Ultraviolet fixed point and generalized flow equation of quantum gravity

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A new exact renormalization group equation for the effective average action of Euclidean quantum gravity is constructed. It is formulated in terms of the component fields appearing in the transverse-traceless decomposition of the metric. It facilitates both the construction of an appropriate infrared cutoff and the projection of the renormalization group flow onto a large class of truncated parameter spaces. The Einstein-Hilbert truncation is investigated in detail and the fixed point structure of the resulting flow is analyzed. Both a Gaussian and a non-Gaussian fixed point are found. If the non-Gaussian fixed point is present in the exact theory, quantum Einstein gravity is likely to be renormalizable at the nonperturbative level. In order to assess the reliability of the truncation a comprehensive analysis of the scheme dependence of universal quantities is performed. We find strong evidence supporting the hypothesis that 4-dimensional Einstein gravity is asymptotically safe, i.e. nonperturbatively renormalizable. The renormalization group improvement of the graviton propagator suggests a kind of dimensional reduction from 4 to 2 dimensions when spacetime is probed at sub-Planckian length scales.

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I. INTRODUCTION

During the past decade, exact renormalization group equations [1], in particular in the context of the effective average action [2], have become a powerful tool for the investigation of nonperturbative phenomena in both quantum field theory and in statistical physics. Those renormalization group (RG) or flow equations may be regarded as the counterpart for the continuum of Wilson's renormalization group of iterated Kadanoff block spin transformations which had been formulated for discrete spin systems originally [3]. In both cases the central idea is to "integrate out" all fluctuations with momenta larger than some cutoff k, and to take account of them by means of a modified dynamics for the remaining fluctuation modes with momenta smaller than k. This modified dynamics is governed by a scale dependent effective Hamiltonian or effective action, Γ_k , whose k dependence is described by a functional differential equation, the exact RG equation.

In quantum field theory this general strategy can be applied to both "effective" and "fundamental" theories. By definition, an effective theory is valid only if all relevant momenta of the process under consideration are close to some specific scale k which characterizes the theory. If Γ_k is the action of an effective theory at scale k we can compute cross sections for the scattering of particles with momenta (or relevant momentum transfers) of the order of k, with all quantum effects included, by simply evaluating the *tree* diagrams of Γ_k . Exact RG equations can be used in order to evolve Γ_k to a smaller scale k' < k by further "coarse graining."

Flow equations may also be used for a complete quantization of fundamental theories. If the latter has the classical action *S* one imposes the initial condition $\Gamma_{\hat{k}} = S$ at the ultraviolet (UV) cutoff scale \hat{k} , uses the exact RG equation to compute Γ_k for all $k < \hat{k}$, and then sends $k \to 0$ and $\hat{k} \to \infty$. Loosely speaking, the defining property of a fundamental theory is that the "continuum limit" $\hat{k} \rightarrow \infty$ actually exists after the "renormalization"—in the traditional sense of the word—of finitely many parameters in the action; only a finite number of generalized couplings in Γ_0 is undetermined and has to be taken from the experiment. This is the case for perturbatively renormalizable theories [4], but there are also perturbatively nonrenormalizable theories which admit a limit $\hat{k} \rightarrow \infty$. The "continuum" limit of those nonperturbatively renormalizable theories is taken at a non-Gaussian fixed point of the RG flow. It replaces the Gaussian fixed point which, at least implicitly, underlies the construction of perturbatively renormalizable theories [1]. Thus knowing its fixed point structure is crucial if one wants to assess whether a given model qualifies as a *fundamental* theory.

In this paper we shall use a formulation where Γ_k is the "effective average action" [2]. It is a coarse grained free energy functional which is constructed in close analogy with the standard effective action Γ to which it reduces in the limit of a vanishing infrared (IR) cutoff, $k \rightarrow 0$. The Euclidean functional integral for the generating functional W is modified by adding an IR cutoff term $\Delta_k S$ to the classical action. It supplies a momentum dependent (mass)²-term $\mathcal{R}_k(p^2)$ for a mode of the quantum field with momentum p. The cutoff function $\mathcal{R}_k(p^2)$ vanishes for $p^2 \gg k^2$; hence the high-momentum modes get integrated out in the usual way. For $p^2 \ll k^2$ it behaves as $\mathcal{R}_k(p^2) \propto k^2$ so that the smallmomentum modes get suppressed in the path integral by a mass term $\propto k^2$ [2]. The scale dependent action Γ_k is closely related to the Legendre transform of the modified generating functional W_k . When regarded as a function of k, Γ_k runs along a RG trajectory in the space of all actions which starts at $\Gamma_{\hat{k}} = S$ and ends at $\Gamma_0 = \Gamma$. In the simplest case, the exact RG equation which describes this trajectory has the following structure:

$$k \partial_k \Gamma_k = \frac{1}{2} \operatorname{Tr} [(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1} k \partial_k \mathcal{R}_k].$$
(1.1)

Here $\Gamma_k^{(2)}$ denotes the infinite dimensional matrix of second functional derivatives of Γ_k with respect to all dynamical fields.

This construction is fairly straightforward for matter fields, the inclusion of gauge fields introduces additional complications though. Using background gauge techniques, a solution to this problem was given in Refs. [5] for Yang-Mills theory and in Refs. [6,7] for gravity. Leaving the Faddeev-Popov ghosts aside, the effective average action for gravity, $\Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}]$, depends not only on the "ordinary" dynamical metric $g_{\mu\nu}$ but also on a background metric $\bar{g}_{\mu\nu}$. The conventional effective action $\Gamma[g]$ is obtained as the $k \rightarrow 0$ limit of the functional $\Gamma_k[g] \equiv \Gamma_k[g, \bar{g} = g]$ with the two metrics identified [8,9]. The motivation for this construction is that in this manner $\Gamma_k[g]$ becomes invariant under general coordinate transformations.

Nonperturbative solutions to the above RG equation which do not require a small expansion parameter can be obtained by the method of "truncations." This means that one projects the RG flow from the infinite dimensional space of all actions onto some finite dimensional subspace which is particularly relevant for the problem at hand. In this manner the functional RG equation becomes an ordinary differential equation for a finite set of generalized couplings which serve as coordinates on this subspace. In Ref. [6] the RG flow of quantum general relativity was projected on the 2-dimensional subspace spanned by the invariants $\int d^d x \sqrt{g}$ and $\int d^d x \sqrt{g}R$. This so-called Einstein-Hilbert truncation amounts to considering only functionals of the form

$$\Gamma_{k}[g,\overline{g}] = (16\pi G_{k})^{-1} \int d^{d}x \sqrt{g} \{-R(g) + 2\overline{\lambda}_{k}\}$$

+ classical gauge fixing. (1.2)

Here G_k and $\overline{\lambda}_k$ are the running Newton constant and cosmological constant, respectively. More general and, therefore, more precise truncations would include higher powers of the curvature tensor as well as nonlocal terms [10] which are not present classically.

Quantum gravity is certainly a particularly interesting topic where exact RG equations can lead to important new insights. As quantized Einstein gravity is perturbatively nonrenormalizable a natural option is to consider it an effective field theory [11]. Already within this setting quantum effects can be studied in a consistent and predictive way. In fact, in Refs. [12,13] the running couplings G_k and $\overline{\lambda}_k$ obtained in [6] were used to investigate how quantum gravity effects modify the structure of black holes, and in [14] the implications for the cosmology of the Planck era in the very early Universe were studied. Along a different line of research it has been proposed [15] that there are strong quantum gravitational effects also in the later stages of the cosmological evolution which even might drive the cosmological constant to zero dynamically; the effective average action would be an ideal tool for exploring such infrared effects.

An even more intriguing possibility is that, despite its perturbative nonrenormalizability, quantized gravity exists *nonperturbatively* as a fundamental theory. It would then be mathematically consistent down to arbitrarily small length scales. A proposal along these lines is Weinberg's "asymptotic safety" scenario [16]. It assumes that there exists a non-Gaussian RG fixed point at which the $\hat{k} \rightarrow \infty$ limit can be taken, i.e. that the theory is "nonperturbatively renormalizable" in Wilson's sense. Asymptotic safety requires that the non-Gaussian fixed point is UV attractive (i.e. attractive for $k \rightarrow \infty$) for finitely many parameters in the action, i.e. that its UV critical hypersurface is finite dimensional. This means that the RG trajectories along which the theory can flow as we send the cutoff \hat{k} to infinity are labeled by only finitely many parameters. Therefore the theory is as predictive as any conventionally renormalizable theory; it is not plagued by the notorious increase of free parameters which is typical of effective theories. The set of generalized couplings for which the non-Gaussian fixed point is UV attractive should include the dimensionless Newton constant, g, and cosmological constant. λ .

Using the ε expansion, Weinberg showed already long ago that gravity in $2+\varepsilon$ dimensions ($0 < \varepsilon \ll 1$) is indeed asymptotically safe [16]. Further progress in this direction, in particular for d=4, was hampered by the lack of an efficient calculational scheme which could be used to search for nonperturbative fixed points.

As a solution to this problem which does not rely on the ε expansion we propose to use the effective average action in order to find nontrivial fixed points $(\lambda_*, g_*, ...)$ of the gravitational RG flow. (The dots stand for the infinitely many other couplings which parametrize a generic action functional.) Using this approach, the case $d=2+\varepsilon$ was reanalyzed in a more general setting and, more importantly, it was shown that the Einstein-Hilbert truncation predicts the existence of a non-Gaussian fixed point (λ_*, g_*) also in dimensions d>2, in particular for d=4 [6,13,17].

The crucial question which arises is whether this result is an artifact of the truncation used, or if it correctly reflects a property of the full theory. It is clear that in order to answer this question one would like to include further invariants into the truncation and to check whether the predictions stabilize.

From the technical point of view such calculations are extremely complicated so that in the present paper we shall use a different method in order to get a first idea about the reliability of the nontrivial fixed point. We are going to analyze to what extent its location in the λ -g plane and its attractivity properties (critical exponents) are scheme dependent. Here "scheme dependence" refers to the dependence on the cutoff operator \mathcal{R}_k used in the derivation of the RG equation.

First of all, \mathcal{R}_k is a matrix in the space of irreducible component fields (see below) which is not uniquely determined by the general principles. Hence we can vary it to some extent. In fact, in the present paper we shall introduce a new cutoff ("cutoff of type B") whose matrix structure is different from the original one of Ref. [6] ("cutoff of type A"). Either of these cutoffs is proportional to a "shape function" $R^{(0)}(p^2/k^2)$ which describes the "thinning out" of degrees of freedom as we pass the threshold $p^2 = k^2$. Also this function can be varied in order to assess the scheme dependence of the fixed point properties.

While in general only the critical exponents but not the location $(\lambda_*, g_*, ...)$ of the fixed point are expected to be universal, i.e. scheme independent [18], we shall argue that the product $g_*\lambda_*$ is an observable quantity as well. For observables the \mathcal{R}_k dependence is a pure truncation artifact; in an exact treatment all \mathcal{R}_k dependencies cancel. The status of the Einstein-Hilbert truncation would be rather questionable if the fixed point was present for some cutoffs but absent for others. Instead, we find that it is actually there for all admissible cutoffs, and moreover that the observable $g_*\lambda_*$ is scheme independent with a quite unexpected precision.

Our results strongly support the conjecture that the non-Gaussian fixed point is present in the exact theory and is not a truncation artifact. Also another prerequisite of asymptotic safety turns out to be satisfied: we find that, for any cutoff, the fixed point is UV attractive in both directions of the λ -g plane.

Ultimately one would like to use more general truncations than Eq. (1.2) in order to study the RG flow in a larger subspace. Typically this requires computations whose algebraic complexity is quite formidable. Assume we make an ansatz $\Gamma_k = \sum_{i=1}^n g_i(k) I_i$ containing *n* diffeomorphism invariant functionals $I_i[g_{\mu\nu}, \overline{g}_{\mu\nu}]$. In order to project the RG flow on the *n*-dimensional space with coordinates g_i we must insert the ansatz into the RHS of the flow equation (1.1). At this point the nontrivial problem, both conceptually and computationally, is to expand the trace with respect to a complete set of actions, $\{\gamma_i[g_{\mu\nu}, \overline{g}_{\mu\nu}]\}$, in such a way that the I_i 's retained in the ansatz are a subset of the γ_i 's. The coefficients of the remaining γ_i 's, those not present in Γ_k , are set to zero by the truncation. In practice the projection on the $\{I_i\}$ subspace is done by inserting a set of metrics $g_{\mu\nu}, \overline{g}_{\mu\nu}$ on both sides of Eq. (1.1) which give a nonzero value only to specific linear combinations of the I_i 's. Provided one manages to compute the functional trace for sufficiently many $g_{\mu\nu}, \overline{g}_{\mu\nu}$ pairs one can then deduce the ordinary differential equations for the generalized couplings $g_i(k)$.

For the Einstein-Hilbert truncation this procedure is fairly simple since (ignoring the running of the gauge parameter) it is sufficient to insert for $g_{\mu\nu} = \bar{g}_{\mu\nu}$ the metric of a family of spheres S^d parametrized by their radius r. Their maximal symmetry facilitates the calculations considerably. With rkept as a free parameter, these metrics are general enough to disentangle $\int d^d x \sqrt{g} \propto r^d$ and $\int d^d x \sqrt{g} R \propto r^{d-2}$. But already when we include invariants with four derivatives of the metric this method fails: the spheres cannot distinguish $\int d^d x \sqrt{g} R^{2} \propto r^{d-4}$ from $\int d^d x \sqrt{g} R_{\mu\nu} R^{\mu\nu} \propto r^{d-4}$, for instance.

These remarks hint at (at least) two major problems which one faces in generalizations of the exact RG approach to gravity. (i) The momentum dependent "mass" term $\Delta_k S[h_{\mu\nu}, \overline{g}_{\mu\nu}]$ depends quadratically on the metric fluctuation $h_{\mu\nu}$, but also, via $\mathcal{R}_k \equiv \mathcal{R}_k[\overline{g}]$ on the background $\overline{g}_{\mu\nu}$. In general it is a quite nontrivial task to construct a cutoff operator $\mathcal{R}_k[\overline{g}]$ which has the desired properties mentioned above for a class of background metrics $\overline{g}_{\mu\nu}$ general enough for the projection on the truncation subspace. (ii) Assume we found an appropriate $\mathcal{R}_k[\bar{g}]$. Then there arises the computational problem of evaluating the trace on the RHS of the RG equation for various $g_{\mu\nu}$'s and $\bar{g}_{\mu\nu}$'s. These metrics do not coincide when we allow for an evolution of the gauge fixing sector. Even if we ignore this complication, the Hessian $\Gamma_k^{(2)}$ under the trace is an extremely complicated nonminimal covariant matrix differential operator constructed from the curvature tensor and covariant derivatives D_{μ} . A priori, even for maximally symmetric backgrounds, not all derivatives D_{μ} are contracted to form powers of the covariant Laplacian $D^2 \equiv D_{\mu}D^{\mu}$, and $\Gamma_k^{(2)}$ is not diagonal in the space of fields with a definite helicity therefore. Hence standard heat kernel techniques or perhaps information about the spectrum of D^2 are of no help at this point.

