-2\Lambda_5(r))\\ &-\sum_{i=0}^{n-1}T_i(r_i)\int d^4x\sqrt{|\bar{g}^{(4)}|}\Bigg|_{r_i}. \end{aligned} \tag{3.1}$$

[8]This model can be easily generalized to the case of an arbitrary number of parallel branes by gluing one-brane manifolds at throats and identifying the two final opposite throats.

[9]It is clear that for a conformally transformed metric the Lanczos-Israel junction conditions will change [see, e.g., Eq. (B8)]. But, at the moment, we do not consider a backreaction of the conformal excitations on the metric, i.e., on the behavior of $a(r)$ and on the junction condition.

With the help of Eqs. (B3) and (2.6), the first term in this action reads

$$\sqrt{|\bar{g}^{(5)}|}R[\bar{g}^{(5)}]=\Omega^5 a^4\sqrt{|\gamma^{(4)}|}\{\Omega^{-2}[a^{-2}R[\gamma^{(4)}]-f_1(r)] \\ -8\Omega^{-3}\Omega_{;M;N}g^{(5)MN} \\ -4\Omega^{-4}\Omega_{,M}\Omega_{,N}g^{(5)MN}\}. \qquad (3.2)$$

For generality, we do not assume the small perturbation limit $\beta\ll 1$ keeping in action all nonlinear perturbation terms. Transition to this limit can be easily performed in the final expression [see Eq. (3.12) below]. In what follows, we shall consider a particular case when the conformal prefactor is a function of 4D space-time coordinates: $\Omega=\Omega(x)\equiv\exp(\beta(x))$. It is well known that conformal excitations of this form behave as scalar fields in 4D space-time (e.g., on branes). Because of the prefactor $\Omega^3(x)$ in front of the 4D scalar curvature $R[\gamma^{(4)}]$ in action, the 4D metric $\gamma^{(4)}$ is written in the Brans-Dicke frame. However, it is more easy to investigate the conformal perturbation stability in the Einstein frame (in the Introduction we mentioned the equivalence of these frames with respect to the stability analysis):

$$\gamma^{(4)}_{\mu\nu}(x)\Rightarrow\tilde{\gamma}^{(4)}_{\mu\nu}(x)=\Omega^3(x)\,\gamma^{(4)}_{\mu\nu}(x). \qquad (3.3)$$

In this frame, the dimensionally reduced action (3.1) reads

$$\overline{S}^{(4)}_{\rm eff}=\frac{1}{2\kappa_4^2}\int_{M^{(4)}}d^4x\sqrt{|\tilde{\gamma}^{(4)}|}R[\tilde{\gamma}^{(4)}]+\frac{1}{2}\int_{M^{(4)}}d^4x\sqrt{|\tilde{\gamma}^{(4)}|} \\ \times(-\tilde{\gamma}^{(4)\mu\nu}\tilde{\beta}_{,\mu}\tilde{\beta}_{,\nu}-2\tilde{U}_{\rm eff}), \qquad (3.4)$$

where $\tilde{\beta}\equiv\sqrt{3/2}(1/\kappa_4)\beta$ and

$$\tilde{U}_{\rm eff}(\Omega)\equiv\frac{1}{\kappa_4^2}U_{\rm eff}(\Omega) \\ =\frac{1}{\kappa_4^2 B_0}[B_1\Omega^{-3}+B_2\Omega^{-1}+B_3\Omega^{-2}]. \qquad (3.5)$$

Here, parameters B_i $(i=0,\dots,3)$ are defined by Eqs. (2.12)–(2.15).

Now, the problem of the background solution (2.5) stability against the conformal excitations is reduced to the existence of a minimum of the effective potential $U_{\rm eff}$ at point $\Omega=1\Leftrightarrow\beta=0$, which corresponds to the absence of the perturbations. In Appendix B, we show additionally that all other values for the minimum lead to metrics which in zero order do not satisfy the same Einstein equation as the background solution (2.5). It is clear also that the effective cosmological constant (2.11) should coincide with $U_{\rm eff}$ at $\Omega=1$: $\Lambda^{(4)}_{\rm eff}=U_{\rm eff}(\Omega=1)$, which we explicitly obtain from Eq. (3.5). The extremum existence condition reads

$$\left.\frac{\partial U_{\rm eff}}{\partial\Omega}\right|_{\Omega=1}=0\Rightarrow 3B_1+B_2+2B_3=0. \qquad (3.6)$$

