Static axially symmetric solutions of Einstein-Yang-Mills equations with a negative cosmological constant: The regular case

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Numerical solutions of the Einstein-Yang-Mills equations with a negative cosmological constant are constructed. These axially symmetric solutions approach asymptotically the anti–de Sitter spacetime and are regular everywhere. They are characterized by the winding number $n>1$, the mass, and the non-Abelian magnetic charge. The main properties of the solutions and the differences with respect to the asymptotically flat case are discussed. The existence of axially symmetric monopole and dyon solutions in fixed anti–de Sitter spacetime is also discussed.

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I. INTRODUCTION

After the discovery by Bartnik and McKinnon (BK) of a nontrivial particlelike solution of the Einstein-Yang-Mills (EYM) equations [1], there has been a great deal of numerical and analytical work on various aspects of EYM theory. A large number of self-gravitating structures with non-Abelian fields have been found (for a review, see [2]). These include black holes with nontrivial hair, thereby leading to the possibility of evading the no-hair conjecture.

Most of these investigations have been carried out on the assumption that spacetime is asymptotically flat. Less is known when the theory is modified to include a cosmological constant Λ which greatly changes the asymptotic structure of spacetime [3].

For a positive cosmological constant, the behavior of the solutions is similar in many respects to that of asymptotically flat geometries [4]. In particular, the configurations are unstable in both cases [5].

If we allow for a negative cosmological constant, the solution of the matter-free Einstein equations possessing the maximal number of symmetries is the anti–de Sitter (AdS) spacetime. Being a maximally symmetric spacetime, it is an excellent model to investigate questions of principle related to the quantization of fields propagating on a curved background, the interaction with the gravitational field, and issues related to the lack of global hyperbolicity. Also, lately there has been a lot of interest in asymptotically AdS spacetimes, connected with string theory and related topics.

Recently, some authors have discussed the properties of soliton and black hole solutions of the EYM system for $\Lambda<0$ (i.e., an asymptotically AdS spacetime [6–8]). They obtained some surprising results, which are strikingly different from the BK-type solutions. First, there is a continuum of solutions in terms of the adjustable shooting parameter that specifies the initial conditions at the origin or at the event horizon, rather than discrete points. The spectrum has a finite number of continuous branches. Secondly, there are nontrivial solutions stable against spherically symmetric linear perturbations, corresponding to stable monopole and dyon configurations. The solutions are classified by non-Abelian electric and magnetic charges and the Arnowitt-Deser-Misner (ADM) mass. When the parameter Λ approaches zero, an already existing branch of monopoles and dyon solutions collapses to a single point in the moduli space. At the same time, new branches of solutions emerge. A fractal structure in the moduli space has been noticed [7,9].

As observed in [7], these solutions may have profound consequences in the evolution of the early universe. In Refs. [10,11], regular gravitating monopole and dyon solutions in Einstein-Yang-Mills-Higgs (EYMH) theory with a Higgs field in the adjoint representation were shown to exist in asymptotically AdS spacetime. As happens in asymptotically flat space, a critical value for the Newton constant exists above which no regular solution can be found. The presence of a cosmological constant enhances this effect, the critical value being smaller than the value found for $\Lambda=0$.

The global existence of a solution of the Cauchy problem for the YMH equations in AdS spacetime is discussed in [12].

However, so far only spherically symmetric solutions have been found.

For $\Lambda=0$, a SU(2) YM theory coupled to a scalar Higgs field [13,14] dilaton [15], or gravity [16] is known to possess also axially symmetric finite-energy solutions. In the 1990s, the numerical calculation of these configurations was one of the most important developments in the domain.

A natural question arises: do non-Abelian axially symmetric solutions also exist for a nonzero cosmological constant? And if this is the case, how does the nonasymptotically flat structure of spacetime affect these configurations?

The aim of this paper is to address the above questions for the SU(2) gauge group, under the assumption of axial symmetry, for an asymptotically AdS geometry.

The corresponding problem for a vanishing cosmological constant and a purely magnetic gauge field has been exhaustively discussed in [16]. Representing generalizations of the BK solutions [2], the solutions obtained by Kleihaus and Kunz in [16] have no non-Abelian charges but are characterized by two integers. These are the node number k of the gauge field functions and the winding number n with respect to the azimuthal angle φ. The spherically symmetric BK solutions have winding number $n=1$. A winding number $n>1$ leads to axially symmetric solutions. As discussed in

[16], these regular axially symmetric solutions have a torus-like shape. With the z axis ($\theta=0$) as symmetry axis, the energy density has a strong peak along the ρ axis ($\theta=\pi/2$) and decreases monotonically along the z axis.

The solutions we are looking for are the asymptotically AdS analogues of these configurations.

Although some common features are present, the results we find are rather different from those valid in the $\Lambda=0$ case. A different behavior is noticeable especially for the lower branch of axially symmetric solutions. These distinctions arise from differences that already exist in the spherically symmetric case.

A negative Λ implies a continuum of axially symmetric solutions. For a fixed winding number, they can be classified in a finite number of branches and have continuous values of mass and non-Abelian magnetic charge. The radial and angular dependence of the metric and gauge functions can also be different from the case discussed in [16].

The paper is structured as follows: in the next section we explain the model and derive the basic equations, while in Sec. III we discuss solutions of YM equations in a fixed AdS background. These solutions were found to be important when discussing the scaling properties of the mass spectrum [9] and have no flat space couterparts. Some features of the axially symmetric solutions possessing a net YM electric charge in a fixed AdS background are also presented in this section. The general properties of the axially symmetric gravitating solutions are presented in Sec. IV, where we show results obtained by numerical calculations. We give our conclusions and remarks in the final section.

II. GENERAL FRAMEWORK AND EQUATIONS OF MOTION

A. Einstein-Yang-Mills action

The basic equations for a static, axially symmetric SU(2) gauge field coupled to Einstein gravity (without cosmological term) are well known (for details see, e.g., [16]). Here we derive them for a nonzero Λ, without going into detail. We will follow most of the conventions and notations used by Kleihaus and Kunz (KK) in their papers.

The starting point is the Einstein-Yang-Mills action

$$S=\int d^4x\sqrt{-g}\left(\frac{1}{16\pi G}(\mathcal{R}-2\Lambda)-\tfrac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\right), \quad (1)$$

where the field strength tensor is

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+ie[A_\mu,A_\nu] \quad (2)$$

and the gauge field

$$A_\mu=\tfrac{1}{2}\tau^a A_\mu^a. \quad (3)$$

Variation of the action (1) with respect to the metric $g^{\mu\nu}$ leads to the Einstein equations

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}=8\pi G T_{\mu\nu}, \quad (4)$$

where the YM stress-energy tensor is

$$T_{\mu\nu}=2\,\mathrm{Tr}(F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta}-\tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}). \quad (5)$$

Variation with respect to the gauge field A_μ leads to the matter field equations

$$\nabla_\mu F^{\mu\nu}+ie[A_\mu,F^{\mu\nu}]=0. \quad (6)$$

B. Static axially symmetric ansatz and gauge condition

We generalize the isotropic axisymmetric line element considered by Kleihaus and Kunz in [16] for a nonzero Λ,

$$ds^2=-f\left(1-\frac{\Lambda}{3}r^2\right)dt^2+\frac{m}{f}\left(\frac{dr^2}{1-\frac{\Lambda}{3}r^2}+r^2d\theta^2\right)+\frac{l}{f}r^2\sin^2\theta\, d\phi^2, \quad (7)$$

where the metric functions f, m, and l are only functions of the coordinates r and θ.

A suitable parametrization of a purely magnetic Yang-Mills connection in terms of spherical coordinates is

$$A_r=\frac{1}{2er}H_1(r,\theta)\tau_\phi^n,$$
$$A_\theta=\frac{1}{2e}[1-H_2(r,\theta)]\tau_\phi^n, \quad (8)$$
$$A_\phi=-n\sin\theta\left[H_3(r,\theta)\frac{\tau_r^n}{2e}+[1-H_4(r,\theta)]\frac{\tau_\theta^n}{2e}\right].$$

Here the symbols τ_r^n, τ_θ^n, and τ_ϕ^n denote the dot products of the Cartesian vector of Pauli matrices, $\vec{\tau}=(\tau_1,\tau_2,\tau_3)$, with the spatial unit vectors

$$\vec{e}_r^{\,n}=(\sin\theta\cos n\phi,\sin\theta\sin n\phi,\cos\theta),$$
$$\vec{e}_\theta^{\,n}=(\cos\theta\cos n\phi,\cos\theta\sin n\phi,-\sin\theta), \quad (9)$$
$$\vec{e}_\varphi^{\,n}=(-\sin n\phi,\cos n\phi,0),$$

respectively. This ansatz is axially symmetric in the sense that a rotation around the z axis can be compensated by a gauge rotation. It satisfies also some additional discrete symmetries [16,17]. To fix the residual Abelian gauge invariance, we choose the usual gauge condition [16]

$$r\partial_r H_1-\partial_\theta H_2=0. \quad (10)$$

This ansatz contains an integer n, representing the winding number with respect to the azimutal angle φ. While φ covers the trigonometric circle once, the fields wind n times around. Note that the spherically symmetric ansatz corresponds to $n=1$, $H_1=H_3=0$, and $H_2=H_4=w(r)$.

C. Field equations

From Eqs. (4) and (6), we obtain a set of seven nonlinear elliptical partial differential equations which can be solved numerically.

