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Scalar measure of the local expansion rate

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We define a scalar measure of the local expansion rate based on how astronomers determine the Hubble constant. Our observable is the inverse conformal d'Alembertian acting on a unit "standard candle." Because this quantity is an integral over the past light cone of the observation point it provides a manifestly causal and covariant technique for averaging over small fluctuations. For an exactly homogeneous and isotropic spacetime our scalar gives minus one-half times the inverse square of the Hubble parameter. Our proposal is that it be assigned this meaning generally and that it be employed to decide the issue of whether or not there is a significant quantum gravitational back reaction on inflation. Several techniques are discussed for promoting the scalar to a full invariant by giving a geometrical description for the point of observation. We work out an explicit formalism for evaluating the invariant in perturbation theory. The results for two simple models are presented in subsequent papers.

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I. INTRODUCTION

Quantum gravitational back reaction offers an attractive model of cosmology. The idea [1] is that there is no fine-tuning of the cosmological constant Λ or of scalar potentials. In fact there need not be any scalars. Inflation begins in the early universe because Λ is positive and not unnaturally small. Inflation eventually ends due to the accumulation of gravitational attraction between long wavelength virtual gravitons which are pulled apart by the rapid expansion of spacetime. Inflation persists for many e-foldings because gravity is a weak interaction, even at typical inflationary scales, and it requires an enormous accumulation of gravitational potential to overcome this. Since the process is infrared it can be studied reliably using quantum general relativity, without regard to the ultraviolet problem [2]. Because the model has only a single free parameter—$G\Lambda$, where G is Newton's constant—it can be used to make unique and testable predictions [3].

The physical mechanism of back reaction requires quanta which are massless on the scale of inflation but not classically conformally invariant. This rules out competition from most ordinary matter, but it does allow an effect from light, minimally coupled scalars. It has been suggested that significant back reaction can occur in scalar-driven inflation, even at one loop [4,5]. It has also been proposed that scalar self-interactions can give a significant back reaction at higher loops in Λ-driven inflation [6]. All these models involve fine tuning to keep the scalar light compared with the scale of inflation, so they are probably not relevant to phenomenology. However, scalars have the great advantage of being comparatively simpler to study than gravitons.

With any model of back reaction one encounters the problem of reliably inferring its impact on the cosmological expansion rate. For a perfectly homogeneous and isotropic geometry one would compute the expansion rate by transforming to co-moving coordinates, reading off the scale factor, and then taking its logarithmic time derivative. But back reaction derives from the gravitational response to quantum fluctuations, and these break homogeneity and isotropy. The notion of a cosmological expansion rate must obviously have a reasonable generalization since the current universe is not perfectly homogeneous and isotropic, yet astronomers mean something by measuring the Hubble constant. However, it is not so clear how to represent this observable in terms of quantum gravitational operators.

Previous studies of back reaction have tried to resolve this problem by averaging over fluctuations to produce an effective geometry which is homogeneous and isotropic. Then the cosmological expansion rate is computed from this effective geometry in the usual way. In one method the averaging is accomplished by taking the expectation value of the gauge fixed metric in the presence of a state which is homogeneous and isotropic [2,6–8]. Then the *expectation value* of the metric must be homogeneous and isotropic even though it is the average over quantum fluctuations which are not. The other technique is to enforce homogeneity and isotropy by spatially averaging the gauge fixed metric over a surface of simultaneity [4,5].

Serious objections have been raised to both techniques. Unruh dislikes using the gauge-fixed metric [9], either in an expectation value or in a spatial average. He argues that certain variations of the gauge fixing condition change the expectation value (or spatial average) of the metric in ways which cannot be subsumed into a coordinate transformation. Unruh therefore maintains that even forming the expectation value (or spatial average) of the metric into coordinate invariant quantities does not purge these quantities of gauge dependence. He would prefer that back reaction be studied with an operator which is itself an invariant, before taking

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the expectation value. He also disbelieves that averaging over a surface of simultaneity can be relevant to what a local observer perceives.

A different objection has been raised by Linde. He is willing to use the gauge fixed metric—and both men accept the validity of quantum field theory in determining the time evolution of the Heisenberg field operators. However, Linde suspects that inferring back reaction with expectation values invites a Schrödinger cat paradox. This is because inflationary particle production leaves the long wavelength modes in highly squeezed states whose behavior is essentially classical. No matter what Heisenberg operator is used to measure the cosmological expansion rate, Linde would prefer to stochastically [10] sample its probability distribution rather than take its expectation value.

The present work is an attempt to address the preceding objections. To avoid potential problems from using the gauge fixed metric we propose to infer the local expansion rate instead from the functional inverse of the conformal d'Alembertian:

$$\square_c \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu t) - \frac{1}{6}R. \tag{1}$$

This operator, acting on a unit "standard candle,"

$$\mathcal{A}[g](x) \equiv \frac{1}{\square_c}1, \tag{2}$$

averages over the past light cone, as astronomers do when compiling a Hubble diagram. In the slow roll approximation the observable gives $-\frac{1}{2}H^{-2}$ for an arbitrary homogeneous and isotropic universe. It is therefore a reasonable candidate for measuring the local expansion rate when the universe is not precisely homogeneous and isotropic. *And* it is a scalar function of the observation point x^μ.

Nothing can be done about the noninvariance associated with the fixed initial value surface upon which the Heisenberg state is defined. However, invariance under the subclass of transformations which preserve the initial value surface can be achieved by geometrically specifying the point at which $\mathcal{A}[g]$ is observed. In scalar-driven inflation this can be done by defining zero-shift surfaces of simultaneity so that the quantum inflaton agrees with its classical value. (Using these coordinates was Unruh's suggestion.) In more general models one can build invariant surfaces of simultaneity using the inverse minimally coupled d'Alembertian. The expectation value of the resulting invariant can then be evaluated, or else its probability distribution can be sampled stochastically.

In Sec. II we motivate the scalar and show that it has the proper correspondence limit for exactly homogeneous and isotropic geometries. Section III discusses the corrections needed to geometrically specify the observation point. In Sec. IV we expand the scalar in powers of the metric fluctuations. Section V concerns the retarded Green's functions which appear in this expansion. We discuss a somewhat more complicated but considerably sharper observable in Sec. VI. Our conclusions comprise Sec. VII. Two subsequent papers give the results of applying the observable to models of scalar-driven [11] and Λ-driven [12] inflation.

II. MOTIVATING THE SCALAR

Since we are interested in the effect of back reaction on inflation it is reasonable to consider perturbations about a background geometry which is homogeneous, isotropic and spatially flat,

$$ds_0^2 = -dt^2 + e^{2b(t)}d\vec{x}\cdot d\vec{x} = a^2(\eta)(-d\eta^2 + d\vec{x}\cdot d\vec{x}). \tag{3}$$

There is general agreement that the cosmological expansion rate for this background is $H=\dot{b}=a'/a^2$. Dots denote comoving time derivatives while primes represent conformal time derivatives. We normalize the initial ($t=0$ or $\eta=\eta_I$) scale factor to unity.