In this paper we outline a general strategy for tackling these problems. It is based upon York's "transverse-traceless (TT) decomposition" [20] which is available on (almost) every spacetime manifold needed for our projection method. The idea is to decompose the fluctuation $h_{\mu\nu}$ into a transverse, traceless tensor $h_{\mu\nu}^T$, a longitudinal-transverse tensor (parametrized by a transverse vector $\hat{\xi}^{\mu}$), a longitudinallongitudinal tensor (parametrized by a scalar $\hat{\sigma}$), and a trace part (parametrized by another scalar ϕ). In the basis of the component fields $\{h_{\mu\nu}^T, \hat{\xi}^{\mu}, \hat{\sigma}, \phi\}$, all D_{μ} 's appear in powers of the Laplacian only, at least for the class of maximally symmetric backgrounds. The important point is that this decomposition can be used in order to simplify the structure of $\Gamma_k^{(2)}$ on essentially all backgrounds, not just on spheres. [On S^d the TT decomposition boils down to the familiar decomposition of $h_{\mu\nu}$ with respect to pieces which are irreducible under the isometry group SO(d+1). In some of the work following the original paper [6] this decomposition on S^d had been used already [19,21-23]. Compared to S^d a certain complication arises, however, because the TT decomposition is nonorthogonal in general.

The TT decomposition also helps in solving the first problem, the construction of \mathcal{R}_k , because $\Delta_k S$ has a much simpler structure when expressed in terms of the component fields rather than the original $h_{\mu\nu}$.

This paper is organized as follows.

In the first part (Secs. II and III) we describe the construction of a new RG equation where the component fields $\{h_{\mu\nu}^T, \hat{\xi}^{\mu}, \hat{\sigma}, \phi\}$ are used from the outset. Along the way we discuss the problems related to a proper identification of $\Delta_k S$. This part of the paper is meant to supply a set of tools which will become indispensable in future investigations when one includes further invariants into the truncation (R^2 terms [24], for instance), if one adds matter fields, or if one allows for a running gauge fixing.

As a first application, we revisit the Einstein-Hilbert truncation in the second part of the paper (Secs. IV and V). We introduce a new cutoff \mathcal{R}_k which is natural in the TT language, and we use an arbitrary gauge parameter. This allows for a nontrivial comparison of the resulting RG equations and their fixed point properties with those of [6] whose cutoff operator \mathcal{R}_k has a rather different structure. We find both a Gaussian and a non-Gaussian fixed point in the (g,λ) system and we perform a detailed analysis of their properties, in particular of their scheme dependence. The chances for realizing the asymptotic safety scenario in 4 dimensions will be discussed in detail.

In Sec. VI we investigate the implications of the non-Gaussian fixed point for the effective graviton propagator at large momenta. A kind of dimensional reduction from 4 to 2 dimensions takes place in the vicinity of this fixed point. The asymptotic form of the propagator suggests that when 4-dimensional spacetime is probed by a very high-energetic graviton it appears to be effectively 2-dimensional.

Various technical results, needed in the present paper but presented also with an eye towards future applications [24], are relegated to a set of Appendixes.

At this point the reader who is mostly interested in the results rather than their derivation can proceed directly to Sec. V.

II. THE EXACT EVOLUTION EQUATION

A. Gauge fixing

Following [6] we define a scale dependent modification of the Euclidean functional integral for the generating functional \mathbf{Z}_k by using the background gauge fixing technique [8,9]. For this purpose we decompose the integration variable in the functional integral over all metrics, $\gamma_{\mu\nu}$, into a fixed background metric $\overline{g}_{\mu\nu}$ and a fluctuation field $h_{\mu\nu}$:

$$\gamma_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x). \tag{2.1}$$

Then we replace the integration over $\gamma_{\mu\nu}$ by an integration over $h_{\mu\nu}$. With the Faddeev-Popov ghosts C^{μ} and \bar{C}_{μ} the generating functional \mathbf{Z}_k may be written as

$$\mathbf{Z}_{k}[\text{sources}] = \int \mathcal{D}h_{\mu\nu}\mathcal{D}C^{\mu}\mathcal{D}\bar{C}_{\mu} \exp[-S[\bar{g}+h] - S_{\text{gf}}[h;\bar{g}] \\ -S_{\text{gh}}[h,C,\bar{C};\bar{g}] \\ -\Delta_{k}S[h,C,\bar{C};\bar{g}] - S_{\text{source}}].$$
(2.2)

The first term in the exponential, $S[\gamma] = S[\overline{g} + h]$, is the classical action which, for the moment, is assumed to be positive definite. It is invariant under arbitrary general coordinate transformations. S_{gf} denotes the gauge fixing term

$$S_{\text{gf}}[h;\overline{g}] = \frac{1}{2\alpha} \int d^d x \sqrt{\overline{g}} \, \overline{g}^{\mu\nu} F_{\mu}[\overline{g},h] F_{\nu}[\overline{g},h]. \quad (2.3)$$

It corresponds to the gauge condition $F_{\mu}[\bar{g},h]=0$. Linear gauge conditions

$$F_{\mu}[\bar{g},h] = \sqrt{2} \kappa \mathcal{F}^{\alpha\beta}_{\mu}[\bar{g}]h_{\alpha\beta}, \qquad (2.4)$$

are particularly convenient. In the present paper we use the harmonic gauge¹ for which

$$\mathcal{F}^{\alpha\beta}_{\mu}[\bar{g}] = \delta^{\beta}_{\mu} \bar{g}^{\alpha\gamma} \bar{D}_{\gamma} - \frac{1}{2} \bar{g}^{\alpha\beta} \bar{D}_{\mu}.$$
(2.5)

Here \bar{D}_{μ} denotes the covariant derivative constructed from the background metric $\bar{g}_{\mu\nu}$, while we shall write D_{μ} for the covariant derivative involving the quantum metric $\gamma_{\mu\nu}$. In Eq. (2.4) we introduced the constant

$$\kappa \equiv (32\pi\bar{G})^{-1/2} \tag{2.6}$$

where \overline{G} denotes the bare Newton constant. The Faddeev-Popov operator associated with the gauge fixing (2.4) with (2.5) takes the form

$$\mathcal{M}[\gamma, \bar{g}]^{\mu}_{\nu} = \bar{g}^{\mu\rho} \bar{g}^{\sigma\lambda} \bar{D}_{\lambda} (\gamma_{\rho\nu} D_{\sigma} + \gamma_{\sigma\nu} D_{\rho}) - \bar{g}^{\rho\sigma} \bar{g}^{\mu\lambda} \bar{D}_{\lambda} \gamma_{\sigma\nu} D_{\rho}.$$
(2.7)

It enters the functional integral (2.2) via the ghost action

$$S_{\rm gh}[h,C,\bar{C};\bar{g}] = -\sqrt{2} \int d^d x \sqrt{\bar{g}} \, \bar{C}_{\mu} \mathcal{M}[\bar{g}+h,\bar{g}]^{\mu}_{\nu} C^{\nu}.$$
(2.8)

Furthermore, $\Delta_k S$ and S_{source} are the cutoff and the source action, respectively. $\Delta_k S$ provides an appropriate infrared cutoff for the integration variables and S_{source} introduces sources for the fields $h_{\mu\nu}$, C^{μ} and \bar{C}_{μ} . Their explicit structure will be discussed later on.

B. Decomposition of the quantum fields

For the calculations in the following sections it turns out to be convenient to decompose the gravitational field $h_{\mu\nu}$ according to (see e.g. [20])

$$h_{\mu\nu} = h_{\mu\nu}^{T} + \bar{D}_{\mu}\hat{\xi}_{\nu} + \bar{D}_{\nu}\hat{\xi}_{\mu} + \bar{D}_{\mu}\bar{D}_{\nu}\hat{\sigma} - \frac{1}{d}\bar{g}_{\mu\nu}\bar{D}^{2}\hat{\sigma} + \frac{1}{d}\bar{g}_{\mu\nu}\phi.$$
(2.9)

To obtain this "TT decomposition" one starts by splitting off the trace part $h_{\mu\nu}^{Tr} \equiv \bar{g}_{\mu\nu} \phi/d$ from $h_{\mu\nu}$. It involves a scalar field ϕ . The remaining symmetric traceless tensor may be decomposed further into a transverse component $h_{\mu\nu}^{T}$ and a longitudinal component $h_{\mu\nu}^{L}$. Introducing a transverse vector field $\hat{\xi}_{\mu}$ and another scalar $\hat{\sigma}$, the longitudinal tensor can be expressed by $h_{\mu\nu}^{L} = h_{\mu\nu}^{LT} + h_{\mu\nu}^{LL}$ with $h_{\mu\nu}^{LT} \equiv \bar{D}_{\mu} \hat{\xi}_{\nu} + \bar{D}_{\nu} \hat{\xi}_{\mu}$ and $h_{\mu\nu}^{LL} \equiv \bar{D}_{\mu} \bar{D}_{\nu} \hat{\sigma} - \bar{g}_{\mu\nu} \bar{D}^2 \hat{\sigma}/d$ thereby ending up with Eq. (2.9). Thus the components of $h_{\mu\nu}$ introduced by this transversetraceless (TT) decomposition obey the relations

$$\bar{g}^{\mu\nu}h^{T}_{\mu\nu} = 0, \quad \bar{D}^{\mu}h^{T}_{\mu\nu} = 0, \quad \bar{D}^{\mu}\hat{\xi}_{\mu} = 0, \quad \phi = \bar{g}_{\mu\nu}h^{\mu\nu}.$$
(2.10)

This decomposition is valid for complete, closed Riemannian d spaces (i.e. compact Riemannian manifolds without boundary). As argued in [20], its domain of validity can be extended to open, asymptotically flat d spaces, certain assumptions concerning the asymptotic behavior of the fields

¹For the flow equation in the conformal gauge (2D Liouville quantum gravity) see Refs. [25,26].

being made. From now on we assume that the gravitational background belongs to one of these classes of spaces.

Obviously $h_{\mu\nu}$ receives no contribution from those $\hat{\xi}_{\mu}$ and $\hat{\sigma}$ modes which satisfy the Killing equation

$$\bar{D}_{\mu}\hat{\xi}_{\nu} + \bar{D}_{\nu}\hat{\xi}_{\mu} = 0 \tag{2.11}$$

and the scalar equation

$$\bar{D}_{\mu}\bar{D}_{\nu}\hat{\sigma} - \frac{1}{d}\bar{g}_{\mu\nu}\bar{D}^{2}\hat{\sigma} = 0, \qquad (2.12)$$

respectively. Therefore such modes, referred to as unphysical $\hat{\xi}_{\mu}$ and $\hat{\sigma}$ modes, have to be excluded from the functional integral. Considering the conformal Killing equation

$$\bar{D}_{\mu}C_{\nu} + \bar{D}_{\nu}C_{\mu} - \frac{2}{d}\bar{g}_{\mu\nu}\bar{D}_{\lambda}C^{\lambda} = 0 \qquad (2.13)$$

we recognize that the unphysical $\hat{\sigma}$ modes correspond to constants or are related via $C_{\mu} = \bar{D}_{\mu}\hat{\sigma}$ to proper conformal Killing vectors (PCKV's), i.e. solutions of Eq. (2.13) which are not at the same time ordinary Killing vectors (KV's), [27].²

By virtue of the decomposition (2.9) the inner product on the space of symmetric tensor fields may be decomposed according to³

$$\langle h^{(1)}, h^{(2)} \rangle \equiv \int d^{d}x \sqrt{\overline{g}} h^{(1)}_{\mu\nu} \overline{g}^{\mu\rho} \overline{g}^{\nu\sigma} h^{(2)}_{\rho\sigma}$$

$$= \int d^{d}x \sqrt{\overline{g}} \left\{ h^{(1)T}_{\mu\nu} h^{(2)T\mu\nu} -2 \hat{\xi}^{(1)}_{\mu} (\overline{g}^{\mu\nu} \overline{D}^{2} + \overline{R}^{\mu\nu}) \hat{\xi}^{(2)}_{\nu} -2 \hat{\xi}^{(1)}_{\mu} \overline{R}^{\mu\nu} \overline{D}_{\nu} \hat{\sigma}^{(2)} -2 \hat{\xi}^{(2)}_{\mu} \overline{R}^{\mu\nu} \overline{D}_{\nu} \hat{\sigma}^{(1)} + \hat{\sigma}^{(1)} \left(\frac{d-1}{d} (\overline{D}^{2})^{2} + \overline{D}_{\mu} \overline{R}^{\mu\nu} \overline{D}_{\nu} \right)$$

$$\times \hat{\sigma}^{(2)} + \frac{1}{d} \phi^{(1)} \phi^{(2)} \right\}.$$

$$(2.14)$$

From Eq. (2.14) we see that, for a general background metric, only $h_{\mu\nu}^T$, $h_{\mu\nu}^L$ and $h_{\mu\nu}^{Tr}$ form an orthogonal set, whereas $h_{\mu\nu}^{LT}$ and $h_{\mu\nu}^{LL}$ are not orthogonal in general. This nonorthogo-

nality manifests itself in the appearance of terms where the components $\hat{\xi}_{\mu}$ and $\hat{\sigma}$ mix. But at least for Einstein spaces, where $\bar{R}_{\mu\nu} = C\bar{g}_{\mu\nu}$ with *C* a constant, we find $\bar{D}_{\mu}\bar{R}^{\mu\nu} = C\bar{D}_{\mu}\bar{g}^{\mu\nu} \equiv 0$ and therefore $\langle h^{LT}, h^{LL} \rangle = 4C\int d^dx \sqrt{\bar{g}} \hat{\sigma} \bar{D}^{\mu} \hat{\xi}_{\mu} = 0$. Thus $\{h_{\mu\nu}^T, h_{\mu\nu}^{LT}, h_{\mu\nu}^{LL}, h_{\mu\nu}^{Tr}\}$ represents an orthogonal set of field components in this case.

In order to determine the Jacobian J_1 which appears in the functional integral (2.2) after performing the transformation of integration variables $h_{\mu\nu} \rightarrow \{h_{\mu\nu}^T, \hat{\xi}_{\mu}, \hat{\sigma}, \phi\}$ we proceed as follows. We consider a Gaussian integral over $h_{\mu\nu}$ and reexpress it in terms of the component fields [27]:

$$\int \mathcal{D}h_{\mu\nu} \exp\left[-\frac{1}{2}\langle h,h\rangle\right]$$
$$= J_{1} \int \mathcal{D}h_{\mu\nu}^{T} \mathcal{D}\hat{\xi}_{\mu} \mathcal{D}\hat{\sigma} \mathcal{D}\phi \exp\left[-\frac{1}{2}\int d^{d}x \sqrt{g}\right]$$
$$\times \left\{h_{\mu\nu}^{T}h^{T\mu\nu} + \frac{1}{d}\phi^{2} + (\hat{\xi}_{\mu},\hat{\sigma})M^{(\mu,\nu)}\begin{pmatrix}\hat{\xi}_{\nu}\\\hat{\sigma}\end{pmatrix}\right\}.$$
(2.15)

Here

$$M^{(\mu,\nu)} \equiv \begin{pmatrix} -2(\bar{g}^{\mu\nu}\bar{D}^2 + \bar{R}^{\mu\nu}) & -2\bar{R}^{\mu\lambda}\bar{D}_{\lambda} \\ 2\bar{D}_{\lambda}\bar{R}^{\lambda\nu} & \frac{d-1}{d}(\bar{D}^2)^2 + \bar{D}_{\lambda}\bar{R}^{\lambda\rho}\bar{D}_{\rho} \end{pmatrix}$$

$$(2.16)$$

is a Hermitian matrix differential operator. Since all functional integrals appearing in Eq. (2.15) are Gaussian they are easily evaluated. This leads to the Jacobian

$$J_1 = N \sqrt{\det'_{(1T,0)} M^{(\mu,\nu)}}.$$
 (2.17)

Here N represents an infinite constant which may be absorbed into the normalization of the measure $Dh_{\mu\nu}$.