Small excitations near a minimum position can be observed on branes as massive scalar field–gravitational excitons with mass squared:

$$m^2=\left.\frac{\partial^2\tilde{U}_{\rm eff}}{\partial\tilde{\beta}^2}\right|_{\tilde{\beta}=0}=\frac{2}{3}\Omega^2\left.\frac{\partial^2 U_{\rm eff}}{\partial\Omega^2}\right|_{\Omega=1} \\ =\frac{2}{3B_0}(12B_1+2B_2+6B_3). \qquad (3.7)$$

Obviously, the original solution (2.5) is stable under these conformal excitations if $m^2>0$, which prevents their runaway behavior from the background solution. As can be easily seen, all four models considered in the previous section satisfy Eq. (3.6). This means that all these solutions are stationary points of $U_{\rm eff}$ if the effective potential is considered as a functional of $a(r)$. For masses squared in the case of the Poincaré, the de Sitter (symmetric solution), the de Sitter (nonsymmetric solution), and the anti–de Sitter branes, we obtain, respectively,

$$m^2=\frac{4}{lB_0}(e^{-4L/l}-1)<0, \qquad (3.8)$$

$$m^2=\frac{4l^3}{B_0}\left(-\sinh^3\frac{L}{l}\cosh\frac{L}{l}-\frac{3}{4}\sinh 2\frac{L}{l}+\frac{3}{2}\frac{L}{l}\right)<0, \qquad (3.9)$$

$$m^2=\frac{2l^3}{B_0}\left(-\sinh^3\frac{L}{l}\cosh\frac{L}{l}-\frac{3}{4}\sinh 2\frac{L}{l}+\frac{3}{2}\frac{L}{l}-2\frac{R_0^3}{l^3}\right)<0, \qquad (3.10)$$

$$m^2=\frac{4l^3}{B_0}\left(-\sinh^3\frac{L}{l}\cosh\frac{L}{l}+\frac{1}{4}\sinh 2\frac{L}{l}+\frac{3}{2}\frac{L}{l}\right)\lesseqgtr 0. \qquad (3.11)$$

Thus, the first three solutions are unstable under the considered conformal excitations: $\Omega=1$ corresponds to the maximum but not to the minimum of the potential (3.5). However, in the AdS brane case, mass squared is positive and decreases from[10] 4 for $L/l\to 0$ to zero for $L/l\to 1$ (more precisely, numerical calculations show that $m^2\to 0$ for $L/l\to 0{,}988$). So, the AdS brane solution is stable with respect to the conformal excitations if the distance between brane and throats of wormholes is less than the AdS radius. As was mentioned in footnote 8, this case can be easily generalized to a number of AdS branes connected with each other via the wormhole throats. Then, in the case of n branes, for the gravexciton mass squared we obtain an expression which is simply an algebraic sum of the type (3.11) (with an evident substitution $L\to L_i$, $i=1,\dots,n$ for each member of the sum) and overall prefactor B_0^{-1}. Here, B_0 is also a generali-

[10]Masses squared of gravexcitons (3.8)–(3.11) are written in dimensionless units. If we take into account footnote 6, then physical gravexciton mass for an observer on the ith brane is $m\to m_{(ph)}=m/a_i(r_i)$.

zation of Eq. (2.35) to an evident sum. This mass squared is positive, e.g., if $L_i/l \lesssim 1$, $i=1,\ldots,n$.

For small fluctuations near the minimum of U_{eff}, action (3.4) reads

$$\overline{S}_{\mathrm{eff}}^{(4)} = \frac{1}{2\kappa_4^2}\int_{M^{(4)}} d^4x\sqrt{|\tilde{\gamma}^{(4)}|}\{R[\tilde{\gamma}^{(4)}]-2\Lambda_{\mathrm{eff}}^{(4)}\} + \frac{1}{2}\int_{M^{(4)}} d^4x\sqrt{|\tilde{\gamma}^{(4)}|}(-\tilde{\gamma}^{(4)\mu\nu}\tilde{\beta}_{,\mu}\tilde{\beta}_{,\nu}-m^2\tilde{\beta}^2), \tag{3.12}$$

where the first integral corresponds to zero-order theory (2.10) (background solution) and the second one describes gravitational excitons. This effective action can be used for investigation of the gravexciton backreaction on the background metric.

If we put in action (3.1) conformally transformed brane tensions $\overline{T}(r_i)=(1/\Omega)T(r_i)$ [see Eqs. (B7) and (B8)] instead of $T(r_i)$, the effective potential reads

$$U_{\mathrm{eff}}(\Omega) = \frac{1}{B_0}[B_1\Omega^{-3}+B_2\Omega^{-1}+B_3\Omega^{-3}]. \tag{3.13}$$

In this case, $\Omega=1$ is not the extremum of the effective potential (3.13) for all of the four considered solutions: they are not stationary points of this potential.