The full metric has the form

$$g_{\mu\nu}(\eta,\vec{x}) \equiv a^2(\eta)\tilde{g}_{\mu\nu}(\eta,\vec{x}) \equiv a^2(\eta)[\eta_{\mu\nu} + \kappa\psi_{\mu\nu}(\eta,\vec{x})], \tag{4}$$

where $\eta_{\mu\nu}$ is the spacelike Lorentz metric and $\kappa^2 \equiv 16\pi G$ is the loop counting parameter of quantum gravity. Fluctuations reside in the pseudo-graviton field, $\psi_{\mu\nu}(\eta,\vec{x})$, whose indices are raised and lowered with the Lorentz metric. What we seek is a scalar functional of the metric which provides a reasonable extrapolation for how a localized observer would measure the cosmological expansion rate when $\psi_{\mu\nu}(\eta,\vec{x}) \neq 0$.

It is worth explaining why the Ricci scalar is not satisfactory. $R(x)$ is certainly scalar, and it is closely related to the Hubble constant for the case of perfect homogeneity and isotropy,

$$R \to 12H^2 + 6a^{-1}H' = 6a^{-3}a''. \tag{5}$$

However, no local curvature invariant can account for the ability of observers to perceive the larger universe at cosmological distances by looking back along their past light cones. Einstein's equations set the Ricci scalar to $-8\pi G$ times the trace of the stress tensor. This actually vanishes during a phase of radiation-dominated expansion. Nor does the local value of $R(x)$ have much to do with what an observer can see at cosmological distances. For example, even "empty" space within our solar system contains about 10 hydrogen atoms per cubic centimeter. Were we to infer the rate of cosmological expansion using $R(x)$ the result would correspond to a Hubble constant about a hundred times larger than the actual value,

$$8\pi G\rho \sim 3\times 10^{-30}\frac{1}{\mathrm{s}^2} \sim \left(5\times 10^4 \frac{\mathrm{km}}{\mathrm{s\ Mpc}}\right)^2. \tag{6}$$

We stress that there must be a reasonable solution to this problem because the current universe is not precisely homogeneous and isotropic, yet human astronomers still claim to be able to measure the Hubble constant. It is instructive to review one of their simpler techniques. Consider light emitted at $(\eta_1,\vec{x}_1)$ and received at $(\eta_0,\vec{x}_0)$. The observed quantities are the redshift z and the flux $\mathcal{F}$. If the source luminosity $\mathcal{L}$ is known, these two quantities can be related under the assumption of perfect homogeneity and isotropy. Astronomers simply define the local Hubble constant so as to make the same relation true in the presence of fluctuations. Then they average over many sources.

Deriving the relation between $\mathcal{F}$ and z is a standard exercise [13]. Assuming perfect homogeneity and isotropy the physical distance between source and observer at time η_0 would be

$$\Delta r \equiv a_0 \|\vec{x}_0-\vec{x}_1\| = a_0(\eta_0-\eta_1). \tag{7}$$

The measured flux is the flat space formula, corrected for the redshifts of energy and rate,

$$\mathcal{F}=\left(\frac{1}{1+z}\right)^2 \frac{\mathcal{L}}{4\pi\Delta r^2}. \tag{8}$$

One inverts this relation to solve for the product of $1+z$ times Δr, which is known as the ''luminosity distance,''

$$d_L \equiv (1+z)\Delta r = \sqrt{\frac{\mathcal{L}}{4\pi\mathcal{F}}}. \tag{9}$$

If both observer and source are at rest in conformal coordinates then the observed redshift would be

$$z=\frac{a_0}{a(\eta_1)}-1. \tag{10}$$

Its relation to Δr comes from the scale factor's Taylor expansion,

$$a(\eta_1)=a_0\left[1-H_0\Delta r+\frac{1}{2}(1-q_0)H_0^2\Delta r^2+\cdots\right], \tag{11}$$

where the current Hubble constant and deceleration parameter are

$$H_0 \equiv \frac{a_0'}{a_0^2}, \quad q_0 \equiv 1-\frac{a_0 a_0''}{a_0'^2}. \tag{12}$$

Inverting to solve for Δr gives

$$H_0\Delta r = z-\frac{1}{2}(1+q_0)z^2+\cdots \tag{13}$$

and multiplication by $1+z$ results in the luminosity distance,

$$H_0 d_L = z+\frac{1}{2}(1-q_0)z^2+\cdots. \tag{14}$$

Plotting z against d_L for relatively small z gives a straight line whose inverse slope is the Hubble constant.

It is not simple to identify invariants which represent the observed quantities z and $\mathcal{F}$ for an arbitrary metric. If one considers the transmission process in terms of individual photons then the redshift could be formulated as follows. Let us denote the worldlines of the emitter and observer as functions of their respective proper times by $X^\mu_{\rm em}(\tau)$ and $X^\mu_{\rm obs}(\tau)$. Recall that proper times are normalized to obey,

$$g_{\alpha\beta}(X(\tau))\dot{X}^\alpha(\tau)\dot{X}^\beta(\tau)=-1, \tag{15}$$

where the overdots stand for differentiation with respect to τ. Now consider a photon which was emitted at proper time τ_1 and reaches the observer at proper time τ_0. Of course the affine parameter σ of the photon's worldline $X^\mu_{\rm ph}(\sigma)$ cannot be a proper time since the 4-velocity must be lightlike,

$$g_{\alpha\beta}(X_{\rm ph}(\sigma))\dot{X}^\alpha_{\rm ph}(\sigma)\dot{X}^\beta_{\rm ph}(\sigma)=0. \tag{16}$$

Given any function $X^\mu_{\rm ph}(\sigma)$ which obeys Eq. (16) as it interpolates from $X^\mu_{\rm em}(\tau_1)=x^\mu_1$ to $X^\mu_{\rm obs}(\tau_0)=x^\mu_0$, one makes a reparametrization of the affine parameter (the new value of which we shall continue to call σ) so as to enforce the geodesic equation,

$$\ddot{X}^\mu_{\rm ph}(\sigma)+\Gamma^\mu{}_{\rho\sigma}(X_{\rm ph}(\sigma))\dot{X}^\rho_{\rm ph}(\sigma)\dot{X}^\sigma_{\rm ph}(\sigma)=0. \tag{17}$$