The notation adopted in Eq. (2.17) has to be interpreted as follows. A prime at the determinant or the trace of an operator \mathcal{A} indicates that all unphysical $\hat{\xi}_{\mu}$ and $\hat{\sigma}$ eigenmodes of \mathcal{A} , characterized by Eqs. (2.11) and (2.12), are to be excluded from the calculation. A subscript at determinants or traces describes on which kind of field the operator \mathcal{A} acts. We use the subscripts (0), (1*T*) and (2*ST*²) for spin-0 fields $\hat{\sigma}$, transverse spin-1 fields $\hat{\xi}_{\mu}$ and symmetric transverse traceless spin-2 fields $h_{\mu\nu}^T$, repectively. The subscript (1*T*,0) appearing in Eq. (2.17) refers to a $(d+1) \times (d+1)$ -matrix differential operator whose first *d* columns act on traceless spin-1 fields $\hat{\xi}^{\nu}$ whereas the last column acts on spin-0 fields $\hat{\sigma}$.

²As a consequence of the linearity of the conformal Killing equation we may add any Killing vector (which is always transversal) to its solutions and obtain another solution. For definiteness we therefore define the PCKV's to be purely longitudinal. Then there is a one-to-one correspondence between the PCKV's and the nonconstant solutions of Eq. (2.12).

³A remark concerning our notation: If not indicated otherwise each covariant derivative acts on everything that stands on the right of it.

Likewise we decompose the ghost and the antighost into their orthogonal components according to

$$\bar{C}_{\mu} = \bar{C}_{\mu}^{T} + \bar{D}_{\mu} \hat{\bar{\eta}}, \quad C^{\mu} = C^{T\mu} + \bar{D}^{\mu} \hat{\eta}$$
 (2.18)

where \bar{C}_{μ}^{T} and $C^{T\mu}$ are the transverse components of \bar{C}_{μ} and C^{μ} : $\bar{D}^{\mu}\bar{C}_{\mu}^{T}=0$, $\bar{D}_{\mu}C^{T\mu}=0$. In order to compute the Jacobian J_{2} induced by the change of variables $\bar{C}_{\mu} \rightarrow \{\bar{C}_{\mu}^{T}, \hat{\bar{\eta}}\}, C^{\mu} \rightarrow \{C^{T\mu}, \hat{\eta}\}$ we write

$$\int \mathcal{D}C^{\mu}\mathcal{D}\bar{C}_{\mu}\exp[-\langle\bar{C},C\rangle]$$

$$=J_{2}\int \mathcal{D}C^{T\mu}\mathcal{D}\bar{C}_{\mu}^{T}\mathcal{D}\hat{\eta}\mathcal{D}\hat{\bar{\eta}}$$

$$\times\exp\left[-\int d^{d}x\sqrt{g}\{\bar{C}_{\mu}^{T}C^{T\mu}+\hat{\bar{\eta}}(-\bar{D}^{2})\hat{\eta}\}\right].$$
(2.19)

and perform the Grassmann functional integrals. The result is

$$J_2 = [\det'_{(0)}(-\bar{D}^2)]^{-1}.$$
 (2.20)

In this case the constant $\hat{\eta}$ mode represents an unphysical mode which has to be excluded.

C. Momentum dependent redefinition of the component fields

It will prove convenient to introduce new variables of integration, ξ_{μ} , σ , $\bar{\eta}$ and η , by means of the momentum dependent (nonlocal) redefinitions

$$\begin{split} \xi^{\mu} &\equiv \sqrt{-\bar{D}^2 - \overline{\operatorname{Ric}}} \ \hat{\xi}^{\mu} \\ \sigma &\equiv \sqrt{(\bar{D}^2)^2 + \frac{d}{d-1}} \ \bar{D}_{\mu} \bar{R}^{\mu\nu} \bar{D}_{\nu} \hat{\sigma} \\ \bar{\eta} &\equiv \sqrt{-\bar{D}^2} \ \hat{\eta}, \quad \eta &\equiv \sqrt{-\bar{D}^2} \ \hat{\eta}. \end{split}$$
(2.21)

Here the operator $\overline{\text{Ric}}$ maps vectors onto vectors according to

$$(\overline{\operatorname{Ric}} \ v)^{\mu} = \overline{R}^{\mu\nu} v_{\nu}. \qquad (2.22)$$

Note that the transformations (2.21) are well defined and invertible since for any (physical) eigenmode the operators under the square roots of Eq. (2.21) have strictly positive eigenvalues.⁴

This is due to the fact that these operators arise from the squares of $h_{\mu\nu}^{LT}$ and $h_{\mu\nu}^{LL}$ and from $(\bar{D}_{\mu}\hat{\eta})(\bar{D}^{\mu}\hat{\eta})$ by shifting all covariant derivatives to the right. Thus they cannot assume negative eigenvalues. For example,

$$\begin{split} \langle h^{LL}, h^{LL} \rangle &= \int d^d x \sqrt{\overline{g}} \bigg(\bar{D}_{\mu} \bar{D}_{\nu} \hat{\sigma} - \frac{1}{d} \overline{\overline{g}}_{\mu\nu} \bar{D}^2 \hat{\sigma} \bigg) \\ &\times \bigg(\bar{D}^{\mu} \bar{D}^{\nu} \hat{\sigma} - \frac{1}{d} \overline{\overline{g}}^{\mu\nu} \bar{D}^2 \hat{\sigma} \bigg) \\ &= \frac{d-1}{d} \int d^d x \sqrt{\overline{g}} \hat{\sigma} \\ &\times \bigg((\bar{D}^2)^2 + \frac{d}{d-1} \bar{D}_{\mu} \bar{R}^{\mu\nu} \bar{D}_{\nu} \bigg) \hat{\sigma}. \quad (2.23) \end{split}$$

Furthermore, the spectra of the operators in Eq. (2.21) do not even contain zeros, since the potential zero modes coincide precisely with the aforementioned unphysical modes which have to be excluded. For instance, ξ^{μ} is a zero mode of $-\overline{D}^2 - \overline{\text{Ric}}$ if and only if ξ^{μ} is a Killing vector.

Along the lines outlined in the previous subsection, we now determine the Jacobians for the transformation of integration variables. We obtain

$$J_{3} = [\det'_{(1[T])}(-\bar{D}^{2} - \overline{\operatorname{Ric}})]^{-1/2}$$

$$J_{4} = \left[\det'_{(0)}\left((\bar{D}^{2})^{2} + \frac{d}{d-1}\bar{D}_{\mu}\bar{R}^{\mu\nu}\bar{D}_{\nu}\right)\right]^{-1/2}$$

$$J_{5} = \det'_{(0)}(-\bar{D}^{2}) \qquad (2.24)$$

for the transformations $\hat{\xi}_{\mu} \rightarrow \xi_{\mu}$, $\hat{\sigma} \rightarrow \sigma$ and $\hat{\eta} \rightarrow \bar{\eta}$, $\hat{\eta} \rightarrow \eta$, respectively. (The integration measures have been chosen such that no additional infinite constants occur in the Jacobians.) The square brackets appearing in the subscript (1,[T]) at J_3 indicate that the operator under consideration acts on spin-one fields which are transverse only for certain background metrics, because the property of transversality is not necessarily transmitted from $\hat{\xi}^{\mu}$ to ξ^{μ} . However, at least for Einstein spaces ξ^{μ} is transverse as well.

After carrying out this change of integration variables, J_1 , J_3 and J_4 are the only Jacobians appearing in the generating functional \mathbf{Z}_k since J_2 and J_5 cancel.

D. The effective average action

By adding an infrared (IR) cutoff $\Delta_k S$ to the classical action under the path integral (2.2) we obtain a scaledependent generating functional \mathbf{Z}_k . The term $\Delta_k S$ is chosen to depend on the fluctuation fields in such a way that their eigenmodes with respect to $-\overline{D}^2$ which correspond to large eigenvalues $p^2 \gg k^2$ are not influenced, whereas contributions from eigenmodes with small eigenvalues $p^2 \ll k^2$ are suppressed. In this sense \mathbf{Z}_k describes an effective theory at the scale $k < \hat{k}$. For technical simplicity we implement the suppression of the low-momentum modes by momentumdependent "mass" terms, i.e. by cutoffs which are quadratic in the fluctuation fields:

$$\Delta_k S[h, C, \overline{C}; \overline{g}] = \frac{1}{2} \langle h, \mathcal{R}_k^{\text{grav}} h \rangle + \langle \overline{C}, \mathcal{R}_k^{\text{gh}} C \rangle. \quad (2.25)$$

⁴In order to make sure that the operators are indeed invertible we also assume that their eigenvalues do not have zero as an accumulation point.

Here the operators $\mathcal{R}_{k}^{\text{grav}}$ and $\mathcal{R}_{k}^{\text{gh}}$ are constructed from the covariant derivative with respect to the background metric, \bar{D}_{μ} . Note that $\mathcal{R}_{k}^{\text{grav}}$ and $\mathcal{R}_{k}^{\text{gh}}$ must not depend on the quantum metric $\gamma_{\mu\nu}$ but only on the background metric $\bar{g}_{\mu\nu}$ since otherwise the cutoff cannot be quadratic. In order to provide the desired behavior these operators must vanish for $p^{2}/k^{2} \rightarrow \infty$ (in particular for $k \rightarrow 0$) and must behave as $\mathcal{R}_{k} \rightarrow \mathcal{Z}_{k}k^{2}$ for $p^{2}/k^{2} \rightarrow 0$. (The meaning of the constant \mathcal{Z}_{k} will be explained later.) As a consequence, all modes with $p^{2} \ll k^{2}$ acquire a mass $\propto k$.

At this stage of the discussion it is not necessary to specify the explicit structure of the cutoff operators. We only mention the following point. According to Appendix A, $\mathcal{R}_{k}^{\text{grav}}$ and $\mathcal{R}_{k}^{\text{gh}}$ can be chosen such that, at the level of the component fields,

$$\Delta_{k}S[h,C,\bar{C};\bar{g}] = \frac{1}{2} \sum_{\zeta_{1},\zeta_{2} \in I_{1}} \langle \zeta_{1},(\mathcal{R}_{k})_{\zeta_{1}\zeta_{2}}\zeta_{2} \rangle + \frac{1}{2} \sum_{\psi_{1},\psi_{2} \in I_{2}} \langle \psi_{1},(\mathcal{R}_{k})_{\psi_{1}\psi_{2}}\psi_{2} \rangle$$
(2.26)

with the index sets $I_1 = \{h^T, \xi, \sigma, \phi\}$, $I_2 = \{\overline{C}^T, C^T, \overline{\eta}, \eta\}$. In contrast to generic cutoffs which are defined in terms of the component fields from the outset, the structure (2.26) allows us to return to the formulation in terms of the fundamental fields, Eq. (2.25), in a straightforward way. (See Appendix A.) The set of operators $(\mathcal{R}_k)_{\zeta_1\zeta_2}$, $(\mathcal{R}_k)_{\psi_1\psi_2}$ introduced by this realization of the cutoff may be fixed later on. Hermiticity demands that they satisfy $(\mathcal{R}_k)_{\zeta_2\zeta_1} = (\mathcal{R}_k)_{\zeta_1\zeta_2}^{\dagger}$ and $(\mathcal{R}_k)_{\psi_2\psi_1} = -(\mathcal{R}_k)_{\psi_1\psi_2}^{\dagger}$. Furthermore, $(\mathcal{R}_k)_{\psi_1\psi_2} \equiv 0$ if both $\psi_1 \in \{\overline{C}^T, \overline{\eta}\}$ and $\psi_2 \in \{\overline{C}^T, \overline{\eta}\}$.

A similar decomposition is applied to S_{source} . The source terms are defined as

$$S_{\text{source}}[h, C, \bar{C}, J, K, \bar{K}; \bar{g}] = -\langle J, h \rangle - \langle \bar{K}, C \rangle - \langle K, \bar{C} \rangle$$
(2.27)

with external sources $J^{\mu\nu}$, K^{μ} and \bar{K}_{μ} for the fundamental fields $h_{\mu\nu}$, \bar{C}_{μ} and C^{μ} , respectively. Proceeding as described in Appendix A, an alternative form of S_{source} may be derived from Eq. (2.27) where each component field is coupled to a certain component of the fundamental sources. Then, in terms of these "component sources," S_{source} takes the form

$$S_{\text{source}}[h, C, \overline{C}, J, K, \overline{K}; \overline{g}] = -\sum_{\zeta \in I_1} \langle J_{\zeta}, \zeta \rangle - \sum_{\psi \in \{C^T, \eta\}} \langle \overline{K}_{\psi}, \psi \rangle$$
$$- \sum_{\psi \in \{\overline{C}^T, \overline{\eta}\}} \langle K_{\psi}, \psi \rangle.$$
(2.28)

As a consequence, the functional

$$\mathbf{Z}_{k}[J,K,K;\overline{g}]$$

$$=J_{1}J_{3}J_{4}\int \mathcal{D}h_{\mu\nu}^{T}\mathcal{D}\xi_{\mu}\mathcal{D}\sigma \mathcal{D}\phi \mathcal{D}C^{T\mu}\mathcal{D}\overline{C}_{\mu}^{T}\mathcal{D}\eta \mathcal{D}\overline{\eta}$$

$$\times \exp[-S[\overline{g}+h] - S_{gf}[h;\overline{g}] - S_{gh}[h,C,\overline{C};\overline{g}]$$

$$-\Delta_{k}S[h,C,\overline{C};\overline{g}] - S_{source}[h,C,\overline{C},J,K,\overline{K};\overline{g}]] \quad (2.29)$$

as well as the scale-dependent generating functional for the connected Green's functions,

$$W_k[J, K, \overline{K}; \overline{g}] = \ln \mathbf{Z}_k[J, K, \overline{K}; \overline{g}], \qquad (2.30)$$

may be viewed as functionals of either the fundamental or the component sources. Furthermore, we may derive *k*-dependent classical fields for both fundamental and component fields in terms of functional derivatives of W_k . In either case the *k*-dependent classical fields represent expectation values $\langle q \rangle$ of quantum fields *q*, in the sense that all degrees of freedom corresponding to momenta with $p^2 > k^2$ have been averaged out. The classical fundamental fields are given by

$$\bar{h}_{\mu\nu} \equiv \langle h_{\mu\nu} \rangle = \frac{1}{\sqrt{\bar{g}}} \frac{\delta W_k}{\delta J^{\mu\nu}}, \quad \bar{v}_\mu \equiv \langle \bar{C}_\mu \rangle = \frac{1}{\sqrt{\bar{g}}} \frac{\delta W_k}{\delta K^\mu},$$
$$v^\mu \equiv \langle C^\mu \rangle = \frac{1}{\sqrt{\bar{g}}} \frac{\delta W_k}{\delta \bar{K}_\mu},$$
(2.31)

and the classical component fields are obtained as

$$\varphi_i \equiv \langle \chi_i \rangle = \frac{1}{\sqrt{g}} \frac{\delta W_k}{\delta \mathcal{J}^i}.$$
(2.32)