Within this specific ansatz and the gauge condition (10), we derive the matter equations

$$\begin{aligned}0=&\Big(r^2H_{1,r,r}+H_{1,\theta,\theta}+H_{2,\theta}-n^2\frac{m}{l}[rH_{4,r}H_3-rH_{3,r}H_4\\&+(H_3^2+H_4^2-1)H_1]\Big)\sin^2\theta+\Big(rH_{2,r}+H_{1,\theta}\\&-n^2\frac{m}{l}(2H_1H_3+rH_{4,r})\Big)\sin\theta\cos\theta-n^2\frac{m}{l}H_1\\&+\sin^2\theta[H_{1,\theta}+rH_{2,r}]\ln\left(\frac{fN\sqrt{l}}{m}\right)\\&+\sin^2\theta[rH_{1,r}-H_{2,\theta}]r\ln\left(\frac{fN\sqrt{l}}{m}\right)_{,r},\end{aligned}\tag{11}$$

$$\begin{aligned}0=&\Big(r^2H_{2,r,r}+H_{2,\theta,\theta}-H_{1,\theta}+n^2\frac{m}{lN}[H_3H_{4,\theta}-H_4H_{3,\theta}\\&-(H_3^2+H_4^2-1)H_2]\Big)\sin^2\theta+\Big(H_{2,\theta}-rH_{1,r}+n^2\frac{m}{lN}\\&\times(-2H_2H_3+H_{4,\theta})\Big)\sin\theta\cos\theta+n^2\frac{m}{lN}(H_4-H_2)\\&-\sin^2\theta[rH_{1,r}-H_{2,\theta}]\ln\left(\frac{fN\sqrt{l}}{m}\right)_{,\theta}\\&+\sin^2\theta[H_{1,\theta}+rH_{2,r}]r\ln\left(\frac{fN\sqrt{l}}{m}\right)_{,r},\end{aligned}\tag{12}$$

$$\begin{aligned}0=&\Big[r^2H_{3,r,r}+\frac{1}{N}H_{3,0,\theta}-H_3\left(H_1^2+\frac{H_2^2}{N}\right)\\&+H_1(H_4-2rH_{4,r})-H_4\left(rH_{1,r}-\frac{H_{2,\theta}}{N}\right)\\&+\frac{2H_2H_{4,\theta}}{N}\Big]\sin^2\theta+\frac{1}{N}(H_{3,\theta}+H_2H_4-NH_l^2-H_2^2)\\&\times\sin\theta\cos\theta-\frac{H_3}{N}+\frac{1}{N}\sin^2\theta[H_{3,\theta}-1+H_2H_4\\&+\cot\theta H_3]\ln\left(\frac{f}{\sqrt{l}}\right)_{,\theta}+\sin^2\\&\times\theta[rH_{3,r}-H_1H_4]r\ln\left(\frac{fN}{\sqrt{l}}\right)_{,r},\end{aligned}\tag{13}$$

$$\begin{aligned}0=&\Big[r^2H_{4,r,r}+\frac{1}{N}H_{4,\theta,\theta}-H_1\left(H_1^2+\frac{H_2^2}{N}\right)-H_1H_3-\frac{H_2}{N}\\&(-1+2H_{3,\theta})+2rH_{3,r}H_1+H_3\left(rH_{1,r}-\frac{H_{2,\theta}}{N}\right)\Big]\\&\times\sin^2\theta+\frac{1}{N}(-H_2H_3-H_1N+rH_{1,r}N-H_{2\theta}+H_{4,\theta})\\&\times\sin\theta\cos\theta+\frac{H_2-H_4}{N}+\frac{1}{N}\sin^2\theta[H_{4,\theta}-H_3H_2\\&-(H_2-H_4)\cot\theta]\ln\left(\frac{f}{\sqrt{l}}\right)_{,\theta}+\sin^2\theta[rH_{1,r}+H_1H_3\\&+\cot\theta H_1]r\ln\left(\frac{fN}{\sqrt{l}}\right)_{,r},\end{aligned}\tag{14}$$

where $N=1-(\Lambda/3)^2$. The resulting equations for the metric functions are

$$\begin{aligned}8\pi G\frac{m}{f}(-2T_t^t)=&\frac{1}{r^2}\Big\{\frac{Nr^2f_{,r,r}+f_{,\theta,\theta}+2Nrf_{,r}}{f}\\&-\left[\left(\frac{f_{,\theta}}{f}\right)^2+N\left(\frac{rf_{,r}}{f}\right)^2\right]+\cot\theta\frac{f_{,\theta}}{f}\\&+\frac{1}{2}\left(N\frac{rf_{,r}}{f}\frac{rl_{,r}}{l}+\frac{f_{,\theta}}{f}\frac{l_{,\theta}}{l}\right)\Big\}\\&-\frac{\Lambda r}{3}\left(\frac{l_{,r}}{l}+2\frac{f_{,r}}{f}\right)+2\Lambda\left(\frac{m}{f}-1\right),\end{aligned}\tag{15}$$

$$\begin{aligned}8\pi G\frac{m}{f}(T_r^r+T_\phi^\phi)=&\frac{1}{4r^2}\Big\{2\Big[\frac{Nr^2m_{,r,r}+Nrm_{,r}+m_{,\theta,\theta}}{m}\\&-\left(\frac{m_{,\theta}}{m}\right)^2-N\left(\frac{rm_{,r}}{m}\right)^2\Big]+2\frac{l_{,\theta,\theta}}{l}\\&+2\left(\frac{f_{,\theta}}{f}\right)^2-\left(\frac{l_{,\theta}}{l}\right)^2+2N\left(\frac{rl_{,r}}{l}+\frac{rm_{,r}}{m}\right)\\&+N\frac{rm_{,r}rl_{,r}}{l}-\frac{m_{,\theta}}{m}\frac{l_{,\theta}}{l}\\&-2\left(\frac{m_{,\theta}}{m}-2\frac{l_{,\theta}}{l}\right)\cot\theta\Big\}\\&-\frac{\Lambda r}{6}\left(\frac{l_{,r}}{l}+2\frac{m_{,r}}{m}\right)+2\Lambda\left(\frac{m}{f}-1\right),\end{aligned}\tag{16}$$

$$8\pi G\frac{m}{f}(T_r^r+T_\theta^\theta)=\frac{1}{4r^2}\left(2\frac{Nr^2l_{,r,r}+l_{,\theta,\theta}}{l}-\left(\frac{l_{,\theta}}{l}\right)^2-N\left(\frac{rl_{,r}}{l}\right)^2+6N\frac{rl_{,r}}{l}+4\frac{l_{,\theta}}{l}\cot\theta\right)-\frac{\Lambda rl_{,r}}{2l}+2\Lambda\left(\frac{m}{f}-1\right). \tag{17}$$

In the above relations, the components of the energy-momentum tensor are

$$T_r^r=\frac{1}{2}\left(\frac{f}{m}\right)^2\frac{N}{r^2}F_{r\theta}^2+\frac{1}{2}\frac{f^2N}{mlr^2\sin^2\theta}F_{r\phi}^2-\frac{1}{2}\frac{f^2}{mlr^4\sin^2\theta}F_{\theta\phi}^2,$$

$$T_\theta^\theta=\frac{1}{2}\left(\frac{f}{m}\right)^2\frac{N}{r^2}F_{r\theta}^2-\frac{1}{2}\frac{f^2N}{mlr^2\sin^2\theta}F_{r\phi}^2+\frac{1}{2}\frac{f^2}{mlr^4\sin^2\theta}F_{\theta\phi}^2,$$

$$T_\phi^\phi=-\frac{1}{2}\left(\frac{f}{m}\right)^2\frac{N}{r^2}F_{r\theta}^2+\frac{1}{2}\frac{f^2N}{mlr^2\sin^2\theta}F_{r\phi}^2+\frac{1}{2}\frac{f^2}{mlr^4\sin^2\theta}F_{\theta\phi}^2,$$

$$-T_t^t=\frac{1}{2}\left(\frac{f}{m}\right)^2\frac{N}{r^2}F_{r\theta}^2+\frac{1}{2}\frac{f^2N}{mlr^2\sin^2\theta}F_{r\phi}^2+\frac{1}{2}\frac{f^2}{mlr^4\sin^2\theta}F_{\theta\phi}^2,$$

where, similar to [16], we define

$$F_{r\theta}^2=\frac{1}{r^2}[(H_{1,\theta}+rH_{2,r})^2+(rH_{1,r}-H_{2,\theta})^2],$$

$$F_{r\phi}^2=\frac{n^2\sin^2\theta}{r^2}[(rH_{3,r}-H_1H_4)^2+(rH_{4,r}+H_1H_3+\cot\theta H_1)^2],$$

$$F_{\theta\phi}^2=n^2\sin^2\theta\{[H_{4,\theta}-H_2H_3-\cot\theta(H_2-H_4)]^2+(H_{3,\theta}-1+H_2H_4+\cot\theta H_3)^2\}.$$

D. Boundary conditions

To obtain asymptotically AdS regular solutions with finite-energy density, the metric functions have to satisfy the boundary conditions

$$\partial_r f|_{r=0}=\partial_r m|_{r=0}=\partial_r l|_{r=0}=0, \tag{18}$$

$$f|_{r=\infty}=m|_{r=\infty}=l|_{r=\infty}=1. \tag{19}$$

We assume also the flatness condition [18]

$$m|_{\theta=0}=l|_{\theta=0}. \tag{20}$$

The boundary conditions satisfied by the matter functions are

$$H_2|_{r=0}=H_4|_{r=0}=1,\quad H_1|_{r=0}=H_3|_{r=0}=0, \tag{21}$$

at the origin and

$$H_2|_{r=\infty}=H_4|_{r=\infty}=\omega_0,\quad H_1|_{r=\infty}=H_3|_{r=\infty}=0, \tag{22}$$

at infinity, where w_0 is an undetermined constant. For a solution with parity reflection symmetry, the boundary conditions along the axes are

$$H_1|_{\theta=0,\pi/2}=H_3|_{\theta=0,\pi/2}=0,\qquad \partial_\theta H_2|_{\theta=0,\pi/2}=\partial_\theta H_4|_{\theta=0,\pi/2}=0, \tag{23}$$

$$\partial_\theta f|_{\theta=0,\pi/2}=\partial_\theta m|_{\theta=0,\pi/2}=\partial_\theta l|_{\theta=0,\pi/2}=0. \tag{24}$$

Therefore, we need to consider the solutions only in the region $0\leqslant\theta\leqslant\pi/2$. Regularity on the z axis requires

$$H_2|_{\theta=0}=H_4|_{\theta=0}. \tag{25}$$

Dimensionless partial differential equations are obtained by the following rescaling: $r\to(\sqrt{4\pi G}/e)r$, $\Lambda\to(e^2/4\pi G)\Lambda$. In Bjoraker-Hosotani conventions, this corresponds to taking a unit value for the parameter $v=4\pi G/e^2$ [7].

E. The mass and the Yang-Mills charges of the solution

At spatial infinity, the line element (7) can be written as

$$ds^2=ds_0^2+h_{\mu\nu}dx^{\mu\nu}, \tag{26}$$

where $h_{\mu\nu}$ are deviations from the background AdS metric ds_0^2. Similar to the asymptotically flat case, one expects the values of conserved quantities to be encoded in the functions $h_{\mu\nu}$. The construction of these quantities for an asymptotically AdS spacetime was addressed for the first time in the 1980s (see, for instance, Refs. [19,20]). However, the generalization of Komar's formula in this case is not straightforward and requires further subtraction of a background configuration in order to render a finite result.