The redshift experienced by such a photon is given by

$$1+z=\frac{g_{\rho\sigma}(x_1)\dot{X}^\rho_{\rm ph}(\sigma_1)\dot{X}^\sigma_{\rm em}(\tau_1)}{g_{\mu\nu}(x_0)\dot{X}^\mu_{\rm ph}(\sigma_0)\dot{X}^\nu_{\rm obs}(\tau_0)}. \tag{18}$$

The flux is essentially the response, at the observer's location, to the (presumed known) source's current density $J^\mu(x)$. One begins by solving Maxwell's equations for the field strength tensor $F_{\mu\nu}(x)$,

$$F_{\mu\nu}{}^{;\nu}=-J_\mu, \quad F_{\alpha\beta;\gamma}+F_{\beta\gamma;\alpha}+F_{\gamma\alpha;\beta}=0. \tag{19}$$

These equations are invariant under local conformal rescalings,

$$F_{\mu\nu}\to F_{\mu\nu}, \quad J_\mu\to\Omega^2 J_\mu, \quad g_{\mu\nu}\to\Omega^2 g_{\mu\nu}, \tag{20}$$

and they can be solved in terms of something we shall call the conformal tensor d'Alembertian,

$${}_{\mu\nu}\Box_c^{\rho\sigma} \equiv {}_{\mu\nu}\Box^{\rho\sigma}-\delta^\rho{}_\mu R^\sigma{}_\nu-\delta^\sigma{}_\nu R^\rho{}_\mu+2R^\rho{}_\mu{}^\sigma{}_\nu. \tag{21}$$

The solution is

$$F_{\mu\nu}={}_{\mu\nu}\left(\frac{1}{\Box_c}\right)^{\rho\sigma}(-J_{\rho;\sigma}+J_{\sigma;\rho}). \tag{22}$$

One gets the stress tensor from the field strength tensor,

$$T_{\mu\nu}=\left(\delta^\alpha{}_\mu\delta^\beta{}_\nu-\frac{1}{4}g_{\mu\nu}g^{\alpha\beta}\right)g^{\rho\sigma}F_{\alpha\rho}F_{\beta\sigma}. \tag{23}$$

The Poynting vector is obtained by contracting the observer's 4-velocity into the electromagnetic stress tensor,

$$S_\mu=T_{\mu\nu}(x_0)\dot{X}^\nu(\tau_0). \tag{24}$$

And the measured flux is the norm of the Poynting vector in the orthogonal projection of the observer's metric,

$$\mathcal{F}^2 = S_\mu S_\nu (g^{\mu\nu} + \dot{X}^\mu_{\rm obs} \dot{X}^\nu_{\rm obs}). \tag{25}$$

Astronomers measure electromagnetic radiation because it is available to them, but this choice of observable complicates the metric dependence of the operators which represent their measurements. For example, the tensor character of Maxwell's field strength is why one has to invert the tensor conformal d'Alembertian in Eq. (22), rather than its simpler scalar cousin. It is also why the response field has to be squared in Eq. (23). Other complications arise from the fact that the source luminosities are not precisely known, and that their distribution throughout space is not uniform.

The preceding complications pose important limitations on observational astronomy but they need not restrict our choice of the operator with which to probe the theory. For us the really essential feature is to measure the response of some long range field to known sources distributed along the observer's past light cone. We can retain this feature and vastly simplify our labor by observing a conformally coupled scalar, rather than a conformally coupled tensor. We can achieve a further simplification by taking the source to be a uniformly distributed monopole, rather than a sparse distribution of dipoles of varying strength. Then a single measurement of the scalar represents a full sky average and we can dispense with the complication of having to tabulate two quantities (z and $\mathcal{F}$) for each source point. We call the scalar $\mathcal{A}$ and define it to obey the equation,

$$\Box_c \mathcal{A} = 1, \tag{26}$$

where $\Box_c$ is the conformal d'Alembertian (1). If we define the scalar and its first time derivative to vanish on the initial value surface the result is just the integral of the retarded conformal Green's function,

$$\mathcal{A}[g](x) = \frac{1}{\Box_c} 1. \tag{27}$$

It remains to show that $\mathcal{A}[g](x)$ has the right correspondence limit for exact homogeneity and isotropy. In this case the conformal d'Alembertian reduces to the form

$$\Box_c \to a^{-3} \partial^2 a, \tag{28}$$

where $\partial^2 \equiv \eta^{\mu\nu}\partial_\mu \partial_\nu$ is the flat space d'Alembertian in conformal coordinates. The operator becomes even simpler acting on spatial constants. One consequence is that Eq. (26) can be solved by simple integration,

$$\begin{aligned}\mathcal{A}_0(\eta,\vec{x}) &= -a^{-1}(\eta)\int_{\eta_I}^{\eta} d\eta' \int_{\eta_I}^{\eta'} d\eta'' a^3(\eta'') \\ &= -e^{-b(t)}\int_0^t dt' e^{-b(t')}\int_0^{t'} dt'' e^{2b(t'')}.\end{aligned} \tag{29}$$

It turns out that Eq. (29) can be evaluated quite generally in what is known as *the slow roll approximation*. This is obeyed by all successful models of inflation and it amounts to neglecting all higher comoving time derivatives of the logarithmic scale factor $b(t)$ with respect to the first,

$$\left|\frac{d^N b}{dt^N}\right| \ll (\dot{b})^N \quad \forall N \geq 2. \tag{30}$$

Most operations of ordinary calculus can be done explicitly in the slow roll approximation. For example, the following trivial rearrangement:

$$e^{2b} = \frac{d}{dt}\left(\frac{e^{2b}}{2\dot{b}}\right) + e^{2b}\frac{\ddot{b}}{2\dot{b}^2}, \tag{31}$$

allows us to express the initial integrand of Eq. (29) as a total derivative plus a term which is negligible in the slow roll approximation. It would be straightforward to develop a series in slow roll corrections but the first is generally sufficient for the inflationary setting in which we wish to employ the new observable. With positive exponents and any significant amount of inflation it is also possible to ignore the lower limit,

$$\int_0^t dt' e^{2b(t')} = \frac{e^{2b(t')}}{2\dot{b}(t')}\left\{1 + \frac{\ddot{b}(t')}{2\dot{b}^2(t')} + \cdots\right\}\Bigg|_0^t \approx \frac{e^{2b(t)}}{2\dot{b}(t)}. \tag{32}$$

