Nonperturbative renormalization in a scalar model within light-front dynamics

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Within the covariant formulation of light-front dynamics, in a scalar model with the interaction Hamiltonian $H=-g\psi^2(x)\varphi(x)$, we calculate nonperturbatively the renormalized state vector of a scalar ''nucleon'' in a truncated Fock space containing the N, $N\pi$ and $N\pi\pi$ sectors. The model gives a simple example of nonperturbative renormalization that is carried out numerically. Though the mass renormalization δm^2 diverges logarithmically with increasing cutoff L, the Fock components of the ''physical'' nucleon are stable when $L\to\infty$.

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I. INTRODUCTION

Knowledge of the hadron properties within the framework of quantum chromo-dynamics (QCD) is one of the main issues in strong interaction physics. Several approaches have been pursued in the last 20 years, in particular lattice gauge calculations. Among the alternatives to these calculations, light-front dynamics (LFD) is of particular interest [1]. It has proven successful in many phenomenological applications involving few-body systems in particle and nuclear physics. However, the application of LFD to field theoretical calculations is still in its infancy [2]. The main issue to be solved is the renormalization procedure [3]. In perturbative calculations, the renormalization of the electron self-energy in QED, is already nontrivial in standard LFD in the sense that it involves nonlocal counterterms [4]. This unpleasant feature is, however, a direct consequence of the choice of a preferential direction, the z axis, in the determination of the quantization plane. This can be well understood in the covariant formulation of light-front dynamics (CLFD) [5], as shown in Ref. [6]. In this formulation, the state vector is defined on the light-front surface given by the equation $\omega\cdot x=0$, where ω is the four-vector with $\omega^2=0$. The particular case where $\omega=(1,0,0,-1)$ corresponds to standard LFD. In the CLFD, the counterterm needed to renormalize the electron self-energy in the first order perturbation expansion is simply dependent on the orientation of the light front, defined by the four-vector ω.

We shall investigate in this article how the question of nonperturbative renormalization can be formulated in the CLFD. We shall first derive the general eigenvalue equation, whose solutions are the Fock state components. As a first example, we shall illustrate our strategy with a simple model involving two coupled scalar particles; namely, the scalar ''nucleon'' N radiates the scalar ''pions'' π. In this simple example, the Fock space is restricted to N, $N\pi$ and $N\pi\pi$ states. Represented as a series of graphs in perturbation theory, it contains an infinite number of irreducible contributions to the self-energy. They diverge and require renormalization. At a large value of the coupling constant this system cannot be solved perturbatively. We show how to determine, in a self-consistent manner, the nonperturbative mass counterterm. This counterterm is then calculated numerically. Models involving two and three constituent bound states were also analyzed in [7].

In a more general field-theoretical framework, in the light-front Tamm-Dancoff approximation involving spin 1/2 particles for instance, renormalization is not reduced to the introduction of a mass counterterm. In this case, one should introduce sector-dependent counterterms, as shown in Ref. [8]. In the two-nucleon sector, for spin 1/2 particles, additional box divergences appear [9]. These divergences and the sector dependent counterterms are absent in the scalar model restricted to the dressed ''one-scalar-nucleon'' system. Therefore, their analysis is beyond the scope of the present paper. In spite of that, the scalar model is still rather instructive, since the renormalization considered here is not reduced to the perturbative one. We will consider a much larger coupling constant, excluding the convergence of the perturbative series.

The plan of the article is as follows. In Sec. II we establish the general equations of motion for the Fock components. In the truncated Fock space, the corresponding system of equations, which determine the Fock components and the mass renormalization, is detailed in Sec. III. The renormal-

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ization of the wave function and of the coupling constant is calculated in Secs. IV and V, respectively. Numerical results are presented and discussed in Sec. VI. We present our conclusions in Sec. VII.

II. EIGENSTATE EQUATION

We start with the general eigenstate equation for the state vector [5]:

$$\hat{P}^2\phi(p)=M^2\phi(p), \tag{1}$$

where

$$\hat{P}_\mu=\hat{P}^0_\mu+\hat{P}^{\rm int}_\mu. \tag{2}$$

Here we have decomposed the momentum operator $\hat{P}_\mu$ into two parts: the free one, $\hat{P}^0_\mu$, and the interacting one $\hat{P}^{\rm int}_\mu$, given by

$$\hat{P}^0_\mu=\sum_i \int d_i^\dagger(\vec{k})d_i(\vec{k})k_\mu\, d^3k,$$

$$\hat{P}^{\rm int}_\mu=\omega_\mu\int H^{int}(x)\,\delta(\omega\cdot x)d^4x$$

$$=\omega_\mu\int_{-\infty}^{+\infty}\tilde{H}^{\rm int}(\omega\tau)\frac{d\tau}{2\pi}, \tag{3}$$

where we have denoted by $\tilde{H}^{\rm int}$ the Fourier transform of the interaction Hamiltonian:

$$\tilde{H}^{\rm int}(p)=\int H^{\rm int}(x)\exp(-ip\cdot x)d^4x, \tag{4}$$

and $d_i^\dagger$ (d_i) corresponds to the creation (destruction) operator for the various particles under consideration. The explicitly covariant formulation of LFD manifests itself in the fact that $\hat{P}^{\rm int}_\mu$ in Eq. (3) is proportional to ω_μ and is determined by the integral over the light-front plane $\omega\cdot x=0$.

The equations for the Fock components can be obtained from Eq. (1) by substituting there the Fock decomposition for the state vector $\phi(p)$ and calculating the matrix elements of $\hat{P}^2$ in the Fock space. With the above expressions for $\hat{P}$, Eq. (1) obtains the form

$$\left[(\hat{P}^0)^2+(\omega\cdot\hat{P}^0)\int\tilde{H}^{\rm int}(\omega\tau)\frac{d\tau}{2\pi}+\int\tilde{H}^{\rm int}(\omega\tau)\frac{d\tau}{2\pi}(\omega\cdot\hat{P}^0)\right]\phi(p)=M^2\phi(p). \tag{5}$$

In order to simplify this equation, we can use the fact that the operators $(\omega\cdot\hat{P}^0)$ and $\int\tilde{H}^{\rm int}(\omega\tau)d\tau$ commute. Indeed, from the commutation relation $[\hat{P}_\mu,\hat{P}_\nu]=0$ we get

$$[\omega\cdot\hat{P},\hat{P}_\nu]=[\omega\cdot\hat{P}^0,\hat{P}^0_\nu+\hat{P}^{\rm int}_\nu]=[\omega\cdot\hat{P}^0,\hat{P}^{\rm int}_\nu]=0.$$

Moreover, since $\omega^2=0$, we can replace $(\omega\cdot\hat{P}^0)$ by $(\omega\cdot\hat{P})$ here and below. We thus obtain

$$(\omega\cdot\hat{P}^0)\int\tilde{H}^{\rm int}(\omega\tau)\phi(p)d\tau=\int(\omega\cdot\hat{P})\tilde{H}^{\rm int}(\omega\tau)\phi(p)d\tau$$

$$=(\omega\cdot p)\int\tilde{H}^{\rm int}(\omega\tau)\phi(p)d\tau,$$

and Eq. (5) is transformed to

$$2(\omega\cdot p)\int\tilde{H}^{\rm int}(\omega\tau)\frac{d\tau}{2\pi}\phi(p)=-[(\hat{P}^0)^2-M^2]\phi(p). \tag{6}$$