Using the Hamiltonian formalism, Henneaux and Teitelboim [20] have computed the mass of an asymptotically AdS spacetime in the following way. They showed that the Hamiltonian must be supplemented by surface terms in order to be consistent with the equations of motion. These surface terms yield conserved charges associated with the Killing vectors of an asymptotic AdS geometry. The general expression of a conserved quantity is

$$J_A=\frac{1}{16\pi}\oint d^2S_i[G^{ijkl}(\xi_A^\perp\mathring{\nabla}_j g_{kl}-h_{kl}\mathring{\nabla}_j\xi_A^\perp)+2\xi_A^k\pi_k^i], \tag{27}$$

where $G^{ijkl}=\frac{1}{2}(-\mathring{g})^{1/2}(\mathring{g}^{ik}\mathring{g}^{jl}+\mathring{g}^{il}\mathring{g}^{jk}-2\mathring{g}^{ij}\mathring{g}^{kl})$, $\xi_A^{\perp}$ is the component of the Killing vector ξ_A in the direction of the unit normal to the hypersurface $t=$const, h_{ik} is the deviation from the AdS metric, π_k^i is the canonical momentum, and $\mathring{\Delta}_j$ is the covariant derivative with respect to the background three-metric $\mathring{g}_{ik}$. The total energy is the charge associated with the Killing vector $\partial/\partial t$.

It has also been checked that the Henneaux-Teitelboim energy, the Brown-York energy [21], and the Einstein and Landau-Lifshitz energy [19] all agree for the Kerr-AdS spacetime. However, for the same spacetime, the generalized Komar mass has a different value [22].

In this work, we use the Henneaux-Teitelboim formalism to compute numerically the mass energy of a gravitating EYM configuration. Substituting the ansatz (7) in Eq. (27) yields the following expression for the total mass:

$$M=\lim_{r\to\infty}\frac{\Lambda}{12}\int_0^{\pi/2}d\theta\sin\theta\left[-\frac{m-l}{rf}+\frac{\partial}{\partial r}\left(\frac{m+l}{f}\right)\right]r^4. \tag{28}$$

The dimensionless mass is obtained by using the rescaling $M\to(eG/\sqrt{4\pi G})M$.

Solutions of the field equations are also classified by the non-Abelian electric and magnetic charges Q_E and Q_M. The non-Abelian charges defined by

$$\begin{pmatrix} Q_E \\ Q_M \end{pmatrix}=\frac{e}{4\pi}\int dS_k\sqrt{-g}\,\mathrm{Tr}\begin{pmatrix} F^{k0} \\ \tilde{F}^{k0} \end{pmatrix}\tau_r \tag{29}$$

are conserved because the Gauss flux theorem [7].

Within our ansatz, $Q_M=n(1-\omega_0^2)$, $Q_E=0$.

III. YANG-MILLS FIELDS IN FIXED AdS BACKGROUND

Because the asymptotic structure of geometry is different, in an AdS spacetime one does not have to couple the YM system to scalar fields or gravity in order to obtain finite-energy solutions. Here the cosmological constant breaks the scale invariance of pure YM theory to give finite-energy solutions.

It is the purpose of this section to present both analytical and numerical arguments for the existence of nontrivial monopoles and dyon solutions of pure YM equations in a four-dimensional AdS spacetime, gravity being regarded as a fixed field (i.e., $f=l=m=1$; also, in this section we do not use the above-discussed rescaling).

Although extremely simple, nevertheless this model appears to contain all the essential features of the Bjoraker-Hosotani solutions. In this way, what might seem surprising (e.g., the existence of stable solutions) finds a natural explanation.

The existence of these nongravitating solutions has recently been noticed in [9] when discussing the scaling behavior of the EYM monopoles and dyons.

A. Spherically symmetric solutions

We start by briefly discussing the solutions obtained for $n=1$ within the ansatz (8). In this case, the equation of motion has the simple form

$$\left[\omega'\left(1-\frac{\Lambda}{3}r^2\right)\right]'=\frac{\omega(\omega^2-1)}{r^2}, \tag{30}$$

where the prime denotes derivative with respect to r. The numerical results show the existence of a one-parameter family of solutions regular at $r=0$ with behavior familiar from the gravitating case,

$$w(r)=1-br^2+O(r^4), \tag{31}$$

where b is an arbitrary constant. The asymptotic expansion at large r is

$$w=w_0+w_1\frac{1}{r}+\cdots, \tag{32}$$

where w_0,w_1 are constants to be determined by numerical calculations. These boundary conditions permit a nonvanishing magnetic charge Q_M.

The overall picture we find is rather similar to the one described in [7] where gravity is taken into account. By varying the parameter b, a continuum of monopole solutions is obtained. As a new feature, we notice the existence of zero- and one-node monopole solutions ($k=0$,1) only. Also, for a fixed value of Λ, we obtain finite-energy solutions for only one interval in parameter space, $b_{\mathrm{min}}<b<b_{\mathrm{max}}$ (with $b_{\mathrm{min}}<0$). The energy of the solutions is an increasing function of the absolute value of b and diverges at the extremities of the interval. The allowed values of b correspond approximately to the lower branch found in [7] when coupling to gravity. This is not an unexpected feature. We recall that, given a flat spacetime soliton solution, one can expect that it will have (asymptotically flat) gravitating generalizations. Apart from the fundamental gravitating solutions, a sequence of radial excitations is likely to exist [2,23]. For example, in flat spacetime the YMH system (with a doublet scalar field) has only one-node solutions; when gravity is included, the solutions exist for all k [24].

For large enough r, it is possible to write an approximate expression of $w(r)$ in terms of elliptic functions. A nontrivial exact solution of the YM equations is

$$\omega=1/(1-\Lambda/3r^2)^{1/2}, \tag{33}$$

describing a monopole in AdS spacetime with unit magnetic charge and mass $\sqrt{(-3\Lambda)}\pi/8e^2$. It should be mentioned that this is not a new result. The existence of this exact solution has been noticed for the first time in [25] for a positive cosmological constant and a different coordinate system.

The arguments presented in [7] for the (linear) stability of the nodeless $n=1$ monopole solutions apply directly to the nongravitating case. In Bjoraker-Hosotani analysis (Ref. [7], Sec. VII), this corresponds to taking $v=0$, $H=1-\Lambda r^2/3$, and $p=1$, while $\delta H=\delta p=\delta m=0$. We start with the most general expression of a spherically symmetric YM connection,

$$A=\frac{1}{2e}\{u(r,t)\tau_3 dt+\nu(r,t)\tau_3 dr+[w(r,t)\tau_1+\tilde{w}(r,t)\tau_2]d\theta +[\cot\theta\tau_3+w(r,t)\tau_2-\tilde{w}(r,t)\tau_1]\sin\theta\, d\phi\}. \tag{34}$$

All field variables are written as the sum of the static equilibrium solution whose stability we are investigating and a time-dependent perturbation. In examining time-dependent fluctuations around finite-energy solutions, it is convenient to work in the $\delta u(r,t)=0$ gauge. By following the standard methods, we derive linearized equations for $\delta w(r,t)$, $\delta\tilde{w}(r,t)$, and $\delta\nu(r,t)$. The linearized equations for $\delta w(r,t)$ and $\delta\nu(r,t)$ are obtained in [7] for the more general gravitating case [as usual, $\delta\tilde{w}(r,t)$ is determined by $\delta\nu(r,t)$]. For a harmonic time dependence $e^{i\Omega t}$, the linearized system implies two standard Schrödinger equations. The analysis of the potential's properties in these Schrödinger equations can be done following Ref. [7]. The standard arguments presented by Bjoraker and Hosotani are still valid and imply that for nodeless solutions there are no negative eigenvalues for Ω^2 and thus no unstable modes.

B. Axially symmetric solutions

In addition to these spherically symmetric solutions, we study their axially symmetric generalizations. Subject to the boundary conditions (18)–(24), we solve the YM equations numerically.

Our methods are similar to those used by the authors of [16] in their works. The KK scheme solves the field equations following an iteration procedure. One starts with a known spherically symmetric configuration and increases the winding number in small steps. The field equations are first discretized on a nonequidistant grid and the resulting system is solved iteratively until convergence is achieved. In this scheme, a new radial variable is introduced which maps the semi-infinite region $[0,\infty)$ to the closed region $[0,1]$. Thus the region of integration is not truncated and the model converges to a higher accuracy. There are various possibilities for this transformation, but a choice which is flexible enough was $x=r/(c+r)$, where c is a properly chosen constant. Typical grids have sizes 170×30, covering the integration region $0\leqslant x\leqslant 1$ and $0\leqslant\theta\leqslant\pi/2$.

The numerical calculations are performed by using the program FIDISOL, based on the iterative Newton-Raphson method. Details on the FIDISOL code are presented in [26]. To obtain axially symmetric solutions, we start with the $n=1$ solution discussed above as an initial guess and increase the value of n slowly. The iterations converge, and repeating the procedure one obtains in this way solutions for arbitrary n. The physical values of n are integers. The numerical error for the functions is estimated to be lower than 10^{-3}.

The energy density of these nongravitating solutions is given by the tt component of the energy momentum tensor T_μ^ν; integration over all space yields their mass energy

$$E=\int\left\{\tfrac{1}{4}F_{ij}^{i}F_{ij}^{a}+\tfrac{1}{2}F_{it}^{a}F_{it}^{a}\right\}\sqrt{-g}\,d^3x =2\pi\int_0^\infty r^2dr\int_0^\pi\sin\theta\, d\theta\{\tfrac{1}{4}F_{ij}^{i}F_{ij}^{a}+\tfrac{1}{2}F_{it}^{a}F_{it}^{a}\}. \tag{35}$$

The energy and the magnetic charge of solutions are proportional to their winding number n.

For every spherically symmetric configuration (i.e., a given value of parameter b) we have obtained higher winding number generalizations. Moreover, the branch structure noticed for $n=1$ is retained for higher winding number solutions. These solutions have very similar properties with the corresponding EYM counterparts. Therefore, the general picture we present here applies also in the next section.

