Cosmology of the Planck era from a renormalization group for quantum gravity

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Homogeneous and isotropic cosmologies of the Planck era before the classical Einstein equations become valid are studied taking quantum gravitational effects into account. The cosmological evolution equations are renormalization group improved by including the scale dependence of Newton's constant and of the cosmological constant as it is given by the flow equation of the effective average action for gravity. It is argued that the Planck regime can be treated reliably in this framework because gravity is found to become asymptotically free at short distances. The epoch immediately after the initial singularity of the Universe is described by an attractor solution of the improved equations which is a direct manifestation of an ultraviolet attractive renormalization group fixed point. It is shown that quantum gravity effects in the very early Universe might provide a resolution to the horizon and flatness problems of standard cosmology, and could generate a scale-free spectrum of primordial density fluctuations.

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I. INTRODUCTION

Two of the most frequently discussed limitations of the cosmological standard model are the flatness and the horizon problem, respectively. These so-called "problems" actually do not endanger the internal consistency of the standard model in the domain where it is applicable but rather express the fact that in order to describe the Universe as we observe it today the standard Friedmann-Robertson-Walker evolution has to start from a set of highly nongeneric initial conditions. Typically these conditions are imposed at some time after the Planck era where the classical Friedmann equations are supposed to become valid. The matter density ρ of the present Universe is very close to the critical density ρ_{crit}. According to the evolution equations of the standard model this implies that the initial value for ρ must have been fine-tuned to the critical density with the enormous precision of about 60 decimal places if the initial conditions are imposed at the Planck time. This phenomenon is referred to as the flatness problem because a generic initial value for the density would never have led to the large and almost flat Universe we observe today. More generally, if one allows for a cosmological constant Λ, it is the total density $\rho_{\text{tot}}=\rho+\rho_\Lambda$ with the vacuum energy density $\rho_\Lambda\equiv\Lambda/8\pi G$ that should be equal to ρ_{crit}.

A similar naturalness problem is posed by the high degree of isotropy of the cosmic microwave background radiation. From the observations we know that even those points on the last scattering hypersurface which, according to the metric of the cosmological standard model, have never been in causal contact emit radiation at a temperature that is constant with a precision of about 10^{-4}. Again, when equipped with sufficiently symmetric initial conditions the cosmological standard model can describe the later evolution of such a highly isotropic universe, but clearly it would be very desirable to identify some causal mechanism that explains why one must start the classical evolution with these very special initial conditions. This is usually called the horizon problem because those Robertson-Walker spacetimes that solve the Friedmann equations have a particle horizon. Because of this horizon, there are points on the last scattering surface whose backward light cones never intersect and which are therefore causally disconnected.

However, strictly speaking this is a "problem" only if one applies the standard model in a domain where it is actually believed not to be valid any more. Whether or not a Robertson-Walker spacetime has a particle horizon depends only on the behavior of its scale factor $a(t)$ in the limit $t\rightarrow 0$. In the ordinary radiation dominated Universe we have $a\propto t^{1/2}$ which does lead to a horizon. However, we expect that for the cosmological time t very close to the big bang ($t=0$) this behavior of $a(t)$ will get modified by some sort of "new physics." If, say, $a\propto t^{\alpha}$ with $\alpha\geqslant 1$ during the very early evolution of the Universe then there is no particle horizon. It might be that a causal mechanism which is operative during this early epoch, before the standard model becomes valid can explain the observed isotropy of the Universe.

It is well known that the above naturalness problems can be addressed and, in a sense, solved within the framework of inflationary cosmology [1], for instance. In the present paper we are going to propose a different physical mechanism which also could lead to a solution of the horizon and the flatness problem. Using renormalization group techniques we determine the leading quantum gravity corrections that modify the standard Friedmann-Robertson-Walker (FRW) cosmology during the first few Planck times after the big bang. Within a certain approximation, which we shall describe in detail below, we find that immediately after the big bang there is a period during which the scale factor increases

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linearly with time, $a \sim t$. This means that the spacetime has no particle horizon. We shall set up a system of quantum corrected cosmological evolution equations for $a(t)$, $\rho(t)$, $p(t)$, and for the now time dependent Newton constant and cosmological constant. We shall argue that, because of a specific form of asymptotic freedom enjoyed by quantum gravity, those equations are reliable even for times infinitesimally close to the big bang where the gravitational coupling constant goes to zero. During the epoch directly after the big bang the quantum corrected equations are uniquely solved by an essentially universal attractor-type solution. For a spatially flat geometry the attractor satisfies $\rho = \rho_\Lambda = \rho_{\rm tot}/2$ and $\rho_{\rm tot} = \rho_{\rm crit}$. For t much larger than the Planck time, the quantum corrected solutions approach those of classical FRW cosmology. Since the quantum solutions are valid for all $t > 0$, they automatically prepare the initial condition $\rho_{\rm tot} = \rho_{\rm crit}$ for the classical regime if one decides for a spatially flat Universe. Hence no fine-tuning is necessary.

In this paper we employ the exact renormalization group approach to quantum gravity which was developed in Ref. [2]. Its basic ingredient is the effective average action $\Gamma_k[g_{\mu\nu}]$, a Wilsonian coarse grained free energy which depends on a momentum scale k. Loosely speaking, Γ_k describes the dynamics of metrics that have been averaged over spacetime volumes of linear dimension k^{-1}; i.e., k is a measure for the resolution of the "microscope" with which a system is observed. The functional $\Gamma_k[g_{\mu\nu}]$ defines an effective field theory appropriate for the scale k. This means that, when evaluated at *tree* level, Γ_k correctly describes all gravitational phenomena, *including all loop effects*, if the typical momenta involved are all of the order of k. The action Γ_k is constructed in a similar way to the ordinary effective action Γ, to which it reduces in the limit $k \to 0$. It has the additional feature of a built-in infrared (IR) cutoff at the momentum k. Quantum fluctuations with momenta $p^2 > k^2$ are integrated out in the usual way, while the contributions coming from large-distance metric fluctuations with $p^2 < k^2$ are not included in Γ_k. When regarded as a function of k, Γ_k describes a renormalization group (RG) trajectory in the space of all action functionals. This trajectory can be determined by solving an exact functional renormalization group equation or "flow equation." The trajectory interpolates between the classical action $S = \Gamma_{k\to\infty}$ and the ordinary effective action $\Gamma = \Gamma_{k\to 0}$. More precisely, in order to quantize a renormalizable fundamental theory with action S one integrates the RG equation from an initial point $\Gamma_{\hat{k}} = S$ down to $\Gamma_0 \equiv \Gamma$. After appropriate renormalizations one then lets $\hat{k} \to \infty$. The RG equation can also be used in order to further evolve (coarse-grain) effective field theory actions from one scale to another. In this case no UV limit $\hat{k} \to \infty$ needs to be taken. The evolution of the effective average action from k_1 down to $k_2 < k_1$ is always well defined even if (as in the case at hand) the model defined by Γ_{k_1} is not perturbatively renormalizable.

Approximate yet nonperturbative solutions to the RG equation that do not require an expansion in a small coupling constant can be obtained by the method of "truncation." The idea is to project the RG flow from the infinite dimensional space of all action functionals onto some finite dimensional subspace which is particularly relevant. In this manner the functional RG equation becomes a system of ordinary differential equations for a finite set of generalized coupling constants which serve as coordinates on this subspace. In Ref. [2] the flow was projected on the 2-dimensional space spanned by the operators $\int \sqrt{g} R$ and $\int \sqrt{g}$ ("Einstein-Hilbert truncation"). The corresponding generalized couplings are the scale dependent ("running") Newton constant $G(k)$ and the cosmological constant $\Lambda(k)$. In the original paper [2] the differential equations governing the k dependence of $G(k)$ and $\Lambda(k)$ were derived, and in [3,4] their solutions were discussed further. In particular one finds that if one increases k from small values (large distances) to higher values (small distances) the value of $G(k)$ decreases, i.e. gravity is asymptotically free, as in non-Abelian gauge theories. For $k \to \infty$ the dimensionless Newton constant $g(k) \equiv k^2 G(k)$ approaches a non-Gaussian UV attractive fixed point $g_*^{\rm UV}$. This means that $G(k)$ vanishes proportionally to $1/k^2$ for $k \to \infty$. The non-Gaussian fixed point of 4-dimensional quantum gravity is similar to the Weinberg fixed point in $2+\epsilon$ dimensions [5].

In the following we shall use the known results about the running of $G(k)$ and $\Lambda(k)$ in order to "renormalization group improve" the Einstein equations that govern the evolution of the Universe. They contain Newton's constant G and the cosmological constant Λ. The improvement is done by substituting $G \to G(k)$, $\Lambda \to \Lambda(k)$, and by expressing k in terms of the geometrically relevant IR cutoff. Considering only homogeneous and isotropic cosmologies we shall argue that the correct identification of the cutoff is $k \propto 1/t$ where t is the cosmological time.

Similar RG improvements are standard tools in particle physics. A first gravitational RG improvement based upon the effective average action was described in Refs. [10,4] where quantum effects in black hole spacetimes were studied.

The applicability of the Wilsonian RG equations is not restricted to renormalizable models. Already, before it was introduced, gravitational RG flows were studied using the familiar RG equations of perturbative renormalization theory which refers to the relevant and marginal couplings only. This framework applies to R^2 gravity [6,7], for instance, but not to ordinary general relativity. The running of $G(k)$ in R^2 gravity was used in [8,9] to explore possible cosmological manifestations of quantum gravity at the kiloparsec scale (rotation curves of galaxies, density perturbations, etc.). Since we are interested in much smaller length scales we shall have nothing to say about such effects.

In the present paper we shall set up a system of differential equations which consists of the RG equations for G and Λ, the improved Einstein equations, an additional consistency condition dictated by the Bianchi identities, and the equation of state of the matter sector. This system determines the evolution of G, Λ, a, ρ and p as a function of the cosmological time t. We shall see that for $t \searrow 0$ all solutions to this system have a simple power law structure. This attractor-type solution fixes $\rho_{\rm tot} = \rho_{\rm crit}$ without any fine-tuning. If the matter system is assumed to obey the equation of state of ordinary radiation, the scale factor expands linearly, $a(t)\propto t$, so that the RG-improved spacetime has no particle horizon. For t much larger than the Planck time the solutions of the RG-improved system approach those of standard FRW cosmology.

The remaining sections of this paper are organized as follows. In Sec. II we review the essential properties of the effective average action for gravity and the solutions of its RG equation which we need in the present context. In Sec. III we describe the derivation of the RG improved Einstein equations and in Sec. IV we obtain solutions to it that are valid for $t\to 0$ and $t\to\infty$, respectively. In Sec. V we investigate the physical properties of solutions that are valid during the entire Planck era. In Sec. VI we discuss the generation of primordial density perturbations and Sec. VII contains the conclusions.

In the main body of this paper we use a specific identification of the cutoff k in terms of the cosmological time ($k \propto 1/t$). In Appendix A we compare the results to those obtained with a different cutoff $[k\propto 1/a(t)]$. In the main part of the paper we improve the basic *equations* for the cosmological evolution. In Appendix B we describe the alternative strategy of improving the *solutions* to the classical equations.

II. THE EFFECTIVE AVERAGE ACTION FOR GRAVITY

In this section we review some properties of the effective average action $\Gamma_k[g_{\mu\nu}]$ and collect various results that we shall need in the present investigation. The average action for gravity was constructed in [2] using an approach which in earlier work [11–14] had already been tested for Yang-Mills theory.

The definition of $\Gamma_k[g_{\mu\nu}]$ is based upon a modified gauge fixed path integral of d-dimensional Euclidean gravity in the background gauge. The crucial new ingredient is an IR cutoff which suppresses the contributions from long-wavelength metric fluctuations with momenta smaller than k. In a second step, the functional Γ_k defined by the modified path integral is shown to satisfy an exact functional differential equation, the flow equation, from which Γ_k, for all values of k, can be computed if it is known at some initial point $\hat{k}$. In order to obtain an action $\Gamma_k[g]$ that is invariant under general coordinate transformations the standard background gauge formulation has been employed. This leads to the complication that we actually have to RG-evolve an action $\Gamma_k[g,\bar{g}]$ that depends on both the "ordinary" metric $g_{\mu\nu}$ and on the background metric $\bar{g}_{\mu\nu}$. The standard action with one argument is recovered by setting $\bar{g}=g$, i.e. $\Gamma_k[g]\equiv\Gamma_k[g,g]$. The flow equation for $\Gamma_k[g,\bar{g}]$ reads[1]

$$\begin{aligned}k\partial_k\Gamma_k[g,\bar{g}]=&\frac{1}{2}\mathrm{Tr}\{\kappa^{-2}(\Gamma_k^{(2)}[g,\bar{g}]\\&+\mathcal{R}_k^{\mathrm{grav}}[\bar{g}])^{-1}k\partial_k\mathcal{R}_k^{\mathrm{grav}}[\bar{g}]\}-\mathrm{Tr}\{(-\mathcal{M}[g,\bar{g}]\\&+\mathcal{R}_k^{\mathrm{gh}}[\bar{g}])^{-1}k\partial_k\mathcal{R}_k^{\mathrm{gh}}[\bar{g}]\}\end{aligned} \tag{2.1}$$

where $\Gamma_k^{(2)}$ stands for the Hessian of Γ_k with respect to $g_{\mu\nu}$ and $\mathcal{M}$ is the Faddeev-Popov ghost operator. The operators $\mathcal{R}_k^{\mathrm{grav}}$ and $\mathcal{R}_k^{\mathrm{gh}}$ implement the IR cutoff in the graviton and the ghost sector. They are defined in terms of a to some extent arbitrary smooth function $\mathcal{R}_k(p^2)\propto k^2R^{(0)}(p^2/k^2)$ by replacing the squared momentum p^2 with the graviton and the ghost kinetic operators, respectively. Inside loops, they suppress the contribution of infrared modes with covariant momenta $p<k$. The function $R^{(0)}(z)$, $z\equiv p^2/k^2$, has to satisfy the conditions $R^{(0)}(0)=1$ and $R^{(0)}(z)\to 0$ for $z\to\infty$. For explicit computations the exponential cutoff

$$R^{(0)}(z)=z[\exp(z)-1]^{-1} \tag{2.2}$$

is particularly convenient.