We can therefore apply the slow roll approximation to Eq. (29) to express the observable in terms of the Hubble constant,

$$\mathcal{A}_0(\eta,\vec{x}) \approx -\frac{1}{2\dot{b}^2(t)}. \tag{33}$$

In analogy to astronomical practice the local Hubble constant in the presence perturbations is defined so as to preserve this relation,

$$\mathcal{A}[g](x) \equiv -\frac{1}{2H^2(x)}. \tag{34}$$

Although we are chiefly interested in applying the new observable during inflation it is worth noting that the slow roll result (34) is valid, up to a number of order one, for quite general geometries. For example, with general power law expansion the logarithmic scale factor and Hubble constant are

$$b(t) = s\ln\left(1 + \frac{H_I t}{s}\right), \quad \dot{b}(t) = H_I\left(1 + \frac{H_I t}{s}\right)^{-1}, \tag{35}$$

where H_I is the initial Hubble constant and s is a constant. With this simple time dependence we can perform the integrals in Eq. (29) exactly,

$$\mathcal{A}_{\rm power} = \frac{-s^2}{\left(s+\frac{1}{2}\right)(s+2)} \frac{1}{2\dot{b}^2(t)} + \frac{s^2}{(s-1)(s+2)} \frac{e^{-b(t)}}{H_I^2} - \frac{s^2}{(s-1)\left(s+\frac{1}{2}\right)} \frac{e^{-2b(t)}}{2H_I\dot{b}(t)}. \tag{36}$$

The second and third terms become insignificant at late times and the first rapidly approaches Eq. (34) for large s. Even for $s=\frac{2}{3}$ the numerical factor is $\frac{1}{7}$.

III. FIXING THE OBSERVATION POINT GEOMETRICALLY

Even scalars depend upon the point at which they are observed. Part of this dependence is physical. The Heisenberg state is specified on a particular initial value surface and the geometrical relation of the observation point to this initial value surface can and should affect the result. The purpose of this section is to formulate the technology for imposing such a relation.

Of course any method of describing points amounts to fixing a gauge; however, there is an important distinction between *ad hoc* gauge conditions and those which exploit some special feature of the particular system under study. For example, if the system includes a Sun then it is geometrically meaningful to take this star's center as the spatial origin. It is therefore necessary to be as precise as possible about the nature of the system under study.

The dynamical variables of our system include the metric $g_{\mu\nu}(x)$ and possibly also a scalar inflaton field $\varphi(x)$. Our goal is to compute the expectation value of the operator $\mathcal{A}[g]$ (or to stochastically sample its probability distribution) in the presence of a Heisenberg state which we shall assume is homogeneous and isotropic. Since the only effective technique for making such a calculation is perturbation theory we shall also assume that the various quantum field operators are perturbations on a background which is homogeneous and isotropic,

$$\varphi(\eta,\vec{x}) = \varphi_0(\eta) + \phi(\eta,\vec{x}), \tag{37}$$

$$g_{\mu\nu}(\eta,\vec{x}) = a^2(\eta)[\eta_{\mu\nu} + \kappa\psi_{\mu\nu}(\eta,\vec{x})]. \tag{38}$$

Because geometrically significant gauge conditions can involve nonlocal and nonlinear functionals of the fields, we wish to preserve the option of carrying out the calculation in more convenient gauge. A simple technique for accomplishing this is to define the observation point as the field-dependent coordinate transformation $Y^\mu[\varphi,g](x)$ such that the transformed scalar (if there is one) and the transformed metric,

$$\varphi'(x) \equiv \varphi(Y(x)), \tag{39}$$

$$g'_{\mu\nu}(x) \equiv \frac{\partial Y^\rho}{\partial x^\mu} \frac{\partial Y^\sigma}{\partial x^\nu} g_{\rho\sigma}(Y(x)), \tag{40}$$

obey the geometrically significant gauge conditions. Then one can evaluate $\mathcal{A}[g](Y(x))$ in any gauge and the result will be the same.

The existence of a fixed initial value surface (at $\eta=\eta_I$) suggests that Y^μ should be expressed as the composition of a temporal transformation $\eta \rightarrow \tau(\eta,\vec{x})$ followed by a purely spatial transformation $x^i \rightarrow \chi^i(\eta,\vec{x})$. Surfaces of simultaneity are defined by the condition $\tau(\eta,\vec{x}) = \text{const}$, while $\chi^i(\eta,\vec{x})$ traces out "the same" space point on the foliation of these surfaces. The full transformation would be

$$Y^0(\eta,\vec{x}) = \tau(\eta,\vec{x}), \quad Y^i(\eta,\vec{x}) = \chi^i(\tau(\eta,\vec{x}),\vec{x}). \tag{41}$$

The problem's homogeneity and isotropy implies that all space points are physically equivalent and we may as well use orthogonal projection to define "the same" space point. This amounts to the condition $g'_{0i}=0$ and hence

$$0 = \frac{\partial \chi^k}{\partial x^i} g_{0k}(\eta,\vec{\chi}) + \frac{\partial \chi^j}{\partial \eta} \frac{\partial \chi^k}{\partial x^i} g_{jk}(\eta,\vec{\chi}). \tag{42}$$

The relation can be simplified by multiplying with the inverse Jacobian,

$$g_{0j}(\eta,\vec{\chi}) + \frac{\partial \chi^i}{\partial \eta} g_{ij}(\eta,\vec{\chi}) = 0. \tag{43}$$

Whereupon multiplication by the inverse 3-metric results in the following first order (but nonlinear) differential equation,

$$\frac{\partial \chi^i}{\partial \eta} = -(g^{-1})^{ij} g_{0j}(\eta,\vec{\chi}) \tag{44}$$

$$= -\kappa\psi_{0i}(\eta,\vec{\chi}) + \kappa^2(\psi_{0j}\psi_{ji})(\eta,\vec{\chi}) - \kappa^3(\psi_{0k}\psi_{kj}\psi_{ji})(\eta,\vec{\chi}) + \cdots. \tag{45}$$

Making the obvious choice of an initial condition gives an integral equation whose iteration to any order is straightforward;

$$\chi^i(\eta,\vec{x}) = x^i - \int_{\eta_I}^{\eta} d\sigma (g^{-1})^{ij} g_{0j}(\sigma,\vec{\chi}) \tag{46}$$

$$= x^i - \kappa \int_{\eta_I}^{\eta} d\sigma \psi_{0i}(\sigma,\vec{x}) + \kappa^2 \int_{\eta_I}^{\eta} d\sigma (\psi_{0j}\psi_{ji})(\sigma,\vec{x}) + \kappa^2 \int_{\eta_I}^{\eta} d\sigma \psi_{0i,j}(\sigma,\vec{x}) \int_{\eta_I}^{\sigma} d\rho \psi_{0j}(\rho,\vec{x}) + O(\kappa^3). \tag{47}$$

Defining surfaces of simultaneity is less subtle for Λ-driven inflation than for its scalar-driven cousin. Without back reaction the cosmological expansion rate is constant in Λ-driven inflation, so the effect is certainly real if one sees

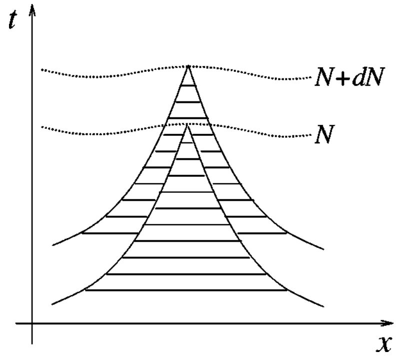

FIG. 1. Invariant procedure to fix the observation point.

progressive slowing under any timelike foliation. In the scalar-driven case there is already slowing as the background scalar rolls down its potential so one must be careful to compare the expansion rate with and without back reaction at the same physical time.