In Fig. 1, we present the gauge functions H_i and the energy density ϵ as a function of the radial coordinate r for the angles $\theta=0$, $\pi/4$, and $\pi/2$ for three different solutions. Here the winding number is $n=3$ and $\Lambda=-0.01$; also the mass is given in units $4\pi/e^2$. The configuration with ($Q_M=3,M=0.332$) represents a higher winding generalization of the exact solution (33) (i.e., $b=0.001\,66$ and $w_0=0$); we suspect the existence of a general analytic form of this solution (valid for $n\geqslant 1$).

The configurations with $\omega_0\neq 0$ have been arbitrarily selected; the solution with ($Q_M=2.391,M=0.576$) is obtained starting with a spherically symmetric solution with $b=0.003$, while for ($Q_M=-13.284,M=1.118$) we have $b=-0.001$.

The functions H_2 and H_4 have a small θ dependence, although the angular dependence of matter functions generally increases with Q_M. We notice also that the gauge field function H_1 remains nodeless and for every solution with $w_0>1$ it takes only negative values [H_1 and H_3 are zero on the axes in Figs. 1(a) and 1(c)].

In Fig. 2, we show the energy density ϵ and the gauge functions H_i for a nodeless solution with $n=3$, $k=0$, total mass $M=5.481$ (in units $4\pi/e^2$), and magnetic charge $Q_M=-37.185$ as a function of the compactified coordinates $\rho=x\sin\theta$ and $z=x\cos\theta$ (for $\Lambda=-0.01$) [here $x=r/(50+r)$]. The parameter b for the corresponding spherically symmetric solution is $b=-0.0015$. In this case, the functions H_2 and H_4 are almost spherical. The function H_1 does not possess a nontrivial node and takes only negative values.

C. Nongravitating dyon solutions

The existence of dyon solutions without a Higgs field is a new feature for AdS spacetime [7]. If $\Lambda\geqslant 0$, the electric part of the gauge fields is forbidden [7,27]. In order for the boundary conditions at infinity to permit the electric fields and maintain a finite ADM mass, we have to add scalar fields to the theory.

The YM ansatz (8) can be generalized to include an electric part (see, e.g., [14]),

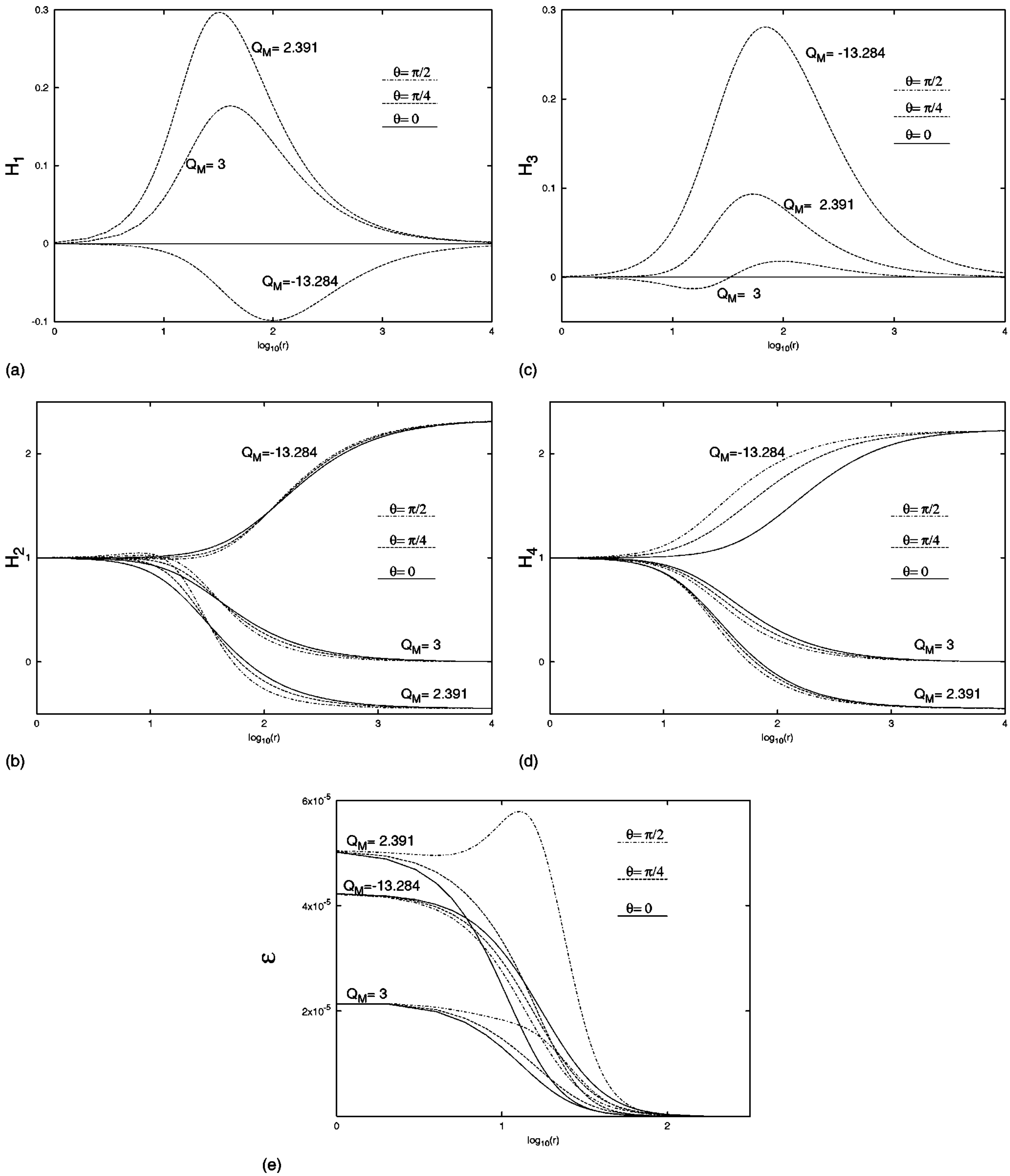

FIG. 1. The gauge functions H_i and the energy density ϵ (in units $4\pi/e^2$) are shown as a function of the radial coordinate r for the angles $\theta=0$, $\pi/4$, and $\pi/2$ for three nongravitating solutions with ($Q_M=2.391, M=0.576$), ($Q_M=3, M=0.332$), and ($Q_M=-13.284, M=1.118$). Here the winding number is $n=3$ and $\Lambda=-0.01$.

$$A_t = H_5(r,\theta)\frac{\tau_r^n}{2e} + H_6(r,\theta)\frac{\tau_\theta^n}{2e}. \quad (36)$$

A possible set of boundary conditions for the electric potentials H_5, H_6 is

$$H_5|_{r=0} = H_6|_{r=0} = 0, \quad H_5|_{r=\infty} = u_0, \quad H_6|_{r=\infty} = 0, \quad (37)$$

and

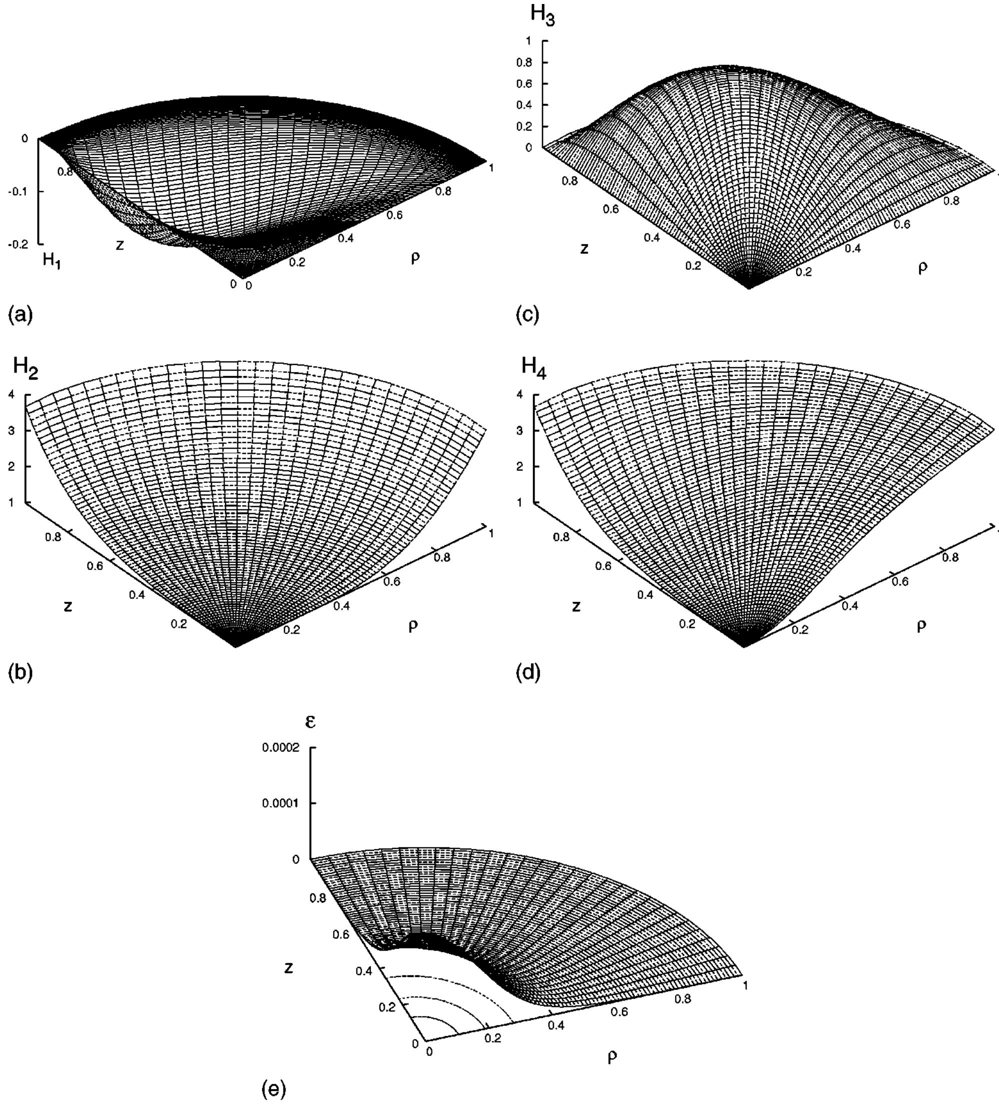

FIG. 2. The gauge functions H_i and the energy density ϵ (in units $4\pi/e^2$) for a monopole solution with $n=3$, $M=5.481$, and $Q_M=-37.185$ are shown as a function of the compactified coordinates z, ρ.