In order to find approximate but nonperturbative solutions to the flow equation the Einstein-Hilbert truncation was adopted in [2]. This means that the RG flow in the space of all actions is projected onto the two-dimensional subspace spanned by $\int\sqrt{g}$ and $\int\sqrt{g}R$. This truncation of the "theory space" amounts to considering only actions of the form[2]

$$\begin{aligned}\Gamma_k[g,\bar{g}]=&[16\pi G(k)]^{-1}\int d^dx\sqrt{g}\{-R(g)+2\Lambda(k)\}\\&+\text{classical gauge fixing}\end{aligned} \tag{2.3}$$

where $G(k)$ and $\Lambda(k)$ denote the running Newton constant and cosmological constant, respectively. More general (and, therefore, more precise) truncations would include higher powers of the curvature tensor as well as nonlocal terms [15], for instance. By inserting Eq. (2.3) into Eq. (2.1) and performing the projection we obtain a coupled system of equations for $G(k)$ and $\Lambda(k)$. It is most conveniently written down in terms of the dimensionless Newton constant

$$g(k)\equiv k^{d-2}G(k) \tag{2.4}$$

and the dimensionless cosmological constant

$$\lambda(k)\equiv\Lambda(k)/k^2. \tag{2.5}$$

One finds

$$k\partial_k g=[d-2+\eta_N]g \tag{2.6}$$

and

[1]This is already a simplified form of the flow equation appropriate for truncations that neglect the running of the ghost term. For its most general form, see [2].

[2]In [2] the notation $G_k\equiv G(k)$ and $\bar{\lambda}_k\equiv\Lambda(k)$ was used.

$$k\partial_k\lambda=-(2-\eta_N)\lambda+\frac{1}{2}g(4\pi)^{1-d/2}[2d(d+1)\Phi^1_{d/2}(-2\lambda)-8d\Phi^1_{d/2}(0)-d(d+1)\eta_N\tilde{\Phi}^1_{d/2}(-2\lambda)]. \quad (2.7)$$

Here

$$\eta_N(g,\lambda)=\frac{gB_1(\lambda)}{1-gB_2(\lambda)} \quad (2.8)$$

is the anomalous dimension of the operator $\sqrt{g}R$, and the functions $B_1(\lambda)$ and $B_2(\lambda)$ are given by

$$B_1(\lambda)\equiv\frac{1}{3}(4\pi)^{1-d/2}[d(d+1)\Phi^1_{d/2-1}(-2\lambda)-6d(d-1)\Phi^2_{d/2}(-2\lambda)-4d\Phi^1_{d/2-1}(0)-24\Phi^2_{d/2}(0)], \quad (2.9)$$

$$B_2(\lambda)\equiv-\frac{1}{6}(4\pi)^{1-d/2}[d(d+1)\tilde{\Phi}^1_{d/2-1}(-2\lambda)-6d(d-1)\tilde{\Phi}^2_{d/2}(-2\lambda)]$$

with the threshold functions ($p=1,2,\dots$)

$$\Phi^p_n(w)=\frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}\frac{R^{(0)}(z)-zR^{(0)\,\prime}(z)}{[z+R^{(0)}(z)+w]^p}, \quad (2.10)$$

$$\tilde{\Phi}^p_n(w)=\frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}\frac{R^{(0)}(z)}{[z+R^{(0)}(z)+w]^p}.$$

These equations are valid for an arbitrary spacetime dimension d. In the following we shall focus on the case $d=4$.

Clearly it is not possible to find solutions to the system (2.6), (2.7) in closed form; for a numerical determination of the phase diagram we refer to [16]. However, for our purposes it will be sufficient to know the behavior of the solutions in the limiting cases $k\to 0$ and $k\to\infty$. For small values of the cutoff the solutions are power series in k. For the dimensionful quantities one obtains [2]

$$G(k)=G_0[1-\omega G_0k^2+O(G_0^2k^4)], \quad (2.11)$$

$$\Lambda(k)=\Lambda_0+\nu G_0k^4[1+O(G_0k^2)] \quad (2.12)$$

with the constants

$$\omega=\frac{1}{6\pi}[24\,\Phi^2_2(0)-\Phi^1_1(0)], \quad (2.13)$$

$$\nu=\frac{1}{4\pi}\Phi^1_2(0). \quad (2.14)$$

As it stands, Eq. (2.12) for $\Lambda(k)$ is correct only if one either neglects the back reaction of the running Λ via the Φ functions, or chooses $\Lambda_0=0$. For $\Lambda_0>0$ and with the back reaction due to the argument of $\Phi^1_2(-2\Lambda/k^2)$ included, the RG trajectory runs into a singularity and cannot be continued below a certain critical value of k. This is probably due to the fact that the Einstein-Hilbert truncation is too simple to describe the IR behavior of quantum gravity with a positive cosmological constant. Since in this paper we are mostly interested in UV physics we avoid this problem by restricting ourselves to the case $\Lambda_0=0$.

The precise values of ω and ν depend on the choice of the cutoff function $R^{(0)}$. For every admissible $R^{(0)}$ both constants are positive, however. In Eqs. (2.11) and (2.12) we wrote $G_0\equiv G(k=0)$ and $\Lambda_0\equiv\Lambda(k=0)$ for the infrared values of G and Λ. At least within the Einstein-Hilbert truncation, $G(k)$ does not run any more between scales where Newton's constant has been determined experimentally (laboratory scale, scale of the solar system, etc.), and the cosmological scale where $k\approx 0$. Therefore we may identify G_0 with the experimentally observed value of Newton's constant. We use G_0 in order to define the (conventional) Planck mass $m_{\rm Pl}$, Planck length $l_{\rm Pl}$, and Planck time $t_{\rm Pl}$:

$$m_{\rm Pl}=G_0^{-1/2},\quad l_{\rm Pl}=t_{\rm Pl}=G_0^{1/2}. \quad (2.15)$$

The solutions (2.11) and (2.12) are expansions in the dimensionless ratio $(k/m_{\rm Pl})^2$. Obviously the renormalization effects become strong only if k is about as large as $m_{\rm Pl}$. We see that $G(k)$ decreases when we increase k, which is a first hint at the asymptotic freedom of pure quantum gravity [2].

In the following we shall say that k is in the *perturbative regime* if the approximations (2.11) and (2.12) are valid, i.e. if $k\lesssim m_{\rm Pl}$, so that the first order in the $(k/m_{\rm Pl})$ expansion is sufficient to describe the running of G and Λ.

Next let us look at the opposite limiting case when $k\gg m_{\rm Pl}$. It turns out [3,4,16,17] that for $k\to\infty$ the physically relevant RG trajectories in (g,λ) space run into a UV-attractive fixed point $(g_*^{\rm UV},\lambda_*^{\rm UV})$. For the exponential cutoff (2.2) the numerical analysis [3,16,17] of Eqs. (2.6), (2.7) yields the values $g_*^{\rm UV}\approx 0.27$ and $\lambda_*^{\rm UV}\approx 0.36$. (If one neglects the running of λ there is still a fixed point for g at $g_*^{\rm UV}\approx 0.71$.) The existence of this fixed point implies that for $k\gg m_{\rm Pl}$ the dimensionful quantities run according to

$$G(k)=\frac{g_*^{\rm UV}}{k^2}, \quad (2.16)$$

$$\Lambda(k)=\lambda_*^{\rm UV}k^2. \quad (2.17)$$

We shall say that k is in the *fixed point regime* if $k\gg m_{\rm Pl}$ so that the asymptotic solutions (2.16), (2.17) apply.

For intermediate values of k the RG equations can be solved numerically only. However, if one neglects the influence of Λ on the running of G [and omits a tiny correction coming from $B_2(0)$] one obtains the following simple formula which is valid for all k [4]:

$$G(k)=\frac{G_0}{1+\omega\, G_0k^2}. \quad (2.18)$$

For k small we recover Eq. (2.11), and for $k^2 \gg G_0^{-1}$ the fixed point behavior sets in, $G(k) \approx 1/\omega k^2$, so that $G(k)$ becomes independent of its IR value G_0.

We observe that for $k \to \infty$ Newton's constant, and hence the strength of the gravitational interaction, decreases very rapidly so that gravity is "asymptotically free." In fact, G runs much faster than the gauge coupling constant in Yang-Mills theory, which depends on k only logarithmically. An asymptotic running of the form (2.16) was conjectured by Polyakov [19]. A similar powerlike running of G was already known to occur in $(2+\epsilon)$-dimensional gravity [5,2]. In fact, the fixed point $(g_*^{\mathrm{UV}}, \lambda_*^{\mathrm{UV}})$ is the 4-dimensional counterpart of Weinberg's fixed point in $2+\epsilon$ dimensions [3]. If the existence of the fixed point can be confirmed by more general truncations this means that Einstein gravity in 4 dimensions is "asymptotically safe" and as well behaved and predictive as a perturbatively renormalizable theory [5].

Newton's constant being an asymptotically free coupling means that the gravitational interaction is "switched off" when we go to very large momenta or small distances. This "tames" the notorious UV divergences one finds in perturbation theory [19]. In principle the running of G could be tested in scattering processes with a large momentum transfer, in complete analogy with deep inelastic scattering in QCD, for instance. Very much like QCD deep inside a proton, say, gravity is very weakly coupled at sub-Planckian length scales. We describe this situation in a formalism where spacetime is still a smooth manifold at short distances, still equipped with a tensor field $g_{\mu\nu}$, but since the coupling constant vanishes the graviton no longer mediates any matter or self-interaction. (As for the absence of local gauge interactions, this "phase" of gravity is reminiscent of a topological field theory, although free propagating gravitons do exist in the present case.)

Recently considerable new evidence has been found that suggests that the UV fixed point is not an artifact of the Einstein-Hilbert truncation but should actually exist in the exact theory. In [17] and [18] a comprehensive analysis of the quality of the Einstein-Hilbert truncation was performed. In [17] the scheme (cutoff) dependence of its predictions was investigated in detail by using two types of cutoff action $\Delta_k S$ of a rather different structure along with different families of shape function $R^{(0)}$. In an exact treatment universal quantities such as critical exponents or, in our case, the product $g_* \lambda_*$ are scheme independent by definition. Approximations spoil this scheme independence though. As a consequence, by looking at the response of universal quantities to a variation of the cutoff function we can judge the quality of the approximation. The Einstein-Hilbert truncation successfully passed these highly nontrivial tests, partly even at a rather surprising level of accuracy [17].

Furthermore, in [18] the truncation was generalized by including an R^2 term. Quite remarkably, it turned out that, within the residual scheme dependence, the results of the Einstein-Hilbert truncation are not changed at all. The fixed point value of the R^2 coefficient was found to be about two orders of magnitude smaller than g_* and λ_*, and one is led to speculate that it might turn out to be zero in an exact treatment. By linearizing the RG flow near the UV fixed point one finds that two of the eigenvalues (critical exponents) of the stability matrix are essentially the same as in the pure Einstein-Hilbert truncation; the plane spanned by the corresponding eigenvectors coincides quite precisely with the g-λ plane. The fixed point is UV attractive in all 3 directions of parameter space. The new, third eigenvalue is such that when a RG trajectory approaches the fixed point from below ($k \to \infty$) it is pushed onto the g-λ plane long before the fixed point is reached. Hence the vicinity of the fixed point is well described by the Einstein-Hilbert truncation alone.

Conversely, when k is lowered, it is not before k approaches the Planck scale *from above* that higher operators such as R^2 are generated. (By approaching the Planck scale *from below* perturbation theory also suggests that higher operators are important *near* the Planck scale, but it fails to discover that they are unimportant again far beyond it.) It appears that both QCD and gravity can be described by simple local actions for $k \to \infty$. Only for sufficiently small values of k, when one leaves the asymptotic scaling region, does the description become very involved because many new operators are generated by the RG flow. In QCD and gravity the scales that mark the lower boundary of the scaling region are Λ_{QCD} and m_{Pl}, respectively.

The fixed point is a typical effect of quantum *field* theory, i.e. it arises due to the presence of *infinitely many* degrees of freedom. It is clear, therefore, that all approximations such as the familiar minisuperspace models which retain only finitely many degrees of freedom cannot see the asymptotic freedom and lead to a different picture.

Up to now we have discussed pure gravity without matter fields. But of course any matter field leads to an additional renormalization of G and Λ [20,21]. In [20] the average action approach was generalized and an arbitrary number of free scalars, spinors, vector fields, and Rarita-Schwinger fields was added. (See also [22,23].) In particular in [22] the connection with the approach based on the scaling of the metric introduced by [24] was discussed and it was shown that, as far as perturbation theory is concerned, the RG evolution of the UV relevant couplings of a scalar field is essentially the same in the two approaches. Depending on the nature and number of the matter fields, either gravity continues to be antiscreening and asymptotically free, or the quantum effects of the matter fields overwhelm those of the metric and destroy asymptotic freedom. (The same happens in QCD with too many quark flavors.) In this paper we assume that the matter system is such that the resulting RG flow for G and Λ is qualitatively the same as in pure gravity. In particular, we assume that there is a non-Gaussian fixed point which is UV attractive for g and λ, but we allow the numerical values of g_*^{UV} and λ_*^{UV} to differ from their pure gravity values. In fact, none of our conclusions will depend on the values g_*^{UV}, λ_*^{UV}, ω, and ν provided all those parameters are strictly positive.