Since the value of the scalar determines the expansion rate without back reaction it seems reasonable to define surfaces of simultaneity so that the full inflaton field agrees with its background value,

$$\varphi(\tau(\eta,\vec{x}),\vec{x}) \equiv \varphi_0(\eta). \tag{48}$$

This can be solved perturbatively by first writing,

$$\tau(\eta,\vec{x}) = \eta + \delta\tau(\eta,\vec{x}), \tag{49}$$

and Taylor expanding,

$$\sum_{n=1}^{\infty} \frac{\varphi_0^{(n)}(\eta)}{n!}(\delta\tau(\eta,\vec{x}))^n = -\sum_{n=0}^{\infty} \frac{\phi^{(n)}(\eta,\vec{x})}{n!}(\delta\tau(\eta,\vec{x}))^n. \tag{50}$$

Inverting results in an expansion for $\delta\tau$ in powers of the quantum scalar ϕ and its derivatives [all evaluated at $(\eta,\vec{x})$],

$$\delta\tau = -\frac{\phi}{\varphi'} + \frac{\phi\phi'}{\varphi_0'^2} - \frac{\varphi_0''\phi^2}{2\varphi_0'^3} + O(\phi^3). \tag{51}$$

Λ-driven inflation can be included within the same scheme by employing a scalar functional of the metric with monotonic time dependence in place of the dynamical scalar. Perhaps the simplest of these "clock functions" makes use of the inverse minimally coupled d'Alembertian acting on the Ricci scalar,

$$\mathcal{N}[g](x) \equiv -\frac{1}{4\Box} R. \tag{52}$$

We define surfaces of simultaneity so as to make the clock agree with its background value, just as relation (48) does for scalar-driven inflation,

$$\mathcal{N}(\tau(\eta,\vec{x}),\vec{x}) \equiv \mathcal{N}_0(\eta). \tag{53}$$

Figure 1 depicts the resulting foliation. To see that $\mathcal{N}[g](x)$ is a good clock in perturbation theory note that $\mathcal{N}_0(\eta) \approx \ln(a(\eta))$ in the slow roll approximation. This follows because the minimally coupled d'Alembertian takes the following form in a homogeneous, isotropic and spatially flat geometry:

$$\Box \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) \rightarrow a^{-4}\partial_\mu(a^2\eta^{\mu\nu}\partial_\nu). \tag{54}$$

When acting on functions of only time the inverse of $\Box$ reduces to

$$\frac{1}{\Box} \rightarrow -\int_{\eta_I}^{\eta} d\eta' a^{-2}(\eta') \int_{\eta_I}^{\eta'} d\eta'' a^4(\eta'')$$

$$= -\int_0^t dt' e^{-3b(t')} \int_0^{t'} dt'' e^{3b(t'')}. \tag{55}$$

With no perturbations we therefore have

$$\mathcal{N}_0(\eta) = \int_0^t dt' e^{-3b(t')} \int_0^{t'} dt'' e^{3b(t'')} \left(3\dot{b}^2(t'') + \frac{3}{2}\ddot{b}(t'')\right). \tag{56}$$

Making the slow roll approximation gives

$$\mathcal{N}_0(\eta) \approx \int_0^t dt' e^{-3b(t')} \int_0^{t'} dt'' \frac{d}{dt''}[\dot{b}(t'')e^{3b(t'')}] \tag{57}$$

$$\approx \int_0^t dt' \dot{b}(t') = b(t). \tag{58}$$

Before closing the section we should comment that there is no problem in perturbation theory about evaluating an operator such as $\mathcal{A}[g](x)$ at a point such as $Y^\mu[\varphi,g](x)$, which is itself an operator. One merely expands in powers of the perturbatively small quantity $Y^\mu(x) - x^\mu$. Only a finite number of terms need be included to reach any fixed order in perturbation theory. Note that the various operator products should be time ordered. This is because the functional integral of the $\mathbb{C}$-number functional $\mathcal{A}[g](Y[\varphi,g](x))$ is manifestly invariant as it is, and gives the expectation value of the time-ordered product of the corresponding operator.

IV. PSEUDO-GRAVITON EXPANSION

The purpose of this section is to expand $\mathcal{A}[g](x)$ in powers of the pseudo-graviton field $\psi_{\mu\nu}(\eta,\vec{x})$. This is most easily accomplished by first expressing $\Box_c$ in terms of the conformally rescaled metric,

$$\tilde{g}_{\mu\nu}(\eta,\vec{x}) \equiv a^{-2}(\eta) g_{\mu\nu}(\eta,\vec{x}) = \eta_{\mu\nu} + \kappa\psi_{\mu\nu}(\eta,\vec{x}). \tag{59}$$

We now write $\Box_c = a^{-3}\mathcal{D}a$, where $\mathcal{D}$ and its expansion are

$$\mathcal{D} \equiv \frac{1}{\sqrt{-\tilde{g}}}\partial_\mu(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu) - \frac{1}{6}\tilde{R} = \partial^2 + \kappa\mathcal{D}_1 + \kappa^2\mathcal{D}_2 + \cdots. \tag{60}$$

The first two operators in the expansion are

$$\mathcal{D}_1 = -\psi^{\mu\nu}\partial_\mu\partial_\nu + \left(-\psi^{\mu\alpha}{}_{,\alpha} + \frac{1}{2}\psi^{,\mu}\right)\partial_\mu - \frac{1}{6}(\psi^{\rho\sigma}{}_{,\rho\sigma} - \psi^{,\rho}{}_{\rho}), \tag{61}$$

$$\mathcal{D}_2 = \psi^{\mu\alpha}\psi^{\nu}_{\alpha}\partial_\mu\partial_\nu + \left((\psi^{\alpha\beta}\psi^{\mu}{}_{\alpha})_{,\beta} - \frac{1}{2}\psi^{\alpha\beta,\mu}\psi_{\alpha\beta} + \frac{1}{2}\psi^{\alpha\mu}\psi_{,\alpha}\right)\partial_\mu - \frac{1}{6}\widetilde{R}_2. \tag{62}$$