$$\partial_\theta H_5|_{\theta=0,\pi/2}=H_6|_{\theta=0,\pi/2}=0 \tag{38}$$

for a solution with parity reflection symmetry. The equations of motion in this case are

$$\begin{aligned}0=&\{r^2H_{1,r,r}+H_{1,\theta,\theta}+H_{2,\theta}-n^2[rH_{4,r}H_3-rH_{3,r}H_4\\&+(H_3^2+H_4^2-1)H_1]\}\sin^2\theta+[rH_{2,r}+H_{1,\theta}\\&-n^2(2H_1H_3+rH_{4,r})]\sin\theta\cos\theta-n^2H_1\\&+\sin^2\theta[rH_{1,r}-H_{2,\theta}]r\ln(N)_{,r}+\frac{\sin^2\theta r^2}{N}\\&\times[rH_{5,r}H_6-rH_{6,r}H_5+H_1(H_5^2+H_6^2)],\end{aligned} \tag{39}$$

$$\begin{aligned}0=&\Big(r^2H_{2,r,r}+H_{2,\theta,\theta}-H_{1,\theta}+\frac{n^2}{N}[H_3H_{4,\theta}-H_4H_{3,\theta}\\&-(H_3^2+H_4^2-1)H_2]\Big)\sin^2\theta+\Big(H_{2,\theta}-rH_{1,r}\\&+\frac{n^2}{N}(-2H_2H_3+H_{4,\theta})\Big)\sin\theta\cos+\frac{n^2}{N}(H_4-H_2)\\&+\sin^2\theta[H_{1,\theta}+rH_{2,r}]r\ln(N)_{,r}+\frac{\sin^2\theta r^2}{N^2}\\&\times[H_5H_{6,\theta}-H_6H_{5,\theta}+H_2(H_5^2+H_6^2)],\end{aligned} \tag{40}$$

$$0=\left(r^2H_{3,r,r}+\frac{1}{N}H_{3,\theta,\theta}-H_3\left(H_1^2+\frac{H_2^2}{N}\right)+H_1(H_4-2rH_{4,r})\right.$$
$$\left.-H_4\left(rH_{1,r}-\frac{H_{2,\theta}}{N}\right)+\frac{2H_2H_{4,\theta}}{N}\right)\sin^2\theta$$
$$+\frac{1}{N}(H_{3,\theta}+H_2H_4-NH_1^2-H_2^2)\sin\theta\cos\theta-\frac{H_3}{N}$$
$$+\sin^2\theta[rH_{3,r}-H_1H_4]r\ln(N)_{,r}+\frac{\sin^2\theta r^2H_6}{N^2}$$
$$\times[H_4H_5+H_6(H_3+\cot\theta)], \tag{41}$$

$$0=\left[r^2H_{4,r,r}+\frac{1}{N}H_{4,\theta,\theta}-H_4\left(H_1^2+\frac{H_2^2}{N}\right)-H_1H_3\right.$$
$$\left.+\frac{H_2}{N}(1-2H_{3,\theta})+2rH_{3,r}H_1+H_3\left(rH_{1,r}-\frac{H_{2,\theta}}{N}\right)\right]$$
$$\times\sin^2\theta+\frac{1}{N}(-H_2H_3-NH_1$$
$$+rH_{1,r}N-H_{2,\theta}+H_{4,\theta})$$
$$\times\sin\theta\cos\theta+\frac{H_2-H_4}{N}+\sin^2\theta[rH_{4,r}+H_1H_3$$
$$+\cot\theta H_1]r\ln(N)_{,r}+\frac{\sin^2\theta r^2H_5}{N^2}$$
$$\times[H_4H_5+H_0(H_3+\cot\theta)], \tag{42}$$

$$0=\left(r^2H_{5,r,r}+\frac{1}{N}H_{5,\theta,\theta}+2rH_{6,r}H_1+rH_{1,r}H_6\right.$$
$$-\frac{1}{N}[2H_{6,\theta}H_2+H_6(H_{2,\theta}+n^2H_3H_4)$$
$$\left.+H_5(NH_1^2+H_2^2+n^2H_4^2)]\right)\sin^2\theta$$
$$+\frac{1}{N}[H_{5,\theta}-H_6(H_2+n^2H_4)]\sin\theta\cos\theta, \tag{43}$$

$$0=\left(r^2H_{6,r,r}+\frac{1}{N}H_{6,\theta,\theta}-rH_{1,r}H_5-2r_{5,r}H_1\right.$$
$$+\frac{1}{N}[2H_{5,\theta}H_2+H_5(H_{2,\theta}-n^2H_3H_4)-H_6(NH_1^2$$
$$\left.+H_2^2+n^2H_3^2-1)]\right)\sin^2\theta-\frac{n^2H_6}{N}+\frac{1}{N}[H_{6,\theta}$$
$$+H_2H_5+n^2(2H_3H_6+H_4H_5)]\sin\theta\cos\theta. \tag{44}$$

Similar to the asymptotically flat case [28], a vanishing u_0 implies a purely magnetic solution. To prove this, we express the electric part in Eq. (35),

$$E_e=\tfrac{1}{2}\int F_{it}^aF_{it}^a\sqrt{-g}\,d^3x \tag{45}$$

as a surface integral at infinity. We use also the existence of the Killing vector $\partial/\partial t$, which implies

$$F_{it}=D_iA_t \tag{46}$$

and the YM equations (6). Thus we obtain the general result

$$-E_e=\mathrm{Tr}\left(\int\{D_i(A_tF^{it}\sqrt{-g})-A_tD_i(F^{it}\sqrt{-g})d^3x\}\right) \tag{47}$$

and, for a regular configuration,

$$E_e=\mathrm{Tr}\left(\oint_\infty A_tF^{rt}dS_r\right). \tag{48}$$

Therefore, for our ansatz

$$E_e=\frac{\pi u_0}{e^2}\lim_{r\to\infty}\int_0^{\pi/2}r^2\partial_rH_5\sin\theta\,d\theta. \tag{49}$$

This result provides also a useful test to verify the accuracy of the numerical calculations.

When taking $n=1$, $H_5=u(r)$, and $H_6=0$, we find spherically symmetric nongravitating dyon solutions. In this case, we have

$$u(r)=ar+\frac{a}{5}(-2b+\tfrac{1}{3}\Lambda)r^3+O(r^5), \tag{50}$$

at the origin (where a and b are arbitrary constants) and

$$u=u_0+u_1\frac{1}{r}+\cdots \tag{51}$$

at large r, where u_0, u_1, w_0, w_1 are constants to be determined by numerical calculations. The expansion for $w(r)$ is still valid. These boundary conditions permit nonvanishing charges Q_M and Q_E. In order to obtain the value of these charges at some distance, we have calculated the integrand in Eq. (29) in the numerical code.

If the shooting parameter a is nonzero, we find dyon solutions. Solutions are found for a continuous set of parameters a and b; for some limiting values of these parameters, solutions blow up. Given (a,b), the general behavior of the gauge functions w,u is similar to the gravitating case; there are also solutions with $Q_M=0$ but $Q_E\neq 0$. The surprising numerical properties noticed for the gravitating case [7] are found also for our solutions (for example, $Q_M\simeq -1/\sqrt{4\pi}$ at $b=0.0061$ independent of the value of a). In fact, we suspect that these properties reside in the nongravitating sector of the theory.

When studying dyon solutions, we notice the existence of higher-node $(k>1)$ configurations. Also, there are solutions where w does not cross the r axis. For a fixed value of b, the

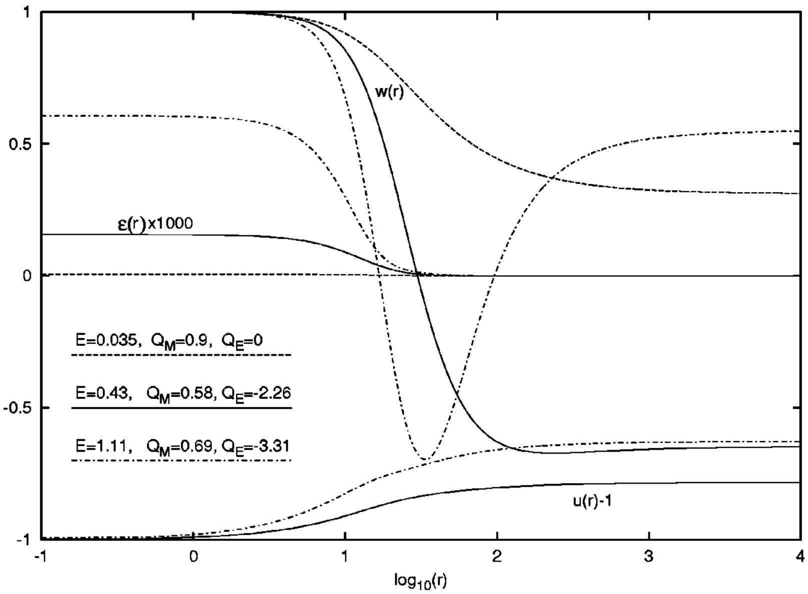

FIG. 3. Typical nongravitating spherically symmetric solutions for $\Lambda=-0.01$, a fixed value of the parameter $b=0.001$, and $a=0.0,0.01,0.02$. The figure for $a=0$ corresponds to a magnetic monopole. The energy density $\epsilon(r)$ is given in units $4\pi/e^2$.

number of nodes is determined by the value of the parameter a. Typical spherically symmetric solutions are displayed in Fig. 3.

The same numerical method described above is used to obtain higher winding number dyon solutions.

Axially symmetric dyon solutions are also known to exist in a SU(2) Yang-Mills-Higgs (when working in flat spacetime background). In the Prasad-Sommerfeld limit, they are known analytically, while for finite Higgs self-coupling they have been recently constructed numerically [14]. To the author's knowledge, there are no known regular axially symmetric dyon solutions (analytical or numerical) in a nonflat geometry. The toroidal shape of the energy density of the monopole solutions is retained for the $n>1$ dyon solutions, as is illustrated in Fig. 4(g). As a typical axially symmetrical configuration, we show in this three-dimensional plot the gauge functions H_i and the energy density ϵ for the solution with $n=2$, $k=1$, total energy $E=1.046$ (in units $4\pi/e^2$), non-Abelian charges $Q_M=-4.124$, and $Q_E=-4.362$ as a function of the compactified coordinates $\rho=x\sin\theta$ and $z=x\cos\theta$ [for a better visualization, we define here $x=r/(100+r)$]. The value of the cosmological constant is $\Lambda=-0.01$. This solution has been obtained starting from a spherically symmetric configuration with the shooting parameters $b=0.01$ and $a=0.005$. As seen in Figs. 4(a)–4(f), gauge functions H_2, H_4, H_5 do not exhibit a strong angular dependence. The H_1, H_3 functions remain nodeless, while the second electric potential H_6 always presents a complicated nodal structure and angular dependence.