In the following we shall write g_* and λ_* for g_*^{UV} and λ_*^{UV}, respectively.

III. THE RG IMPROVED EINSTEIN EQUATIONS

We consider homogeneous, isotropic cosmologies described by Robertson-Walker metrics of the form

$$ds^2=-dt^2+a(t)^2\left[\frac{dr^2}{1-Kr^2}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right]. \tag{3.1}$$

For $K=0$ the 3-spaces of constant cosmological time t are flat, and for $K=+1$ and -1 they are spheres and pseudospheres, respectively. In standard FRW cosmology the dynamics of the scale factor $a(t)$ is determined by Einstein's equations

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=-\Lambda g_{\mu\nu}+8\pi G\, T_{\mu\nu} \tag{3.2}$$

where G and Λ are constant parameters. In order to take the leading quantum corrections into account we now "improve" Eq. (3.2) by replacing G and Λ with the scale dependent quantities $G(k)$ and $\Lambda(k)$.

In general it is a difficult task to identify the actual physical cutoff mechanism which, in a concrete situation, stops the running in the infrared. Typically this involves expressing k in terms of all scales that are relevant to the problem under consideration, such as the momenta of particles, field strengths, or the curvature of the spacetime, for instance. In the case at hand the situation simplifies because the conditions of homogeneity and isotropy imply that k can be a function of the cosmological time only: $k=k(t)$. Provided we know how k depends on t we can turn the solutions of the RG equation, $G(k)$ and $\Lambda(k)$, into functions of time:

$$G(t)\equiv G(k=k(t)), \quad \Lambda(t)\equiv\Lambda(k=k(t)). \tag{3.3}$$

There are two plausible scales that could determine the identification of k in terms of t. The first one is $k\propto 1/t$. In fact, the temporal proper distance of some point $P(t,r,\theta,\phi)$ to the big bang (which will still be present in the improved spacetime) is directly given by t itself. If we want to construct an effective field theory Γ_k that is valid near P we may not integrate out quantum fluctuations with momenta smaller than $1/t$ because, by the time the age of the Universe is t, fluctuations with frequencies smaller than $1/t$ cannot have played any role yet. By this argument we are indeed led to the identification

$$k(t)=\frac{\xi}{t} \tag{3.4}$$

where ξ is a positive constant. (Note that t and a have mass dimension -1, while r,θ,ϕ,K and ξ are dimensionless.) As it stands, Eq. (3.4) refers to the t,r,θ,ϕ coordinate system, but it has an invariant meaning. At any point P we set

$$k(P)=\frac{\xi}{d(P)} \tag{3.5}$$

where $d(P)\equiv\int_{\mathcal{C}(P)}\sqrt{ds^2}$ is the proper length of the curve $\mathcal{C}(P)$ as given by the metric (3.1). With respect to the t,r,θ,ϕ system, $\mathcal{C}(P)$ is defined by $\lambda\mapsto(\lambda,r,\theta,\phi)$ with $\lambda\in[0,t]$ where (t,r,θ,ϕ) are the coordinates of P. Both the metric and the curve can be reexpressed in a generic coordinate system x^μ, so that the cutoff is actually a scalar function $k(x^\mu)$.

Another momentum scale which appears natural at first sight is

$$k(t)=\frac{\xi}{a(t)}, \tag{3.6}$$

but in particular for the most important case of $K=0$ it is not obvious why the RG flow should be stopped at this point. In fact, it will turn out that for the perturbative regime the improved system of equations has no consistent solution if one uses Eq. (3.6). On the other hand, for the fixed point regime of a radiation-dominated Universe Eqs. (3.4) and (3.6) lead to exactly the same answers so that our predictions are particularly robust in this case. A third scale one might invoke is the Hubble parameter

$$H(t)=\frac{\dot{a}(t)}{a(t)}. \tag{3.7}$$

However, in the present context only power laws $a\propto t^\alpha$ are of interest. For them H is proportional to $1/t$ and does not define an independent scale.

While we believe that the leading effects are correctly described by the $1/t$ cutoff, the more subtle subleading effects most probably require more complicated cutoffs which, apart from an *explicit* time dependence, also have an *implicit* time dependence via $a(t)$ and its derivatives:

$$k=k(t,a(t),\dot{a}(t),\ddot{a}(t),\ldots). \tag{3.8}$$

In this paper we discard those subleading effects. From now on we assume that $k\propto 1/t$ is indeed the correct first order approximation and we shall use Eq. (3.4) in the main body of the paper. For comparison we also investigate the consequences of the $1/a$ cutoff (3.6) in Appendix A.

Upon inserting Eq. (3.4) into Eqs. (2.11) and (2.12) we obtain for the time dependent Newton constant and cosmological constant in the perturbative regime

$$G(t)=G_0\left[1-\tilde{\omega}\left(\frac{t_{\rm Pl}}{t}\right)^2+O\left(\frac{t_{\rm Pl}^4}{t^4}\right)\right], \tag{3.9}$$

$$\Lambda(t)=\Lambda_0+\tilde{\nu}m_{\rm Pl}^2\left(\frac{t_{\rm Pl}}{t}\right)^4\left[1+O\left(\frac{t_{\rm Pl}^2}{t^2}\right)\right] \tag{3.10}$$

with the positive constants

$$\tilde{\omega}\equiv\omega\,\xi^2, \quad \tilde{\nu}\equiv\nu\,\xi^4. \tag{3.11}$$

In the fixed point regime we get from Eqs. (2.16), (2.17)

$$G(t)=\tilde{g}_*t^2, \tag{3.12}$$

$$\Lambda(t)=\tilde{\lambda}_*t^{-2} \tag{3.13}$$

with

$$\tilde{g}_* \equiv g_* \xi^{-2}, \quad \tilde{\lambda}_* \equiv \lambda_* \xi^2. \tag{3.14}$$

In order to find the functions $G(t)$ and $\Lambda(t)$ that interpolate between the behaviors (3.9), (3.10) and (3.12), (3.13) one must solve the RG equation numerically.

At this point several comments might be in order. The first one concerns the logical status of the improved Einstein equation (3.2). We emphasize that, conceptually, it is a completely quantum mechanical equation which happens to look like its classical counterpart only because of the specific approximations made (Einstein-Hilbert truncation). Quite generally, if $\Gamma[\varphi]$ is the effective action for an arbitrary set of fields φ, the equation of motion for the expectation value $\varphi \equiv \langle \hat{\varphi} \rangle$ is given by $\delta\Gamma/\delta\varphi = 0$. This is a quantum mechanically exact equation, the analogue of the classical $\delta S/\delta\varphi = 0$ with all modifications due to the quantum fluctuations included. The exact quantum mechanical amplitudes are obtained by evaluating Γ at tree level. The same remarks apply to Γ_k with the only difference that $\delta\Gamma_k/\delta\varphi = 0$ does not yet contain the effects of fluctuations with momenta smaller than k. Now, if there is a physical cutoff mechanism that stops the RG running at some value of k so that Γ_k at this value of k already coincides with the ordinary effective action $\Gamma \equiv \Gamma_{k=0}$, the functional Γ_k with k identified appropriately takes all quantum effects into account [14].

Up to now we have assumed that we know the exact Γ_k. We argued already that for $k \to \infty$ the only important invariants in Γ_k are $\int \sqrt{g}$ and $\int \sqrt{g} R$, albeit with strongly k-dependent coefficients. Thus, in the domain where the Einstein-Hilbert truncation of theory space is reliable, the classical equation $\delta S/\delta h_{\mu\nu} = 0$, and the quantum one $\delta\Gamma_k/\delta h_{\mu\nu} = 0$ have the same structure formally. The latter equation is precisely Eq. (3.2) with the k-dependent constants Λ and G. It is clear, therefore, that the RG-improved Einstein equation for the expectation value field $g_{\mu\nu}(x)$ has the status of a fully quantum mechanical equation in any regime where the Einstein-Hilbert truncation applies, in particular close to the fixed point. The familiar appearance of the equation of motion does not mean that gravity or the geometry of spacetime is treated classically in any sense, or that there are classes of quantum effects which are not accounted for.

We need the running couplings on Lorentzian spacetimes. *A priori* the RG equations are derived within a covariant Euclidean formalism so that the problems typical of the Hamiltonian approach (notion of time, choice of spacetime foliation, etc.) are not encountered. In contrast to the Euclidean path integral, the flow equation allows for a rather simple ‘‘Wick rotation’’ to the Lorentzian signature.[3] While the RG flows in the Euclidean and Lorentzian cases might be different for $k \to 0$ where topological issues play a role, the large-k behavior is the same in both cases, and this is all we need for the present investigation.

The next issue is the energy momentum tensor $T_{\mu\nu}$ to be used on the RHS of the improved Einstein equations. Because of the imposed homogeneity and isotropy it can always be transformed to the form

$$T_\mu^\nu = \mathrm{diag}(-\rho, p, p, p) \tag{3.15}$$

where the density ρ and the pressure p depend on t only. As in standard cosmology we assume that the energy-momentum tensor is covariantly conserved,[4]

$$D_\nu T_\mu^\nu = 0, \tag{3.16}$$

so that for the Robertson-Walker metric

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{3.17}$$

The physical picture behind $T_{\mu\nu}$ is not necessarily that of a perfect classical fluid as in the familiar FRW case. We rather interpret it as the functional derivative of some *effective* action $\Gamma^{\mathrm{M}}[g_{\mu\nu}]$ for the matter system in the background of the metric $g_{\mu\nu}$. For the equation of state relating p to ρ we shall use the linear ansatz

$$p(t) = w\,\rho(t) \tag{3.18}$$

where w is an arbitrary constant. It includes the case of a perfect fluid consisting of classical dust $(w=0)$ or radiation $(w=1/3)$, but we emphasize that Γ^{M} is by no means restricted to describing classical matter. In particular, w may be different from its classical value.

The energy-momentum tensor for a quantum field in a curved spacetime is a very complicated object, containing information about vacuum polarization, particle creation, or the trace anomaly. In general this can give rise to a complicated equation of state. Even leaving calculational problems aside, we are facing a problem of principle here. Unless we know all the matter fields in the the Universe (which we do not) we cannot determine $\Gamma^{\mathrm{M}}[g_{\mu\nu}]$ and the resulting equation of state from first principles. However, it is almost certain that the matter content influences the RG improved cosmology even at the qualitative level. We mentioned already that certain matter systems destroy the antiscreening character of pure gravity. They can also destroy the UV fixed point and lead to completely different cosmologies. Thus, in the absence of a complete matter theory, the best thing one can do is to work out the cosmology resulting from a specific set of assumptions about the matter system. In the present paper the assumptions are the equation of state (3.18), and that the fixed point of pure gravity is not destroyed by the matter system. The form (3.18) is motivated by its mathematical simplicity and the absence of explicit dimensionful parameters, which seems natural at very high energies.

[3] This is due to the fact that the functional traces on the right-hand side (RHS) of the flow equation are always convergent and well defined [2].

[4] See for instance Ref. [25] for a class of cosmologies with a time dependent Λ where $T_{\mu\nu}$ as defined here is not conserved.

Let us return to the Einstein equation (3.2) now. By virtue of Bianchi's identity its LHS is covariantly conserved, so for consistency the RHS must be conserved too:

$$D_\nu[-\Lambda g^\nu_\mu + 8\pi G T^\nu_\mu] = 0. \tag{3.19}$$

Because Λ and G depend on t, this equation is not automatically satisfied if $T_{\mu\nu}$ is conserved. Instead we obtain the following consistency condition which relates the time dependencies of Λ, G and ρ:

$$\dot{\Lambda} + 8\pi\rho\,\dot{G} = 0. \tag{3.20}$$

Sometimes it is convenient to rewrite Eq. (3.20) in the form

$$\frac{d}{dt}(\Lambda + 8\pi G\,\rho) = 8\pi G\,\dot{\rho}. \tag{3.21}$$

When we insert the Robertson-Walker metric (3.1) into Einstein's equation (3.2) we obtain two independent equations:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{1}{3}\Lambda + \frac{8\pi}{3}G\,\rho \tag{3.22}$$

from the 00 component, and

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \Lambda - 8\pi G\,\rho \tag{3.23}$$

from the ii components. As in the classical case, these two field equations are consistent only if $T_{\mu\nu}$ is conserved. After multiplying Eq. (3.22) by a^2, taking its time derivative, and combining it with Eq. (3.23) one obtains the conservation law (3.17) as an integrability condition for the improved Einstein equations. In this calculation essential use is made of the new consistency condition (3.21). We see that its role is completely analogous to that of the conservation equation for $T_{\mu\nu}$: both of them constrain the sources to which gravity can be coupled consistently. Thus only 2 of the 3 equations (3.17), (3.22) and (3.23) are independent; in the following we shall use the conservation law (3.17) and the improved Friedmann equation (3.22) as independent equations.

To summarize: We would like to write down a set of (differential) equations that determine a,ρ,p,G and Λ as a function of time. This set includes Friedmann's equation, the conservation law for $T_{\mu\nu}$, the equation of state, the new consistency condition, and the RG equations for G and Λ. More precisely, we shall always assume that the RG equations are already solved so that we can simply replace the constant k by $k(t)$ in the solution. Eliminating the pressure by virtue of the equation of state, this system of equations reads

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{1}{3}\Lambda + \frac{8\pi}{3}G\rho, \tag{3.24a}$$

$$\dot{\rho} + 3(1+w)\frac{\dot{a}}{a}\rho = 0, \tag{3.24b}$$

$$\dot{\Lambda} + 8\pi\rho\,\dot{G} = 0, \tag{3.24c}$$

$$G(t) = G(k(t)), \quad \Lambda(t) = \Lambda(k(t)). \tag{3.24d}$$

These are 5 equations for the 4 functions $a(t),\rho(t),G(t)$ and $\Lambda(t)$. [Of course we could immediately insert Eq. (3.24d) into the first 3 equations. Then Eqs. (3.24a,b,c) are 3 equations for the 2 unknowns a and ρ. For the time being we shall not adopt this point of view.]