The state vector $\phi(p)$ is now decomposed in Fock components according to

$$\phi(p)=(2\pi)^{3/2}\int\phi_1(k_1,p,\omega\tau)a^\dagger(\vec{k}_1)|0\rangle\delta^{(4)}(k_1-p-\omega\tau)2(\omega\cdot p)d\tau\frac{d^3k_1}{(2\pi)^{3/2}\sqrt{2\varepsilon_{k_1}}}$$

$$+(2\pi)^{3/2}\int\phi_2(k_1,k_2,p,\omega\tau)a^\dagger(\vec{k}_1)b^\dagger(\vec{k}_2)|0\rangle\delta^{(4)}(k_1+k_2-p-\omega\tau)2(\omega\cdot p)d\tau\frac{d^3k_1}{(2\pi)^{3/2}\sqrt{2\varepsilon_{k_1}}}\frac{d^3k_2}{(2\pi)^{3/2}\sqrt{2\varepsilon_{k_2}}}$$

$$+(2\pi)^{3/2}\int\phi_3(k_1,k_2,k_3,p,\omega\tau)a^\dagger(\vec{k}_1)b^\dagger(\vec{k}_2)b^\dagger(\vec{k}_3)|0\rangle\delta^{(4)}(k_1+k_2+k_2-p-\omega\tau)$$

$$\times2(\omega\cdot p)d\tau\frac{d^3k_1}{(2\pi)^{3/2}\sqrt{2\varepsilon_{k_1}}}\frac{d^3k_2}{(2\pi)^{3/2}\sqrt{2\varepsilon_{k_2}}}\frac{d^3k_3}{(2\pi)^{3/2}\sqrt{2\varepsilon_{k_3}}}+\cdots, \tag{7}$$

where $\varepsilon_{k_i}=\sqrt{\vec{k}_i^2+m_i^2}$ and m_i is the mass of the particle i of momentum k_i. We introduce in Eq. (7) spinless particles of two types: the ''nucleon'' (with creation operator $a^\dagger$) and the ''pion'' (with creation operator $b^\dagger$). The state vector (7) represents the ''dressed nucleon,'' consisting of the ''bare nucleon'' and the admixture of one, two, . . . , many pions. Our dressed nucleon is also scalar; therefore the state vector (6) corresponds to zero total angular momentum. This means that the Fock components are scalars and depend only on scalar products of all available four-vectors.

The conservation law for the momenta in each Fock component has the form

$$k_1+k_2+\cdots+k_n=p+\omega\tau.$$

Hence, the action of the operator $(\hat{P}^0)^2-M^2$ in Eq. (6) on the state vector $\phi(p)$ is reduced to the multiplication of each Fock component by the factor $(\Sigma k_i)^2-M^2=2(\omega\cdot p)\tau$.

We introduce the notation

$$\mathcal{G}(p)=2(\omega\cdot p)\hat{\tau}\phi(p),$$

where $\hat{\tau}$ is the operator that, acting on a given component ϕ_i of ϕ, gives $\tau\phi_i$. $\mathcal{G}$ has therefore a Fock decomposition that differs from Eq. (7) by the replacement of the wave functions ϕ_i by the vertex parts Γ_i given by

$$\Gamma_i\equiv 2(\omega\cdot p)\tau\phi_i=(s-M^2)\phi_i, \tag{8}$$

where $s=(\Sigma k_i)^2$. We thus find the eigenvalue equation

$$\frac{1}{2\pi}\int \tilde{H}^{\rm int}(\omega\tau)\frac{d\tau}{\tau}\mathcal{G}(p)=-\mathcal{G}(p)\equiv-\lambda(M^2)\mathcal{G}(p). \tag{9}$$

We introduce in Eq. (9) the factor $\lambda(M^2)$ depending on M^2. The eigenvalue M^2 is found from the condition $\lambda(M^2)=1$. This equation is quite general and equivalent to the eigenstate equation (1).

III. EQUATION FOR THE FOCK COMPONENTS

A. System of coupled integral equations

For the simplified model we consider in this study, we take the following interaction Hamiltonian:

$$H=-g\psi^2(x)\varphi(x) \tag{10}$$

where the scalar field ψ with mass m corresponds to the scalar nucleon and the field φ with mass μ corresponds to the scalar pion.

The system of equations for these vertex parts is shown graphically in Fig. 1. In order to write down this system of equations, it is enough to apply to the diagrams of Fig. 1 the rules of the graph technique detailed in Ref. [5]. We thus find

$$\begin{aligned}\Gamma_1(k_1,p,\omega\tau_1)=\delta m^2\int \Gamma_1(k_1',p,\omega\tau')\,\delta^{(4)}(k_1'-p-\omega\tau')\,\theta(\omega\cdot k_1')\,\delta(k_1'^2-m^2)\,d^4k_1'\frac{d\tau'}{\tau'-i0}\\ +g\frac{1}{(2\pi)^3}\int \Gamma_2(k_1',k_2',p,\omega\tau')\,\delta^{(4)}(k_1'+k_2'-p-\omega\tau')\,\theta(\omega\cdot k_1')\,\delta(k_1'^2-m^2)\\ \times d^4k_1'\,\theta(\omega\cdot k_2')\,\delta(k_2'^2-\mu^2)\,d^4k_2'\frac{d\tau'}{\tau'-i0},\end{aligned} \tag{11a}$$

$$\begin{aligned}\Gamma_2(k_1,k_2,p,\omega\tau_2)=g\int \Gamma_1(k_1',p,\omega\tau')\,\delta^{(4)}(k_1'-p-\omega\tau')\,\theta(\omega\cdot k_1')\,\delta(k_1'^2-m^2)\,d^4k_1'\frac{d\tau'}{\tau'-i0}\\ +\delta m^2\int \Gamma_2(k_1',k_2,p,\omega\tau')\,\delta^{(4)}(k_1'+k_2-p-\omega\tau')\,\theta(\omega\cdot k_1')\,\delta(k_1'^2-m^2)\,d^4k_1'\frac{d\tau'}{\tau'-i0}\\ +g\frac{1}{(2\pi)^3}\int \Gamma_3(k_1',k_2,k_3',p,\omega\tau')\,\delta^{(4)}(k_1'+k_2+k_3'-p-\omega\tau')\,\theta(\omega\cdot k_1')\\ \times\delta(k_1'^2-m^2)\,d^4k_1'\,\theta(\omega\cdot k_3')\,\delta(k_3'^2-\mu^2)\,d^4k_3'\frac{d\tau'}{\tau'-i0},\end{aligned} \tag{11b}$$

$$\begin{aligned}\Gamma_3(k_1,k_2,k_3,p,\omega\tau_3)=g\int \Gamma_2(k_1',k_2,p,\omega\tau')\,\delta^{(4)}(k_1'+k_2-p-\omega\tau')\,\theta(\omega\cdot k_1')\,\delta(k_1'^2-m^2)\,d^4k_1'\frac{d\tau'}{\tau'-i0}\\ +g\int \Gamma_2(k_1',k_3,p,\omega\tau')\,\delta^{(4)}(k_1'+k_3-p-\omega\tau')\,\theta(\omega\cdot k_1')\,\delta(k_1'^2-m^2)\,d^4k_1'\frac{d\tau'}{\tau'-i0}.\end{aligned} \tag{11c}$$

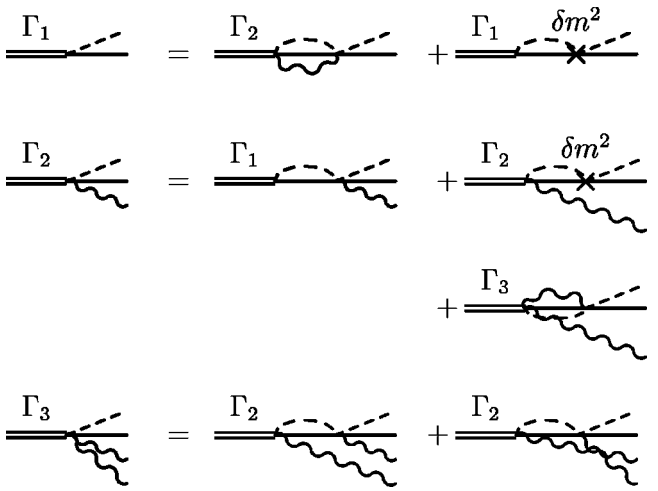

FIG. 1. Diagrammatical representation of the eigenvalue equation projected onto the Fock components of the state vector (restricted here to $N=3$). For the sake of clarity, we remove all kinematical variables.