The effect of the presence of the YM electric charge is seen in Fig. 5, where the energy density is shown as a function of the radial coordinate for several values of the angle θ, both for an axially symmetric dyon solution and a monopole solution with the same values of magnetic charge and winding number ($\Lambda=-0.01$).

IV. AXIALLY SYMMETRIC SOLUTIONS IN THE PRESENCE OF GRAVITY

In their paper [7], Bjoraker and Hosotani have used Schwarzschild-like coordinates with a line element

$$ds^2=-\frac{H(\tilde r)}{p(\tilde r)^2}dt^2+\frac{1}{H(\tilde r)}d\tilde r^2+\tilde r^2(d\theta^2+\sin^2\theta\, d\phi^2), \tag{52}$$

where

$$H(\tilde r)=1-\frac{2\tilde m(\tilde r)}{\tilde r}-\frac{\Lambda}{3}\tilde r^2. \tag{53}$$

Since we want to use the spherically symmetric solution as the starting point for the calculation of axially symmetric configurations, it is appropriate to transform the known Bjoraker-Hosotani solution to the coordinates which appear in our general metric (7).

Given the presence of a cosmological constant, this coordinate transformation is more complicated than the transformation in [16].

A. Coordinate transformation

By requiring $l=m$ and the metric functions f and m to be only functions of the coordinate r, the axially symmetric isotropic metric (7) reduces to the form

$$ds^2=-f\left(1-\frac{\Lambda}{3}r^2\right)dt^2+\frac{m}{f}\left(\frac{dr^2}{1-(\Lambda/3)r^2}+r^2(d\theta^2+\sin^2\theta\, d\phi^2)\right). \tag{54}$$

The relations (52)–(54) yield

$$\frac{f(r)}{m(r)}=\frac{r^2}{\tilde r^2}=\beta^2 \tag{55}$$

and

$$\frac{dr}{r\sqrt{1-(\Lambda/3)r^2}}=\frac{1}{\sqrt{H(\tilde r)}}\frac{d\tilde r}{\tilde r}. \tag{56}$$

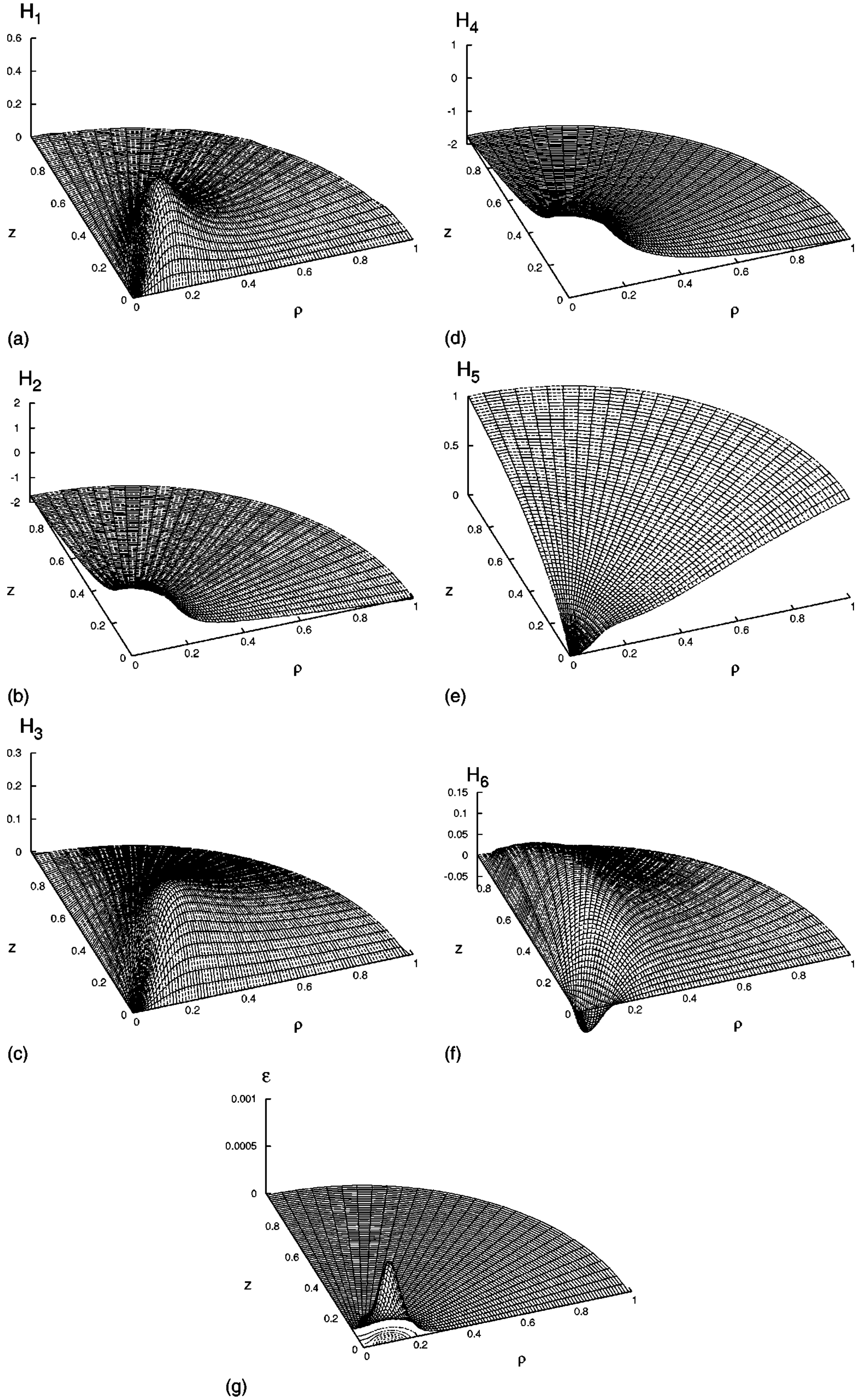

FIG. 4. The gauge functions H_i and the energy density ϵ (in units $4\pi/e^2$) for a nongravitating dyon solution with $n=2$, $k=1$, $E=1.046$, $Q_M=-4.124$, and $Q_E=-4.36$ are shown as a function of the compactified coordinates z, ρ.

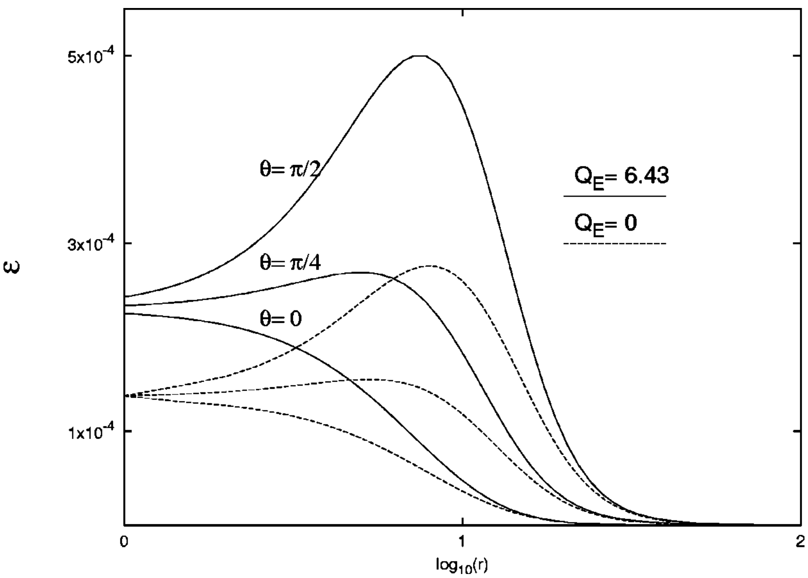

FIG. 5. The energy density ϵ (in units $4\pi/e^2$) for a dyon solution with $E=1.116$, $Q_M=-2.272$, $Q_E=6.43$, and for a monopole solution with $E=0.607$, and the same value of Q_M is shown for several values of the angle θ $(n=2)$.

Since the mass function $\tilde{m}(\tilde{r})$ is only known numerically, we have to numerically integrate Eq. (56) to obtain $r(\tilde{r})$. Therefore, we find

$$\beta(\tilde{r})=\frac{2\Psi}{1+(\Lambda/3)\tilde{r}^2\Psi^2}, \tag{57}$$

with

$$\Psi(\tilde{r})=\exp\left[-\int_{\tilde{r}}^{\infty}\frac{1}{\tilde{r}'}\left(\frac{1}{\sqrt{H}}-1\right)d\tilde{r}'\right]. \tag{58}$$

The integration constant is adjusted such that at infinity $\beta=1$, i.e., $r=\tilde{r}$. The integrand in Eq. (58) is well behaved at the origin, since $m(\tilde{r})\sim\tilde{r}^3$ [7].

For a first branch solution, the values of the metric functions are close to 1. Figure 6(a) demonstrates the coordinate transformation for higher branch spherically symmetric solutions with $k=1-3$ and different magnetic charges. The metric functions f, m, and the gauge field function w are shown in Fig. 6(b). These solutions resemble those obtained for $\Lambda=0$, with quantitative differences only. For example, a smaller value of metric function m at the origin has to be noticed.

B. Numerical method

We employ the same numerical algorithm as for the YM solutions in the fixed AdS background presented above. To obtain axially symmetric solutions, we start with an $n=1$ EYM solution as an initial guess and increase the value of n slowly (for a fixed ω_0). A second procedure, also employed for the first branch solutions, is to start with a known axially symmetric YM solution (with $n=2,3,\ldots$) as an initial guess for the full system.

The numerical error for the functions is estimated to be on the order of 10^{-3} or lower for first branch solutions and 10^{-2} in the rest. This error depends also on the magnetic charge and mass of the solutions. Axially symmetric generalizations of the $n=1$ solutions with a large mass are difficult to obtain.

A set of $\Lambda=0$ test runs was carried out, primarily designed to evaluate the code's ability to reproduce the KK results. In this case, we have obtained an excellent agreement with the results of [16].