We remind the reader that pseudo-graviton indices are raised and lowered by the Lorentz metric, $\eta_{\mu\nu}$. Other notational points are that the trace of the pseudo-graviton field is $\psi \equiv \psi^{\rho}{}_{\rho}$ and that the second order, conformally rescaled Ricci scalar is

$$\widetilde{R}_2 \equiv \psi^{\alpha\beta}(\psi_{\alpha\beta}{}^{,\gamma}{}_{\gamma} + \psi_{,\alpha\beta} - 2\psi^{\gamma}{}_{\alpha,\beta\gamma}) + \frac{3}{4}\psi^{\alpha\beta,\gamma}\psi_{\alpha\beta,\gamma} - \frac{1}{2}\psi^{\alpha\beta,\gamma}\psi_{\gamma\beta,\alpha} - \psi^{\alpha\beta}{}_{,\beta}\psi^{\gamma}{}_{\alpha,\gamma} + \psi^{\alpha\beta}{}_{,\beta}\psi_{,\alpha} - \frac{1}{4}\psi^{,\alpha}\psi_{,\alpha}. \tag{63}$$

The next step is to factor ∂^2 out of $\mathcal{D}$,

$$\mathcal{D} = \partial^2\left(1 + \frac{1}{\partial^2}\kappa\mathcal{D}_1 + \frac{1}{\partial^2}\kappa^2\mathcal{D}_2 + O(\kappa^3)\right). \tag{64}$$

Inverting $\mathcal{D}$ is now straightforward,

$$\frac{1}{\mathcal{D}} = \frac{1}{\partial^2} - \frac{1}{\partial^2}\kappa\mathcal{D}_1\frac{1}{\partial^2} + \frac{1}{\partial^2}\kappa\mathcal{D}_1\frac{1}{\partial^2}\kappa\mathcal{D}_1\frac{1}{\partial^2} - \frac{1}{\partial^2}\kappa^2\mathcal{D}_2\frac{1}{\partial^2} + O(\kappa^3). \tag{65}$$

All this implies the following expansion for the scalar observable,

$$\mathcal{A}[g] = a^{-1}\frac{1}{\mathcal{D}}a^3 = \mathcal{A}_0 + \kappa\mathcal{A}_1 + \kappa^2\mathcal{A}_2 + O(\kappa^3). \tag{66}$$

$\mathcal{A}_0$ was worked out at the end of Sec. II. The next two terms are

$$\mathcal{A}_1 \equiv -a^{-1}\frac{1}{\partial^2}\mathcal{D}_1\frac{1}{\partial^2}a^3, \tag{67}$$

$$\mathcal{A}_2 \equiv -a^{-1}\frac{1}{\partial^2}\mathcal{D}_2\frac{1}{\partial^2}a^3 + a^{-1}\frac{1}{\partial^2}\mathcal{D}_1\frac{1}{\partial^2}\mathcal{D}_1\frac{1}{\partial^2}a^3. \tag{68}$$

It should be noted that the pseudo-graviton field is not free, nor are all of its components dynamical. The next step after this would be to expand $\psi_{\mu\nu}(\eta,\vec{x})$ in terms of the fundamental dynamical degrees of freedom, whatever they happen to be. This obviously depends upon selecting a particular model and must be postponed until this has been done [11,12].

V. RETARDED GREEN'S FUNCTIONS

The pseudo-graviton expansion of the previous section results in a series of terms which involve the inverse differential operator $1/\partial^2$. The purpose of this section is to precisely define the action of this operator. We also apply the slow roll approximation.

The first task is easily accomplished. The retarded Green's function for the operator ∂^2 is well known,

$$G(x;x') = -\frac{\theta(\Delta\eta)}{4\pi\Delta x}\delta(\Delta\eta - \Delta x), \tag{69}$$

where $\Delta\eta \equiv \eta - \eta'$ and $\Delta x \equiv \|\vec{x} - \vec{x}'\|$. Since the initial value surface is at $\eta = \eta_I$ we define the result of acting $1/\partial^2$ on an arbitrary function $f(\eta,\vec{x})$ as

$$\left[\frac{1}{\partial^2}f\right](\eta,\vec{x}) \equiv -\int_{\eta_I}^{\eta} d\eta' \int d^3x' \frac{\delta(\Delta\eta - \Delta x)}{4\pi\Delta x} f(\eta',\vec{x}') \tag{70}$$

$$= -\int_{\eta_I}^{\eta} d\eta' \Delta\eta \int \frac{d^2\hat{n}}{4\pi} f(\eta', \vec{x} + \Delta\eta\hat{n}). \tag{71}$$

When the function depends only upon time we can reach a form similar to that of Sec. II. Making the substitution $f(\eta,\vec{x}) \rightarrow F(\eta)$ gives

$$\left[\frac{1}{\partial^2}F\right](\eta,\vec{x}) = -\int_{\eta_I}^{\eta} d\eta' \Delta\eta F(\eta') \tag{72}$$

$$= -\int_{\eta_I}^{\eta} d\eta' (\eta - \eta') \frac{d}{d\eta'} \int_{\eta_I}^{\eta'} d\eta'' F(\eta'') \tag{73}$$

$$= -\int_{\eta_I}^{\eta} d\eta' \int_{\eta_I}^{\eta'} d\eta'' F(\eta''). \tag{74}$$

An important example is provided by the rightmost term for each of the $\mathcal{A}_n$'s—$\partial^{-2}a^3$. We can explicitly evaluate these terms by making use of the slow roll approximation,

$$\left[\frac{1}{\partial^2}a^3\right](\eta,\vec{x}) = -\int_0^t dt' e^{-b(t')} \int_0^{t'} dt'' e^{2b(t'')} \approx \frac{-e^{b(t)}}{2\dot{b}^2(t)}. \tag{75}$$

The fact that this depends only upon η allows one to simplify expressions in which derivatives act upon it. For example, the contribution from the first term in Eq. (61) is

$$a^{-1}\frac{1}{\partial^2}\kappa\psi^{\mu\nu}\partial_\mu\partial_\nu\frac{1}{\partial^2}a^3=-a^{-1}\frac{1}{\partial^2}\kappa\psi_{00}a^3 \tag{76}$$

$$=a^{-1}(\eta)\int_{\eta_I}^{\eta}d\eta' a^3(\eta')\Delta\eta \times\int\frac{d^2\hat{n}}{4\pi}\kappa\psi_{00}(\eta',\vec{x}+\Delta\eta\hat{n}). \tag{77}$$

Further progress requires using the slow roll approximation. It turns out, as one evaluates the various factors of $1/\partial^2$ from left to right, that the various integrands upon which they act are always dominated by the universal initial factor of a^3. In typical gauges the pseudo-graviton field can grow at most like powers of ln(a). Although derivatives can sometimes result in a net loss of powers of the scale factor, they can never add such powers. Further, whenever even a single power of a is lost the contribution which finally results to $\mathcal{A}[g](x)$ is exponentially suppressed and hence irrelevant. It therefore suffices to consider terms of the form $\partial^{-2}(a^3f)$ for functions $f(\eta,\vec{x})$ which grow less rapidly that $a(\eta)$,

$$\left[\frac{1}{\partial^2}a^3f\right](\eta,\vec{x})=-\int_0^t dt' e^{2b(t')}\Delta\eta\int\frac{d^2\hat{n}}{4\pi}f(\eta',\vec{x}+\Delta\eta\hat{n}). \tag{78}$$