The origin of the mass counterterm δm^2 in these equations is explained below. Since we truncate the Fock space to three particles, we omit in the last equation (11c) the coupling of Γ_3 to the four-body component Γ_4. Note that our approximation is not based on a perturbative expansion in terms of the coupling constant. It is based on a decomposition over intermediate states with an increasing number of particles. Once this number is fixed, we solve a nonperturbative problem in terms of g. If we iterate the system (11), i.e., express Γ_3 and Γ_2 from the third and the second equations and substitute them in the first one, we get for Γ_1 the sum of the diagrams shown in Fig. 2. Of course, this series of graphs corresponds to all the irreducible contributions to the self-energy with intermediate states up to $N\pi\pi$. These contributions are iterated again, i.e., they appear repetitively on the nucleon line. In contrast to the case of the intermediate state $N\pi$, which generates only one self-energy diagram shown in Fig. 3, the number of irreducible contributions generated by the intermediate states up to $N\pi\pi$ is infinite. The system of equations (11) corresponds to the sum of all of them.

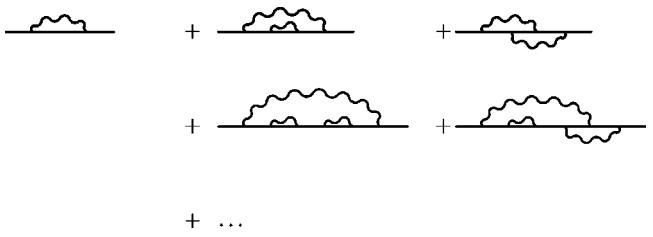

FIG. 2. Perturbative expansion, in terms of the pion-nucleon coupling constant g, of the nucleon self-energy.

In the right-hand side of Eqs. (11a) and (11b) we introduced the counterterm δm^2 corresponding to the interaction Hamiltonian $H=-\delta m^2\psi^2(x)$. This term provides mass renormalization. The counterterm is not introduced, however, in the third equation containing the three-body intermediate state. At this point we have some freedom in the definition of the model Hamiltonian. In principle, the counterterm could also be introduced into the three-body intermediate state, since formally it does not increase the number of particles in the intermediate state. Our motivation in choosing the Hamiltonian without the counterterm in the last nth Fock sector is the following. Consider, for example, the mass operator $-g^2\Sigma(p^2)$ in second order perturbation theory. It is given by the diagram of Fig. 3 with two particles (N and π) in the intermediate state. The counterterm δm^2, which is shown by a cross on the nucleon line (one particle N in the intermediate state) just renormalizes this mass operator. So, one ''pion'' is deleted when the counterterm appears. Therefore, for the renormalization of the mass operator determined by the sum of (an infinite number of) irreducible diagrams with n particles in the intermediate state, one should consider the graphs, with the counterterm insertions, with $n-1$ particles in the intermediate states only [4]. These diagrams are just those generated by the Hamiltonian without the counterterm in the nth Fock sector.

We can easily transform Eqs. (11) by performing the integrations which do not involve loops, keeping the loop integrals untouched. The result is the following:

$$\Gamma_1(k_1,p,\omega\tau_1)=\frac{\delta m^2}{2(\omega\cdot p)\tau_1}\Gamma_1(k_1,p,\omega\tau_1)+g\frac{1}{(2\pi)^3}\int\Gamma_2(k_1',k_2',p,\omega\tau')\,\delta^{(4)}$$

$$\times(k_1'+k_2'-p-\omega\tau')\,\theta(\omega\cdot k_1')\,\delta(k_1'^2-m^2)d^4k_1'\;\theta(\omega\cdot k_2')\,\delta(k_2'^2-\mu^2)d^4k_2'\frac{d\tau'}{\tau'-i0}, \tag{12a}$$

$$\Gamma_2(k_1,k_2,p,\omega\tau_2)=\frac{g}{2(\omega\cdot p)\tau_1}\Gamma_1(p_1,p,\omega\tau_1)+\frac{\delta m^2}{2(\omega\cdot p)\tau_2x_1}\Gamma_2(k_1,k_2,p,\omega\tau_2)$$

$$+g\frac{1}{(2\pi)^3}\int\Gamma_3(k_1',k_2,k_3',p,\omega\tau')\,\delta^{(4)}(k_1'+k_2+k_3'-p-\omega\tau')\,\theta(\omega\cdot k_1')\,\delta(k_1'^2-m^2)$$

$$\times d^4k_1'\,\theta(\omega\cdot k_3')\,\delta(k_3'^2-\mu^2)d^4k_3'\frac{d\tau'}{\tau'-i0}, \tag{12b}$$

$$\Gamma_3(k_1,k_2,k_3,p,\omega\tau_3)=\frac{g}{2(\omega\cdot p)\tau_2'(1-x_2)}\Gamma_2(k_1',k_2,p,\omega\tau_2')+\frac{g}{2(\omega\cdot p)\tau_2''(1-x_3)}\Gamma_2(k_1'',k_3,p,\omega\tau_2''). \tag{12c}$$

In Eqs. (12a) and (12b), $2(\omega\cdot p)\tau_1=k_1^2-p^2=m^2-p^2\to 0$ when $p^2\to m^2$. Since τ_1 is in the denominator, we keep $p^2\neq m^2$ and take the limit $p^2\to m^2$ in the final equation. In Eq. (12b), $p_1=p-\omega\tau_1$ with $2(\omega\cdot p)\tau_2=s_{12}-m^2$ and $s_{12}=(k_1+k_2)^2$. In Eq. (12c) we use the notation

$$2(\omega\cdot p)\tau_2'=s_{12}'-m^2,\quad 2(\omega\cdot p)\tau_2''=s_{12}''-m^2,$$

where

$$s_{12}'=(k_1'+k_2)^2,\quad s_{12}''=(k_1''+k_3)^2,$$

and k_1',k_1'' are determined by the conservation laws

$$k_1'=k_1+k_3+\omega\tau_2'-\omega\tau_{123},\quad k_1''=k_1+k_2+\omega\tau_2''-\omega\tau_{123}, \tag{13}$$

where $\omega\tau_{123}$ is the momentum of the spurion line entering the diagram. Everywhere we note that $x_i=\omega\cdot k_i/\omega\cdot p$.

B. Reduction to two-particle Fock states

We consider first the approximation in which the state vector (7) contains only the bare nucleon N and the state $N\pi$. In this approximation, the system of equations for one- and two-body Fock components is represented diagrammatically in Fig. 4. It is easily obtained by omitting Γ_3 in Eq. (11c), together with the counterterm in the equation that determines the last Fock sector Γ_2. This truncation of the Fock space, retaining the minimal number of components, is equivalent to second order perturbation theory.