The system (3.24a), (3.24b), (3.24c) without the last equations coming from the renormalization group has already been studied in the literature [26,27]. It consists of only 3 equations for 4 unknowns and is underdetermined therefore. As a way out, the authors made an *ad hoc* assumption about one of the functions, typically $G(t)$, and checked if there are interesting cosmologies consistent with, but not uniquely determined by, Eqs. (3.24a), (3.24b), (3.24c).

In our case with Eq. (3.24d) included we seem to be in the opposite situation because the 5 equations might overdetermine the 4 unknowns and no consistent solution might exist. In order to see that this is not actually the case we must return to the RG equation from which Eq. (3.24d) is derived. The flow equation contains the function $R^{(0)}$ which is completely arbitrary up to the two conditions $R^{(0)}(0)=1$ and $R^{(0)}(z\to\infty)=0$. This function describes the details of the cutoff mechanism, i.e. how quickly the modes with different momenta p get suppressed when p approaches k. Only if one uses the flow equation in order to compute quantities that are "universal" in the sense of statistical mechanics are the answers independent of the shape of $R^{(0)}$. In general Γ_k, for intermediate values of k, does depend on $R^{(0)}$. (Only the limit $k\to 0$ is $R^{(0)}$ independent because the cutoff drops out.) Therefore the RG trajectory $k\mapsto(G(k),\Lambda(k))$ is also $R^{(0)}$ dependent. This is obvious from Eqs. (2.13),(2.14), for instance: the coefficients ω and ν depend on $R^{(0)}$ via the Φ-integrals. This means that, if we want to give a physical meaning to $G(k)$ and $\Lambda(k)$ at intermediate values of k, the function $R^{(0)}$ should be chosen in such a way that it models the actual *physical* cutoff mechanism as accurately as possible.

Similarly, the identification of the scale k in terms of the actual physical parameters of the system also depends on the system under consideration. In our case we have $k=\xi/t$ with an unknown constant ξ. If we change $R^{(0)}$ the optimal value for ξ also changes. Typically combinations of parameters in the RG equation $(\omega,\nu,\dots)$ and in the cutoff identification (ξ) such as $\tilde{\omega}=\omega\xi^2$, for instance, are much less $R^{(0)}$ dependent, i.e. more "physical," than those parameters separately. (For the RG improved Newton potential it can be checked that the $R^{(0)}$ dependences of ω and an analogously defined ξ^2 mutually cancel, and that $\tilde{\omega}$ is a physical, i.e. observable, quantity [4].) However, even measurable combinations similar to $\tilde{\omega}$ cannot be calculated by RG techniques alone.

In this situation it is a virtue of the system (3.24) rather than a disadvantage that it is seemingly overdetermined because in this manner it also places restrictions on $R^{(0)}$ and on the cutoff identification. In fact, we may regard it as a system of 5 integro-differential equations for the 5 functions

a,ρ,G,Λ and $R^{(0)}$. In the next section we shall solve this system in the perturbative and in the fixed point regimes, and we shall see that solutions exist only if certain relations among the parameters $\tilde{\omega}$, $\tilde{g}_*$, etc. are satisfied. They are implicit conditions on $R^{(0)}$ and/or ξ. This shows that the system (3.24) is quite powerful in the sense that it also teaches us something about how to consistently model the IR cutoff for the concrete system "expanding Universe."

This enhanced degree of predictability is also one of the reasons why we are RG improving *equations* rather than *solutions*. Improving solutions means that we take some fixed solution $a(t),\rho(t)$ of standard cosmology that depends parametrically on the constants G and Λ and then substitute $G\to G(t)$, $\Lambda\to\Lambda(t)$. In general this simple approach is reliable only if the improved solution is close to the classical one. (See [4] for a detailed discussion in the context of black holes.) The main advantage of improving the underlying equations is that their solutions may well be quite different from the classical ones without necessarily lying in a domain where the entire approach has become unreliable. In Appendix B we describe the improvement of the classical FRW solutions. Where they are valid, the results are consistent with the approach of improving equations. They are less predictive, however, in particular because they do not reproduce the relations among $\tilde{\omega}$, $\tilde{g}_*$, etc. mentioned above.

It is important to understand how many constants of integration occur in the process of solving the system (3.24). Let us pick some $R^{(0)}$ and a function $k=k(t)$ with an explicit t dependence only. Then $G(k)$ and $\Lambda(k)$ can be obtained by solving 2 coupled RG equations which are of first order and therefore lead to 2 constants of integration. We choose them to be the $k=0$ values G_0 and Λ_0. As a consequence, the functions $G(t)$ and $\Lambda(t)$ in Eq. (3.24d) depend parametrically on G_0 and Λ_0, i.e. on the RG trajectory selected. In a first step we may insert Eq. (3.24d) into Eq. (3.24c) and obtain the energy density as

$$\rho(t)=-\frac{1}{8\pi}\frac{\dot{\Lambda}}{\dot{G}}. \tag{3.25}$$

The time dependence of ρ is completely determined once $\Lambda(t)$ and $G(t)$ are fixed, and no new constant of integration arises. In a second step we insert ρ of Eq. (3.25) into Eq. (3.24b) and solve the resulting differential equation for $a(t)$. Equation (3.24b) is easily integrated:

$$\rho(t)[a(t)]^{3+3w}=\mathcal{M}/8\pi=\text{const.} \tag{3.26}$$

Here we encounter a further constant of integration, $\mathcal{M}$. Its mass dimension is $1-3w$. For a radiation dominated Universe $\mathcal{M}$ is dimensionless, while it has the dimension of a mass in the matter dominated case. Combining Eqs. (3.25) and (3.26) we obtain the scale factor

$$a(t)=\left[-\frac{\mathcal{M}\dot{G}}{\dot{\Lambda}}\right]^{1/(3+3w)}. \tag{3.27}$$

Already at this point all 4 functions G, Λ, ρ and a are completely determined. They depend on 3 constants of integration: G_0, Λ_0, and $\mathcal{M}$. The last and crucial step is to insert the solution we found into Eq. (3.24a) and check if this equation is satisfied too. In general it will be satisfied only for appropriately chosen cutoff functions $R^{(0)}$ and $k(t)$, and for special values of the constants of integration and of the parameter w.

We note that the Hubble parameter also has a simple representation directly in terms of G and Λ:

$$H=\frac{\dot{a}}{a}=\frac{1}{3+3w}\left(\frac{\ddot{G}}{\dot{G}}-\frac{\ddot{\Lambda}}{\dot{\Lambda}}\right). \tag{3.28}$$

It is clear that the system (3.24) can be solved in this simple manner only in the special case when $k(t)$ has no implicit time dependence via $a(t)$. For a generic $k=k(t,a(t),\dots)$ the situation is much more involved; see for instance Appendix A for the ansatz $k=\xi/a$.

Before closing this section let us introduce a few convenient definitions. We define the vacuum energy density ρ_Λ, the total energy density ρ_{tot} and the critical energy density ρ_{crit} according to

$$\rho_\Lambda(t)\equiv\frac{\Lambda(t)}{8\pi G(t)}, \tag{3.29}$$

$$\rho_{\text{tot}}(t)\equiv\rho+\rho_\Lambda, \tag{3.30}$$

$$\rho_{\text{crit}}(t)\equiv\frac{3}{8\pi G(t)}\left(\frac{\dot{a}}{a}\right)^2. \tag{3.31}$$

The definitions (3.29) and (3.31) are the same as usual except that $G(t)$ and $\Lambda(t)$ appear in place of G_0 and Λ_0. This means in particular that, for very late times when the running Newton constant assumes its IR value G_0, the quantity ρ_{crit} is exactly the standard critical density of classical FRW cosmology. It is also customary to introduce

$$\Omega_{\text{M}}\equiv\frac{\rho}{\rho_{\text{crit}}},\qquad \Omega_\Lambda\equiv\frac{\rho_\Lambda}{\rho_{\text{crit}}}, \tag{3.32}$$

$$\Omega_{\text{tot}}\equiv\Omega_{\text{M}}+\Omega_\Lambda=\frac{\rho_{\text{tot}}}{\rho_{\text{crit}}}, \tag{3.33}$$

so that we may rewrite Friedmann's equation (3.24a) either as

$$\frac{\dot{a}^2+K}{a^2}=\frac{8\pi}{3}G\,\rho_{\text{tot}} \tag{3.34}$$

or as

$$K=\dot{a}^2\left[\frac{\rho_{\text{tot}}}{\rho_{\text{crit}}}-1\right]=\dot{a}^2[\Omega_{\text{tot}}-1]. \tag{3.35}$$

As a trivial consequence of its definition, the critical density satisfies

$$\rho_{\rm crit}(t)G(t)H(t)^{-2}=\frac{3}{8\pi}. \tag{3.36}$$

By Eq. (3.35), an expanding Universe with $K=0$ has

$$\rho_{\rm tot}(t)=\rho_{\rm crit}(t) \quad (K=0) \tag{3.37}$$

at any time. In this case

$$\rho_{\rm tot}(t)G(t)H(t)^{-2}=\frac{3}{8\pi} \quad (K=0). \tag{3.38}$$

Sometimes the flatness problem is rephrased as the cosmological ''coincidence puzzle:'' Why does the product of the observed matter density of the Universe, the square of its age t, and Newton's constant give rise to a number of order unity,

$$(\rho\, G\, t^2)_{\rm today}=O(1) \quad ? \tag{3.39}$$

It is clear that Eq. (3.39) is essentially the same statement as Eq. (3.38) if ρ_Λ is negligible or at most of the same order of magnitude as ρ, and if the age of the Universe is of the order of $H(t)^{-1}$. The ''coincidence'' (3.39) has also been regarded as a manifestation of Mach's principle [28].

IV. PERTURBATIVE AND FIXED POINT SOLUTIONS

In this section we solve the system of equations (3.24) using the approximate RG equations that are valid in the perturbative and in the fixed point regimes, respectively.

A. The perturbative regime

The perturbative approximation is valid for $k \ll m_{\rm Pl}$, i.e. for $t \gg t_{\rm Pl}$. The corresponding solutions to the RG equations are given by Eqs. (3.9),(3.10) from where we obtain

$$\dot G(t)=\frac{2\tilde\omega\, G_0^2}{t^3}\left\{1+O\left(\frac{t_{\rm Pl}^2}{t^2}\right)\right\}, \tag{4.1}$$

$$\dot\Lambda(t)=-\frac{4\,\tilde\nu\, G_0}{t^5}\left\{1+O\left(\frac{t_{\rm Pl}^2}{t^2}\right)\right\}. \tag{4.2}$$

Hence Eq. (3.25) for the energy density and Eq. (3.27) for the scale factor lead to

$$\rho(t)=\frac{1}{4\pi}\left(\frac{\tilde\nu}{\tilde\omega}\right)\frac{1}{G_0 t^2}\left\{1+O\left(\frac{t_{\rm Pl}^2}{t^2}\right)\right\} \tag{4.3}$$

and

$$a(t)=\left[\frac{1}{2}\left(\frac{\tilde\omega}{\tilde\nu}\right)\mathcal{M}\,G_0\right]^{1/(3+3w)} t^{2/(3+3w)}\left\{1+O\left(\frac{t_{\rm Pl}^2}{t^2}\right)\right\}, \tag{4.4}$$

respectively. Now we must insert Eqs. (4.3) and (4.4) along with $G(t)$ and $\Lambda(t)$ from Eqs. (3.9) and (3.10) into the Friedmann equation (3.24a) in order to check whether the above solutions are consistent. Omitting subleading terms, consistency requires that

$$\left(\frac{2}{3+3w}\right)^2\frac{1}{t^2}+K\left[\frac{1}{2}\left(\frac{\tilde\omega}{\tilde\nu}\right)\mathcal{M}G_0\right]^{-2/(3+3w)}\frac{1}{t^{4/(3+3w)}}$$
$$=\frac{\Lambda_0}{3}+\left(\frac{8\pi G_0}{3}\right)\frac{1}{4\pi}\left(\frac{\tilde\nu}{\tilde\omega}\right)\frac{1}{G_0t^2}+\cdots. \tag{4.5}$$

Note that on the RHS of Eq. (4.5) it is sufficient to set $G=G_0+\cdots$ and $\Lambda=\Lambda_0+\cdots$ because the (known) corrections to these approximations have the same time dependence as the (unknown) second order corrections on the LHS. In order to analyze Eq. (4.5) we must distinguish the cases $K=0$ and $K=\pm1$.