(2.33)

Here we are making use of the shorthand notation $\chi \equiv (h^T, \xi, \sigma, \phi, \overline{C}^T, C^T, \overline{\eta}, \eta)$ for the quantum component fields, $\mathcal{J} \equiv (J_{h^T}, J_{\xi}, J_{\sigma}, J_{\phi}, K_{\overline{C}^T}, \overline{K}_{C^T}, \overline{K}_{\eta})$ for their sources and $\varphi \equiv (\overline{h}^T, \overline{\xi}, \overline{\sigma}, \overline{\phi}, \overline{v}^T, v^T, \overline{\varrho}, \varrho)$ for the classical component fields. We may reconstruct the classical fundamental fields from φ according to

$$\begin{split} \bar{h}_{\mu\nu} &= \bar{h}_{\mu\nu}^{T} + \bar{D}_{\mu} [-\bar{D}^{2} - \overline{\mathrm{Ric}}]^{-1/2} \bar{\xi}_{\nu} + \bar{D}_{\nu} [-\bar{D}^{2} - \overline{\mathrm{Ric}}]^{-1/2} \bar{\xi}_{\mu} \\ &+ \frac{1}{d} \bar{g}_{\mu\nu} \bar{\phi} + \bar{D}_{\mu} \bar{D}_{\nu} \Big[(\bar{D}^{2})^{2} + \frac{d-1}{d} \bar{D}_{\rho} \bar{R}^{\rho\lambda} \bar{D}_{\lambda} \Big]^{-1/2} \bar{\sigma} \\ &- \frac{1}{d} \bar{g}_{\mu\nu} \bar{D}^{2} \Big[(\bar{D}^{2})^{2} + \frac{d-1}{d} \bar{D}_{\rho} \bar{R}^{\rho\lambda} \bar{D}_{\lambda} \Big]^{-1/2} \bar{\sigma}, \\ \bar{v}_{\mu} &= \bar{v}_{\mu}^{T} + \bar{D}_{\mu} (-\bar{D}^{2})^{-1/2} \bar{\varrho}, \quad v^{\mu} = v^{T\mu} + \bar{D}^{\mu} (-\bar{D}^{2})^{-1/2} \varrho. \end{split}$$

Performing a Legendre transformation on W_k with respect to $J_{\mu\nu}$, K^{μ} and \bar{K}_{μ} leads to the following scale-dependent modification of the effective action:

$$\widetilde{\Gamma}_{k}[\bar{h}, v, \bar{v}; \bar{g}] = \langle J, \bar{h} \rangle + \langle \bar{K}, v \rangle + \langle K, \bar{v} \rangle - W_{k}[J, K, \bar{K}; \bar{g}].$$
(2.34)

Since Eqs. (2.27), (2.28) imply

$$\langle \mathcal{J}, \varphi \rangle = \langle J, \bar{h} \rangle + \langle \bar{K}, v \rangle + \langle K, \bar{v} \rangle$$
 (2.35)

it is clear that Eq. (2.34) also would result from Legendre transforming W_k with respect to the component sources. Denoting the corresponding Legendre transform by $\tilde{\Gamma}_k^{\text{comp}}$ we have $\tilde{\Gamma}_k^{\text{comp}}[\bar{h}^T, \bar{\xi}, \bar{\sigma}, \bar{\phi}, v^T, \bar{v}^T, \varrho, \bar{\varrho}; \bar{g}] \equiv \tilde{\Gamma}_k[\bar{h}, v, \bar{v}; \bar{g}]$ where the arguments of $\tilde{\Gamma}_k^{\text{comp}}$ and $\tilde{\Gamma}_k$ are related by Eq. (2.33).

The effective average action proper, Γ_k , is defined as the difference between $\tilde{\Gamma}_k$ and the cutoff action with the classical fields inserted [28,5]:

$$\Gamma_k[g,\overline{g},v,\overline{v}] \equiv \widetilde{\Gamma}_k[g-\overline{g},v,\overline{v};\overline{g}] - \Delta_k S[g-\overline{g},v,\overline{v};\overline{g}].$$
(2.36)

Here we expressed $\bar{h}_{\mu\nu}$ in terms of the classical counterpart $g_{\mu\nu}$ of the quantum metric $\gamma_{\mu\nu} \equiv \bar{g}_{\mu\nu} + h_{\mu\nu}$ which, by definition, is given by

$$g_{\mu\nu} \equiv \overline{g}_{\mu\nu} + \overline{h}_{\mu\nu}. \tag{2.37}$$

The main advantage of the background gauge is that it makes Γ_k a gauge invariant functional of its agruments [6]. It is invariant under general coordinate transformations of the form

$$\Gamma_k[\Phi] = \Gamma_k[\Phi + \mathcal{L}_u\Phi], \quad \Phi = (g_{\mu\nu}, \overline{g}_{\mu\nu}, v^{\mu}, \overline{v}_{\mu}) \quad (2.38)$$

where \mathcal{L}_u is the Lie derivative with respect to the generating vector field $u^{\mu}(x)$. Since general coordinate invariance ensures that no symmetry violating terms occur in the course of the evolution of Γ_k the class of consistent truncations is restricted to those which involve only invariant field combinations. This is important for practical applications of the evolution equation.

We are mainly interested in the exclusively $g_{\mu\nu}$ -dependent functional

$$\bar{\Gamma}_k[g] \equiv \Gamma_k[g,g,0,0]. \tag{2.39}$$

In the limit $k \to 0$ it coincides with the conventional effective action $\Gamma[g_{\mu\nu}]$, the generator of the 1PI graviton Green's functions [8]: $\Gamma[g] = \lim_{k\to 0} \overline{\Gamma}_k[g]$. However, in order to derive an exact evolution equation it is necessary to retain the dependence on the ghost fields and $\overline{g}_{\mu\nu}$.

From the definition of the effective average action it follows that Γ_k satisfies the integro-differential equation

$$= \int \mathcal{D}h_{\mu\nu}\mathcal{D}C^{\mu}\mathcal{D}\bar{C}_{\mu}\exp\left[-\tilde{S}[h,C,\bar{C};\bar{g}] + \int d^{d}x \\ \times \left\{(h_{\mu\nu} - g_{\mu\nu} + \bar{g}_{\mu\nu})\frac{\delta\Gamma_{k}}{\delta\bar{h}_{\mu\nu}} + (C^{\mu} - v^{\mu})\frac{\delta\Gamma_{k}}{\delta v^{\mu}} \\ + (\bar{C}_{\mu} - \bar{v}_{\mu})\frac{\delta\Gamma_{k}}{\delta\bar{v}_{\mu}}\right\}\right] \\ \times \exp\{-\Delta_{k}S[h - g + \bar{g}, C - v, \bar{C} - \bar{v};\bar{g}]\} \quad (2.40)$$

with

 $\exp\{-\Gamma_{i}[g,\overline{g},v,\overline{v}]\}$

$$\widetilde{S}[h,C,\overline{C};\overline{g}] \equiv S[\overline{g}+h] + S_{gf}[h;\overline{g}] + S_{gh}[h,C,\overline{C};\overline{g}].$$
(2.41)

Equation (2.40) may be derived by inserting the definition of Γ_k into Eq. (2.2) and replacing the sources according to

$$J^{\mu\nu} = \frac{1}{\sqrt{\overline{g}}} \frac{\delta \widetilde{\Gamma}_k}{\delta \overline{h}_{\mu\nu}}, \quad K^{\mu} = -\frac{1}{\sqrt{\overline{g}}} \frac{\delta \widetilde{\Gamma}_k}{\delta \overline{v}_{\mu}}, \quad \overline{K}_{\mu} = -\frac{1}{\sqrt{\overline{g}}} \frac{\delta \widetilde{\Gamma}_k}{\delta v^{\mu}}.$$
(2.42)

E. Derivation of the exact evolution equation

The exact renormalization group equation describes the change of the action functional Γ_k induced by a change in the scale k. It may be obtained as follows. Differentiating the functional integral (2.29) with respect to $t \equiv \ln k$ leads to

$$-\partial_{t}W_{k} = \frac{1}{2}\mathrm{Tr}' \left[\sum_{\zeta_{1},\zeta_{2} \in I_{1}} \langle \zeta_{1} \otimes \zeta_{2} \rangle \partial_{t}(\mathcal{R}_{k})_{\zeta_{1}\zeta_{2}} \right] \\ + \frac{1}{2}\mathrm{Tr}' \left[\sum_{\psi_{1},\psi_{2} \in I_{2}} \langle \psi_{1} \otimes \psi_{2} \rangle \partial_{t}(\mathcal{R}_{k})_{\psi_{1}\psi_{2}} \right].$$

$$(2.43)$$

Here we used Eq. (2.30) and adopted the matrix notation on the RHS of Eq. (2.43) which, in turn, can be expressed in terms of the Hessian

$$(\tilde{\Gamma}_{k}^{(2)})^{ij}(x,y) \equiv (-1)^{[j]} \frac{1}{\sqrt{\bar{g}(x)\bar{g}(y)}} \frac{\delta^{2}\tilde{\Gamma}_{k}}{\delta\varphi_{i}(x)\delta\varphi_{j}(y)}$$
(2.44)

with [j]=0 for commuting fields φ_j and [j]=1 for Grassmann fields φ_j . Since the connected two point function

$$(G_k)_{ij}(x,y) \equiv \langle \chi_i(x)\chi_j(y) \rangle - \varphi_i(x)\varphi_j(y)$$
$$= \frac{1}{\sqrt{\overline{g}(x)\overline{g}(y)}} \frac{\delta^2 W_k}{\delta \mathcal{J}^i(x)\delta \mathcal{J}^j(y)} \quad (2.45)$$

and $\widetilde{\Gamma}_{k}^{(2)}$ are inverse matrices in the sense that

$$\int d^d y \sqrt{\overline{g}(y)} (G_k)_{ij}(x,y) (\widetilde{\Gamma}_k^{(2)})^{jl}(y,z) = \delta_i^l \frac{\delta(x-z)}{\sqrt{\overline{g}(z)}}$$
(2.46)

we may replace the expectation values $\langle \chi_i(x)\chi_j(y)\rangle$ appearing in Eq. (2.43) with $(\tilde{\Gamma}_k^{(2)})_{ij}^{-1}(x,y) + \varphi_i(x)\varphi_j(y)$. Then performing a Legendre transformation according to Eq. (2.34) and subtracting the cutoff action $\Delta_k S[\bar{h}, v, \bar{v}; \bar{g}]$ yields the desired exact renormalization group equation:

$$\partial_{t}\Gamma_{k}[g,\overline{g},v,\overline{v}] = \frac{1}{2}\mathrm{Tr}' \bigg[\sum_{\zeta_{1},\zeta_{2}\in\overline{I}_{1}} \left(\Gamma_{k}^{(2)}[g,\overline{g},v,\overline{v}]\right) \\ + \mathcal{R}_{k} \Big)_{\zeta_{1}\zeta_{2}}^{-1} \partial_{t}(\mathcal{R}_{k})_{\zeta_{2}\zeta_{1}} \bigg] \\ + \frac{1}{2}\mathrm{Tr}' \bigg[\sum_{\psi_{1},\psi_{2}\in\overline{I}_{2}} \left(\Gamma_{k}^{(2)}[g,\overline{g},v,\overline{v}]\right) \\ + \mathcal{R}_{k} \Big)_{\psi_{1}\psi_{2}}^{-1} \partial_{t}(\mathcal{R}_{k})_{\psi_{2}\psi_{1}} \bigg].$$
(2.47)

Here we wrote $(\mathcal{R}_k)_{\zeta_1\zeta_2} \equiv (\mathcal{R}_k)_{\langle\zeta_1\rangle\langle\zeta_2\rangle}, \quad (\mathcal{R}_k)_{\psi_1\psi_2} \equiv (\mathcal{R}_k)_{\langle\psi_1\rangle\langle\psi_2\rangle}$ and introduced the index sets

$$\overline{I}_1 \equiv \{ \overline{h}^T, \overline{\xi}, \overline{\sigma}, \overline{\phi} \}, \quad \overline{I}_2 \equiv \{ \overline{v}^T, v^T, \overline{\varrho}, \varrho \}.$$
(2.48)

In a position space representation, the operators appearing on the RHS of the flow equation are given by matrix elements whose traces are evaluated according to

$$\int d^d x d^d y \sqrt{\overline{g}(x)} \sqrt{\overline{g}(y)} \left(\left(\Gamma_k^{(2)} + \mathcal{R}_k \right)_v^{-1} \tau_{\overline{v}}^{-1} \right)_{\mu x}^{\nu y} (\partial_t (\mathcal{R}_k)_{\overline{v}} \tau_v \tau)_{\nu y}^{\mu x},$$
(2.49)

for instance. The notation adopted for the matrix elements is similar to Eq. (2.44); for example,

$$\left(\left(\Gamma_{k}^{(2)}\right)_{\overline{v}^{T}v}\tau\right)^{\mu x}{}_{\nu y} = \frac{1}{\sqrt{\overline{g}(y)}} \frac{\delta}{\delta v^{T\nu}(y)} \frac{1}{\sqrt{\overline{g}(x)}} \frac{\delta\Gamma_{k}}{\delta \overline{v}_{\mu}^{T}(x)}.$$
(2.50)

By virtue of the properties of \mathcal{R}_k discussed above the traces appearing in the flow equation (2.47) are perfectly convergent for all values of $k \leq \hat{k}$.