Rewritten in terms of the variables $\vec{q}$ and $\vec{n}$ (see Appendix A), this system of equations obtains the simple form

$$\Gamma_1=\frac{\delta m_0^2}{m^2-p^2}\Gamma_1+g\int\sigma(\vec{q}\,',p^2)\Gamma_2(\vec{q}\,',\vec{n})\frac{d^3q'}{(2\pi)^3}, \tag{14a}$$

$$\Gamma_2(\vec{q},\vec{n})=\frac{g}{m^2-p^2}\Gamma_1, \tag{14b}$$

where $\sigma(\vec{q},p^2)$ is the integrand that determines the self-energy $\Sigma(p^2)$:

$$\Sigma(p^2)=\int\sigma(\vec{q}\,',p^2)\frac{d^3q'}{(2\pi)^3},$$

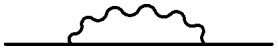

FIG. 3. First order perturbative expansion of the nucleon self-energy.

$$\sigma(\vec{q},p^2)=\frac{1}{2}\frac{\sqrt{s_{12}}}{(s_{12}-p^2-i0)}\frac{1}{\varepsilon(q,m)\varepsilon(q,\mu)}, \tag{15}$$

with

$$s_{12}=(k_1+k_2)^2=[\varepsilon(\vec{q},m)+\varepsilon(\vec{q},\mu)]^2,$$

$$\varepsilon(\vec{q},m)=\sqrt{m^2+\vec{q}^2}, \tag{16}$$

and similarly for $\varepsilon(\vec{q},\mu)$. Since Γ_1 does not depend on the relative momentum, it follows from Eqs. (14) that $\Gamma_2(\vec{q},\vec{n})$ does not depend on the relative momentum either, i.e., $\Gamma_2(\vec{q},\vec{n})=\text{const}$. Hence, we get

$$\left(-1+\frac{\delta m_0^2}{m^2-p^2}\right)\Gamma_1+g\Sigma(p^2)\Gamma_2=0,$$

$$\frac{g}{m^2-p^2}\Gamma_1-\Gamma_2=0. \tag{17}$$

From Eq. (17) we find the eigenvalue equation:

$$p^2=m^2-\delta m_0^2-g^2\Sigma(p^2). \tag{18}$$

The counterterm δm_0^2 is determined from the on-shell condition $p^2=m^2$, where m is the physical mass of the nucleon. This gives

$$\delta m_0^2=-g^2\Sigma(m^2), \tag{19}$$

as expected in a second order calculation of mass renormalization.

C. Solution for the three particle system

Consider now the system of equations that incorporates the three-body Fock component Γ_3. It was shown graphically in Fig. 1. We can easily express Γ_3 through Γ_2. The system of equations is thus transformed as shown in Fig. 5. Its analytical representation has the form

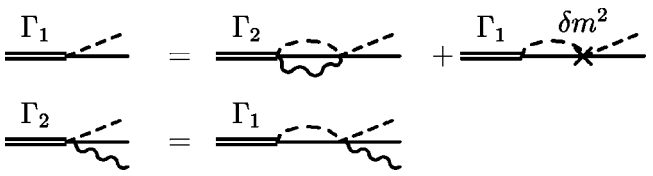

FIG. 4. Diagrammatical representation of the eigenvalue equation in first order perturbation theory.

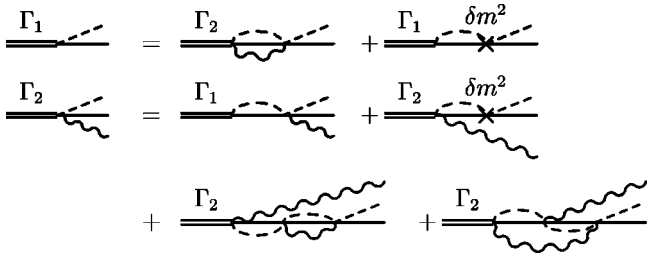

FIG. 5. Same as Fig. 1, but where the last Fock sector ($n=3$) has been expressed in terms of the two-body one.

$$\Gamma_1=\frac{\delta m^2}{m^2-p^2}\Gamma_1+g\int \sigma(\vec{q}',p^2)\Gamma_2(\vec{q}',\vec{n})\frac{d^3q'}{(2\pi)^3}, \tag{20a}$$

$$\Gamma_2(\vec{q},\vec{n})=\frac{g}{m^2-p^2}\Gamma_1+\frac{\delta m^2+g^2\Sigma(s_1)}{(s_{12}-p^2)x_{k_1}}\Gamma_2(\vec{q},\vec{n}) +g^2\int \Pi(\vec{q},\vec{q}',\vec{n},p^2)\sigma(\vec{q}',p^2)\Gamma_2(\vec{q}',\vec{n})\frac{d^3q'}{(2\pi)^3}. \tag{20b}$$

Each term in Eqs. (20) is represented by a graph on Fig. 5. In these equations, s is given by Eq. (16), and

$$s_1=(k_1-\omega\tau)^2=m^2-x_{k_1}(s_{12}-p^2),$$

$$x_{k_1}=\frac{\omega\cdot k_1}{\omega\cdot p}=\frac{1}{\sqrt{s_{12}}}[\varepsilon(\vec{q},m)-\vec{n}\cdot\vec{q}].$$

In this equation, $\Pi(\vec{q},\vec{q}',\vec{n},p^2)$ is the propagator of the second intermediate state:

$$\Pi(\vec{q},\vec{q}',\vec{n},p^2) =\int \theta[\omega\cdot(k_1-k_2')]\delta[(k_1-k_2' +\omega\tau_2-\omega\tau'')^2-m^2]\frac{d\tau''}{\tau''-i0} =\frac{\theta[\omega\cdot(k_1'-k_2)]}{m^2-(k_1'-k_2-\omega\tau')^2}. \tag{21}$$

Its expression in terms of the variables $\vec{q},\vec{q}',\vec{n}$ can be easily calculated with the kinematics detailed in Appendix A.

The system of equations (20) can now be solved by two independent methods.

(i) The first one consists in the elimination of Γ_1 in the two equations in (20). The result is an equation for Γ_2 which, at $p^2=m^2$, obtains the form

$$\Gamma_2(\vec{q},\vec{n})=-\frac{g^2}{\delta m^2}\int \sigma(\vec{q}',p^2)\Gamma_2(\vec{q}',\vec{n})\frac{d^3q'}{(2\pi)^3} +\frac{g^2\Sigma(s_1)+\delta m^2}{(s_{12}-p^2)x_{k_1}}\Gamma_2(\vec{q},\vec{n}) +g^2\int \Pi(\vec{q},\vec{q}',\vec{n},p^2)\sigma(\vec{q}',p^2)\Gamma_2(\vec{q}',\vec{n})\frac{d^3q'}{(2\pi)^3} \equiv\lambda(\delta m^2)\Gamma_2(\vec{q},\vec{n}). \tag{22}$$

As in Eq. (9), we introduce in Eq. (22) the factor $\lambda(\delta m^2)$. The mass renormalization δm^2 will be found from the nonperturbative condition $\lambda(\delta m^2)=1$.