IV. CONCLUSIONS

In the present paper, we investigated the stability of 5D brane-world solutions against conformal perturbations. For these models, the five-dimensional metric contains a four-dimensional metric component multiplied by a warp (scale) factor $a(r)$ which is a function of the additional dimension. Models contain n parallel branes "transversal" to the additional coordinate. As a point of interest we consider bulk cosmological constants between branes and tensions ("vacuum energies") on the branes. The scale factor is a continuous piecewise function while its derivative has jumps on the branes. There are a number of well-known exact solutions which belong to this class of model (e.g., [8,12,17,24–28]). We investigated the stability of some of these models under the conformal excitations, which are functions of 4D space-time. Such excitations are of special interest because they behave as massive minimal scalar fields in 4D space-time, for example they can be observed as massive scalar particle–gravitational excitons on branes, such as takes place in the Kaluza-Klein approach [1,6].

To perform the stability analysis, we put the perturbed metric in the original 5D action, took into account the background metric solution, and integrated the action over the extra dimension (the dimension reduction). The obtained 4D effective action describes the dynamical behavior of the perturbations on the fixed background. The extremum of the effective potential in this action corresponds to the background solution. If this extremum is a minimum, the conformal perturbations oscillate around the background solution providing its stability. In this case, small excitations around this minimum are observed as gravitational excitons on branes. However, if this extremum is a maximum, the conformal perturbations have the runaway behavior, and the background solution is unstable against such excitations.

We have shown that in the case of one and two Poincaré branes, one de Sitter brane (symmetric solution), and one de Sitter brane (nonsymmetric solution), all these solutions are unstable with respect to these excitations because the effective potential has a maximum but not a minimum at the point corresponding to the original (background) solutions. In these models, the 4D effective cosmological constant is nonnegative [see Eqs. (2.21), (2.27), and (2.33)]. However, one AdS brane solution is stable if the distance between brane and throats of wormholes is less than the AdS radius. The effective 4D cosmological constant is negative in this model [see Eq. (2.39)]. The latter case is easily generalized to a stable model with a number of parallel AdS branes connected with each other via the wormhole throats. It is necessary to note that we found stability or instability against gravexcitons for models with the same kind of branes, which are correspondingly stable or unstable against radions considered in [11,12], although a direct comparison between the models can only be made for cases with two or more branes.

Another remark involves the analogy between the stability under conformal perturbations in the brane-world models considered here and the standard Kaluza-Klein models. The situation with these four solutions is similar to one we have in the pure geometrical case in the standard Kaluza-Klein approach [1]. Here, the stability also takes place when the 4D effective cosmological constant is negative. If the effective cosmological constant is positive, we have maximum of the effective potential instead of the minimum. To shift the minimum of the effective potential to positive values, we should include matter in the model.

We found also that 4D effective cosmological and gravitational constants on branes as well as gravexciton masses undergo a hierarchy. It was shown that for observers on different branes with different warp factors, these parameters have different values. A similar result with respect to the effective 4D Newton constant was obtained in [10].

There are a number of possible generalizations that are worth investigating. First, it is of interest to include richer types of matter in the model, e.g., perfect fluid in bulk as well as on branes, which simulates different forms of matter in the Universe. The presence of matter can stabilize radions in the brane-world models with non-negative 4D effective cosmological constant on branes, as was shown in [9,15–20]. As we mentioned above, stabilization of gravexcitons in models with the 4D positive effective cosmological constant takes place also in the standard Kaluza-Klein approach if we include matter here [1]. Thus, we expect a similar stabilization effect for gravexcitons in the brane-world models. The second possibility consists in a generalization of the model to a multidimensional case with $D>5$. It will give us the opportunity to include into consideration already obtained exact brane-world solutions with $D=6$ and more dimensions. Additionally, as we wrote in Sec. III, the investigation of the background solution stability against the conformal perturbations is only the first problem. When we found the stable solutions, the second problem consists in the investigation of the perturbation backreaction on the background solution. We leave these issues for future work.