1. The case K=0

In the case $K=0$, Eq. (4.5) is satisfied provided that the consistency conditions

$$\Lambda_0=0 \quad \text{and} \quad \frac{\tilde\omega}{\tilde\nu}=\frac{3}{2}(1+w)^2 \tag{4.6}$$

are satisfied. The condition $\Lambda_0=0$ does not come as a surprise because the formula (2.12) for $\Lambda(k)$ from which we started is accurate for $k\to0$ only if $\Lambda_0=0$. Recalling that $\tilde\nu/\tilde\omega=(\nu/\omega)\xi^2$ we see that the second condition puts a constraint on the cutoff $R^{(0)}$ which affects ω and ν, as well as the function $k=k(t)$, i.e. ξ in our case. We use this condition in order to express ξ in terms of ω and ν which are not then subject to any further condition:

$$\xi^2=\frac{2\omega}{3\nu(1+w)^2}. \tag{4.7}$$

Thus, upon inserting Eq. (4.7) into Eqs. (3.9) and (3.10), the time dependences of Newton's constant and of the cosmological constant are now completely determined. Moreover, using Eq. (4.6) for the ratio $\tilde\omega/\tilde\nu$ in Eq. (4.3) and Eq. (4.4) we see that $\rho(t)$ and $a(t)$ are actually completely independent of ω and ν. As a consequence, the consistent solution we found is given by the following four equations:

$$a(t)=\left[\frac{3}{4}(1+w)^2\mathcal{M}G_0\right]^{1/(3+3w)} t^{2/(3+3w)}\left\{1+O\left(\frac{t_{\rm Pl}^2}{t^2}\right)\right\}, \tag{4.8a}$$

$$\rho(t)=\frac{1}{6\pi(1+w)^2G_0t^2}+O\left(\frac{1}{t^4}\right), \tag{4.8b}$$

$$G(t)=G_0\left[1-\frac{2\omega^2}{3\nu(1+w)^2}\left(\frac{t_{\rm Pl}}{t}\right)^2+O\left(\frac{t_{\rm Pl}^4}{t^4}\right)\right], \tag{4.8c}$$

$$\Lambda(t)=\frac{4\omega^2m_{\rm Pl}^2}{9\nu(1+w)^4}\left(\frac{t_{\rm Pl}}{t}\right)^4+O\left(\frac{t_{\rm Pl}^4}{t^6}\right). \tag{4.8d}$$

We observe that the leading terms of the above expressions for $a(t)$ and $\rho(t)$ coincide *exactly* with the corresponding solutions of the classical FRW equations. [See Eqs. (B3) and (B4) in Appendix B.] This coincidence is quite remarkable because in our approach, by Eqs. (3.25) and (3.27), a and ρ arise from the *time dependent*, i.e. higher order, terms in $G(t)$ and $\Lambda(t)$, which clearly have no counterpart in the classical situation.

The vacuum energy density and the critical energy density for the cosmology (4.8) are

$$\rho_\Lambda = 0 + O\left(\frac{1}{t^4}\right), \quad \rho_{\text{crit}} = \rho + O\left(\frac{1}{t^4}\right) \tag{4.9}$$

so that, in leading order, $\rho_{\text{tot}} = \rho = \rho_{\text{crit}}$, or

$$\Omega_{\text{M}} = 1, \quad \Omega_\Lambda = 0, \quad \Omega_{\text{tot}} = 1. \tag{4.10}$$

2. The case $K = \pm 1$

Equation (4.5) has a chance of being consistent only if all terms can be given a time dependence proportional to $1/t^2$. If $K \neq 0$ this is possible only for an "exotic" equation of state with $w = -1/3$. Indeed the consistency conditions implied by Eq. (4.5) are

$$\Lambda_0 = 0, \quad w = -\frac{1}{3}, \quad \frac{\tilde{\omega}}{\tilde{\nu}} = \frac{2}{3} - \frac{2K}{\mathcal{M} G_0}. \tag{4.11}$$

Again we use the last condition in order to eliminate ξ:

$$\xi^2 = \frac{\omega}{\nu}\left(\frac{2}{3} - \frac{2K}{\mathcal{M} G_0}\right)^{-1}. \tag{4.12}$$

Note that in the present case ξ depends also on the constants of integration $\mathcal{M}$ and G_0. Proceeding as above we find the following consistent solution for $w = -1/3$:

$$a(t) = \left[\frac{1}{3}\mathcal{M} G_0 - K\right]^{1/2} t \left\{1 + O\left(\frac{t_{\text{Pl}}^2}{t^2}\right)\right\}, \tag{4.13a}$$

$$\rho(t) = \frac{\mathcal{M}}{8\pi}\left[\frac{1}{3}\mathcal{M} G_0 - K\right]^{-1} \frac{1}{t^2}\left\{1 + O\left(\frac{t_{\text{Pl}}^2}{t^2}\right)\right\}, \tag{4.13b}$$

$$G(t) = G_0\left[1 - \frac{\omega^2}{\nu}\left(\frac{2}{3} - \frac{2K}{\mathcal{M} G_0}\right)^{-1}\left(\frac{t_{\text{Pl}}}{t}\right)^2 + O\left(\frac{t_{\text{Pl}}^4}{t^4}\right)\right], \tag{4.13c}$$

$$\Lambda(t) = \frac{\omega^2}{\nu} m_{\text{Pl}}^2 \left(\frac{2}{3} - \frac{2K}{\mathcal{M} G_0}\right)^{-2}\left(\frac{t_{\text{Pl}}}{t}\right)^4 + O\left(\frac{t_{\text{Pl}}^4}{t^6}\right). \tag{4.13d}$$

The leading terms in Eqs. (4.13a) and (4.13b) coincide with the corresponding classical FRW solutions for $w = -1/3$. The cosmology (4.13) gives rise to

$$\rho_\Lambda(t) = 0 + O\left(\frac{1}{t^4}\right), \quad \rho_{\text{crit}}(t) = \frac{3}{8\pi G_0 t^2} + O\left(\frac{1}{t^4}\right) \tag{4.14}$$

so that, in leading order, $\Omega_\Lambda = 0$ and $\Omega_{\text{M}} = \Omega_{\text{tot}}$ with

$$\Omega_{\text{tot}} = \frac{\mathcal{M} G_0}{\mathcal{M} G_0 - 3K}\left\{1 + O\left(\frac{t_{\text{Pl}}^2}{t^2}\right)\right\}. \tag{4.15}$$

As expected, Ω_{tot} depends on the constants of integration in the case $K = \pm 1$.

B. The fixed point regime

The fixed point approximation is valid when $k \gg m_{\text{Pl}}$ or $t \ll t_{\text{Pl}}$. In this regime the time dependence of G and Λ is given by Eqs. (3.12) and (3.13), respectively. From Eqs. (3.25) and (3.27) we obtain

$$a(t) = \left(\frac{\tilde{g}_* \mathcal{M}}{\tilde{\lambda}_*}\right)^{1/(3+3w)} t^{4/(3+3w)}, \tag{4.16}$$

$$\rho(t) = \frac{\tilde{\lambda}_*}{8\pi \tilde{g}_*}\frac{1}{t^4}. \tag{4.17}$$

The next step is to check the consistency of Eq. (3.24a). Inserting G, Λ and the above expressions for a and ρ we have

$$\left(\frac{4}{3+3w}\right)^2 \frac{1}{t^2} + K\left[\frac{\tilde{g}_* \mathcal{M}}{\tilde{\lambda}_*}\right]^{-2/(3+3w)} \frac{1}{t^{8/(3+3w)}} = \frac{2\tilde{\lambda}_*}{3t^2}. \tag{4.18}$$

We shall discuss this equation for $K=0$ and $K=\pm 1$ separately.

1. The case $K=0$

For $K=0$, Eq. (4.18) implies only a single consistency condition:

$$\tilde{\lambda}_* = \frac{8}{3(1+w)^2}. \tag{4.19}$$

If we use this condition in order to eliminate $\tilde{\lambda}_*$ in all equations we are led to

$$a(t) = \left[\frac{3}{8}(1+w)^2 \tilde{g}_* \mathcal{M}\right]^{1/(3+3w)} t^{4/(3+3w)}, \tag{4.20a}$$

$$\rho(t) = \frac{1}{3\pi(1+w)^2 \tilde{g}_*}\frac{1}{t^4}, \tag{4.20b}$$

$$G(t) = \tilde{g}_* t^2, \tag{4.20c}$$

$$\Lambda(t) = \frac{8}{3(1+w)^2}\frac{1}{t^2}. \tag{4.20d}$$

This family of solutions, one for each value of $\tilde{g}_*$ and w, was already found in Ref. [27]. In this work, the RG equations (3.24d) were not used. Since the system (3.24a), (3.24b), (3.24c) is underdetermined, the time dependence for $G(t)$, Eq. (4.20c) above, was postulated on an *ad hoc* basis in order to obtain a unique solution. In this manner the analogue of $\tilde{g}_*$ appears as a free parameter while $\tilde{\lambda}_*$ is fixed. In our case it is more natural to use the consistency condition (4.19) in order to express ξ in terms of λ_* which is given by the renormalization group. Because $\tilde{\lambda}_* \equiv \lambda_* \xi^2$ we have then

$$\xi^2 = \frac{8}{3(1+w)^2 \lambda_*}. \tag{4.21}$$

When expressed in terms of the fixed point values, the solutions read

$$a(t) = \left[\left(\frac{3}{8} \right)^2 (1+w)^4 g_* \lambda_* \mathcal{M} \right]^{1/(3+3w)} t^{4/(3+3w)}, \tag{4.22a}$$

$$\rho(t) = \frac{8}{9\pi(1+w)^4 g_* \lambda_*} \frac{1}{t^4}, \tag{4.22b}$$

$$G(t) = \frac{3}{8}(1+w)^2 g_* \lambda_* t^2, \tag{4.22c}$$

$$\Lambda(t) = \frac{8}{3(1+w)^2} \frac{1}{t^2}. \tag{4.22d}$$

Since g_*, λ_* and w are given by the renormalization group and the equation of state, respectively, Eqs. (4.22) represent a one-parameter family of solutions parametrized by the constant $\mathcal{M}$. The solutions (4.22) reflect the renormalization group flow in the vicinity of the UV attractive fixed point where the RG trajectories have "forgotten" their IR values G_0 and Λ_0. Because of this universality, these solutions are independent of the constants of integration G_0 and Λ_0. This means that (4.22) is an attractor solution for $t \searrow 0$ in the sense that *every* consistent solution to Eq. (3.24), characterized by arbitrary constants of integration (G_0, Λ_0, $\mathcal{M}$), looks like (4.22) in the limit $t \searrow 0$. Actually the $\mathcal{M}$ dependence of the solutions (4.22) is quite trivial: ρ, G and Λ are $\mathcal{M}$ independent, while $a(t)$ responds to a change of $\mathcal{M}$ by a simple constant rescaling.

It is very remarkable and a nontrivial confirmation of our approach that after the elimination of ξ the RG data enter the attractor solution only via the product $g_* \lambda_*$. This product is *universal* (scheme independent) in the sense that it does not depend on the function $R^{(0)}$ [17]. Hence Eqs. (4.22) are free from any numerical ambiguities.

For the cosmologies (4.22) we find that $\rho_\Lambda(t) = \rho(t)$ and $\rho_{\text{crit}}(t) = 2\rho(t)$ so that

$$\rho = \rho_\Lambda = \frac{1}{2}\rho_{\text{crit}}, \quad \rho_{\text{tot}} = \rho_{\text{crit}} \tag{4.23}$$

or

$$\Omega_{\text{M}} = \Omega_\Lambda = \frac{1}{2}, \quad \Omega_{\text{tot}} = 1. \tag{4.24}$$

We also read off the Hubble parameter

$$H = \frac{4}{3+3w} \frac{1}{t} \tag{4.25}$$

and observe that

$$\rho(t) G(t) t^2 = \frac{1}{3\pi(1+w)^2} \tag{4.26}$$

is a time-independent fixed number that depends only on the equation of state.

The solutions (4.22) exists for every equation of state of the type considered, i.e. for every value of the parameter w. Since at least immediately after the Planck era during which (4.22) is valid the Universe is radiation dominated, a particularly plausible choice is $w = 1/3$. In the case of a "radiation dominated Planck era" with $w = 1/3$ we have

$$a(t) = \left[\frac{4}{9} g_* \lambda_* \mathcal{M} \right]^{1/4} t, \tag{4.27a}$$

$$\rho(t) = \frac{9}{32\pi g_* \lambda_*} \frac{1}{t^4}, \tag{4.27b}$$

$$G(t) = \frac{2}{3} g_* \lambda_* t^2, \tag{4.27c}$$

$$\Lambda(t) = \frac{3}{2} \frac{1}{t^2}. \tag{4.27d}$$

The most interesting property of this solution is that it is perfectly *scale free*. Because $\mathcal{M}$ is dimensionless for $w = 1/3$ and because G_0 and Λ_0 do not occur due to the fixed point behavior, the only dimensionful quantity available is the cosmological time t itself. As a consequence, the various exponents of t appearing in Eqs. (4.27) are completely fixed by the canonical mass dimensions of a, ρ, G and Λ, which are -1, $+4$, -2, and $+2$, respectively. In particular, the linear expansion law $a \propto t$ is a direct consequence of this type of scale invariance. Since $w = 1/3$ corresponds to a traceless energy momentum tensor, this solution is realized if Γ^{M} is the effective action of a quantum conformal field theory, for instance. It is interesting in this respect that there are indications from semiclassical gravity [24] that the effective matter action could be asymptotically scale invariant.