Note that $\Delta\eta$ and η' are the following functions of t and t':

$$\Delta\eta=\int_{t'}^{t}dt'' e^{-b(t'')},\qquad \eta'=\eta_I+\int_0^{t'}dt'' e^{-b(t'')}. \tag{79}$$

Because $\Delta\eta$ vanishes at $t'=t$ a single partial integration fails to extract the leading order term in the slow roll approximation,

$$\int_0^t dt' e^{2b(t')}\Delta\eta f(\eta',\vec{x}+\Delta\eta\hat{n})$$

$$=e^{2b}\frac{\Delta\eta f}{2\dot{b}}\Big|_0^t-\int_0^t dt' e^{2b}\frac{d}{dt'}\left[\frac{\Delta\eta f}{2\dot{b}}\right]. \tag{80}$$

The surface term vanishes at the upper limit and is exponentially suppressed at the lower limit. The remaining integrand is

$$e^{2b}\frac{d}{dt'}\left(\frac{\Delta\eta f}{2\dot{b}}\right)=-\frac{\ddot{b}}{2\dot{b}^2}e^{2b}\Delta\eta f+\frac{e^b}{2\dot{b}}\{-f+\Delta\eta\partial_0 f -\Delta\eta\hat{n}\cdot\vec{\nabla}f\}. \tag{81}$$

The first term on the right is the original integrand times a term which is negligible in the slow roll approximation. Of the remaining terms only the one without the factor of $\Delta\eta$ survives at the upper limit after another partial integration. Since each additional partial integration produces either a factor of $\ddot{b}/\dot{b}^2$ or of e^{-b} the slow roll approximation of Eq. (78) is

$$\left[\frac{1}{\partial^2}a^3f\right](\eta,\vec{x})\approx\int_0^t dt'\frac{e^{b(t')}}{2\dot{b}(t')}\int\frac{d^2\hat{n}}{4\pi}f(\eta',\vec{x}+\Delta\eta\hat{n}), \tag{82}$$

$$\approx-\frac{a(\eta)}{2H^2(\eta)}f(\eta,\vec{x}). \tag{83}$$

The vast simplification inherent in Eq. (83) derives from the fact that the response of a conformally coupled scalar at $\eta=\eta_0$ to that part of the source at $\eta=\eta_1$ redshifts as $a(\eta_0)/a(\eta_1)$. Hence only the most recent sources contribute effectively. One can recapture the slow roll approximation much more simply by rewriting the differential equation (26) which defines $\mathcal{A}[g](x)$,

$$\mathcal{A}=-\frac{6}{R}\left(1-\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\mathcal{A}\right). \tag{84}$$

Since the Ricci scalar is almost constant during inflation the leading order slow roll term—to *all* orders in the pseudo-graviton expansion—is contained in the reduction, $\mathcal{A}[g](x)\rightarrow-6/R(x)$.

VI. A BETTER OBSERVABLE

Simple expressions are nice but the results of the previous section are too much of a good thing. To leading order in the slow roll approximation our new observable has turned out to be nothing more than -6 over the Ricci scalar. The next order terms disrupt this correspondence but still give expressions which are local in the observation point. We criticized this sort of locality in Sec. II. A reasonable measure of the cosmological expansion rate should not be dominated by local fluctuations. Of course small fluctuations are, by definition, subdominant to the background, so one can still use $\mathcal{A}[g](x)$ to measure back reaction during the first stages of inflation. However, the defect of too much locality has a relatively simple fix which we shall present in this section. We shall also refine the observable so that it gives the Hubble constant exactly for any homogeneous and isotropic geometry, without recourse to the slow roll approximation.

The effective locality of $\mathcal{A}[g](x)$ derives from the fact that conformal scalars redshift like the inverse scale factor. During inflation the scale factor grows so rapidly that only the most recent sources matter much. A straightforward way of avoiding this is by breaking conformal invariance. Suppose we measure a minimally coupled scalar $\mathcal{B}[g](x)$, whose value and whose first derivative vanish on the initial value surface, and which is driven by a source $S(x)$,

$$\Box\mathcal{B}(x)=S(x). \tag{85}$$

For a homogeneous and isotropic geometry (3) the slow roll approximation results in an almost uniformly weighted average over comoving time,

$$\frac{1}{\Box}S \to -\int_0^t dt' e^{3b(t')}\int_0^t dt'' e^{3b(t'')}S(t'') \tag{86}$$

$$\approx -\int_0^t dt' \frac{S(t')}{3\dot{b}(t')}. \tag{87}$$

It remains to identify a suitable source. Note that for a homogeneous and isotropic geometry (3) the nonzero components of the Ricci tensor are

$$R_{00} \to -3\ddot{b} - 3\dot{b}^2, \quad R_{ij} \to (\ddot{b} + 3\dot{b}^2)g_{ij}. \tag{88}$$

The trace of the spatial part has the curious property of giving a total derivative when multiplied by e^{3b},

$$3[\ddot{b}(t) + 3\dot{b}^2(t)]e^{3b(t)} = \frac{d}{dt}[3\dot{b}(t)e^{2b(t)}]. \tag{89}$$

If the source $S(x)$ reduces to minus one-third times this spatial trace, for a homogeneous and isotropic geometry, then the minimally coupled scalar will reduce to the logarithmic scale factor exactly,

$$\mathcal{B}[g](x) \to \int_0^t dt' e^{3b(t')}\int_0^{t'} dt'' e^{3b(t'')}[\ddot{b}(t'') + 3\dot{b}^2(t'')]$$

$$= b(t). \tag{90}$$

We can then obtain the Hubble constant by differentiation with respect to comoving time.