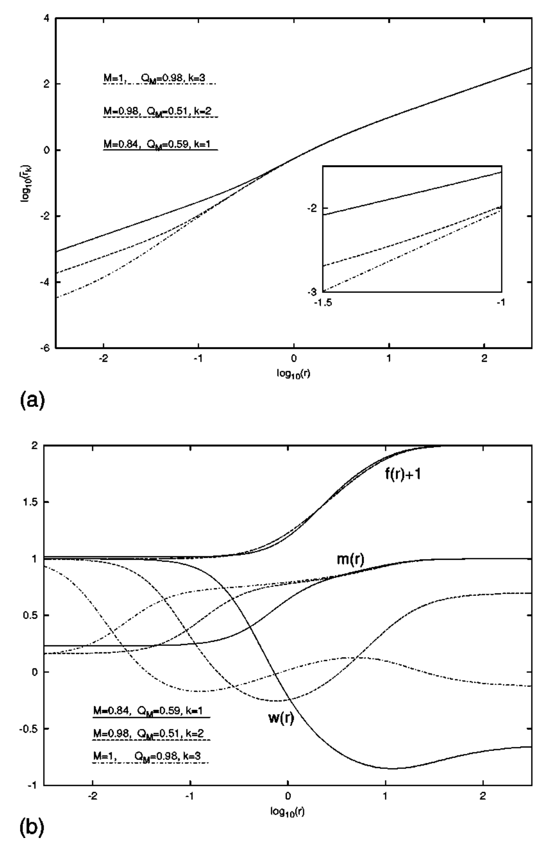

FIG. 6. (a) The coordinate transformation between the isotropic coordinate r and the Schwarzschild-like coordinate $\tilde{r}$ is shown for spherically symmetric solutions with $k=1-3$. (b) The metric functions f,l and the gauge field function w are shown as a function of the isotropic coordinate r for the solutions presented in (a).

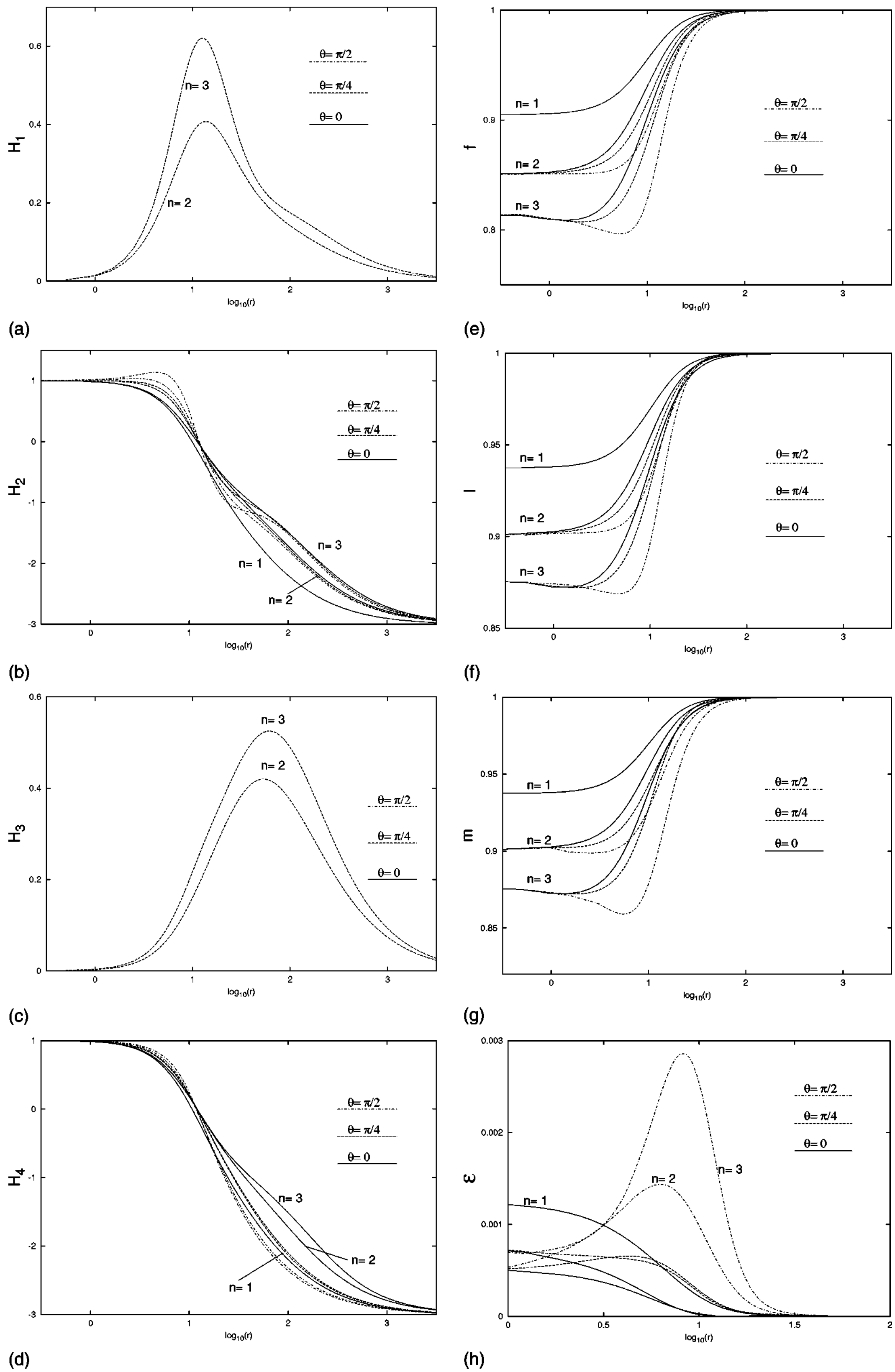

FIG. 7. The metric functions f,l,m, the gauge functions H_i, and the mass density are shown as a function of the radial coordinate r for the angles $\theta=0$, $\pi/4$, and $\pi/2$. Here $n=1$, 2, and 3, $k=1$, $Q_M/n=-8$, $M(n=1)=0.955$, $M(n=2)=2.351$, and $M(n=3)=4.179$.

C. Properties of the solutions

For all the solutions we present, we take a cosmological constant $\Lambda = -0.01$, also the main value considered in Ref. [7]. However, a similar general behavior has been found for other negative values of Λ.

Starting from a spherically symmetric configuration, we obtain higher winding number generalizations with many similar properties. For a fixed winding number, the solutions can also be indexed in a finite number of branches classified by the mass and the non-Abelian magnetic charge. This is in sharp contrast to the $\Lambda = 0$ case, where only a discrete set of solutions is found [16].

The metric functions *f,m,l* are completely regular and show no sign of an apparent horizon.

We begin with a description of the lowest branch axially symmetric regular solutions. In this case, the winding number is $n > 1$ and nodeless or one-node solutions are allowed. These solutions are of particular interest because they are likely to be stable against linear perturbations (for $k = 0$). As expected, the gauge functions H_i look very similar to those of the corresponding (pure) YM solutions. The general picture presented in Fig. 2 is valid in this case, too. The typical values of the metric functions *m,f,l* are closed to 1. These functions do not exhibit a strong angular dependence, while *m* and *l* have a rather similar shape.

To see the change of the functions for an increasing *n*, we exhibit in Fig. 7 first branch solutions with $k = 1$, $\omega_0 = -3$, and $n = 1$, 2, and 3. In Figs. 7(a)–7(d), the gauge field functions are shown, in Figs. 7(e)–7(g), the metric functions, and in Fig. 7(h) the energy density of the matter fields. These two-dimensional plots exhibit the r dependence for three fixed angles $\theta = 0$, $\pi/4$, and $\pi/2$. Note that the H_1, H_3 functions remain nodeless [H_1 and H_3 are zero on the axes in Figs. 7(a) and 7(c) as required by the boundary conditions (23)]. As expected, the angular dependence of the metric and matter functions increases with *n*. However, the location of the nodes of the gauge field functions H_2, H_4 does not move farther outward with increasing *n*. At the same time, the peak of the energy density along the *r* axis slightly shifts outward with increasing *n* and increases in height. At the origin, the values of the metric functions decrease with *n*.

This behavior contrasts with the picture obtained in an asymptotically flat spacetime.

In Fig. 8, the mass of the first branch solutions *M* is plotted as a function of the non-Abelian magnetic charge Q_M for various winding numbers. For the studied configurations, we find that the total mass of a gravitating solution has a smaller value than the mass of the corresponding solution in a fixed AdS background (for fixed Q_M, Λ). This inequality is of course in accord with our intuition that gravity tends to reduce the mass. A similar property has been noticed for monopole solutions in a spontaneously broken gauge theory (without cosmological constant) [29].

Beside these fundamental gravitating solutions, EYM theory possesses also excited solutions not presented in a fixed AdS background.

In this case, the metric functions of this EYM solution are considerably smaller at the origin, and the gauge field functions have their peaks and nodes shifted inwards, as compared to the first branch solutions. The energy density of the matter fields has higher peaks, which are shifted inwards, compared to the fundamental gravitating configurations. Otherwise, many properties of the branch axially symmetric solutions are similar to those of their asymptotically flat counterparts.

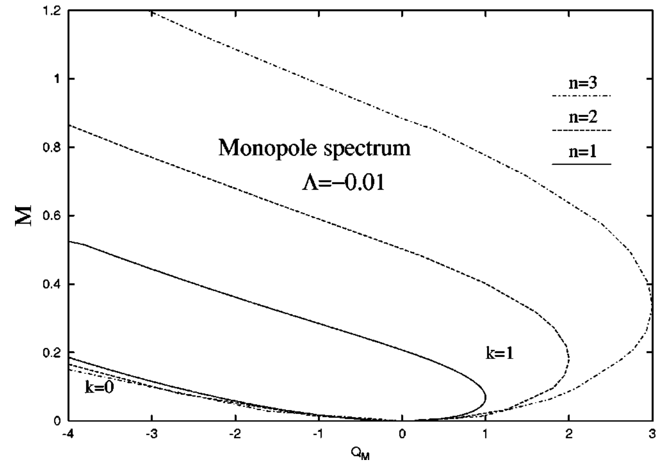

FIG. 8. Mass M is plotted as a function of magnetic charge Q_M for first branch gravitating monopole solutions at $\Lambda = -0.01$. The winding number n is also marked.