2. The case $K = \pm 1$

In this case Eq. (4.18) can be made consistent only for a specific choice of the equation of state, namely, for $w = +1/3$. Equation (4.18) is satisfied if

$$w = +\frac{1}{3} \quad \text{and} \quad 1 + K \left(\frac{\tilde{\lambda}_*}{\tilde{g}_* \mathcal{M}} \right)^{1/2} = \frac{2}{3} \tilde{\lambda}_*. \tag{4.28}$$

We use the second consistency condition in order to eliminate ξ in favor of g_*, λ_* and $\mathcal{M}$:

$$\xi^2 = \left(g_* \frac{\mathcal{M}}{\lambda_*} \right)^{1/2} \left[\frac{2}{3}\sqrt{g_*\lambda_*\mathcal{M}} - K \right]^{-1}. \tag{4.29}$$

This leads to the following solutions for $w = +1/3$:

$$a(t) = \left[\frac{2}{3}\sqrt{g_*\lambda_*\mathcal{M}} - K \right]^{1/2} t, \tag{4.30a}$$

$$\rho(t) = \frac{\mathcal{M}}{8\pi} \left[\frac{2}{3}\sqrt{g_*\lambda_*\mathcal{M}} - K \right]^{-2} \frac{1}{t^4}, \tag{4.30b}$$

$$G(t) = \left(\frac{g_*\lambda_*}{\mathcal{M}} \right)^{1/2} \left[\frac{2}{3}\sqrt{g_*\lambda_*\mathcal{M}} - K \right] t^2, \tag{4.30c}$$

$$\Lambda(t) = \sqrt{g_*\lambda_*\mathcal{M}} \left[\frac{2}{3}\sqrt{g_*\lambda_*\mathcal{M}} - K \right]^{-1} \frac{1}{t^2}. \tag{4.30d}$$

This family of solutions, again parametrized by a dimensionless constant $\mathcal{M}$, is scale free as well. All solutions have the property that their vacuum energy density equals the matter density:

$$\rho_\Lambda(t) = \rho(t) = \frac{1}{2}\rho_{\text{tot}}(t). \tag{4.31}$$

Furthermore, their critical density reads

$$\rho_{\text{crit}}(t) = \frac{3}{8\pi} \left(\frac{\mathcal{M}}{g_*\lambda_*} \right)^{1/2} \left[\frac{2}{3}\sqrt{g_*\lambda_*\mathcal{M}} - K \right]^{-1} \frac{1}{t^4} \tag{4.32}$$

from which one obtains

$$\Omega_{\text{M}} = \Omega_\Lambda = \frac{1}{2}\Omega_{\text{tot}} = \frac{1}{3}\sqrt{g_*\lambda_*\mathcal{M}} \left[\frac{2}{3}\sqrt{g_*\lambda_*\mathcal{M}} - K \right]^{-1}. \tag{4.33}$$

If $K = +1$, solutions of the form (4.30) exist only if $\mathcal{M}$ is such that $\sqrt{g_*\lambda_*\mathcal{M}} > 3/2$. It is also important to note that for $K = \pm 1$ the quantity

$$\rho(t)G(t)t^2 = \frac{1}{8\pi}\sqrt{g_*\lambda_*\mathcal{M}} \left[\frac{2}{3}\sqrt{g_*\lambda_*\mathcal{M}} - K \right]^{-1} \tag{4.34}$$

is not a universal number but depends on $\mathcal{M}$.

V. COMPLETE SOLUTIONS FOR THE PLANCK ERA

A. Early versus late stages of the Planck era

In the previous section we found solutions to the RG improved system of cosmological evolution equations that are valid for $t \searrow 0$ and for $t \gtrsim t_{\text{Pl}}$, respectively. In particular, it turned out that the improved cosmologies, too, start from a "big bang," i.e. there exists a time (conveniently chosen as $t=0$) at which the scale factor vanishes. We also saw that there is a certain transition time t_{class} such that for $t > t_{\text{class}}$ quantum gravitational effects become negligible so that the evolution of the Universe is correctly described by the classical FRW models. The time t_{class} is of the order of a few Planck times, $t_{\text{class}} \gtrsim t_{\text{Pl}}$. We shall refer to the epoch between $t=0$ and $t = t_{\text{class}}$ as the *Planck era*. At the beginning of the Planck era, immediately after the big bang, we are in the fixed point regime of the RG equations, while the end of the Planck era and its transition to classical cosmology corresponds to the perturbative regime.

We were able to find analytic solutions to the improved equations only for the very early and the very late parts of the Planck era. Let us now discuss how those solutions can be fitted together to obtain complete solutions that are valid during the entire Planck era.

For a spatially flat geometry, $K=0$, and for every value of w, there exist exact solutions of Eqs. (3.24) both in the fixed point and in the perturbative regimes, see Eqs. (4.22) and (4.8), respectively. We expect that those two limiting solutions possess a continuous interpolation that satisfies Eqs. (3.24) for all $t \in (0, t_{\text{class}})$. Generically this interpolating solution should exist, because we have considerable freedom in adjusting the functions $R^{(0)}$ and $k(t, a(t), \ldots)$ without changing their qualitative features. We shall refer to this solution $\{a(t), \rho(t), G(t), \Lambda(t)\}$, $t \in (0, t_{\text{class}})$, as the *complete $K=0$ solution*. Actually this is a whole family of solutions labeled by the constants of integration $(G_0, \Lambda_0, \mathcal{M})$. (Within the present approximation, only solutions with $\Lambda_0 = 0$ were found.)

It is the main assumption of this paper that the RG improved system (3.24) and its complete $K=0$ solution are valid throughout the Planck era, i.e. even immediately after the big bang. The reason why we think that our approximations are valid even for $t \searrow 0$ is the asymptotic freedom we found for quantum gravity. It entails gravity in the very early Universe being weakly coupled. In fact, the coupling constant, i.e. Newton's constant, vanishes very rapidly as we approach the initial singularity: $G \propto t^2$. For $k \to \infty$ the RG flow in (g, λ) space is dominated by a fixed point that is UV attractive for both g and λ. By the RG improvement, this fixed point translates into the attractor solution (4.22) for a, ρ, G and Λ. In the vicinity of the attractor, all solutions have the same universal behavior.

The w value of the perturbative regime must coincide with that of the following classical era, most plausibly $w = 1/3$. In principle it is conceivable that the interpolation from the fixed point to the perturbative regime involves an adiabatic change of w.

For the spatially curved geometries with $K = +1$ or -1 we found a solution in the fixed point regime only if $w = +1/3$, and a solution in the perturbative regime only for $w = -1/3$. Hence, at least within the present approximation, there exists no consistent interpolating solution for $t \in (0, t_{\text{class}})$ with a constant w.

As for the interpretation of this result, we must be very careful. Clearly it would be premature to conclude that the RG approach predicts $K=0$ as the only possibility. In particular the nonexistence of perturbative solutions with $w \neq -1/3$ is quite likely to be an artifact of our approximations. We mentioned already that the simple perturbative form of

$\Lambda(t)$, Eq. (2.12), is correct only if one either neglects the back reaction of the running Λ contained in the Φ functions or specializes to $\Lambda_0=0$. In general the situation is similar to QCD [29] where, thanks to asymptotic freedom, simple truncations are sufficient for large values of k, but at small k they necessarily become very complicated because they have to describe all sorts of nonperturbative effects. On the basis of this analogy we expect that in quantum gravity also it is much more difficult to describe the IR behavior correctly. It is intriguing that in our approach this problem is particularly pressing if $\Lambda_0\neq 0$. In fact, it has been suggested [30] that there are strong renormalization effects in the IR which might solve the cosmological constant problem [31] in a dynamical way.

It is less obvious why for $K=\pm 1$ there seem to be no solutions with $w\neq +1/3$ in the fixed point regime. It would be tempting to speculate that this reflects a property of the exact theory in which case the slightest deviation from the classical value $w=+1/3$ would lead to the prediction that $K=0$.

B. "Naturalness" of the solutions

Let us now make more precise in what sense the existence of the complete $(K=0)$ RG improved solution removes the flatness problem. We emphasize that the reason is *not* that we found no solutions for $K=\pm 1$ and that $\rho_{\rm tot}=\rho_{\rm crit}$ is automatic if $K=0$. In fact, for the sake of argument, let us suppose that there is some better approximation (an exact treatment) such that there are complete solutions for $K=\pm 1$ and perhaps also for $K=0$, $\Lambda_0\neq 0$. Then, both the classical and the RG improved theories describe cosmologies with all 3 types of spatial geometry: flat $(K=0)$, spherical $(K=+1)$, and pseudospherical $(K=-1)$. Let us select one out of these 3 options, $K=0$ say, and let us compare what the two theories have to say about the evolution of the Universe.

Classical FRW cosmology has a limited domain of applicability. It is valid only for $t\geqslant t_{\rm i}$ where $t_{\rm i}\gtrsim t_{\rm class}$ is some initial time at which one must specify initial conditions for the classical differential equations. They include the initial density $\rho(t_{\rm i})$ and the Hubble parameter $H(t_{\rm i})$ from which one can deduce the initial critical density $\rho_{\rm crit}(t_{\rm i}) \equiv 3H(t_{\rm i})^2/8\pi G_0$. Since we opted for $K=0$, the classical differential equations tell us that there is a solution only if the initial conditions are such that $\rho(t_{\rm i})=\rho_{\rm crit}(t_{\rm i})$. Thus, in order to be in the $K=0$ sector, an infinite fine-tuning of the initial data is necessary, and this is what is referred to as the flatness problem.

Because gravity is weakly coupled for $t\searrow 0$, *RG improved cosmology* has the ambition of being valid for all $t>0$, i.e. already directly after the big bang. At $t=0$ the spacetime is singular, and there is no such thing as a $t=t_{\rm i}$ hypersurface at which initial data are to be imposed. There is a family of complete consistent $K=0$ cosmologies labeled by the parameters $(G_0,\Lambda_0,\mathcal{M})$. For any value of the parameters, $\rho_{\rm tot}(t) =\rho_{\rm crit}(t)$ is automatically satisfied for all $t>0$. For $t\searrow 0$ all solutions approach an essentially universal attractor solution which is independent of $(G_0,\Lambda_0,\mathcal{M})$ except for an overall $\mathcal{M}$ dependence of a. It is precisely this attractor that makes it not only unnecessary but even impossible to specify initial conditions in a standard way. Thus, by the time the classical solution emerges from the quantum solution, *the condition* $\rho_{\rm tot}=\rho_{\rm crit}$ *is imposed automatically*.

To summarize: At present the RG improvement provides no strong theoretical arguments against $K=+1$ or -1. However, if one selects the $K=0$ option "by hand," no naturalness problem occurs.

C. Particle horizons

Let us consider an observer in a Robertson-Walker spacetime who, at cosmological time t, receives a light signal that was emitted by some distant galaxy at time $\tau<t$. Then, at time t, the proper distance between this galaxy and the observer is given by [32]

$$R(t,\tau)=a(t)\int_\tau^t \frac{dt'}{a(t')}. \qquad (5.1)$$

In a spacetime with a singularity at time zero, the most distant galaxies from which the observer can receive a light signal at time t have the proper distance $R(t,0)\equiv d_H(t)$. If this distance is finite, i.e. if the integral (5.1) converges for $\tau\to 0$,

$$d_H(t)=a(t)\int_0^t \frac{dt'}{a(t')}, \qquad (5.2)$$

we say that the spacetime has a particle horizon at the distance d_H. Hence it is the $t\searrow 0$ behavior of the scale factor that decides the presence or absence of a particle horizon. For instance, if

$$a(t)\propto t^\alpha \quad (\alpha>0) \qquad (5.3)$$

there is a horizon at $d_H(t)=t/(1-\alpha)$ for $\alpha\in(0,1)$ but there is no horizon if $\alpha\geqslant 1$.

For $t\ll 1/\sqrt{\Lambda}$ all classical FRW solutions are power laws of the type (5.3) with the exponent

$$\alpha_{\rm class}=\frac{2}{3+3w} \qquad (5.4)$$

(see Appendix B). If we take these solutions at face value even for $t\searrow 0$, there appears to be a horizon in both the physically relevant cases of the radiation and the matter dominated Universe with $w=1/3$ and $w=0$, respectively. However, since the classical equations become invalid for $t\searrow 0$ there is no compelling reason why these horizons actually should exist in nature.

In the RG improved cosmology for $K=0$ the early part of the Planck era is governed by the attractor solution (4.22) with

$$a(t)\propto t^{4/(3+3w)}. \qquad (5.5)$$

Since we believe that this attractor provides a valid description for $t\searrow 0$, even very close to the big bang, we may use

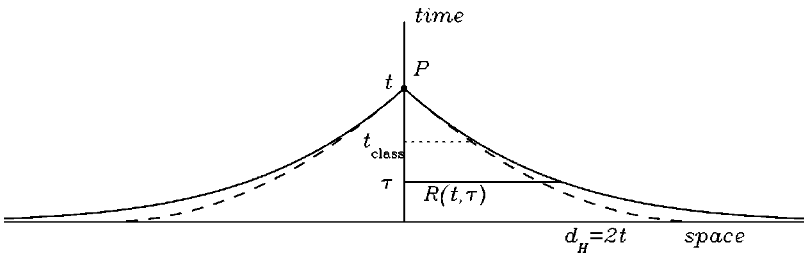

FIG. 1. Graphical representation of the proper distance $R(t,\tau)$ as a function of τ for fixed t. Only light signals emitted from points below the solid line can reach the spacetime point P. The dashed line shows $R_{\rm class}(t,\tau)$ which gives rise to a horizon at $d_H=2t$. The deviation of R from $R_{\rm class}$ becomes appreciable only for $\tau<t_{\rm class}$.