Provided we impose the correct initial condition at the UV scale $k = \hat{k}$ we can, in principle, determine the functional integral (2.2) by integrating the flow equation from \hat{k} down to k and letting $k \rightarrow 0$, $\hat{k} \rightarrow \infty$ after appropriate renormalizations. The initial condition $\Gamma_{\hat{k}}$ can be obtained from the integro-differential equation (2.40). For sufficiently large

values of k, the cutoff term in Eq. (2.40) strongly suppresses fluctuations with $(h, C, \overline{C}) \neq (\overline{h}, v, \overline{v})$ so that the main contribution to the functional integral results from small fluctuations about $(h, C, \overline{C}) = (\overline{h}, v, \overline{v})$. This field configuration corresponds to the global minimum of the total action in the exponential of Eq. (2.40). Performing a saddle point expansion of the functional integral about this minimum leads to

$$\Gamma_{k}[g,\overline{g},v,\overline{v}] = \widetilde{S}[g-\overline{g},v,\overline{v};\overline{g}] - \frac{1}{2}\ln \operatorname{Det}'(\widetilde{S}^{(2)} + \mathcal{R}_{k})$$
(2.51)

where the second term contains one-loop effects. For $k = \hat{k} \rightarrow \infty$ they amount to an often unimportant shift in the bare parameters of \tilde{S} which can be ignored usually. For finite \hat{k} , additional contributions from the determinant occur which are suppressed by inverse powers of \hat{k} [25]. Therefore we obtain the initial value for $\hat{k} \rightarrow \infty$:

$$\Gamma_{\hat{k}}[g,\overline{g},v,\overline{v}] = S[g] + S_{gf}[g-\overline{g};\overline{g}] + S_{gh}[g-\overline{g},v,\overline{v};\overline{g}].$$
(2.52)

At the level of the functional $\overline{\Gamma}_k[g_{\mu\nu}]$ this initial condition boils down to

$$\overline{\Gamma}_{\hat{k}}[g] = S[g]. \tag{2.53}$$

So far we assumed the fundamental action to be positive definite. However, the Einstein-Hilbert action, for instance, does not have this property which is due to the appearance of a "wrong-sign" kinetic term associated with the conformal factor. In such cases it is nevertheless possible to formulate a well-defined evolution equation if the signs of the cutoff operators \mathcal{R}_k are properly adjusted [6]. We will return to this point in the next section.

F. A special case: Einstein backgrounds

Before continuing we summarize the simplifications that occur for Einstein backgrounds, for which $\bar{R}_{\mu\nu} = C\bar{g}_{\mu\nu}$ with *C* a constant. In this case the decomposition (2.9) of $h_{\mu\nu}$ is completely orthogonal. In fact, thanks to the Einstein condition, the $\hat{\xi}_{\mu}$ - $\hat{\sigma}$ mixing terms in the inner product (2.14) vanish so that $h_{\mu\nu}^T$, $h_{\mu\nu}^{LT}$, $h_{\mu\nu}^{LL}$ and $h_{\mu\nu}^{Tr}$ form an orthogonal set.

Furthermore, for Einstein spaces the Jacobians appearing in the path integral (2.29) cancel, at least up to an (infinite) constant which can be absorbed into the normalization of the integration measure. This can be seen as follows:

$$J_{1} = N\sqrt{\det'_{(1T,0)}M^{(\mu,\nu)}} = \left(\int \mathcal{D}\hat{\xi}_{\mu}\mathcal{D}\hat{\sigma}\exp\left[-\int d^{d}x\sqrt{g}\left\{-2\hat{\xi}_{\mu}(\bar{D}^{2}+C)\hat{\xi}^{\mu}+\hat{\sigma}\left(\frac{d-1}{d}(\bar{D}^{2})^{2}+C\bar{D}^{2}\right)\hat{\sigma}\right\}\right]\right)^{-1}$$
$$= N_{1}\sqrt{\det'_{(1T)}(-\bar{D}^{2}-C)}\sqrt{\det'_{(0)}(-\bar{D}^{2})}\sqrt{\det'_{(0)}\left(-\bar{D}^{2}-\frac{dC}{d-1}\right)} = N_{2}J_{3}^{-1}J_{4}^{-1}.$$
(2.54)

Here N_1 and N_2 are unimportant constants so that $J_1J_3J_4$ is indeed field independent.

Finally, for Einstein spaces the field redefinitions in the gravitational sector take the form

$$\xi^{\mu} = \sqrt{-\bar{D}^2 - C} \hat{\xi}^{\mu}, \quad \sigma = \sqrt{-\bar{D}^2} \sqrt{-\bar{D}^2 - \frac{dC}{d-1}} \hat{\sigma}.$$
(2.55)

As a consequence, we find that $\bar{D}_{\mu}\xi^{\mu} = \sqrt{-\bar{D}^2 - 2C}\bar{D}_{\mu}\xi^{\mu}$. Thus, transversality of $\hat{\xi}^{\mu}$ implies that ξ^{μ} is transverse as well.

III. TRUNCATIONS AND CUTOFFS

A. A general class of truncations

In practical applications of the exact evolution equation one encounters the problem of dealing with an infinite system of coupled differential equations since the evolution equation describes trajectories in an infinite dimensional space of action functionals. In general it is impossible to find an exact solution so that we are forced to rely on approximations. A powerful nonperturbative approximation scheme is the truncation of the parameter space, i.e. only a finite number of couplings is considered. In this manner the renormalization group flow of Γ_k is projected onto a finitedimensional subspace of action functionals. In practice one makes an ansatz for Γ_k that comprises only a few couplings and inserts it on both sides of Eq. (2.47), thereby obtaining a truncated evolution equation. By projecting the RHS of this equation onto the space of operators appearing on the LHS one arrives at a set of coupled differential equations for the couplings taken into account.

As discussed in Refs. [25,29], Ward identities provide an important tool for judging the admissability and quantitative reliability of a given truncation; approximate solutions of the flow equation are not necessarily consistent with the Ward identities, in contrast to the exact solution. Therefore, only those truncations which are indeed consistent with the Ward identities, at least up to a certain degree of accuracy, will yield reliable results. The Ward identities to be considered here are modified by additional terms coming from the cutoff which are not present in the ordinary identities. Since $\Delta_k S$ vanishes as $k \rightarrow 0$ the ordinary Ward identities are recovered in this limit.

In Ref. [6] the modified Ward identities were derived for the gravitational effective average action $\Gamma_k[g,\bar{g},v,\bar{v};\beta,\tau]$ where β and τ are auxiliary sources for the Becchi-Rouet-Stora (BRS) variations of the graviton and the ghosts which are needed in order to formulate the Ward identities. Setting $\beta = \tau = 0$ in the argument of this more general functional we obtain the action $\Gamma_k[g, \overline{g}, v, \overline{v}]$ discussed in the present paper. (It would be straightforward to include the β , τ -sources also in the new formulation of the flow equation, but we will not need them in the following.)

In [6] the Ward identities were used to test the consistency of truncations of the form

$$\Gamma_{k}[g,\overline{g},v,\overline{v}] = \overline{\Gamma}_{k}[g] + \hat{\Gamma}_{k}[g,\overline{g}] + S_{gf}[g-\overline{g};\overline{g}] + S_{gh}[g-\overline{g},v,\overline{v};\overline{g}]$$
(3.1)

with $\overline{\Gamma}_k[g]$ defined as in Eq. (2.39). The term $\widehat{\Gamma}_k[g,\overline{g}]$ encodes the quantum corrections to the gauge fixing term. This interpretation of $\hat{\Gamma}_k[g,\overline{g}]$ is obvious because for $\overline{g} \neq g$ the purely gravitational part of Eq. (3.1) implies $\Gamma_k[g,\overline{g},0,0]$ $-\Gamma_k[g,g,0,0] = \hat{\Gamma}_k[g,\overline{g}] + S_{gf}[g-\overline{g};\overline{g}].$ By definition, $\hat{\Gamma}_{k}[g,g]=0$. In the ansatz (3.1) the ghost dependence has been extracted in terms of the classical $S_{\rm gh}$, thereby neglecting the evolution of the ghost action. This guarantees that the initial condition (2.52) is satisfied automatically in the ghost sector. In the gravitational sector it requires $\overline{\Gamma}_{\hat{k}} = S$, $\hat{\Gamma}_{\hat{k}} = 0$. For truncations of the type (3.1) the Ward identities demand that $\overline{\Gamma}_k[g]$ is a gauge invariant functional of $g_{\mu\nu}$ and they yield a constraint equation for $\hat{\Gamma}_k[g,\overline{g}]$. To lowest order, this equation is solved by $\hat{\Gamma}_k = 0 \quad \forall k \leq \hat{k}$. In the Einstein-Hilbert truncation we go beyond this approximation and set $\hat{\Gamma}_k$ $\propto S_{\rm gf}$ with a constant of proportionality which vanishes at k $=\hat{k}$; it takes the running of the graviton's wave function normalization into account (see below).

Inserting the ansatz (3.1) into the exact evolution equation (2.47) leads to a truncated renormalization group equation which describes the evolution of Γ_k in the subspace of action functionals spanned by Eq. (3.1). The equation governing the evolution of the purely gravitational action

$$\Gamma_k[g,\overline{g}] \equiv \Gamma_k[g,\overline{g},0,0] = \overline{\Gamma}_k[g] + S_{\text{gf}}[g-\overline{g};\overline{g}] + \widehat{\Gamma}_k[g,\overline{g}]$$
(3.2)

reads

$$\partial_{t}\Gamma_{k}[g,\overline{g}] = \frac{1}{2} \mathrm{Tr}' \Biggl[\sum_{\zeta_{1},\zeta_{2} \in \overline{I}_{1}} (\Gamma_{k}^{(2)}[g,\overline{g}] + \mathcal{R}_{k})_{\zeta_{1}\zeta_{2}}^{-1} \partial_{t}(\mathcal{R}_{k})_{\zeta_{2}\zeta_{1}} \Biggr] + \frac{1}{2} \mathrm{Tr}' \Biggl[\sum_{\psi_{1},\psi_{2} \in \overline{I}_{2}} (S_{\mathrm{gh}}^{(2)}[g,\overline{g}] + \mathcal{R}_{k})_{\psi_{1}\psi_{2}}^{-1} \partial_{t}(\mathcal{R}_{k})_{\psi_{2}\psi_{1}} \Biggr].$$
(3.3)

Here $\Gamma_k^{(2)}$ and $S_{\rm gh}^{(2)}$ are the Hessians of $\Gamma_k[g,\overline{g}]$ and $S_{\rm gh}[\overline{h}, v, \overline{v}; \overline{g}]$ with respect to the gravitational and the ghost component fields, respectively. They are taken at fixed $\overline{g}_{\mu\nu}$.

B. Specification of the cutoff

In order to obtain a tractable evolution equation for a given truncation it is convenient to use a cutoff which is adapted to this truncation but still has the general suppression properties described in Sec. II D. It is desirable to start from a definition of $\Delta_k S$ that brings about the correct suppression of low-momentum modes for a class of truncations and of gravitational backgrounds which is as large as possible.

A convenient, adapted cutoff can be found by the following rule [6,21]. Given a truncation, we assume that for $\bar{g} = g$ the kinetic operators of all modes with a definite helicity are of the form $(\Gamma_k^{(2)})_{ij} = f_{ij}(-\bar{D}^2, k, ...)$ where $\{f_{ij}\}$ is a set of c-number functions and the indices i, j refer to the different types of fields. (The difficulty of bringing $\Gamma_k^{(2)}$ to this form is one of the main reasons for using the TT decomposition. At least for maximally symmetric spaces it allows us to eliminate all covariant derivatives which do not appear as a Laplacian $\bar{D}^2 \equiv \bar{g}^{\mu\nu} \bar{D}_{\mu} \bar{D}_{\nu}$.) Then we choose the cutoff in such a way that the structure

$$(\Gamma_k^{(2)} + \mathcal{R}_k)_{ij} = f_{ij}(-\bar{D}^2 + k^2 R^{(0)}(-\bar{D}^2/k^2), k, \dots)$$
(3.4)

is achieved. Here the function $R^{(0)}(y)$, $y = -\overline{D}^2/k^2$, describes the details of the mode suppression; it is required to satisfy the boundary conditions $R^{(0)}(0)=1$ and $\lim_{y\to\infty} R^{(0)}(y)=0$, but is arbitrary otherwise. By virtue of Eq. (3.4), the inverse propagator of a field mode with covariant momentum square $p^2 = -\overline{D}^2$ is given by $p^2 + k^2 R^{(0)}(p^2/k^2)$ which equals p^2 for $p^2 \gg k^2$ and $p^2 + k^2$ for $p^2 \ll k^2$. This means that the small- p^2 modes, and only those, have acquired a mass $\propto k$ which leads to the desired suppression.

In the next section we shall see in detail that for the truncations used in the present paper we can comply with the above rule by using the following cutoff operator:

$$\begin{split} (\mathcal{R}_{k})_{\bar{h}\bar{h}\bar{h}\bar{h}}^{\mu\nu\alpha\beta} &= \frac{1}{2} \mathcal{Z}_{k}^{\bar{h}\bar{h}\bar{h}} \kappa^{2} (\bar{g}^{\mu\alpha} \bar{g}^{\nu\beta} + \bar{g}^{\mu\beta} \bar{g}^{\nu\alpha}) k^{2} R^{(0)} (-\bar{D}^{2}/k^{2}), \\ (\mathcal{R}_{k})_{\bar{\xi}\bar{\xi}}^{\mu\nu} &= \mathcal{Z}_{k}^{\bar{\xi}\bar{\xi}} \kappa^{2} \bar{g}^{\mu\nu} k^{2} R^{(0)} (-\bar{D}^{2}/k^{2}), \\ (\mathcal{R}_{k})_{\bar{\sigma}\bar{\sigma}} &= \mathcal{Z}_{k}^{\bar{\sigma}\bar{\sigma}} \kappa^{2} k^{2} R^{(0)} (-\bar{D}^{2}/k^{2}), \\ (\mathcal{R}_{k})_{\bar{\phi}\bar{\sigma}} &= (\mathcal{R}_{k})_{\bar{\sigma}\bar{\phi}}^{\dagger} \\ &= \mathcal{Z}_{k}^{\bar{\phi}\bar{\sigma}} \kappa^{2} \bigg[\sqrt{\bigg(\bar{P}_{k} + \frac{d}{d-1} \bar{D}_{\mu} \bar{R}^{\mu\nu} \bar{D}_{\nu} (-\bar{D}^{2})^{-1} \bigg) \bar{P}_{k}} \\ &- \sqrt{(\bar{D}^{2})^{2} + \frac{d}{d-1} \bar{D}_{\mu} \bar{R}^{\mu\nu} \bar{D}_{\nu}} \bigg], \end{split}$$

$$(\mathcal{R}_{k})_{\bar{\phi}\bar{\phi}} = \mathcal{Z}_{k}^{\bar{\phi}\bar{\phi}}\kappa^{2}k^{2}R^{(0)}(-\bar{D}^{2}/k^{2}),$$

$$(\mathcal{R}_{k})_{\bar{v}}^{\mu\nu}{}_{v}T = -(\mathcal{R}_{k})_{v}^{\mu\nu}{}_{\bar{v}}T = \mathcal{Z}_{k}^{\bar{v}}{}^{T}\bar{g}^{\mu\nu}k^{2}R^{(0)}(-\bar{D}^{2}/k^{2}),$$

$$(\mathcal{R}_{k})_{\bar{\varrho}\bar{\varrho}} = -(\mathcal{R}_{k})_{\varrho\bar{\varrho}} = \mathcal{Z}_{k}^{\bar{\varrho}\varrho}k^{2}R^{(0)}(-\bar{D}^{2}/k^{2}).$$
(3.5)

Here \overline{P}_k is defined as

$$\bar{P}_k \equiv -\bar{D}^2 + k^2 R^{(0)} (-\bar{D}^2/k^2).$$
(3.6)

The remaining cutoff operators not listed in Eq. (3.5) are set to zero. The \mathbb{Z}_k 's are constants which, again by using Eq. (3.4), will be fixed in terms of the generalized couplings appearing in the ansatz for Γ_k . The cutoff (3.5) is inspired by the \mathcal{R}_k used in [21] for S^d .