(ii) The second method consists in the direct elimination of Γ_2 in the two equations (20). We can indeed rewrite Eq. (20b) in the following form:

$$\left[\Gamma_2(\vec{q},\vec{n})-\frac{\delta m^2+g^2\Sigma(s_1)}{(s_{12}-p^2)x_{k_1}}\Gamma_2(\vec{q},\vec{n}) -g^2\int \Pi(\vec{q},\vec{q}',\vec{n},p^2)\sigma(\vec{q}',p^2)\Gamma_2(\vec{q}',\vec{n})\frac{d^3q'}{(2\pi)^3}\right] =\frac{g}{m^2-p^2}\Gamma_1. \tag{23}$$

After discretization of the momenta, this equation can be written schematically:

$$A_{ij}\Gamma_2^j=\frac{g}{m^2-p^2}\Gamma_1[1]^i, \tag{24}$$

where $[1]$ is the vector whose every component is 1, and A_{ij} is a two-dimensional matrix obtained after discretization of the momenta (index i for $q,\vec{n}\cdot\vec{q}$ and index j for $q',\vec{n}\cdot\vec{q}'$). The vertex function Γ_2 can now be expressed in terms of Γ_1 after a simple matrix inversion:

$$\Gamma_2^i=\frac{g}{m^2-p^2}\Gamma_1 A_{ij}^{-1}[1]^j. \tag{25}$$

After insertion in Eq. (20a), we end up with an equation involving Γ_1 only. Since Γ_1 is a nonzero constant, it can be removed from the equation, leading to an equation for δm^2 generalizing Eq. (19). In the perturbative limit, the matrix A reduces to the unit matrix, and we therefore recover Eq. (19) exactly.

IV. RENORMALIZATION OF THE WAVE FUNCTION

A. Nonperturbative case

We schematically rewrite the state vector $|p\rangle$, given by Eq. (7), as

$$|p\rangle=\phi_1|N\rangle+\phi_2|N\pi\rangle+\phi_3|N\pi\pi\rangle.$$

It is normalized as follows [5]:

$$\langle p'|p\rangle = 2p_0\delta^{(3)}(\vec{p}-\vec{p}'). \tag{26}$$

The Fock components are thus normalized in order to provide the condition (26). Substituting the state vector (7) in the left-hand side of Eq. (26), we get, before normalization of the Fock components,

$$\langle p'|p\rangle = Z\,2p_0\delta^{(3)}(\vec{p}-\vec{p}'), \tag{27}$$

where

$$Z = N_1 + N_2 + N_3$$

with

$$N_1 = \phi_1^2, \tag{28a}$$

$$N_2 = \frac{1}{2(2\pi)^3}\int \phi_2^2(\vec{q},\vec{n})\frac{[\varepsilon(q,m)+\varepsilon(q,\mu)]}{\varepsilon(q,m)\varepsilon(q,\mu)}d^3q, \tag{28b}$$

$$N_3 = \frac{1}{4(2\pi)^6}\int \phi_3^2(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{n})\,\delta^{(3)}(\vec{q}_1+\vec{q}_2+\vec{q}_3) \times\frac{[\varepsilon(q_1,m)+\varepsilon(q_2,\mu)+\varepsilon(q_3,\mu)]}{\varepsilon(q_1,m)\varepsilon(q_2,\mu)\varepsilon(q_3,\mu)}d^3q_1\,d^3q_2\,d^3q_3. \tag{28c}$$

From Γ_2 calculated with Eq. (22), we can find ϕ_2 according to

$$\phi_2(\vec{q},\vec{n}) = \frac{\Gamma_2(\vec{q},\vec{n})}{s_{12}-m^2}. \tag{29}$$

The calculation of N_2 is then straightforward. To calculate N_1 and N_3 we should know ϕ_1 and ϕ_3. The calculation of ϕ_1 and ϕ_3 is explained in Appendix B. Note that ϕ_1 does not depend on any relative momentum.

The normalized state vector satisfying the condition (26) obtains the form

$$|p\rangle = \phi_1^{\rm ren}|N\rangle + \phi_2^{\rm ren}|N\pi\rangle + \phi_3^{\rm ren}|N\pi\pi\rangle,$$

where

$$\phi_{1,2,3}^{\rm ren} = \phi_{1,2,3}/\sqrt{Z}. \tag{30}$$

We can introduce the creation operator of the new, composed field, directly creating the state $|p\rangle$. It is written schematically as

$$A^\dagger(\vec{p}) = \phi_1^{\rm ren}a^\dagger + \phi_2^{\rm ren}a^\dagger b^\dagger + \phi_3^{\rm ren}a^\dagger b^\dagger b^\dagger, \tag{31}$$

so that $|p\rangle = A^\dagger(\vec{p})|0\rangle$. With the renormalized wave functions $\phi^{\rm ren}$, the vacuum expectation value of the commutator, $\langle 0|[A(\vec{p}),A^\dagger(\vec{p})]|0\rangle = 2p_0\delta^{(3)}(\vec{p}-\vec{p}')$, is the same as the one-body operators $a,a^\dagger$ (except for the normalization factor $2p_0$). So the state $|p\rangle$, which is a dressed state in terms of the bare operators $a^\dagger,b^\dagger$, can be interpreted, in this sense, as an elementary particle in terms of the operator $A^\dagger$.

B. Perturbative case

The perturbative case is simply obtained from the preceding one by omitting ϕ_3. Taking into account that from Eq. (14) we have $\Gamma_1/(m^2-p^2) = \phi_1$, we find $\Gamma_2 = g\phi_1$. Substituting it into Eq. (29), we find ϕ_2 and then, by Eq. (28b), we obtain N_2:

$$N_2 = \phi_1^2 g^2 I_2$$

where

$$I_2 = \frac{1}{16\pi^3}\int \frac{\sqrt{s}}{(s_{12}-m^2)^2}\frac{d^3q}{\varepsilon(q,m)\varepsilon(q,\mu)}. \tag{32}$$

We thus find the ratio

$$\frac{N_2}{N_1} = 16\pi m^2\alpha I_2 \approx 0.38\alpha \tag{33}$$

with $\alpha = g^2/16\pi m^2$. The numerical value of I_2 is given for $m = 0.94$, $\mu = 0.14$. The integral I_2 is logarithmically divergent for $\mu \to 0$. The two-body contribution is rapidly decreasing, when the mass of the intermediate particle increases.

V. RENORMALIZATION OF THE COUPLING CONSTANT

As mentioned above, the mass renormalization counterterm δm^2 for the state vector incorporating N and $N\pi$ states only is given by Eq. (19) and coincides with the perturbative result. For the state vector incorporating the states N, $N\pi$ and $N\pi\pi$, it is determined by Eq. (22). This renormalization constant is infinite when the cutoff tends to infinity.

The coupling constant is also renormalized, although its renormalization is finite for the particular scalar system we are interested in the present study. We show below how this renormalization can be carried out. This renormalization is a by-product of the renormalization of the wave function fulfilled in the previous section. Let us consider first the case of the state vector containing N and $N\pi$ states only.

A. Truncation to N and $N\pi$ states

In this approximation, the normalization factor N_2 is given by Eq. (32). We thus get

$$Z = N_1 + N_2 = \phi_1^2 Z_1 \quad \text{with} \quad Z_1 = 1 + g^2 I_2, \tag{34}$$

where I_2 is given by Eq. (32). After renormalization the wave function ϕ_2 therefore turns into

$$\phi_2 = \frac{g\phi_1}{s-m^2} \to \phi_2^{\rm ren} = \frac{g}{\sqrt{Z_1}(s-m^2)} = \frac{g_{\rm ren}}{s-m^2} \tag{35}$$

where we introduced the renormalized coupling constant

$$g_{\rm ren}=g/\sqrt{Z_1}. \tag{36}$$

This value of $g_{\rm ren}$ can also be represented as the residue of the two-body wave function at $s=m^2$, i.e. the value of $\Gamma_2(q,z)$ at the nonphysical value of q corresponding to $s=m^2$.