Eq. (5.5) in order to check for the existence of horizons. We observe that *the RG improved spacetime has no particle horizon provided* $w\leqslant 1/3$.[5]

During the following discussion we assume that the matter system is such that $w\leqslant 1/3$ so that there is indeed no horizon. However, as we shall see now, this fact by itself is not yet a solution to the horizon problem. For the sake of simplicity we consider a ''radiation dominated Planck era'' with $w=1/3$ followed by a classical radiation dominated era, again with $w=1/3$. In this case we have a linear expansion at early times and the familiar square-root expansion at late times:

$$a(t)\propto\begin{cases} t & \text{for} \quad t\ll t_{\rm class}, \\ t^{1/2} & \text{for} \quad t\gg t_{\rm class}. \end{cases} \tag{5.6}$$

In order to visualize the causal properties of this Robertson-Walker spacetime we consider a simple toy model which interpolates smoothly between $a\propto t$ for $t\ll t_{\rm class}$ and $a\propto t^{1/2}$ for $t\gg t_{\rm class}$:

$$a(t)=\frac{At}{1+\sqrt{t/t_{\rm class}}}. \tag{5.7}$$

Here A is an arbitrary positive constant. It is easy to calculate the proper distance (5.1) for Eq. (5.7):

$$R(t,\tau)=\frac{t}{1+\sqrt{t/t_{\rm class}}}\left[\ln\left(\frac{t}{\tau}\right)+2\sqrt{\frac{t}{t_{\rm class}}}-2\sqrt{\frac{\tau}{t_{\rm class}}}\right]. \tag{5.8}$$

As expected, this distance diverges for $\tau\to 0$ and t fixed. In Fig. 1 it is represented graphically as a kind of gravitationally distorted backward light cone of the point P. It is compared to its classical counterpart

$$R_{\rm class}(t,\tau)=2\sqrt{t}(\sqrt{t}-\sqrt{\tau}) \tag{5.9}$$

which results from $a\propto t^{1/2}$ and gives rise to the familiar horizon at $d_H=2t$.

In Fig. 2 we show two spacetime points P_1 and P_2 at the same cosmological time t. In classical cosmology those two points would be causally disconnected because their ''light cones'' given by $R_{\rm class}$ do not intersect. However, in the RG improved spacetime, the light cones become infinitely broad for $t\searrow 0$. This means that events which take place at sufficiently early times τ can causally influence both P_1 and P_2. Because of this quantum gravity induced broadening of the backward light cones, the light cones of all events P_i at a given time t overlap for some small enough τ. Since this broadening sets in only for $\tau\lesssim t_{\rm class}=O(t_{\rm Pl})$ we see that only events P_0 in the Planck era can causally influence *all* points P on the hypersurface at time t.

Let us imagine, for instance, that the two points P_1 and P_2 are located in opposite directions in the sky. Two microwave antennas pointing in these directions receive radiation that has been emitted at the time $t_{\rm r}$ of the hydrogen recombination when the cosmological plasma had just become optically thin to radiation, about 10^5 years after the big bang. In the standard FRW spacetime the number of horizon distances separating the two sources in opposite directions is given by

$$N=\frac{2R(t_0,t_{\rm r})}{d_H(t_{\rm r})}=\lim_{\tau\to 0}\frac{2R(t_0,t_{\rm r})}{R(t_{\rm r},t_{\rm e})+R(t_{\rm e},\tau)} \tag{5.10}$$

where t_0 denotes the present time, and $t_{\rm e}\gtrsim t_{\rm class}$ is in the equivalence era, when matter and radiation were in local thermodynamic equilibrium. However, since both $R(t_0,t_{\rm r})$ and $R(t_{\rm r},t_{\rm e})$ are finite, it is clear that, in the quantum gravity improved spacetime, eventually $N<1$ for sufficiently small $\tau<t_{\rm Pl}$.

In view of the above discussion we propose that the isotropy of the cosmic microwave background radiation on large angular scales is a consequence of the quantum gravity effects in the Planck era which remove the particle horizon and hence allow for causal mechanisms giving rise to approximately the same temperature everywhere on the last scattering surface. The important point of this discussion is that since the broadening of the light cones becomes significant only for $t<t_{\rm Pl}$, it is necessary that those causal mechanisms are already operative during the Planck era. In the following section we outline a scenario for the generation of primordial density fluctuations where this is actually the case.

VI. DENSITY FLUCTUATIONS

It is a fascinating idea that the structure formation in the Universe started out from primordial density fluctuations $\delta\rho(\mathbf{x})$ which were triggered by quantum mechanical fluctuations. As the Universe expanded, these density fluctuations became amplified and magnified, and finally gave rise to the large-scale structures that we observe today. This idea has been worked out in the framework of inflationary cosmology. Here instead we consider the possibility that the primordial density fluctuations were already generated during the

[5]Within the phenomenological applications [26,27] of the system (3.24a), (3.24b), (3.24c) this was already pointed out earlier [27].

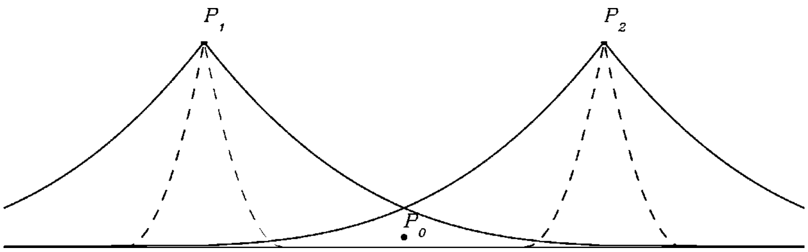

FIG. 2. While the points P_1 and P_2 are causally disconnected classically, the quantum gravity induced broadening of the backward light cones allows for events P_0 in the Planck era that can causally influence both P_1 and P_2.

Planck era as the aftermath of the big bang. This hypothesis allows us to invoke the broadening of the light cones for $t < t_{\rm class}$ that we found above in order to explain the high degree of isotropy of the fluctuations at later times.

In our approach the most natural assumption about the quantum origin of $\delta\rho$ is that, before $t \approx t_{\rm class}$, the quantum fluctuations of the metric itself generated the primordial density fluctuations by some decoherence mechanism. As we shall argue now, this assumption naturally leads to a scale free (Harrison-Zeldovich) fluctuation spectrum.

We need to know the two-point correlation function [33]

$$\xi(\mathbf{x}) = \langle \delta(\mathbf{x}+\mathbf{y})\, \delta(\mathbf{y}) \rangle \tag{6.1}$$

of the density contrast $\delta(\mathbf{x}) \equiv \delta\rho(\mathbf{x})/\langle \rho \rangle_t$ at some *fixed* time $t \lesssim t_{\rm class}$ close to the end of the Planck era when the spectrum is ''handed over'' from the quantum gravity to the classical regime. We define the power spectrum by

$$|\delta_k|^2 \equiv V \int d^3x\, \xi(\mathbf{x})\, e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{6.2}$$

and we say that the fluctuation spectrum has the spectral index n if $|\delta_k|^2$ has the form of a power law $|\delta_k|^2 \propto |\mathbf{k}|^n$. ($V$ denotes the normalization volume.) What is the prediction for $|\delta_k|^2$ if our above hypothesis is correct?

In [17] it was shown that, on a flat background, the effective graviton propagator for the fixed point regime is proportional to $\widetilde{\mathcal{G}}(p) \propto 1/p^4$, which amounts to $\mathcal{G}(x,y) \propto \ln(x-y)^2$ in position space. This form of the propagator is valid for $p^2 \gg m_{\rm Pl}^2$ or $(x-y)^2 \ll l_{\rm Pl}^2$, respectively. The logarithmic two-point function may be understood as a limiting case of the familiar ''critical'' propagator $\mathcal{G}(x,y) \propto 1/|x-y|^{d-2+\eta}$ for $d=4$ and the anomalous dimension $\eta \equiv \eta_N(g_*, \lambda_*) = -2$ which characterizes the UV fixed point [17]. Let us look at the curvature fluctuation $\delta\mathbf{R} \propto \partial\partial h$ caused by a fluctuation $h_{\mu\nu}(x)$ of the metric. (We use a symbolic notation where $\mathbf{R}$ stands for the curvature scalar or for any component of the Riemann or Einstein tensor.) Because $\langle h_{\mu\nu}(x) h_{\lambda\tau}(y) \rangle \propto \ln(x-y)^2$, the curvature correlation function is $\langle \delta\mathbf{R}(x)\, \delta\mathbf{R}(y) \rangle \propto 1/(x-y)^4$, rather than $\propto 1/(x-y)^6$ as implied by the tree level propagator. Therefore the leading short distance singularity in a curved spacetime is given by $\langle \delta\mathbf{R}(x)\, \delta\mathbf{R}(y) \rangle \propto 1/d(x,y)^4$ where $d(x,y)$ is the geodesic distance of x and y. This formula is applicable when the spacetime curvature is small compared to $1/d(x,y)^2$.

Now we consider the background of a Robertson-Walker spacetime and we put x and y on the same time slice. Hence $d(x,y) = a(t)|\mathbf{x}-\mathbf{y}|$ where $\mathbf{x}$ and $\mathbf{y}$ are the comoving Cartesian coordinates of x and y, respectively. This leads to the important result

$$\langle \delta\mathbf{R}(\mathbf{x},t)\, \delta\mathbf{R}(\mathbf{y},t) \rangle \propto \frac{1}{|\mathbf{x}-\mathbf{y}|^4}. \tag{6.3}$$

The constant of proportionality implicit in Eq. (6.3) is time dependent but for the derivation of the spectrum this is unimportant.

In the scenario where the primordial density fluctuations are generated by quantum fluctuations one assumes [33] that the classical statistical expectation value (6.1) is proportional to a quantum mechanical expectation value $\langle \Psi | \hat{\phi}(\mathbf{x}+\mathbf{y})\, \hat{\phi}(\mathbf{y}) | \Psi \rangle$ where $\hat{\phi}$ is the operator whose fluctuations are supposed to become classical. In the case at hand where we assume that $\delta\rho$ originates from the fluctuations of the spacetime geometry itself the natural choice for $\hat{\phi}$ is $\hat{\phi} \propto \mathbf{R}$, i.e., a to some extent arbitrary linear combination of curvature components. In fact, classically the Einstein equation (3.2) already implies $8\pi G\, \delta\rho = -\delta G_0^0$ where G_μ^ν is the Einstein tensor.[6] As a consequence, the two-point function of $\hat{\phi}$ is proportional to the $\delta\mathbf{R}$ correlator (6.3). Therefore the correlation function of $\delta\rho$ behaves as

$$\xi(\mathbf{x}) \propto \frac{1}{|\mathbf{x}|^4} \tag{6.4}$$

provided the physical distance $a(t)|\mathbf{x}|$ is smaller than $l_{\rm Pl}$. The power spectrum of the modes with physical momenta $|\mathbf{k}|/a(t) \lesssim m_{\rm Pl}$ (at fixed time $t \lesssim t_{\rm class}$) is given by the 3-dimensional Fourier transform of Eq. (6.4):

$$|\delta_k|^2 \propto |\mathbf{k}|. \tag{6.5}$$

This is precisely the Harrison-Zeldovich scale invariant spectrum with the spectral index $n=1$.

We can thus imagine that ''sub-Hubble scale'' modes evolve according to the standard theory of cosmological perturbations starting with a scale-invariant spectrum immediately after the quantum gravity epoch, $t \gtrsim t_{\rm Pl}$. A more complete treatment would also include the contribution from ''super-Hubble scale'' modes in a gauge-invariant framework, but this is beyond the scope of the present paper.

[6] See Ref. [34] for a similar discussion.

VII. CONCLUSION

In this paper we studied homogeneous, isotropic cosmologies in the Planck era before the classical Einstein equations become valid. We performed a RG improvement of the cosmological evolution equations by taking into account the running of G and Λ as it follows from the flow equation of the effective average action. For a spatially flat geometry we found solutions to the improved equations that are mathematically consistent even for $t\searrow 0$, i.e., immediately after the initial singularity of the the Universe. We believe that calculations can be done reliably in this regime because gravity becomes asymptotically free at high momentum scales so that Newton's constant is very small close to the big bang: $G\propto t^2$. The situation is comparable to QCD where physics at small length scales is simple but becomes increasingly complex as one probes larger distance scales. For $t\searrow 0$ the cosmological evolution is described by an attractor-type solution in (a,ρ,G,Λ) space which is a direct manifestation of the UV fixed point of the RG flow in (g,λ) space.

For a radiation dominated Planck era the attractor is perfectly scale free, the only dimensionful parameter being the cosmological time t. The RG improved solutions are "natural" in the sense that no fine-tuning is required, and for a broad class of equations of state ($w\leqslant 1/3$) they are free from particle horizons. Thus they offer an intriguing possibility for overcoming the flatness and the horizon problem of standard cosmology. We also found a natural mechanism for generating a scale free spectrum of primordial density fluctuations.

It is important to keep in mind which assumptions went into our derivation. They enter at different stages of the construction:

(i) We assume that for $k\rightarrow\infty$ the RG flow in (g,λ) space is governed by an UV attractive fixed point with $g_*>0$ and $\lambda_*>0$ so that gravity becomes asymptotically free in this limit. This UV fixed point is known to exist within the Einstein-Hilbert truncation of pure gravity. The assumption is that the coupled system of gravity plus matter behaves qualitatively in the same way.

(ii) We assume that the system of RG improved cosmological evolution equations (3.24) with $k\propto 1/t$ is valid for all times t after the big bang. This assumption means that the dominant quantum corrections are correctly incorporated by substituting $G_0\rightarrow G(t)$, $\Lambda_0\rightarrow\Lambda(t)$ in Einstein's equations and that no further modifications need to be taken into account explicitly (higher curvature terms, etc.). This assumption is consistent with (i) where it is also assumed that the Einstein-Hilbert action is sufficient to describe physics for $k\rightarrow\infty$ or $t\searrow 0$.

(iii) We assume that all matter fields can be integrated out completely before solving the gravitational equations. This is supposed to lead to an effective conserved energy momentum tensor $T_{\mu\nu}$ with a linear equation of state, $p=w\rho$. (However, quantum effects in the matter sector can influence g_* and λ_*, and they may shift w away from its classical value.) This assumption means that, consistently with (i), there are no renormalization effects coming from the matter sector that would be more important than those of pure quantum gravity.

In conclusion it is clear that cosmologies of the kind found in this paper are certainly extremely interesting and promising candidates for an extrapolation of classical FRW cosmology toward earlier cosmological times and for a possible solution of its problems and limitations. Their most attractive feature is that the resolution of those problems is obtained at a very low price. No *ad hoc* additional geometric structures, matter fields or cosmological eras have to be invoked. All that is needed is the quantization of the fields that are present anyway.