If Eq. (3.4) allows us to choose $\mathcal{Z}_k^{\zeta\zeta} > 0$ for all ζ $\in \{\bar{h}^T, \bar{\xi}, \bar{\sigma}, \bar{\phi}\}$ and $\mathcal{Z}_k^{\bar{\phi}\bar{\sigma}} = 0$ one obtains a positive definite $\Delta_k S$ in the gravitational sector. In this case $\exp(-\Delta_k S)$ is a damped exponential which indeed suppresses the contributions from the low-momentum modes. In the following sections we shall focus on the Einstein-Hilbert truncation for Γ_k which suffers from the conformal factor problem: its kinetic term for $\overline{\phi}$ is negative definite. As a consequence, Eq. (3.4) forces us to work with a $\mathcal{Z}_{k}^{\overline{\phi}\overline{\phi}} < 0$. Hence, in the ϕ sector, $\Delta_k S$ is negative definite and, at least at a naive level, $\exp(-\Delta_k S)$ seems to enhance rather than suppress the lowmomentum modes. As we discussed in detail in Ref. [6] we nevertheless believe that the rule (3.4), i.e. allowing for $\mathcal{Z}_{\mu}^{\phi\phi} < 0$, is correct also in this case. We emphasize that the RHS of the flow equation, contrary to the Euclidean path integral, is perfectly well-defined even if S and Γ_k are not positive definite.

At this point it should be mentioned that the situation with respect to the positivity of the action improves considerably by including higher-derivative terms in *S* and the truncated Γ_k since these actions are bounded below, provided we choose the correct sign in front of these higher-derivative terms. Furthermore, their quadratic forms are positive definite at least for sufficiently large momenta, and so is the cutoff. For a study of the evolution equation for R^2 -gravity we refer to [24].

As compared to the original paper [6], the cutoff (3.5) has a rather different structure which is due to the fact that it is formulated in terms of the component fields arising form the TT decomposition. Contrary to the original one of Ref. [6], the new cutoff (3.5) is defined for *all* values of α . This is one of the main advantages of the new approach.

Note that in Refs. [22,19] where the TT decomposition was used on S^d the actual construction of the effective average action and its RG equation was omitted and has been replaced by an *ad hoc* modification of the standard one-loop determinants. No $\Delta_k S$ has been specified at the component field level. Hence the scale dependent action constructed in this manner has no reason to respect the general properties of an effective average action [2]. Despite the use of the component fields in [22,19] their cutoff seems to be more similar to the original one in [6] than to the new one of the present paper. In fact, it represents an α -dependent generalization of the cutoff in [6], in the sense that the latter is recovered from the one of Refs. [22,19] by setting $\alpha = 1$.

From now on we will refer to the cutoff used in the original paper [6] and in [17,22,19] as the cutoff of *type A*. However, one has to keep in mind that the existence of a corresponding $\Delta_k S$ is guaranteed only for $\alpha = 1$, i.e. the case considered in [6,17]. Furthermore the cutoff (3.5) of the present paper will be referred to as the cutoff *type B*; it is defined for all values of α .

Each cutoff type contains the shape function $R^{(0)}$. A particularly suitable choice is the exponential shape function

$$R^{(0)}(y) = y [\exp(y) - 1]^{-1}.$$
 (3.7)

In order to check the scheme independence of universal quantities we employ a one-parameter "deformation" of Eq. (3.7), the *class of exponential shape functions*,

$$R^{(0)}(y;s) = sy[\exp(sy) - 1]^{-1}, \qquad (3.8)$$

with *s* parametrizing the profile of $R^{(0)}$ [19]. Another admissible choice is the following *class of shape functions with compact support*:

$$R^{(0)}(y;b) = \begin{cases} 1, & y \le b, \\ \exp[(y-1.5)^{-1}\exp[(b-y)^{-1}]], & b < y < 1.5, \\ 0, & y \ge 1.5. \end{cases}$$
(3.9)

Here $b \in [0, 1.5)$ parametrizes the profile of $R^{(0)}$.

For our analysis of the flow equation in Sec. V we shall use both cutoff types with both classes of shape functions.

IV. THE EINSTEIN-HILBERT TRUNCATION

A. The ansatz

In this section we use a simple truncation to derive the renormalization group flow of the Newton and the cosmological "constant" by means of the truncated flow equation (3.3). In our example we assume that, at the UV scale $\hat{k} \rightarrow \infty$, gravity is described by the classical Einstein-Hilbert action in *d* dimensions,

$$\bar{\Gamma}_{k}[g] = S[g] = \frac{1}{16\pi\bar{G}} \int d^{d}x \sqrt{g} \{-R(g) + 2\bar{\lambda}\}.$$
 (4.1)

For the investigation of the evolution of $\Gamma_k[g,\overline{g}]$ towards smaller scales $k < \hat{k}$ we consider a truncated action functional of the following form:

$$\Gamma_{k}[g,\overline{g}] = 2\kappa^{2}Z_{Nk}\int d^{d}x\sqrt{g}\{-R(g)+2\overline{\lambda}_{k}\} + \kappa^{2}\frac{Z_{Nk}}{\alpha}\int d^{d}x\sqrt{g}\,\overline{g}\,\overline{\mu}^{\mu\nu}(\mathcal{F}^{\alpha\beta}_{\mu}g_{\alpha\beta})(\mathcal{F}^{\rho\sigma}_{\nu}g_{\rho\sigma}).$$

$$(4.2)$$

$$\bar{G} \to G_k \equiv Z_{Nk}^{-1} \bar{G}, \quad \bar{\lambda} \to \bar{\lambda}_k, \quad \alpha \to Z_{Nk}^{-1} \alpha$$
(4.3)

so that its form agrees with that of the gravitational sector of the ansatz (3.1) with

$$\hat{\Gamma}_{k}[g,\overline{g}] = \kappa^{2} \frac{Z_{Nk} - 1}{\alpha} \int d^{d}x \sqrt{\overline{g}} \,\overline{g}^{\mu\nu} (\mathcal{F}^{\alpha\beta}_{\mu} g_{\alpha\beta}) (\mathcal{F}^{\rho\sigma}_{\nu} g_{\rho\sigma}).$$
(4.4)

Generally speaking also the gauge fixing parameter α should be treated as a running quantity, $\alpha \rightarrow \alpha_k$. Fortunately there is a simple shortcut which avoids an explicit computation of the corresponding β -function. In fact, there are general arguments showing that α should have a (IR attractive) fixed point at $\alpha_*=0$. This means that the initial condition $\alpha_k=0$ leads to $\alpha_k=0$ for all $k \le \hat{k}$. Thus, even using the truncation with a constant α , we can take the correct "flow" of the gauge fixing term into account simply by setting $\alpha = 0$.

In Yang-Mills theory the existence of the fixed point α_* =0 has been demonstrated for a truncation containing a covariant gauge fixing [29], while for the axial gauge a nonperturbative proof is available [30]. The following general argument⁵ suggests that this fixed point should exist in any gauge theory, including gravity [30]. In the ordinary functional integral, the limit $\alpha \rightarrow 0$ corresponds to a sharp implementation of the gauge fixing condition, i.e. $exp(-S_{gf})$ becomes proportional to $\delta[F_{\mu}]$. The domain of the $\int \mathcal{D}h_{\mu\nu}$ -integration consists of those $h_{\mu\nu}$'s which satisfy the gauge condition exactly, $F_{\mu} = 0$. Adding the IR cutoff at k amounts to suppressing some of the $h_{\mu\nu}$ -modes while retaining the others. But since all of them satisfy $F_{\mu} = 0$, it is clear that a variation of k cannot change the domain of the $h_{\mu\nu}$ integration. The delta-functional $\delta[F_{\mu}]$ continues to be present for any value of k if it was there originally. Hence α vanishes for arbitrary k.

B. Projecting the flow equation

The *k*-dependent couplings in Eq. (4.3) satisfy the initial conditions $\bar{\lambda}_{\hat{k}} = \bar{\lambda}$ and $Z_{N\hat{k}} = 1$ which implies $G_{\hat{k}} = \bar{G}$. Here the UV scale \hat{k} is taken to be large but finite. The evolution of Z_{Nk} and $\bar{\lambda}_k$ towards smaller scales may now be determined as follows. As a first step the ansatz (4.2) is inserted into both sides of the truncated flow equation (3.3). Then we may set $g_{\mu\nu} = \bar{g}_{\mu\nu}$. As a consequence, the gauge fixing term drops out from the LHS which then reads

$$\partial_t \Gamma_k[\bar{g},\bar{g}] = 2\kappa^2 \int d^d x \sqrt{\bar{g}}[-\bar{R}(\bar{g})\partial_t Z_{Nk} + 2\partial_t (Z_{Nk}\bar{\lambda}_k)].$$
(4.5)

Performing a derivative expansion on the RHS we may extract those contributions which are proportional to the operators spanning the LHS, i.e. $\int d^d x \sqrt{g}$ and $\int d^d x \sqrt{gR}$. Then, comparing the coefficients of these operators yields a system

⁵We are grateful to J. M. Pawlowski for a discussion of this point.

of coupled differential equations for Z_{Nk} and $\overline{\lambda}_k$. It describes the projection of the renormalization group flow onto the two-dimensional subspace of the space of all action functionals which is spanned by $\int d^d x \sqrt{g}$ and $\int d^d x \sqrt{gR}$.

It is important to note that during this calculation we may insert any metric $\overline{g}_{\mu\nu}$ that is general enough to allow for a unique identification of the operators $\int d^d x \sqrt{g}$ and $\int d^d x \sqrt{g} R$. In practice it proves particularly convenient to exploit this freedom by choosing the gravitational background to be maximally symmetric. Such spaces form a special class of Einstein spaces and are characterized by

$$\bar{R}_{\mu\nu\rho\sigma} = \frac{\bar{R}}{d(d-1)} (\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}), \quad \bar{R}_{\mu\nu} = \frac{\bar{R}}{d}\bar{g}_{\mu\nu}$$
(4.6)

with the curvature scalar \overline{R} considered a constant number rather than a functional of the metric. In the following we restrict our considerations to maximally symmetric spaces with positive curvature scalar $\overline{R} > 0$, i.e. *d*-spheres S^d . For *d* fixed, S^d is parametrized by the radius *r* of the spheres, which is related to the curvature scalar and the volume in the usual way,

$$\bar{R} = \frac{d(d-1)}{r^2}, \quad \int d^d x \sqrt{\bar{g}} = \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma(d)} (4\pi r^2)^{d/2}. \quad (4.7)$$

Before continuing with the evaluation of the RHS of the flow equation we have to comment on the properties of fields defined on spherical backgrounds. According to Appendix D, we may expand the quantum and classical component fields in terms of spherical harmonics $T^{lm}_{\mu\nu}$, T^{lm}_{μ} , T^{lm} , which form complete sets of orthogonal eigenfunctions with respect to the corresponding covariant Laplacians. The expansions of $h^T_{\mu\nu}$, ϕ , C^{μ} , \bar{C}_{μ} and their classical counterparts can be inferred directly from Eq. (D3). The remaining component fields are expanded according to

$$\xi_{\mu}(x) = \sum_{l=2}^{\infty} \sum_{m=1}^{D_{l}(d,1)} \xi_{lm} T_{\mu}^{lm}(x),$$

$$\sigma(x) = \sum_{l=2}^{\infty} \sum_{m=1}^{D_{l}(d,0)} \sigma_{lm} T^{lm}(x),$$

$$\eta(x) = \sum_{l=1}^{\infty} \sum_{m=1}^{D_{l}(d,0)} \eta_{lm} T^{lm}(x),$$

$$\bar{\eta}(x) = \sum_{l=1}^{\infty} \sum_{m=1}^{D_{l}(d,0)} \bar{\eta}_{lm} T^{lm}(x).$$
(4.8)

Similar expansions hold for the associated classical fields.

Note that in Eq. (4.8) the summations do *not* start at l = 1 for vectors and at l=0 for scalars as in Eq. (D3), but at l=2 for ξ_{μ} and σ , and at l=1 for the scalar ghost fields. The

modes omitted here are the KV's $(T_{\mu}^{l=1,m})$, the solutions of the scalar equation (2.12) which are in one-to-one correspondence with the PCKV's $(T^{l=1,m})$, and the constants $(T^{l=0,m=1})$. As discussed in Sec. II B, the fundamental fields obtain no contribution from these modes. Therefore they have to be excluded from Eq. (4.8).

This exclusion is also of importance for the momentum dependent field redefinitions (2.21) because they would not be well-defined otherwise, as can be seen e.g. from

$$\hat{\sigma}(x) = \sum_{l=2}^{\infty} \sum_{m=1}^{D_l(d,0)} \frac{\sigma_{lm}}{\sqrt{\Lambda_l(d,0) \left(\Lambda_l(d,0) - \frac{\bar{R}}{d-1}\right)}} T^{lm}(x).$$
(4.9)

Equation (4.9) follows from inverting Eq. (2.21) and then inserting Eq. (4.8). The eigenvalues corresponding to the modes excluded, i.e. $\Lambda_0(d,0) = 0$ and $\Lambda_1(d,0) = \overline{R}/(d-1)$ (see Table I in Appendix D), would lead to a vanishing denominator in Eq. (4.9). Similar arguments hold for the other fields in Eq. (2.21).

We may now split the quantum field ϕ into a part ϕ_1 spanned by the same set of eigenfunctions as σ , and a part ϕ_0 containing the contributions from the remaining modes:

$$\phi(x) = \phi_0(x) + \phi_1(x),$$

$$\phi_0(x) = \sum_{l=0}^{1} \sum_{m=1}^{D_l(d,0)} \phi_{lm} T^{lm}(x),$$

$$\phi_1(x) = \sum_{l=2}^{\infty} \sum_{m=1}^{D_l(d,0)} \phi_{lm} T^{lm}(x).$$

(4.10)

The orthogonality of the spherical harmonics implies $\langle \phi_1, \phi_0 \rangle = \langle \sigma, \phi_0 \rangle = 0$ so that $\langle \phi, \phi \rangle = \langle \phi_0, \phi_0 \rangle + \langle \phi_1, \phi_1 \rangle$ and in particular $\langle \sigma, \phi \rangle = \langle \sigma, \phi_1 \rangle$. As a consequence, decomposing ϕ according to Eq. (4.10) ensures that any nonzero term bilinear in the scalar fields is of such a form that the scalars involved are spanned by the same set of eigenfunctions. The same is true for the corresponding classical fields $\overline{\phi}_0$ and $\overline{\phi}_1$.