One can alternatively define $g_{\rm ren}$ from the πN scattering amplitude determined by N exchange in the s channel:

$$F=\frac{g^2}{m^2-p^2}+\frac{g^2}{m^2-p^2}[\,\delta m^2+g^2\Sigma(p^2)]\frac{g^2}{m^2-p^2}+\cdots$$

$$=\frac{g^2}{m^2-p^2-\delta m^2-g^2\Sigma(p^2)} \tag{37}$$

where $p^2=(k_1+k_2)^2-2(\omega\cdot p)\tau$. Near the pole $p^2=m^2$ we get

$$F=\frac{g^2}{(m^2-p^2)[1+g^2 d\Sigma(p^2)/dp^2|_{p^2=m^2}]}=\frac{g_{\rm ren}^2}{m^2-p^2} \tag{38}$$

where we introduced $g_{\rm ren}$ by Eq. (36), but with Z_1 given by

$$Z_1=1+g^2\left.\frac{d\Sigma(p^2)}{dp^2}\right|_{p^2=m^2}. \tag{39}$$

Taking Eqs. (15) for Σ one can easily check that Eqs. (34) and (39) determine the same Z_1.

Note that the renormalized coupling constant $g_{\rm ren}$ is finite and it is always smaller than the bare value g. When the bare constant g tends to infinity, the constant $g_{\rm ren}$ remains finite, however, but it reaches its maximal value. Expressed in terms of $\alpha=g^2/(16\pi m^2)$ it has the form

$$\alpha_{\rm ren}^{\rm max}=1/(16\pi m^2 I_2).$$

It does not depend on g. According to Eq. (33), its numerical value for $m=0.94$ and $\mu=0.14$ is: $\alpha_{\rm ren}^{\rm max}=1/0.38=2.63$.

B. Truncation to N, $N\pi$ and $N\pi\pi$ states

In this case the renormalized wave functions are given by Eq. (30) with Z determined by Eqs. (27),(28). Note that the renormalization constant Z can still be represented in the form (39) with $\Sigma(p^2)$ determined by all the irreducible contributions (see Appendix C).

Restricting ourselves to the system considered above and having found the renormalized state vector $|p\rangle$, we can get all the physical information (we can for example calculate the electromagnetic form factors, if the particles are charged). So we do not need in practice to define and calculate the renormalized coupling constant $g_{\rm ren}$. However, it is useful to calculate it for further generalization to the case of particles with spin, when the charge renormalization constant will become infinite.

As we already mentioned, the standard definition of the coupling constant is the residue of the wave function ϕ_2 at $s=m^2$, i.e., the value of $\Gamma_2(q,z)$ at the nonphysical value of $q=i\kappa$ corresponding to $s=m^2$. For a nonrelativistic bound state calculation for instance, it is given by $\kappa=\sqrt{m|\epsilon_b|}$, where ϵ_b is the binding energy of the bound state. In coordinate space it is the coefficient of the asymptotical behavior of the wave function $\psi(r\to\infty)\propto\exp(-\kappa r)$. Since we calculate $\Gamma_2(q,z)$ numerically in the physical region for q, it is not easy to find its interpolation into the nonphysical region numerically with enough accuracy. However, we do not need to choose the renormalization point $s=m^2$; we can choose any other renormalization point. We can define for instance the renormalized coupling constant as the value of $\Gamma_2(q,z)$ at $q=0$:

$$\tilde{g}_{\rm ren}^2=\Gamma_2^{\rm ren}(q=0,z). \tag{40}$$

Note that $\Gamma_2(q=0,z)$ does not depend on z. In the case of $N+N\pi$ intermediate states Γ_2 does not depend on q and the renormalized coupling constants $\tilde{g}_{\rm ren}$ and $g_{\rm ren}$ coincide with each other.

TABLE I. Numerical results for $\alpha=3$, as a function of the cutoff L.

L	1	5	10	50	100	200
δm_0^2	−1.234	−3.67	−4.82	−7.54	−8.71	−9.88
$\delta m^2/\delta m_0^2$	1.095	1.052	1.040	1.025	1.022	1.020
N_1	0.377	0.329	0.327	0.326	0.326	0.326
N_2	0.441	0.455	0.456	0.458	0.458	0.458
N_2/N_1	1.183	1.380	1.395	1.407	1.405	1.405
N_3	0.182	0.216	0.217	0.216	0.216	0.216
$\langle q\rangle$	0.366	0.538	0.565	0.588	0.591	0.593

VI. NUMERICAL RESULTS

Both methods to solve Eqs. (11) are used to cross-check our results. For regularization purposes, we introduce a cutoff L, i.e. integrate in Eq. (22) over moduli of all the relative three-momenta q until $q\leqslant L$. Note that this cutoff procedure preserves rotational invariance. For the masses, we choose the nucleon and pion masses $m=0.94$ GeV and $\mu=0.14$ GeV. The integration over the azimuthal angle is done analytically. The equation is reduced to a matrix form by discretizing the integral. Convergence of the integrals is already obtained for 30 points in the variable q and 15 points in the variable z. The points in q were not taken equally spaced, but with a spacing proportional to h^2, where h is the equal spacing in the variable $\sqrt{q}$. In the first method, the eigenvalue $\lambda(\delta m^2)$ of the matrix is found numerically and δm^2 is fixed to get $\lambda\equiv 1$ to less than 0.5%. For the second method, δm^2 is calculated by a standard iteration procedure, starting from the perturbative result. Since both methods give identical results to less than 1% we quote only the results obtained with the second method.

The results for the dimensionless coupling constant $\alpha=g^2/(16\pi m^2)=3$ and for different values of the cutoff parameter L are shown in Table I. We denote by $\delta m^2/\delta m_0^2$ the

TABLE II. Same as Table I, but with L fixed to 200, and α varied.

α	1	3	10	30	100	1000
$\delta m^2/\delta m_0^2$	1.011	1.019	1.029	1.035	1.040	1.043
N_1	0.661	0.325	0.081	0.016	2×10^{-3}	3×10^{-5}
N_2	0.292	0.457	0.366	0.186	0.068	8×10^{-3}
N_3	0.047	0.218	0.553	0.798	0.930	0.992
N_2/N_1	0.441	1.41	4.51	11.91	32.9	230
$(N_2/N_1)_0$	0.38	1.14	3.8	11.4	38	380

ratio of the value δm^2 found from Eq. (22), to the perturbative value δm_0^2 given by Eq. (19). The values $N_{1,2,3}$ denote the contributions of the corresponding Fock sectors to the normalization of the state vector. In order to characterize the wave function quantitatively, we calculate also the average value of the (absolute) relative momentum $\langle q\rangle$, normalized to the two-body Fock component:

$$\langle q\rangle = \frac{1}{N_2}\frac{1}{2(2\pi)^3}\int \phi_2^2(\vec{q},\vec{n})\frac{[\varepsilon(q,m)+\varepsilon(q,\mu)]}{\varepsilon(q,m)\varepsilon(q,\mu)} q\, d^3q. \tag{41}$$

One clearly sees that although δm^2 increases logarithmically, the contributions of the Fock components N_1, N_2, N_3 as well as the average momentum $\langle q\rangle$ in the two-body Fock component become stable after $L=5$. This means that we have indeed found numerically the renormalized solution for the wave function.

One can see also that the nonperturbative value of δm^2 is very close to the perturbative one δm_0^2. We have checked that the function $\Gamma_2(q,z)$ is almost constant, as expected for the perturbative solution. The solution for δm^2 remains very close to the perturbative result also for higher values of the coupling constant. The results for $L=200$ and for different α between 1 and 1000 are shown in Table II. One can see that δm^2 is very close to the perturbative value even for extremely large coupling constant $\alpha=1000$. As expected, the three-body sector dominates when the coupling constant increases. However, the ratio N_2/N_1 is still close to its perturbative value $(N_2/N_1)_0$, given by Eq. (33).