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APPENDIX A: THE CUTOFF $k\propto 1/a$

In this appendix we analyze the system of differential equations (3.24) under the assumption that the relevant cutoff momentum is given by the inverse scale function:

$$k(t)=\frac{\xi}{a(t)}. \tag{A1}$$

Since this cutoff functionally depends on the unknown function $a(t)$, it is less straightforward to find solutions than for the $1/t$ cutoff. We begin by solving the conservation law (3.24b) for the density ρ. From Eq. (3.26) we have

$$\rho(t)=\frac{\mathcal{M}}{8\pi a(t)^{3+3w}}. \tag{A2}$$

Next we insert Eq. (A2) into Eqs. (3.24a) and (3.24c) and reexpress the time derivatives in the latter equation according to $\dot{G}=(dG/da)\dot{a}$, $\dot{\Lambda}=(d\Lambda/da)\dot{a}$. Clearly this trick is possible only for cutoffs such as (A1) for which the time dependence of k is purely implicit. Thus we have to solve the system (for $\dot{a}\neq 0$)

$$\left(\frac{\dot{a}}{a}\right)^2+\frac{K}{a^2}=\frac{\Lambda}{3}+\frac{\mathcal{M}G}{3a^{3+3w}}, \tag{A3a}$$

$$\frac{d\Lambda}{da}+\frac{\mathcal{M}}{a^{3+3w}}\frac{dG}{da}=0, \tag{A3b}$$

$$G(t)=G(k=\xi/a),$$

$$\Lambda(t)=\Lambda(k=\xi/a). \tag{A3c}$$

It is interesting that Eq. (A3b) can be rewritten directly in terms of the RG beta functions:

$$k\frac{d\Lambda}{dk}+\mathcal{M}\left(\frac{k}{\xi}\right)^{3+3w}k\frac{dG}{dk}=0. \tag{A4}$$

Let us look at the fixed point regime and the perturbative regime separately.

1. The fixed point regime

In the fixed point regime Eq. (A3c) assumes the form

$$G(t)=\tilde{g}_*a^2, \quad \Lambda(t)=\tilde{\lambda}_*a^{-2}. \tag{A5}$$

Again we set

$$\tilde{g}_*\equiv g_*\xi^{-2}, \quad \tilde{\lambda}_*\equiv\lambda_*\xi^2 \tag{A6}$$

but the constant ξ differs from the one occurring in the $1/t$ cutoff. If we now insert Eq. (A5) into Eq. (A3b) we find that this equation is satisfied provided

$$w=+\frac{1}{3} \quad \text{and} \quad \tilde{\lambda}_*=\mathcal{M}\tilde{g}_*. \tag{A7}$$

A consistent solution can be obtained only for the $w=1/3$ equation of state, satisfied by classical radiation for instance. The second condition of Eq. (A7) will be used in order to determine ξ:

$$\xi^2=\sqrt{\frac{g_*\mathcal{M}}{\lambda_*}}. \tag{A8}$$

The last equation to be checked is Eq. (A3a). Substituting in $w=1/3$, and Eqs. (A5) and (A8) it reduces to the trivial differential equation $\dot{a}=\text{const}$ which, for the initial condition $a(0)=0$, is solved by $a\propto t$. Taking everything together we see that for $w=+1/3$ there exists the following consistent solution for all three cases $K=0,\ -1$, and $+1$:

$$a(t)=\left[\frac{2}{3}\sqrt{g_*\lambda_*\mathcal{M}}-K\right]^{1/2}t, \tag{A9a}$$

$$\rho(t)=\frac{\mathcal{M}}{8\pi}\left[\frac{2}{3}\sqrt{g_*\lambda_*\mathcal{M}}-K\right]^{-2}t^{-4}, \tag{A9b}$$

$$G(t)=\left(\frac{g_*\lambda_*}{\mathcal{M}}\right)^{1/2}\left[\frac{2}{3}\sqrt{g_*\lambda_*\mathcal{M}}-K\right]t^2, \tag{A9c}$$

$$\Lambda(t)=\sqrt{g_*\lambda_*\mathcal{M}}\left[\frac{2}{3}\sqrt{g_*\lambda_*\mathcal{M}}-K\right]^{-1}t^{-2}. \tag{A9d}$$

We observe that Eqs. (A9) coincide precisely with Eq. (4.27) for $K=0$ and with Eq. (4.30) derived for $K=\pm 1$. Contrary to the situation with the $1/t$ cutoff, no solution exists for $w\neq 1/3$, not even if $K=0$.

2. The perturbative regime

In the perturbative regime we have

$$G(t)=G_0-\tilde{\omega}G_0^2a^{-2}+\cdots, \tag{A10}$$

$$\Lambda(t)=\Lambda_0+\tilde{\nu}G_0a^{-4}+\cdots$$

with $\tilde{\omega}\equiv\omega\xi^2$ and $\tilde{\nu}\equiv\nu\xi^4$. By using Eq. (A10) in Eq. (A3b) the following conditions arise:

$$w=-\frac{1}{3} \quad \text{and} \quad 2\tilde{\nu}=\tilde{\omega}\mathcal{M}G_0. \tag{A11}$$

Consistency can be achieved only for rather exotic matter with $w=-1/3$ but not for the physically relevant cases with $w=+1/3$ or $w=0$, for instance. If we insert Eqs. (A10) and (A11) into Eq. (A3a) we obtain the differential equation that determines $a(t)$:

$$\dot{a}^2+K=\frac{1}{3}\Lambda_0a^2+\frac{1}{3}\mathcal{M}G_0-\frac{1}{6}\tilde{\omega}\mathcal{M}G_0^2a^{-2}+\cdots. \tag{A12}$$

To lowest order in $1/a$, the solution to this equation is precisely the classical FRW solution for $w=-1/3$.

To summarize: In the fixed point regime and for $w=+1/3$ the $1/a$ cutoff leads to precisely the same cosmology as the $1/t$ cutoff. For $w\neq+1/3$ there are no solutions in the fixed point regime. In the perturbative regime solutions exist only for the exotic equation of state with $w=-1/3$. Because the fixed point regime and the perturbative regime describe the limiting cases of $t\searrow 0$ and $t\to\infty$, respectively, we must conclude that, at least with the (perhaps too poor) approximations we used, there exists no solution with constant w, valid from $t=0$ up to the beginning of the classical era, which would connect to a standard radiation dominated FRW cosmology.

APPENDIX B: RG IMPROVEMENT OF THE CLASSICAL FRW SOLUTIONS

In the main body of the paper we made the improvement $G_0\to G(t)$, $\Lambda_0\to\Lambda(t)$ in the *equations* which determine the time evolution of $a(t)$ and the other quantities of cosmological interest. In this appendix we discuss an alternative strategy: the improvement of the *solutions* to the classical equations. In this second approach one first solves the differential equations containing G_0 and Λ_0, and then one makes the replacements $G_0\to G(t)$, $\Lambda_0\to\Lambda(t)$ in their solutions. If $k(t)$ has an implicit time dependence, $G(t)$ and $\Lambda(t)$ will depend on the classical solution $a_{\text{class}}(t)$ through $k=k(t,a_{\text{class}}(t),\dot{a}_{\text{class}}(t),\ldots)$. It seems clear, and we shall demonstrate this in detail, that the method of improving equations is superior to the improvement of solutions. In the latter case only small quantum corrections that do not change the behavior of the solution too strongly can be dealt with reliably, while with the first method solutions that are qualitatively different from the classical ones can also be investigated.

The starting point is the classical Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2+\frac{K}{a^2}=\frac{\Lambda_0}{3}+\frac{\mathcal{M}G_0}{3a^{3+3w}} \tag{B1}$$

from which ρ has been eliminated via the conservation law (3.26),

$$\rho=\frac{\mathcal{M}}{8\pi a^{3+3w}}. \tag{B2}$$

We restrict our analysis to the case $K=0$ for which the solutions to Eq. (B1) can be expressed in terms of elementary functions. Omitting the subscript "class," they read [as always, for the initial condition $a(0)=0$]:

(i) For $K=0,\ \Lambda_0=0$:

$$a(t)=\left[\frac{3}{4}(1+w)^2\mathcal{M}G_0\right]^{1/(3+3w)} t^{2/(3+3w)}. \tag{B3}$$

Hence, for any w,

$$\rho(t)=\frac{1}{6\pi(1+w)^2 G_0 t^2}. \tag{B4}$$

(ii) For $K=0,\ \Lambda_0>0$:

$$a(t)=\left[\frac{\mathcal{M}G_0}{2\Lambda_0}\{\cosh[(1+w)\sqrt{3\Lambda_0}t]-1\}\right]^{1/(3+3w)}. \tag{B5}$$

We shall need the Taylor expansion of this scale factor for early times $t\ll 1/\sqrt{\Lambda_0}$:

$$a(t)=\left[\frac{3}{4}(1+w)^2\mathcal{M}G_0 t^2\right]^{1/(3+3w)} \times\left\{1+\frac{1+w}{12}\Lambda_0 t^2+O(\Lambda_0^2 t^4)\right\}. \tag{B6}$$

(iii) For $K=0,\ \Lambda_0<0$:

$$a(t)=\left[\frac{\mathcal{M}G_0}{2|\Lambda_0|}\{1-\cos[(1+w)\sqrt{3|\Lambda_0|}t]\}\right]^{1/(3+3w)}. \tag{B7}$$

Next we shall discuss the improvement of these solutions in the perturbative and in the fixed point regimes, respectively. We use the identification $k=\xi/t$ throughout.

1. The perturbative regime

In this regime, t may be close to the Planck time so that quantum effects are important, but it is assumed that the lowest order terms in the $t_{\rm Pl}/t$ expansion are sufficient to describe them: $t\gtrsim t_{\rm Pl}$. Furthermore we assume that, as in the real Universe, Λ_0 is small: $\Lambda_0\ll m_{\rm Pl}^2$. The epoch we are interested in is characterized by[7]

$$t_{\rm Pl}\lesssim t\ll 1/\sqrt{\Lambda_0}. \tag{B8}$$

This interval contains the late part of the Planck era where quantum gravity still plays a role, as well as the classical era before the effect of the cosmological constant becomes dominant. During this epoch the product $\Lambda_0 t^2$ is small so that it is legitimate to base the improvement on the expanded form of the classical solution, Eq. (B6). Thus the RG improved scale factor reads

$$a_{\rm imp}(t)=\left[\frac{3}{4}(1+w)^2\mathcal{M}t^2\right]^{1/(3+3w)}[G(t)]^{1/(3+3w)} \times\left\{1+\frac{1+w}{12}\Lambda(t)t^2+\cdots\right\}. \tag{B9}$$

Using the $1/t$ cutoff, $G(t)$ and $\Lambda(t)$ are given by Eqs. (3.9) and (3.10), respectively. Hence we find the result

$$a_{\rm imp}(t)=\left[\frac{3}{4}(1+w)^2\mathcal{M}G_0 t^2\right]^{1/(3+3w)}\left\{1+\frac{1+w}{12}\Lambda_0 t^2 +\left(\frac{(1+w)\tilde{\nu}}{12}-\frac{\tilde{\omega}}{3(1+w)}\right)\left(\frac{t_{\rm Pl}}{t}\right)^2+\cdots\right\}. \tag{B10}$$

The leading quantum correction is a modification of $a(t)$ by a term of order $(t_{\rm Pl}/t)^2$. Within the present approach, its prefactor is completely undetermined, however. It involves the parameter ξ which cannot be fixed by renormalization group arguments alone. The method of improving equations is much more powerful in this respect; it allows us to express ξ in terms of ω and ν. In a kind of hybrid calculation we could use this result in order to rewrite Eq. (B10). This would change the terms inside the curly brackets of Eq. (B10) to

$$1-\frac{5\omega^2}{27\nu(1+w)^3}\left(\frac{t_{\rm Pl}}{t}\right)^2+\cdots. \tag{B11}$$

[We also took the other one of the consistency conditions (4.6), $\Lambda_0=0$, into account.]

It is important to note that a correction term of the type (B11) could not have been found as a solution to the improved *equation* unless one included in $G(t)$ and $\Lambda(t)$ higher orders of the $t_{\rm Pl}/t$ expansion. The reason is the remarkable fact, discussed in Sec. IV, that the classical $a(t)$ arises as a consequence of the lowest order nontrivial time dependence in $G(t)$ and $\Lambda(t)$.

2. The fixed point regime

Let us look at the improvement for $t\ll t_{\rm Pl}$. In this regime the renormalization effects are strong and strictly speaking it is not clear if the results are reliable. We start from the clas-

[7]For definiteness we assume that $\Lambda_0>0$.

sical $K=0$, $\Lambda_0>0$ solution (B5) and substitute $G_0 \to G(t)$, $\Lambda_0 \to \Lambda(t)$ according to Eq. (3.12) and Eq. (3.13), respectively. This substitution turns the $\cosh(t)$ time dependence into a purely algebraic one:

$$a_{\rm imp}(t) = \left[\frac{\tilde{g}_* \mathcal{M}}{2\tilde{\lambda}_*} \{ \cosh[(1+w)\sqrt{3\tilde{\lambda}_*}] - 1 \} \right]^{1/(3+3w)} \times t^{4/(3+3w)}. \tag{B12}$$

It is reassuring that apart from the details of the prefactor Eq. (B12) coincides with our previous result obtained by improving equations, Eq. (4.20a).

To summarize: Improving the classical FRW solutions shows that for $t \searrow 0$ the onset of the Planck era is characterized by a $(t_{\rm Pl}/t)^2$ correction to the scale factor. In the fixed point regime, this approach provides an independent confirmation of the picture we obtained by RG improving the equations.

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