C. Evaluation of the functional trace

Let us now return to the evaluation of the flow equation. On the RHS we need the operator $\Gamma_k^{(2)}[g,\overline{g}]$. For our purposes it is sufficient to determine this operator at $g_{\mu\nu} = \overline{g}_{\mu\nu}$. It may be derived by expanding $\Gamma_k[g,\overline{g}]$ according to

$$\Gamma_{k}[\bar{g}+\bar{h},\bar{g}] = \Gamma_{k}[\bar{g},\bar{g}] + \mathcal{O}(\bar{h}) + \Gamma_{k}^{\text{quad}}[\bar{h};\bar{g}] + \mathcal{O}(\bar{h}^{3})$$

$$(4.11)$$

and retaining only the part quadratic in $\bar{h}_{\mu\nu}$, i.e. $\Gamma_k^{\text{quad}}[\bar{h};\bar{g}]$. For our truncation it takes the form

$$\Gamma_{k}^{\text{quad}}[\bar{h};\bar{g}] = \kappa^{2} Z_{Nk} \int d^{d}x \sqrt{\bar{g}} \,\bar{h}_{\mu\nu} \Biggl\{ -\Biggl[\frac{1}{2} \,\delta_{\rho}^{\mu} \delta_{\sigma}^{\nu} + \frac{1-2\alpha}{4\alpha} \,\bar{g}^{\mu\nu} \bar{g}_{\rho\sigma}\Biggr] \bar{D}^{2} + \frac{1}{4} [2 \,\delta_{\rho}^{\mu} \delta_{\sigma}^{\nu} - \bar{g}^{\mu\nu} \bar{g}_{\rho\sigma}] (\bar{R} - 2\bar{\lambda}_{k}) + \bar{g}^{\mu\nu} \bar{R}_{\rho\sigma} - \delta_{\sigma}^{\mu} \bar{R}^{\nu}{}_{\rho} - \bar{R}^{\nu}{}_{\rho}{}^{\mu}{}_{\sigma} + \frac{1-\alpha}{\alpha} [\bar{g}^{\mu\nu} \bar{D}_{\rho} \bar{D}_{\sigma} - \delta_{\sigma}^{\mu} \bar{D}^{\nu} \bar{D}_{\rho}] \Biggr\} \bar{h}^{\rho\sigma}$$

$$(4.12)$$

where $\bar{g}_{\mu\nu}$ is fixed but still arbitrary. In order to (partially) diagonalize this quadratic form we insert the family of spherical background metrics into Eq. (4.12) and decompose $\bar{h}_{\mu\nu}$ according to Eq. (2.33). Then we use the classical analog of Eq. (4.10) to decompose $\bar{\phi}$ as well. This leads to

$$\Gamma_{k}^{\text{quad}}[\bar{h};\bar{g}] = \kappa^{2} Z_{Nk} \int d^{d}x \sqrt{\bar{g}} \frac{1}{2} \Biggl\{ \bar{h}_{\mu\nu}^{T} [-\bar{D}^{2} + A_{T}(d)\bar{R} - 2\bar{\lambda}_{k}] \bar{h}^{T\mu\nu} + \frac{2}{\alpha} \bar{\xi}_{\mu} [-\bar{D}^{2} + A_{V}(d,\alpha)\bar{R} - 2\alpha\bar{\lambda}_{k}] \bar{\xi}^{\mu} + C_{S2}(d,\alpha) \Biggl(\bar{\sigma} [-\bar{D}^{2} + A_{S2}(d,\alpha)\bar{R} + B_{S2}(d,\alpha)\bar{\lambda}_{k}] \bar{\sigma} + 2C_{S3}(d,\alpha)\bar{\phi}_{1}\sqrt{-\bar{D}^{2}} \sqrt{-\bar{D}^{2} - \frac{\bar{R}}{d-1}} \bar{\sigma} + C_{S1}(d,\alpha) \sum_{\bar{\phi} \in \{\bar{\phi}_{0}, \bar{\phi}_{1}\}} \bar{\phi} [-\bar{D}^{2} + A_{S1}(d,\alpha)\bar{R} + B_{S1}(d,\alpha)\bar{\lambda}_{k}] \bar{\phi} \Biggr) \Biggr\}.$$

$$(4.13)$$

Here the *A*'s, *B*'s, and *C*'s are functions of the dimensionality *d* and the gauge parameter α . The explicit expressions for these coefficients can be found in Appendix F.

Note that this partial diagonalization simplifies further calculations considerably, and this is the main reason for using the decomposition (2.33) and specifying a concrete background. In contrast to the case $\alpha = 1$ considered in [6], a complete diagonalization cannot be achieved by merely splitting off the trace part from $\bar{h}_{\mu\nu}$ since Eq. (4.12) contains additional terms proportional to $1 - \alpha$ which introduce mixings between the traceless part and ϕ . To be more precise, it is the term $\int d^d x \sqrt{\bar{g}} \bar{h}_{\mu\nu} [\bar{g}^{\mu\nu} \bar{D}_{\rho} \bar{D}_{\sigma} - \delta^{\mu}_{\sigma} \bar{D}^{\nu} \bar{D}_{\rho}] \bar{h}^{\rho\sigma}$ that gives rise to such cross terms.

In terms of the component fields these cross terms boil down to a purely scalar $\bar{\sigma}$ - $\bar{\phi}$ mixing term that vanishes for the spherical harmonics $T^{l=0,m=1}$ and $T^{l=1,m}$. Since these modes contribute to $\overline{\phi}$ (but not to $\overline{\sigma}$) we cannot directly invert the associated matrix differential operator $((\Gamma_k^{(2)})_{ij})_{i,j \in \{\bar{h}^T, \bar{\xi}, \bar{\sigma}, \bar{\phi}\}}$. As a way out, we split $\bar{\phi}$ according to Eq. (4.10) into $\overline{\phi}_0$ and $\overline{\phi}_1$. This has the effect that only mixings between the scalars $\bar{\sigma}$ and $\bar{\phi}_1$ survive, which have the same set of eigenfunctions T^{lm} starting at l=2. Hence the resulting matrix differential operator $((\Gamma_k^{(2)})_{ij})_{i,j \in \{\bar{h}^T, \bar{\xi}, \bar{\phi}_0, \bar{\sigma}, \bar{\phi}_1\}}$ is invertible, but since this additional split of ϕ affects the matrix structure of this operator it leads to a slightly modified flow equation. In fact, on the RHS of Eq. (3.3) the summation in the gravitational sector now runs over the set of fields $\{\bar{h}^T, \bar{\xi}, \bar{\phi}_0, \bar{\sigma}, \bar{\phi}_1\}$, with $(\mathcal{R}_k)_{\bar{\phi}_0\bar{\phi}_0} \equiv (\mathcal{R}_k)_{\bar{\phi}_1\bar{\phi}_1} \equiv (\mathcal{R}_k)_{\bar{\phi}\bar{\phi}} \text{ and } (\mathcal{R}_k)_{\bar{\sigma}\bar{\phi}_1} \equiv (\mathcal{R}_k)_{\bar{\sigma}\bar{\phi}}.$

In the context of the Einstein-Hilbert truncation it is only the α dependence that introduces mixings of $\overline{\phi}$ and the traceless part of $\overline{h}_{\mu\nu}$ and therefore necessitates the decompositions (2.9), (4.10). In general the inclusion of higher derivative terms like $\int d^d x \sqrt{g} R^2$ and of matter fields leads to similar mixings.

In order to determine the contributions from the ghosts appearing on the RHS of Eq. (3.3) we set $g_{\mu\nu} = \bar{g}_{\mu\nu}$ in $S_{\rm gh}$ and assume that $\bar{g}_{\mu\nu}$ corresponds to a spherical background. Then we decompose the ghost fields according to Eq. (2.33) which leads to

$$S_{\rm gh}[0,v,\bar{v};g] = \sqrt{2} \int d^d x \sqrt{g} \left\{ \bar{v}^T_{\mu} \left[-D^2 - \frac{R}{d} \right] v^{T\mu} + \bar{\varrho} \left[-D^2 - 2\frac{R}{d} \right] \varrho \right\}.$$
(4.14)

From now on the bars are omitted from the metric, the curvature and the operators D^2 and P_k . Note that the decomposition of the ghosts is not really necessary, but it allows for a comparison with the results obtained in [21].

At this point we can continue with the adaption of the cutoff to the operators $\Gamma_k^{(2)}$ and $S_{\rm gh}^{(2)}$ of Eqs. (4.13), (4.14). According to the rule (3.4) the Z_k 's have to be chosen as

$$\mathcal{Z}_{k}^{\bar{h}T_{h}T} = Z_{Nk}, \qquad \mathcal{Z}_{k}^{\bar{\xi}\bar{\xi}} = \frac{2}{\alpha} Z_{Nk},$$

$$\mathcal{Z}_{k}^{\bar{\phi}_{1}\bar{\sigma}} = C_{S2}(d,\alpha) C_{S3}(d,\alpha) Z_{Nk},$$

$$\mathcal{Z}_{k}^{\bar{\sigma}\bar{\sigma}} = C_{S2}(d,\alpha) Z_{Nk},$$

$$\mathcal{Z}_{k}^{\bar{\phi}_{0}\bar{\phi}_{0}} = \mathcal{Z}_{k}^{\bar{\phi}_{1}\bar{\phi}_{1}} = C_{S2}(d,\alpha) C_{S1}(d,\alpha) Z_{Nk},$$

$$\mathcal{Z}_{k}^{\bar{v}^{T}v^{T}} = \mathcal{Z}_{k}^{\bar{\varrho}\varrho} = \sqrt{2}. \qquad (4.15)$$

Thus, for $g = \overline{g}$ the nonvanishing entries of the matrix differential operators $\Gamma_k^{(2)} + \mathcal{R}_k$ and $S_{\rm gh}^{(2)} + \mathcal{R}_k$ take the form

$$(\Gamma_{k}^{(2)}[g,g] + \mathcal{R}_{k})_{\bar{h}^{T}\bar{h}^{T}} = Z_{Nk}\kappa^{2}[P_{k} + A_{T}(d)R - 2\bar{\lambda}_{k}],$$
$$(\Gamma_{k}^{(2)}[g,g] + \mathcal{R}_{k})_{\bar{\xi}\bar{\xi}} = Z_{Nk}\kappa^{2}\frac{2}{\alpha}[P_{k} + A_{V}(d,\alpha)R - 2\alpha\bar{\lambda}_{k}].$$

$$(\Gamma_k^{(2)}[g,g] + \mathcal{R}_k)_{\bar{\sigma}\bar{\sigma}} = Z_{Nk} \kappa^2 C_{S2}(d,\alpha) [P_k + A_{S2}(d,\alpha)R + B_{S2}(d,\alpha)\bar{\lambda}_k],$$

$$(\Gamma_k^{(2)}[g,g] + \mathcal{R}_k)_{\bar{\phi}_1\bar{\sigma}} = (\Gamma_k^{(2)}[g,g] + \mathcal{R}_k)_{\bar{\sigma}\bar{\phi}_1}$$

$$= Z_{Nk} \kappa^2 C_{S2}(d,\alpha) C_{S3}(d,\alpha) \sqrt{P_k}$$
$$\times \sqrt{P_k - \frac{R}{d-1}},$$

$$\begin{aligned} (\Gamma_{k}^{(2)}[g,g] + \mathcal{R}_{k})_{\bar{\phi}_{0}\bar{\phi}_{0}} &= (\Gamma_{k}^{(2)}[g,g] + \mathcal{R}_{k})_{\bar{\phi}_{1}}\bar{\phi}_{1} \\ &= Z_{Nk}\kappa^{2}C_{S2}(d,\alpha)C_{S1}(d,\alpha)[P_{k} \\ &+ A_{S1}(d,\alpha)R + B_{S1}(d,\alpha)\bar{\lambda}_{k}], \\ (S_{gh}^{(2)}[g,g] + \mathcal{R}_{k})_{\bar{v}}\tau_{v}\tau &= -(S_{gh}^{(2)}[g,g] + \mathcal{R}_{k})_{v}\tau_{\bar{v}}\tau \\ &= \sqrt{2} \bigg[P_{k} - \frac{R}{d}\bigg], \\ (S_{gh}^{(2)}[g,g] + \mathcal{R}_{k})_{\bar{\varrho}\varrho} &= -(S_{gh}^{(2)}[g,g] + \mathcal{R}_{k})_{\varrho\bar{\varrho}} \\ &= \sqrt{2} \bigg[P_{k} - 2\frac{R}{d}\bigg]. \end{aligned}$$
(4.16)

Here we set $(S_{gh}^{(2)}[0,v,\bar{v};g])_{\psi_1\psi_2} \equiv (S_{gh}^{(2)}[g,g])_{\psi_1\psi_2}$ for $\psi_1, \psi_2 \in \overline{I}_2$.

Now we are in a position to write down the RHS of the flow equation with $g = \overline{g}$. We shall denote it $S_k(R)$ in the following. In $S_k(R)$ we need the inverse operators $(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}$ and $(S_{\text{gh}}^{(2)} + \mathcal{R}_k)^{-1}$. The inversion is carried out in Appendix B 1. Inserting the inverse operators into $S_k(R)$ leads to

$$S_{k}(R) = \operatorname{Tr}_{(2ST^{2})}[(P_{k} + A_{T}(d)R - 2\bar{\lambda}_{k})^{-1}\mathcal{N}] + \operatorname{Tr}'_{(1T)}[(P_{k} + A_{V}(d,\alpha)R - 2\alpha\bar{\lambda}_{k})^{-1}\mathcal{N}] + \operatorname{Tr}''_{(0)}\left[(P_{k} + A_{S3}(d)R - 2\bar{\lambda}_{k})^{-1}(P_{k} + A_{S4}(d,\alpha)R - 2\alpha\bar{\lambda}_{k})^{-1}\left\{(F_{S1}(d,\alpha)P_{k} + A_{S5}(d,\alpha)R - 2(\alpha+1)\bar{\lambda}_{k})\mathcal{N} + F_{S2}(d,\alpha)\sqrt{P_{k}}\sqrt{P_{k} - \frac{R}{d-1}} \frac{1}{2Z_{Nk}}\partial_{t}\left[Z_{Nk}\left(\sqrt{P_{k}}\sqrt{P_{k} - \frac{R}{d-1}} - \sqrt{-D^{2}}\sqrt{-D^{2} - \frac{R}{d-1}}\right)\right]\right\}\right] - 2\operatorname{Tr}_{(1T)}\left[\left(P_{k} - \frac{R}{d}\right)^{-1}\mathcal{N}_{0}\right] - 2\operatorname{Tr}'_{(0)}\left[\left(P_{k} - 2\frac{R}{d}\right)^{-1}\mathcal{N}_{0}\right] + \frac{1}{2Z_{Nk}}\sum_{l=0}^{1}\left[D_{l}(d,0)\frac{\partial_{l}[Z_{Nk}k^{2}R^{(0)}(\Lambda_{l}(d,0)/k^{2})]}{\Lambda_{l}(d,0) + k^{2}R^{(0)}(\Lambda_{l}(d,0)/k^{2}) + A_{S1}(d,\alpha)R + B_{S1}(d,\alpha)\bar{\lambda}_{k}}\right].$$
(4.17)

Here we set

$$\mathcal{N} = (2Z_{Nk})^{-1} \partial_t [Z_{Nk} k^2 R^{(0)} (-D^2/k^2)]$$

= $\left[1 - \frac{1}{2} \eta_N(k) \right] k^2 R^{(0)} (-D^2/k^2) + D^2 R^{(0)'} (-D^2/k^2),$
$$\mathcal{N}_0 = 2^{-1} \partial_t [k^2 R^{(0)} (-D^2/k^2)] = k^2 R^{(0)} (-D^2/k^2) + D^2 R^{(0)'} (-D^2/k^2)$$
(4.18)

where

$$\eta_N(k) \equiv -\partial_t \ln Z_{Nk} \tag{4.19}$$

is the anomalous dimension of the operator $\int d^d x \sqrt{gR}$ and the prime at $R^{(0)}$ denotes the derivative with respect to the argument. Furthermore, the new A's and F's introduced above are again functions of d and α , tabulated in Appendix F.