The reason that the solution is close to the perturbative one lies in the superrenormalizability of the scalar theory. The third term in Eq. (22) converges and does not require any cutoff, whereas the first two terms are divergent and dominate. Therefore, a very small departure of δm^2 from its perturbative value δm_0^2 can accommodate a finite higher order correction. If the third term can be neglected, the equation is approximately satisfied with the perturbative value of δm^2. In order to show that it is indeed so, we introduce the cutoff in the two divergent terms, but do not introduce it in the third, convergent one. The results for $\alpha=3$ are shown in Table III. For small enough values of L the two first terms are suppressed, and the solution differs drastically from the perturbative one. When L increases, the solution becomes closer and closer to the perturbative one.

TABLE III. Test calculation in which the cutoff L is introduced only in the two first terms in Eq. (22) (see text), for $\alpha=3$.

L	0.1	1	10	100
$\delta m^2/\delta m_0^2$	36.3	1.55	1.07	1.03

We emphasize that the small deviation of the solution from the perturbative one, even for very large values of the coupling constant, is just a property of the nonperturbative Eq. (22), which includes the contributions (with one-, two- and three-body intermediate states) to all orders of g. This result cannot be justified in any perturbative expansion in terms of g.

VII. CONCLUSION

In a first attempt to address the question of nonperturbative renormalization in the CLFD, we have investigated in this study a simple, but nevertheless meaningful, model based on two scalar particles. This model is reminiscent of the structure of the physical nucleon in the low energy regime, in terms of bare nucleons coupled to pions.

Using the nice features of the CLFD, we have first derived the general eigenstate equation that should be used in order to calculate any physical state vector. We emphasize here that this equation is quite general and is not restricted to the case of scalar particles nor to the restricted Fock space we consider in this study. It should therefore also be used when solving more complex systems such as QED or QCD.

The results we obtained for the simple scalar model, with a restricted Fock space expansion up to three particles, are quite encouraging. We obtained, numerically, a renormalized solution using a simple mass counterterm. Surprisingly enough, this counterterm is not very different from the perturbative, logarithmically divergent, mass counterterm, even for very large values of the coupling constant. We traced this feature back to the nature of the scalar model we started from, which is superrenormalizable. This result, however, does not imply that higher Fock states are negligible. We find that the first nontrivial Fock component gets larger and larger as the coupling constant increases.

The direct generalization of this study is the investigation of nonperturbative renormalization in scalar QED, following the study of perturbative renormalization in QED already done in Ref. [6] in the CLFD. This will be the subject of a forthcoming publication.

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APPENDIX A: KINEMATICS

We show in this appendix how to express the propagator (21) in terms of the variables $\vec{q},\vec{q}',\vec{n}$.

The two-body wave function depends on the following four-vectors:

$$\phi=\phi(k_1,k_2,p,\omega\tau),\quad k_1+k_2=p+\omega\tau. \tag{A1}$$

We introduce the variables

$$\vec{q}=L^{-1}(\mathcal{P})\vec{k}_1=\vec{k}_1-\frac{\vec{\mathcal{P}}}{\sqrt{\mathcal{P}^2}}\left[k_{10}-\frac{\vec{k}_1\cdot\vec{\mathcal{P}}}{\sqrt{\mathcal{P}^2}+\mathcal{P}_0}\right], \tag{A2}$$

$$\vec{n}=L^{-1}(\mathcal{P})\vec{\omega}/|L^{-1}(\mathcal{P})\vec{\omega}|=\sqrt{\mathcal{P}^2}L^{-1}(\mathcal{P})\vec{\omega}/\omega\cdot p, \tag{A3}$$

where

$$\mathcal{P}=p+\omega\tau, \tag{A4}$$

and $L^{-1}(\mathcal{P})$ is the Lorentz boost. The wave function under the integral (see the last diagram in Fig. 3) depends on $k_1',k_2',p,\omega\tau'$, and, correspondingly, on the variable

$$\vec{q}'=L^{-1}(\mathcal{P}')\vec{k}_1', \tag{A5}$$

where

$$\mathcal{P}'=p+\omega\tau'. \tag{A6}$$

In order to obtain $\Pi(\vec{q},\vec{q}',\vec{n},p^2)$, we should express the four-momenta $k_1',k_2,\omega\tau'$ in $(k_1'-k_2-\omega\tau')^2$ in terms of $\vec{q},\vec{q}',\vec{n}$. Since under the Lorentz transformations and the rotations of the four-vectors the variables $\vec{q},\vec{q}',\vec{n}$ are rotated only, the expression for the scalar $\Pi(\vec{q},\vec{q}',\vec{n},p^2)$ does not depend on the system of reference, and hence can be found in the most convenient one. We find it in the system where

$$\vec{\mathcal{P}}'=\vec{k}_1'+\vec{k}_2'=\vec{p}+\vec{\omega}\tau'=0.$$

In this system $\vec{k}_1'=\vec{q}'$ and $k_{10}'=\varepsilon(\vec{q}',m)$. From the conservation law the vector $\vec{\mathcal{P}}$ is expressed as $\vec{\mathcal{P}}=\vec{k}_1+\vec{k}_2=\vec{\omega}(\tau-\tau')$. We have

$$\tau=\frac{s-p^2}{2(\omega\cdot p)},\quad \tau'=\frac{s'-p^2}{2(\omega\cdot p)}, \tag{A7}$$

where $\sqrt{s}=\varepsilon(\vec{q},m)+\varepsilon(\vec{q},\mu)$, $\sqrt{s'}=\varepsilon(\vec{q}',m)+\varepsilon(\vec{q}',\mu)$. Since $\vec{\omega}=\omega_0\vec{n}$ and $\omega\cdot p=\omega\cdot(k_1'+k_2')=\omega_0[\varepsilon(\vec{q}',m)+\varepsilon(\vec{q}',\mu)]=\omega_0\sqrt{s'}$, we find

$$\vec{\mathcal{P}}=\vec{n}\frac{s-s'}{2\sqrt{s'}},\quad \mathcal{P}_0=\frac{s+s'}{2\sqrt{s'}}. \tag{A8}$$

One can check that $\mathcal{P}^2=s$. According to Eqs. (A2) and (A5), the variables $\vec{q}$ and $\vec{q}'$ are defined by different Lorentz boosts. From Eq. (A8) one can see that this difference is the boost in the direction of $\vec{n}$. This boost does not change the unit vector $\vec{n}$.

Now let us find $\vec{k}_2$. It is obtained by reverting to Eq. (A2):

$$\vec{k}_2=L(\mathcal{P})(-\vec{q})=-\vec{q}+\frac{\vec{\mathcal{P}}}{\sqrt{\mathcal{P}^2}}\left[\varepsilon(\vec{q},\mu)-\frac{\vec{q}\cdot\vec{\mathcal{P}}}{\sqrt{\mathcal{P}^2}+\mathcal{P}_0}\right]. \tag{A9}$$

Equation (A9) is obtained from Eq. (A2) by replacing $\vec{k}_1$ in the right-hand side (RHS) by $-\vec{q}$ and by changing the sign of $\vec{\mathcal{P}}$.