The object we have just described is not quite a scalar because the spatial trace of the Ricci tensor is not. However, we can give the latter an invariant formulation by exploiting the technology of Sec. III to define it in a special coordinate system which reduces to conformal coordinates for a homogeneous and isotropic geometry. With the transformation $Y^\mu[g](x)$ we can define the spatial components of the metric and the Ricci tensor,

$$g'_{ij}(x) \equiv \frac{\partial Y^\rho}{\partial x^i}\frac{\partial Y^\sigma}{\partial x^j} g_{\rho\sigma}(Y(x)),$$

$$R'_{ij}(x) \equiv \frac{\partial Y^\rho}{\partial x^i}\frac{\partial Y^\sigma}{\partial x^j} R_{\rho\sigma}(Y(x)). \tag{91}$$

The 3-curvature is just the inverse of the first contracted into the second,

$$\mathcal{R}[g](x) \equiv (g'^{-1})^{ij} R'_{ij}(x), \tag{92}$$

and the minimally coupled scalar is

$$\mathcal{B}[g](x) = \frac{1}{\Box}\left(-\frac{1}{3}\mathcal{R}\right). \tag{93}$$

Since conformal coordinates have zero shift we obviously wish to use the same spatial transformation (46) as in Sec. III. The temporal transformation requires a scalar clock function, the most uniformly applicable choice of which is $\mathcal{V}[g](x)$, the invariant volume of the past light cone as seen from the point x^μ back to the inital value surface. Note that it must be a good clock generally, not just in perturbation theory, because the volume of the past light cone increases monotonically under any timelike foliation.

We define $\mathcal{V}[g](x)$ as the invariant integral over all points which are connected to x'^μ by any future-directed, non-space-like path.[1] For a homogeneous and isotropic geometry this reduces to a single integral,

$$\mathcal{V}[g](x) \to \mathcal{V}_0(\eta) \equiv \frac{4}{3}\pi \int_{\eta_I}^{\eta} d\eta' \Omega^4(\eta')(\eta - \eta')^3. \tag{94}$$

As in Sec. III, we define surfaces of simultaneity to make this relation persist in the presence of perturbations,

$$\mathcal{V}(\tau(\eta,\vec{x}),\vec{x}) \equiv \mathcal{V}_0(\eta). \tag{95}$$

We define the general conformal factor as the square root of the 00 component of the metric in these coordinates,

$$\Omega[g](x) \equiv \sqrt{g'_{00}(x)}, \tag{96}$$

$$= \frac{\partial \tau}{\partial \eta}[g_{00}(Y(x)) - (g^{-1})^{ij} g_{0i} g_{0j}(Y(x))]^{1/2}. \tag{97}$$

Since the coordinates have zero shift, $g'_{0i} = 0$, and it follows that Ω is precisely the factor needed to scale from conformal to comoving time. One possible definition for the Hubble constant is therefore the comoving time derivative of the scalar $\mathcal{B}$ evaluated in these coordinates,

$$\mathcal{H}_1[g](x) \equiv \Omega^{-1}[g](x)\frac{\partial}{\partial \eta}\mathcal{B}[g](Y[g](x)). \tag{98}$$

Of course one might equally well base the observable on $\Omega[g](x)$ now that we have it,

$$\mathcal{H}_2[g](x) \equiv \Omega^{-1}[g](x)\frac{\partial}{\partial \eta}\ln(\Omega[g](x)). \tag{99}$$

One might instead employ the third root of the determinant of $g'_{ij}(x)$. All of these are plausible measures for the cosmological expansion rate, all reduce exactly to the Hubble constant for homogeneous and isotropic geometries, and we anticipate that all will give the same result as regards the existence or non-existence of a significant back reaction. We emphasize this multiplicity of plausible observables is as it should be because a similar situation exists in the many different methods by which astronomers attempt to measure the Hubble constant.

[1]The path need not be a geodesic, nor does it have to be the sole path which connects x'^μ and x^μ.

VII. DISCUSSION

Reliably quantifying the effect of back reaction on inflation poses a frustrating paradox. The possibility of an effect derives from fluctuations in homogeneity and isotropy, but these call into question precisely what is meant by the rate of cosmological expansion. Previous work has attempted to resolve the issue by averaging the gauge fixed metric, either over a surface of simultaneity [4,5] or over the range of quantum fluctuations in a homogeneous and isotropic state [2,6–8]. It has been objected that neither technique is manifestly invariant, and also that the former procedure involves superposing data unavailable to a local observer on the surface of simultaneity [9].

In Sec. II we argued that both problems can be avoided by measuring the response of a noninteracting, conformally coupled scalar to a constant source. The scalar $\mathcal{A}[g](x)$ is a nonlocal functional of the metric which is obtained by superposing over the past light cone of x^μ, just as astronomers do in measuring the Hubble constant. Its phenomenological interpretation also follows the standard practice in astronomy: we define the locally observed rate of cosmological expansion to bear the same relation—Eq. (34)—to $\mathcal{A}[g](x)$ for a general metric as it does for a homogeneous and isotropic one. Of course the result will be a little different at different locations, just as we must expect the Hubble constant measured by human astronomers to disagree slightly with the value obtained from the different field of view available to their opposite numbers in the Coma Cluster. But nearby observers will tend to agree because their past light cones largely overlap.

It should be noted that we are not adding a conformal scalar to whatever model of inflation is being probed. The observable is only a functional of the metric used to pose invariant questions about the expansion rate; it does not change the dynamics of the model. Even if one insists that the scalar represents a sort of measuring device whose effect must be included, the strength of the constant source can still be adjusted so as to make this effect negligible. The required constant would simply appear on the right-hand sides of both Eqs. (27) and (34),

$$\mathcal{A}[g](x) \rightarrow \frac{1}{\square_c} K \equiv \frac{-K}{2H^2(x)}, \tag{100}$$

so that the magnitude of the scalar could be made arbitrarily small without affecting our determination of the local Hubble constant.

One can either compute the expectation value of $\mathcal{A}[g](x)$—or else stochastically sample its probability distribution—in the presence of a Heisenberg state which we assume to be homogeneous and isotropic. The passage from a scalar to an invariant can be achieved by geometrically fixing the observation point relative to the initial value surface on which the Heisenberg state is defined. Because all points on the initial value surface are physically equivalent the problem reduces to orthogonally projecting between geometrically specified surfaces of simultaneity. Two definitions for such surfaces were presented in Sec. III, along with perturbative expansions for the field-dependent observation point $Y^\mu[\varphi,g](x)$ which can be used to fix the observation point when working an arbitrary gauge.

Sections IV and V developed the general machinery necessary to evaluate the new observable perturbatively. We emphasize that these computations are imminently doable in the slow roll approximation. It remains to apply the technology to simple models of scalar-driven [11] and Λ-driven [12] inflation.

An embarrassing postscript to these labors is that the slow roll approximation purges $\mathcal{A}[g](x)$ of its nonlocality. In fact it reduces to $-6/R(x)$, which we initially rejected as being dominated by local fluctuations. There is actually no obstacle to making use of $\mathcal{A}[g](x)$ in perturbation theory because small fluctuations are, by definition, subdominant to the homogeneous background. However, one would still prefer an observable which represents a more evenly weighted average over the past light cone. Several alternatives are discussed in Sec. VI. We have taken the additional trouble to construct them to reduce exactly to the Hubble constant for homogeneous and isotropic geometries, without recourse to the slow roll approximation.

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