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APPENDIX A: BLOCK-DIAGONAL METRICS

In this appendix, we present some useful formulas (see also [29]) for curvature tensors in the case of a block-diagonal metric of the form

$$(g_{MN}^{(D_0+D_1)}(x,y))=\begin{pmatrix} e^{2\sigma(y)}\gamma_{\mu\nu}^{(D_0)}(x) & 0 \\ 0 & g_{mn}^{(D_1)}(y)\end{pmatrix}. \quad \text{(A1)}$$

For this metric, the Ricci tensor (everywhere in this paper we use the Misner-Thorne-Wheeler book conventions [30]) reads

$$R_{\mu\nu}[g^{(D)}]=R_{\mu\nu}[\gamma^{(D_0)}]-e^{2\sigma}\gamma_{\mu\nu}^{(D_0)}[D_0 g^{(D_1)mn}(\partial_m\sigma)(\partial_n\sigma)+g^{(D_1)mn}\nabla_m^{(D_1)}(\partial_n\sigma)],$$

$$R_{\mu n}[g^{(D)}]=R_{n\mu}[g^{(D)}]=0, \quad \text{(A2)}$$

$$R_{mn}[g^{(D)}]=R_{mn}[g^{(D_1)}]-D_0[(\partial_m\sigma)(\partial_n\sigma)+\nabla_m^{(D_1)}(\partial_n\sigma)],$$

where $D=D_0+D_1$ and $\nabla_m^{(D_1)}$ is a covariant derivative with respect to the metric $g^{(D_1)}$. The scalar curvature reads, correspondingly,

$$R[g^{(D)}]=e^{-2\sigma}R[\gamma^{(D_0)}]+R[g^{(D_1)}]-D_0[(D_0+1)g^{(D_1)mn}\times(\partial_m\sigma)(\partial_n\sigma)+2g^{(D_1)mn}\nabla_m^{(D_1)}(\partial_n\sigma)]. \quad \text{(A3)}$$

APPENDIX B: CONFORMAL TRANSFORMATION

For a conformally transformed metric,

$$\bar{g}_{MN}^{(D)}(X)=\Omega^2(X)g_{MN}^{(D)}(X)\equiv e^{2\beta(X)}g_{MN}^{(D)}(X), \quad \text{(B1)}$$

the Ricci tensor and the scalar curvature read, correspondingly,

$$R_{MN}[\bar{g}^{(D)}]=R_{MN}[g^{(D)}]-(D-2)\beta_{;M;N}-g_{MN}^{(D)}g^{(D)KL}\beta_{;K;L}+(D-2)\beta_{;M}\beta_{;N}-(D-2)g_{MN}^{(D)}g^{(D)KL}\beta_{;K}\beta_{;L} \quad \text{(B2)}$$

and

$$R[\bar{g}^{(D)}]=\Omega^{-2}R[g^{(D)}]-2(D-1)\Omega^{-3}\Omega_{;M;N}g^{(D)MN}-(D-1)(D-4)\Omega^{-4}\Omega_{;M}\Omega_{;N}g^{(D)MN}, \quad \text{(B3)}$$

where in Eqs. (B2) and (B3), covariant derivatives are taken with respect to the metric $g^{(D)}$.

We suppose now that the metric $\bar{g}^{(D)}$ is a solution of the Einstein equation

$$R_{MN}[\bar{g}^{(D)}]-\tfrac{1}{2}\bar{g}_{MN}^{(D)}R[\bar{g}^{(D)}]=-\bar{\Lambda}_D\bar{g}_{MN}^{(D)}-\frac{\kappa_D^2}{\sqrt{|\bar{g}^{(D)}|}}\sum_{i=0}^{n-1}\bar{T}_i(y_i)\sqrt{|\bar{g}^{(D_0)}(x,y_i)|}\times\bar{g}_{\mu\nu}^{(D_0)}(x,y_i)\,\delta_M^\mu\delta_N^\nu\delta(y-y_i), \quad \text{(B4)}$$

which describes a model with the bulk cosmological constant $\bar{\Lambda}_D$ and n branes of tension $\bar{T}_i(y_i)$. Let us consider a particular case of the constant conformal transformation (B1): $\Omega\equiv$const. Then, with the help of Eqs. (B2) and (B3) for the conformally transformed metric $g^{(D)}$, we obtain

$$R_{MN}[g^{(D)}]-\tfrac{1}{2}g_{MN}^{(D)}R[g^{(D)}]=-\bar{\Lambda}_D\Omega^2 g_{MN}^{(D)}-\frac{\kappa_D^2}{\sqrt{|g^{(D)}|}}\sum_{i=0}^{n-1}\bar{T}_i(y_i)\times\Omega^{-D+D_0+2}\sqrt{|g^{(D_0)}(x,y_i)|}g_{\mu\nu}^{(D_0)}(x,y_i)\times\delta_M^\mu\delta_N^\nu\delta(y-y_i). \quad \text{(B5)}$$

This equation shows that the conformally transformed metric $g^{(D)}$ is the solution for the model with the cosmological constant $\Lambda_D\equiv\Omega^2\bar{\Lambda}_D$ and the brane tensions $T_i(y_i)\equiv\Omega^{(-D+D_0+2)}\bar{T}_i(y_i)$. The latter one is invariant for $D=D_0+2$, which certainly is not the case for $D=5$ if $D_0=4$. Thus, if we want the solution to correspond to a minimum of the effective potential for the conformal excitations describing the model with the original cosmological constant and the brane tensions, this minimum should take place at $\Omega=1$.