To see the change of the functions for a second branch solution, we exhibit in Fig. 9 a three-dimensional plot for a configuration with winding number $n = 2$, node number $k = 1$, magnetic charge $Q_M = 1.18$, and total mass $M = 1.498$. Figure 9(h) presents the energy density of the matter fields ϵ, showing a pronounced peak along the ρ axis and decreasing monotonically along the z axis [here $x = r/(1+r)$]. Equal density contours presented here reveal a toruslike shape of the solutions.

The same general behavior is obtained for two-node solutions.

We do not address in this paper the problem of limiting solutions, which is still unclear even in the spherically symmetric case. Using the metric form (52), Bjoraker and Hosotani have observed that, as the parameter b [in Eq. (31)] is increased, the function $H(\tilde{r})$ hits zero from above for some values of $\tilde{r}$. Also, when $b = b_c$, k has a finite value, $\omega(\tilde{r}_h) = p(\tilde{r}_h) = H'(\tilde{r}_h) = 0$, and the space ends at $r = \tilde{r}_h$. There is a universality in the behavior of the critical solutions [7]. The meaning of the critical spacetime is yet to be clarified.

This behavior strongly contrast with the asymptotically flat case. There are no restrictions on the node number k and, as $k \to \infty$, the BK solutions tend to a configuration that is the union of two parts. A nontrivial part for $\tilde{r} < 1$ represents an oscillating solution, and a simple part for $\tilde{r} > 1$ represents the exterior of an extremal Reissner-Nordstrøm (RN) solution with mass $M = 1$ and charge $Q_M = 1$ [30]. The limiting axially symmetric configuration represents the exterior of an extremal RN solution with mass n and charge $Q_M = n$ [16].

We have found it difficult to obtain axially symmetric generalizations of the spherically symmetric solutions near the critical spacetime, with large errors for the functions. A different metric parametrization appears to be necessary.

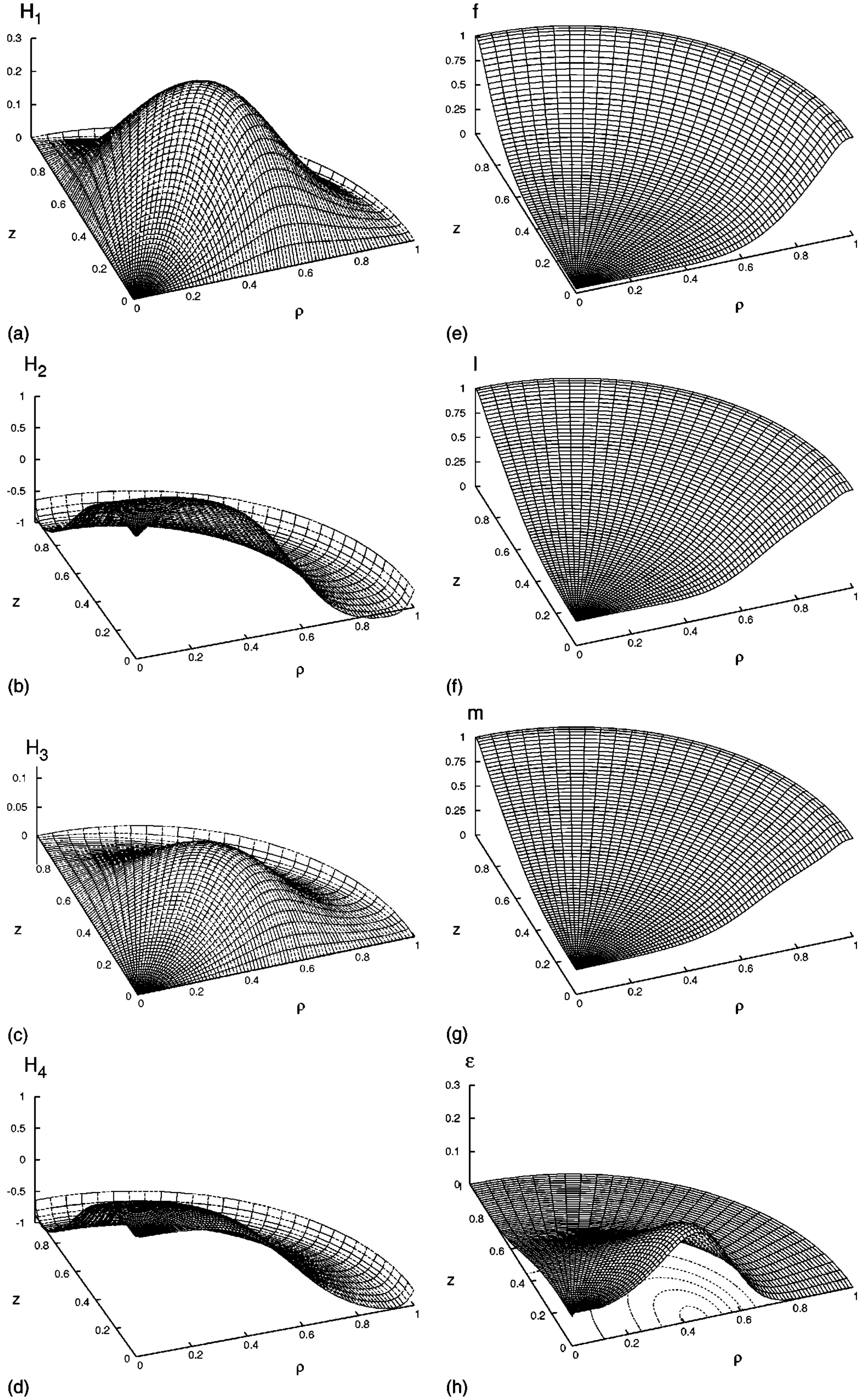

FIG. 9. The metric functions f,l,m, the gauge functions H_i, and the mass density ϵ for a monopole solution with $n=2$, $M=1.498$, and $Q_M=1.18$, are shown as a function of the compactified coordinates z, ρ.

V. CONCLUDING REMARKS

In this paper, we have presented numerical arguments that EYM theory with a negative cosmological constant possesses regular static axially symmetric solutions. They generalize to higher winding number the known spherically symmetric solutions.

We started by presenting arguments that SU(2)-YM theory possesses solutions with nonvanishing magnetic and electric charges and arbitrary winding number when AdS spacetime replaces the Minkowski space as the ground state of the theory. The spherically symmetric solutions we found have properties similar to the lower branch of their known gravitating counterparts.

When including gravity, we have presented results suggesting the existence of axially symmetric solutions. These configurations have continuous values of mass and non-Abelian magnetic charges and present a branch structure. As seen from the figures, the distributions of the mass-energy density $-T_t^t$ can be different from those of spherical configurations (i.e., almost toroidal distributions). As a result of these distributions of mass-energy density, the spacetime structure of our solutions can be considerably nonspherical and strongly axisymmetric. We have noticed a somewhat different behavior of the $k=1$ fundamental gravitating solutions as compared to higher-node excitations.

We have not considered the question of stability for higher winding number solutions. However, we expect that the nodeless, lower branch of axially symmetric solutions is stable. A rigorous proof is, however, desirable, analogous to the proof given for the spherically symmetric case.

In Ref. [9], a scaling law is derived for the mass spectrum of the spherically symmetric solutions with respect to their non-Abelian charges Q_E and Q_M, Λ, and the parameter $v=4\pi G/e^2$. The mass of monopoles and dyons is expressed in terms of a universal function $f(Q_M,Q_E)$ independent of v, Λ, and also k. The monopole and dyon solutions in the lowest branch $(k=0)$ are essentially the solutions in the fixed AdS background metric and will be stable. The solutions in the higher branches $(k>0)$ are obtained by dressing monopole and dyon solutions in the fixed AdS background metric around the BK solutions in the asymptotically flat space. As all BK solutions are unstable, the monopole and dyon solutions in the higher branches are also unstable.

We suspect the existence of a similar behavior for the axially symmetric monopole solutions discussed in this paper. We have found already that for $n>1$, the lowest branch monopole solutions are essentially solutions in a pure YM theory. Here the BK solutions are replaced with the KK solutions. The axial symmetry will introduce a new parameter to the universal scaling function, namely the winding number n.

As discussed in [7,9], the soliton solutions depend nontrivially on the value of Λ. A fractal structure in the moduli space of the solutions has been observed [7,31]. New branches emerge as $|\Lambda|$ becomes smaller and the shape of the branches has approximate self-similarity. As $\Lambda\to 0$, solutions on a branch collapse to a point; the BK solution and the nodeless solutions disappear as their ADM mass vanishes. We did not investigate this point in the present work, restricting ourselves to $\Lambda=-0.01$. However, it is very probable that this general picture remains valid for the axially symmetric configurations and the continuum of solutions becomes a discrete set as $\Lambda\to 0$. Here also, in this limit, the BK solutions are replaced with the KK generalizations.

Actually, due to conformal invariance of the YM equations and the fact that the AdS metric is reduced to the flat metric by a conformal transformation, any solution on AdS background corresponds to some solution in the flat space. In Minkowski coordinates, these are nonstationary solutions. For example, the monopole solution (33) corresponds to the well-known flat-space meron solution [32].

Analogous solutions with higher winding number should also exist when including a Higgs or a dilaton field in a theory with $\Lambda<0$. We conjecture the existence of axially symmetric gravitating YM black hole solutions with a nonvanishing cosmological constant. These would be the AdS spacetime generalizations of the asymptotically flat solutions discussed in [33]. The axially symmetric nodeless solutions are of particular interest, because they are likely to be stable against linear perturbations. For $\Lambda>0$, axially symmetric EYM solutions generalizing the spherically symmetric configurations found in [4] should exist also.

Finally, this is not the complete story: the investigations can be extended to the gravitating axially symmetric dyon solutions. This can be an important issue, since in the asymptotically flat case this problem has not been solved yet. For $\Lambda\geqslant 0$, there are no-go theorems forbidding the spherically symmetric dyon regular solutions [27,7]. Also, the authors of [28] conjectured the absence of charged regular EYM solutions. However, as shown in [34], the BK solutions admit slowly rotating charged generalizations. The total angular momentum of these solutions is proportional to the non-Abelian electric charge. Therefore, it is natural to look for (gravitating) axially symmetric dyon solutions, which, however, have not been found so far within a nonperturbative approach.

In a theory with a negative cosmological constant, the boundary conditions at infinity allow for spherically symmetric dyon solutions. Starting from the spherically symmetric $n=1$ solutions, we have obtained nongravitating axially symmetric dyon YM configurations. We suppose that these configurations will survive when coupling to gravity.

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