Substituting here Eqs. (A8) for $\mathcal{P}$, we find

$$\vec{k}_2=-\vec{q}+\vec{n}\frac{s-s'}{2\sqrt{ss'}}\left(\varepsilon(\vec{q},\mu)-\vec{n}\cdot\vec{q}\frac{\sqrt{s}-\sqrt{s'}}{\sqrt{s}+\sqrt{s'}}\right), \tag{A10}$$

and similarly for k_{20}:

$$k_{20}=\frac{\varepsilon(\vec{q},\mu)\mathcal{P}_0}{\sqrt{\mathcal{P}^2}}-\frac{\vec{q}\cdot\vec{\mathcal{P}}}{\sqrt{\mathcal{P}^2}}=\varepsilon(\vec{q},\mu)\frac{s+s'}{2\sqrt{ss'}}-\vec{n}\cdot\vec{q}\frac{s-s'}{2\sqrt{ss'}}. \tag{A11}$$

From Eq. (A7) we get

$$\vec{\omega}\tau'=\vec{n}\frac{s'-p^2}{2\sqrt{s'}},\quad \omega_0\tau'=\frac{s'-p^2}{2\sqrt{s'}}.$$

Substituting the above expressions for the four-momenta into $(k_1'-k_2-\omega\tau')^2$, we find that the four-vector squared in the denominator of the propagator (21) is expressed in terms of the variables $\vec{q},\vec{q}',\vec{n}$ as follows:

$$\begin{aligned}(k_1'-k_2-\omega\tau')^2=&\left[\varepsilon(\vec{q}',m)-\varepsilon(\vec{q},\mu)\frac{s+s'}{2\sqrt{ss'}}+\vec{n}\cdot\vec{q}\frac{s-s'}{2\sqrt{ss'}}\right.\\&\left.-\frac{s'-p^2}{2\sqrt{s'}}\right]^2-\left[\vec{q}'+\vec{q}-\vec{n}\frac{s-s'}{2\sqrt{ss'}}\left(\varepsilon(\vec{q},\mu)\right.\right.\\&\left.\left.-\vec{n}\cdot\vec{q}\frac{\sqrt{s}-\sqrt{s'}}{\sqrt{s}+\sqrt{s'}}\right)-\vec{n}\frac{s'-p^2}{2\sqrt{s'}}\right]^2.\end{aligned} \tag{A12}$$

APPENDIX B: CALCULATION OF ϕ_1 AND ϕ_3

The Fock components ϕ_1 and ϕ_3 are calculated as follows. The wave function ϕ_2 in Eq. (28b) is related to the vertex functions Γ_2 by Eq. (29). Having found the vertex function $\Gamma_2(\vec{q},\vec{n})=\Gamma_2(q,z)$ numerically from Eq. (22), we then find by Eq. (29) the Fock component ϕ_2.

With Eqs. (12a), reduced to the first equation in Eq. (14), we express Γ_1 through Γ_2 in the limit $p^2\to m^2$, and then find the component ϕ_1:

$$\phi_1=-\frac{g}{\delta m^2}\int\Sigma_i(\vec{q}',p^2)\Gamma_2(\vec{q}',\vec{n})\frac{d^3q'}{(2\pi)^3}. \tag{B1}$$

With Eq. (12c), we express Γ_3 through Γ_2 and then find ϕ_3:

$$\phi_3(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{n})=\frac{g\phi_2(q',z')}{(s_{123}-m^2)(1-x_2)}+\frac{g\phi_2(q'',z'')}{(s_{123}-m^2)(1-x_3)}. \quad \text{(B2)}$$

Here $\phi_2(q',z')$ is $\phi_2(k_1',k_2,p,\omega\tau_2')$ represented through the relative momentum, and $\phi_2(q'',z'')$ is $\phi_2(k_1'',k_2,p,\omega\tau_2'')$.

The wave function ϕ_2 in the RHS of Eq. (B2) depends on the variables defined in the center of mass of the two-body subsystem. We shall now express these variables in terms of the three-body relative momenta $\vec{q}_1,\vec{q}_2,\vec{q}_3$.

The variable q' is related to s_{12}' by $s_{12}'=[\varepsilon(q',m)+\varepsilon(q',\mu)]^2$. We thus get

$$q'^2=[m^4+(\mu^2-s_{12}')^2-2m^2(\mu^2+s_{12}')]/(4s_{12}'),$$

where, by taking the first of Eqs. (13) squared,

$$s_{12}'=s_{123}-\frac{s_{13}-m^2}{1-x_2},$$

with

$$\begin{aligned} s_{123}&=[\varepsilon(q_1,m)+\varepsilon(q_2,\mu)+\varepsilon(q_3,\mu)]^2,\\ s_{13}&=(k_1+k_3)^2=[\varepsilon(q_1,m)+\varepsilon(q_3,\mu)]^2-q_2^2,\\ x_1&=\frac{\varepsilon(q_1,m)-\vec{n}\cdot\vec{q}_1}{\sqrt{s_{123}}},\\ x_{2,3}&=\frac{\varepsilon(q_{2,3},\mu)-\vec{n}\cdot\vec{q}_{2,3}}{\sqrt{s_{123}}}. \end{aligned} \quad \text{(B3)}$$

We use the fact that $\vec{q}_1+\vec{q}_2+\vec{q}_3=\vec{0}$.

The value of z' is found by comparing expressions (B3) for x_1 with

$$x_1=\frac{\varepsilon(q',m)-z'q'}{\varepsilon(q',m)+\varepsilon(q',\mu)}.$$

This gives

$$z'=\{\varepsilon(q',m)-x_1[\varepsilon(q',m)+\varepsilon(q',mu)]\}/q',$$

where x_1 is given by Eq. (B3) in terms of $\vec{q}_1,\vec{q}_2,\vec{q}_3$.

The values of q'',z'' are found from the equations

$$q''^2=[m^4+(\mu^2-s_{12}'')^2-2m^2(\mu^2+s_{12}'')]/(4s_{12}''),$$

$$z''=\{\varepsilon(q'',m)-x_1[\varepsilon(q'',m)+\varepsilon(q'',\mu)]\}/q'',$$

where

$$s_{12}''=s_{123}-\frac{s_{12}-m^2}{1-x_3},$$

with $s_{12}=(k_1+k_2)^2=[\varepsilon(q_1,m)+\varepsilon(q_2,\mu)]^2-q_3^2$.

APPENDIX C: PROOF OF EQ. (39) IN THE GENERAL CASE

We show below that the renormalization constant Z can still be represented in the form (39) with $\Sigma(p^2)$ determined by all the irreducible contributions. In this case, the number of irreducible diagrams containing two- and three-body intermediate states is infinite. Some of them are shown in Fig. 2. Consider, for example, a contribution containing n successive intermediate states. The corresponding amplitude contains the factor $1/\tau_i$ for any of these states. As in Eq. (15), any factor $1/\tau_i$ turns into $1/(s_i-p^2)$, where s_i is the invariant energy of a given intermediate state and p is the external incoming/outgoing momentum. So, indicating only these factors, we represent this contribution to $\Sigma(p^2)$ as

$$\Sigma_n(p^2)=\int\prod_{i=1,\ldots,n}\frac{1}{s_i-p^2}\cdots$$

where the ellipsis includes all the integrations with the corresponding measures. The derivative over p^2 gives the factor $1/(s_i-p^2)^2$.

Now consider graphs that are the same at the right to a given ith intermediate state and differ from each other by the contributions at the left to this state. The infinite sum of them determines the amplitude of the virtual transition from the initial state N to the states $N\pi$ or $N\pi\pi$. It is the vertex Γ_2 (if the state i is the two-body state), or Γ_3 (if the state i is the three-body state). We can then take the sum over all the contributions to the right of this given state i and again obtain Γ_2 or Γ_3. So the result has the form

$$\frac{d\Sigma(p^2)}{dp^2}=\int\frac{\Gamma_2^2}{(s_i-p^2)^2}\cdots+\int\frac{\Gamma_3^2}{(s_i-p^2)^2}\cdots=\int\phi_2^2\cdots+\int\phi_3^2\cdots=N_2+N_3$$

and, after extraction of the common factor N_1, we recover Eq. (39).

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