Let us consider transformation of the trace of the extrinsic curvature evoked by conformal transformation of the metric. In the case of 5D metric (2.5) written in Gaussian normal coordinates, the trace of the extrinsic curvature of the hypersurface Σ: $r=r_i=$const reads

$$K(r_i)=-\nabla_M n^M\big|_{r_i}=-\frac{1}{2}g^{(4)\mu\nu}\frac{\partial g_{\mu\nu}^{(4)}}{\partial r}\bigg|_{r_i}=-4\frac{1}{a}\frac{da}{dr}\bigg|_{r_i}, \quad \text{(B6)}$$

where $n^M = \delta_r^M$ is the unit vector field orthogonal to Σ. If the extrinsic curvature has a jump at this hypersurface, $\hat{K}(r_i) \equiv K(r_i^+) - K(r_i^-) \neq 0$, then it results in the Lanczos-Israel junction condition,

$$T(r_i) = \frac{1}{\kappa_D^2}\frac{3}{4}\hat{K}(r_i), \tag{B7}$$

where $T(r_i)$ is the tension of the brane which causes the jump of the extrinsic curvature.

For the metric $\overline{g}^{(D)}$, obtained with the help of the conformal transformation (B1) of the metric (2.5), the unit vector field orthogonal to Σ is $\overline{n}^M = \Omega^{-1}\delta_r^M \Rightarrow \overline{n}_M = \Omega\,\delta_M^r$. Here, we consider the case when $\Omega = \Omega(x)$ does not depend on the extra dimension r. Then, we obtain for the trace of the extrinsic curvature of the conformal space-time

$$\overline{K}(r_i) = -\overline{\nabla}_M \overline{n}^M|_{r_i} = -\frac{4}{\Omega}\frac{1}{a}\frac{da}{dr}\bigg|_{r_i}. \tag{B8}$$

Correspondingly, the tensions of the brane in conformal and original space-times are connected with each other as follows: $\overline{T}(r_i) = \Omega^{-1}T(r_i)$ in accordance with Eq. (B5).

APPENDIX C: TRUNCATED CONFORMAL TRANSFORMATION

In this appendix, we shall show that our results do not change if only additional dimension undergoes conformal perturbations in metric (2.5):

$$g^{(5)}(X) \Rightarrow \overline{g}^{(5)}(X) = \Omega^2(x)dr\otimes dr + a^2(r)\,\gamma^{(4)}_{\mu\nu}(x)dx^\mu \otimes dx^\nu. \tag{C1}$$

Subsequent application of appropriate formulas from Appendixes A and B yields

$$\begin{aligned}\sqrt{|\overline{g}^{(5)}|}R[\overline{g}^{(5)}] &= \Omega a^4\sqrt{|\gamma^{(4)}|}\times\{a^{-2}[R[\gamma^{(4)}] - 2\Omega^{-1}\Omega_{;\mu;\nu}\gamma^{(4)\mu\nu}] \\ &\quad - \Omega^{-2}f_1(r)\},\end{aligned} \tag{C2}$$

where $f_1(r)$ is defined in Eq. (2.7). To get this expression, it is useful to go first to a new coordinate $R{:}dR = a^{-1}(r)dr$ and then, after using conformal transformation formulas, come back to r again. It can be easily seen that after conformal transformation to the Einstein frame,

$$\gamma^{(4)}_{\mu\nu}(x) \Rightarrow \tilde{\gamma}^{(4)}_{\mu\nu}(x) = \Omega(x)\,\gamma^{(4)}_{\mu\nu}(x), \tag{C3}$$

and the dimensional reduction, action (3.1), is exactly reduced to the effective action (3.4). Thus, gravexcitons have exactly the same masses (3.7)–(3.11). This result shows that geometry (gravitational field) under conformal transformations behaves as an elastic media. For an elastic body, the eigenfrequencies of its oscillations do not depend on the manner of excitation.

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