In Eq. (4.17) we refined our notation concerning the primes at the traces. From now on one prime indicates the subtraction of the contribution from the lowest eigenvalue, while two primes indicate that the modes corresponding to the lowest two eigenvalues have to be excluded.

The next step is to extract the contributions proportional to $\int d^d x \sqrt{g}$ and $\int d^d x \sqrt{g}R$ by expanding $S_k(R)$ with respect to R or r, respectively. Since $\int d^d x \sqrt{g} \propto r^d$, $\int d^d x \sqrt{g} R \propto r^{d-2}$, only terms of order r^d and r^{d-2} are needed. This leads to

$$S_{k}(R) = \operatorname{Tr}_{(2ST^{2})}[(P_{k}-2\bar{\lambda}_{k})^{-1}\mathcal{N}] + \operatorname{Tr}'_{(1T)}[(P_{k}-2\alpha\bar{\lambda}_{k})^{-1}\mathcal{N}] + \operatorname{Tr}'_{(0)}[(P_{k}-2\bar{\lambda}_{k})^{-1}\mathcal{N}] + \operatorname{Tr}'_{(0)}[(P_{k}-2\alpha\bar{\lambda}_{k})^{-1}\mathcal{N}] \\ - 2\operatorname{Tr}_{(1T)}[P_{k}^{-1}\mathcal{N}_{0}] - 2\operatorname{Tr}'_{(0)}[P_{k}^{-1}\mathcal{N}_{0}] - R \left\{ A_{T}(d)\operatorname{Tr}_{(2ST^{2})}[(P_{k}-2\bar{\lambda}_{k})^{-2}\mathcal{N}] + A_{V}(d,\alpha)\operatorname{Tr}'_{(1T)}[(P_{k}-2\alpha\bar{\lambda}_{k})^{-2}\mathcal{N}] \right. \\ \left. + A_{S3}(d)\operatorname{Tr}''_{(0)}[(P_{k}-2\bar{\lambda}_{k})^{-2}\mathcal{N}] + A_{S4}(d,\alpha)\operatorname{Tr}''_{(0)}[(P_{k}-2\alpha\bar{\lambda}_{k})^{-2}\mathcal{N}] + \frac{2}{d}\operatorname{Tr}_{(1T)}[P_{k}^{-2}\mathcal{N}_{0}] + \frac{4}{d}\operatorname{Tr}'_{(0)}[P_{k}^{-2}\mathcal{N}_{0}] \\ \left. - \frac{\delta_{d,2}}{4\pi} \int d^{2}x\sqrt{g} \frac{\partial_{t}(Z_{Nk}k^{2})}{Z_{Nk}(k^{2}-2\bar{\lambda}_{k})} \right\} + \mathcal{O}(r^{

$$(4.20)$$$$

Here $\mathcal{O}(r^{\leq d-2})$ means that terms $\propto r^n$ with powers $n \leq d-2$ are neglected.

The term in Eq. (4.20) proportional to $\delta_{d,2}$ arises from the last term in Eq. (4.17). Contrary to the other terms of Eq. (4.17), its expansion does not contain *d*-dependent powers of *r*, but is of the form $\sum_{m=0}^{\infty} b_{2m}r^{-2m}$ with $\{b_{2m}\}$ a set of *r*-independent coefficients. As for comparing powers of *r*, this has the following consequence. Since, for all $m \ge 0$ and $d \ge 0$, -2m = d - 2 is satisfied only if m = 0 and d = 2, and since -2m = d cannot be satisfied at all, this term contributes to the evolution equation only in the two-dimensional case. Using Eq. (4.7) the piece contributing, i.e. $b_{2m=0}r^0$, may be expressed in terms of the operator $\int d^d x \sqrt{gR}$ which yields the last term in Eq. (4.20).

The traces appearing in Eq. (4.20) are evaluated in Appendix B 2 using heat kernel techniques. Then combining the result with the LHS of the flow equation, Eq. (4.5), and comparing the coefficients of the invariants $\int d^d x \sqrt{g}$ and $\int d^d x \sqrt{g} R$ leads to the desired system of coupled differential equations for Z_{NK} and $\bar{\lambda}_k$. We obtain

$$\partial_{t}(Z_{Nk}\bar{\lambda}_{k}) = (4\kappa^{2})^{-1}(4\pi)^{-d/2}k^{d} \left\{ \frac{1}{2}d(d-1)\Phi_{d/2}^{1}(-2\bar{\lambda}_{k}/k^{2}) + d\Phi_{d/2}^{1}(-2\alpha\bar{\lambda}_{k}/k^{2}) - \frac{1}{2}\eta_{N}(k) \left[\frac{1}{2}d(d-1)\tilde{\Phi}_{d/2}^{1}(-2\bar{\lambda}_{k}/k^{2}) + d\Phi_{d/2}^{1}(-2\bar{\lambda}_{k}/k^{2}) + d\Phi_{d/2}^{1}(-2\bar{\lambda}_{k}/k^{2}) \right] \right\}$$

$$(4.21)$$

$$\partial_{t} Z_{Nk} = -(2\kappa^{2})^{-1} (4\pi)^{-d/2} k^{d-2} \Biggl\{ c_{1}(d) \Phi_{d/2-1}^{1}(-2\bar{\lambda}_{k}/k^{2}) + c_{2}(d) \Phi_{d/2-1}^{1}(-2\alpha\bar{\lambda}_{k}/k^{2}) + c_{3}(d) \Phi_{d/2}^{2}(-2\bar{\lambda}_{k}/k^{2}) + c_{4}(d,\alpha) \Phi_{d/2}^{2}(-2\alpha\bar{\lambda}_{k}/k^{2}) - \frac{1}{2} \eta_{N}(k) [c_{1}(d)\tilde{\Phi}_{d/2-1}^{1}(-2\bar{\lambda}_{k}/k^{2}) + c_{2}(d)\tilde{\Phi}_{d/2-1}^{1}(-2\alpha\bar{\lambda}_{k}/k^{2}) + c_{3}(d) \\ \times \tilde{\Phi}_{d/2}^{2}(-2\bar{\lambda}_{k}/k^{2}) + c_{4}(d,\alpha)\tilde{\Phi}_{d/2}^{2}(-2\alpha\bar{\lambda}_{k}/k^{2})] - 2c_{2}(d)\Phi_{d/2-1}^{1}(0) + c_{5}(d)\Phi_{d/2}^{2}(0) + 3\delta_{d,2}\Biggl(1 - \frac{1}{2}\eta_{N}(k)\Biggr) \\ \times \Biggl[\frac{1}{1 - 2\bar{\lambda}_{k}/k^{2}} - \frac{1}{1 - 2\alpha\bar{\lambda}_{k}/k^{2}}\Biggr]\Biggr\}.$$

$$(4.22)$$

Here Φ_n^p , $\tilde{\Phi}_n^p$ are cutoff-dependent "threshold" functions defined as

$$\Phi_n^p(w) \equiv \begin{cases} \frac{1}{\Gamma(n)} \int_0^\infty dy \, y^{n-1} \frac{R^{(0)}(y) - y R^{(0)'}(y)}{(y+R^{(0)}(y)+w)^p}, & n > 0, \\ (1+w)^{-p}, & n = 0, \end{cases}$$

$$\widetilde{\Phi}_{n}^{p}(w) \equiv \begin{cases} \frac{1}{\Gamma(n)} \int_{0}^{\infty} dy \, y^{n-1} \frac{R^{(0)}(y)}{(y+R^{(0)}(y)+w)^{p}}, & n > 0, \\ (1+w)^{-p}, & n = 0. \end{cases}$$
(4.23)

The coefficients c_i are given by

$$c_{1}(d) \equiv \frac{d^{3} - 2d^{2} - 11d - 12}{12(d - 1)}, \quad c_{2}(d) \equiv \frac{d^{2} - 6}{6d},$$

$$c_{3}(d) \equiv -\frac{d^{3} - 4d^{2} + 7d - 8}{2(d - 1)},$$

$$c_{4}(d, \alpha) \equiv -\frac{\alpha d(d - 2) - d - 1}{d}, \quad c_{5}(d) \equiv -\frac{2(d + 1)}{d}.$$
(4.24)

In Eq. (4.22) the terms proportional to $\delta_{d,2}$ arise not only from the last term of Eq. (4.20), but also by evaluating the "primed" traces, i.e., by subtracting the contributions coming from unphysical modes, see Appendix B 2 for details. All these contributions are obtained by expanding various functions f(R) with respect to R and retaining only the term of zeroth order, f(0). As we argued above, these are the only pieces of f which may contribute to the evolution in the truncated parameter space. Furthermore, the heat kernel expansions of the traces corresponding to differentially constrained fields introduce additional contributions $\propto \delta_{d,2}$ into Eq. (4.22).

In Appendix C we concentrate on the 4-dimensional case and compare our result for $S_k(R)$ and for the corresponding RG flow of Z_{Nk} and $\overline{\lambda}_k$ with the one of Ref. [21] where a cutoff of type B is used, too.

D. The system of flow equations for g_k and λ_k

Now we introduce the dimensionless, renormalized Newton constant

$$g_k \equiv k^{d-2} G_k \equiv k^{d-2} Z_{Nk}^{-1} \bar{G}$$
 (4.25)

and the dimensionless, renormalized cosmological constant

$$\lambda_k \equiv k^{-2} \bar{\lambda}_k \tag{4.26}$$

where G_k denotes the corresponding dimensionful, renormalized Newton constant at scale k. Inserting Eq. (4.26) into $\partial_t(Z_{Nk}\overline{\lambda}_k)$ leads to the relation

$$\partial_t \lambda_k = -(2 - \eta_N(k))\lambda_k + 32\pi g_k \kappa^2 k^{-d} \partial_t (Z_{Nk} \bar{\lambda}_k).$$
(4.27)

Then, by using Eq. (4.21), we obtain the following differential equation for the dimensionless cosmological constant:

$$\partial_t \lambda_k = \boldsymbol{\beta}_{\lambda}(\lambda_k, g_k; \alpha, d)$$

= $A_1(\lambda_k, g_k; \alpha, d) + \eta_N(k) A_2(\lambda_k, g_k; \alpha, d)$
(4.28)

The $\boldsymbol{\beta}$ -function $\boldsymbol{\beta}_{\lambda}$ contains the quantities A_1 and A_2 which are defined as

$$A_{1}(\lambda_{k},g_{k};\alpha,d) \equiv -2\lambda_{k} + (4\pi)^{1-d/2}g_{k}\{d(d-1) \\ \times \Phi_{d/2}^{1}(-2\lambda_{k}) + 2d\Phi_{d/2}^{1}(-2\alpha\lambda_{k}) \\ -4d\Phi_{d/2}^{1}(0)\}, \\ A_{2}(\lambda_{k},g_{k};\alpha,d) \equiv \lambda_{k} - (4\pi)^{1-d/2}g_{k}\left\{\frac{1}{2}d(d-1) \\ \times \tilde{\Phi}_{d/2}^{1}(-2\lambda_{k}) + d\tilde{\Phi}_{d/2}^{1}(-2\alpha\lambda_{k})\right\}.$$

$$(4.29)$$

The corresponding β -function for g_k may be determined as follows. Taking the scale derivative of Eq. (4.25) leads to

$$\partial_t g_k = \boldsymbol{\beta}_g(\lambda_k, g_k; \alpha, d) \equiv [d - 2 + \eta_N(k)]g_k. \quad (4.30)$$

For the anomalous dimension $\eta_N(k)$ we obtain from Eq. (4.22)

$$\eta_N(k) = g_k B_1(\lambda_k; \alpha, d) + \eta_N(k) g_k B_2(\lambda_k; \alpha, d)$$
(4.31)

with B_1 , B_2 functions of λ_k , d and α given by

$$B_{1}(\lambda_{k};\alpha,d) \equiv 4(4\pi)^{1-d/2} \Biggl\{ c_{1}(d) \Phi_{d/2-1}^{1}(-2\lambda_{k}) + c_{2}(d) \Phi_{d/2-1}^{1}(-2\alpha\lambda_{k}) + c_{3}(d) \Phi_{d/2}^{2}(-2\lambda_{k}) + c_{4}(d,\alpha) \Phi_{d/2}^{2}(-2\alpha\lambda_{k}) - 2c_{2}(d) \Phi_{d/2-1}^{1}(0) + c_{5}(d) \Phi_{d/2}^{2}(0) + 3\delta_{d,2} \\ \times \Biggl[\frac{1}{1-2\lambda_{k}} - \frac{1}{1-2\alpha\lambda_{k}} \Biggr] \Biggr\},$$

$$B_{2}(\lambda_{k};\alpha,d) \equiv -2(4\pi)^{1-d/2} \Biggl\{ c_{1}(d) \tilde{\Phi}_{d/2-1}^{1}(-2\lambda_{k}) + c_{2}(d) \tilde{\Phi}_{d/2-1}^{1}(-2\alpha\lambda_{k}) + c_{3}(d) \tilde{\Phi}_{d/2}^{2}(-2\lambda_{k}) + c_{4}(d,\alpha) \tilde{\Phi}_{d/2}^{2}(-2\alpha\lambda_{k}) + 3\delta_{d,2} \\ \times \Biggl[\frac{1}{1-2\lambda_{k}} - \frac{1}{1-2\alpha\lambda_{k}} \Biggr] \Biggr\}.$$

$$(4.32)$$

Equation (4.31) may now be solved for the anomalous dimension in terms of λ_k , g_k , α and d:

$$\eta_N(k) = \frac{g_k B_1(\lambda_k; \alpha, d)}{1 - g_k B_2(\lambda_k; \alpha, d)}.$$
(4.33)

The system of coupled flow equations (4.28), (4.30) is the main result of this section.

E. Comparing the cutoffs A and B

Let us compare the flow equations (4.28), (4.30) of the present paper with those obtained in Ref. [6]. Reference [6] covers the case $\alpha = 1$ only, and the cutoff used there has a

different structure than the present one. In Ref. [6], the $\Delta_k S$ for the cutoff A is formulated at the level of the *complete* field $h_{\mu\nu}$, i.e., symbolically, $\Delta_k S \propto \int h_{\mu\nu} \mathcal{R}_k h^{\mu\nu}$, while the cutoff B of the present paper is based upon a similar action for the component fields: $\Delta_k S \propto \int h_{\mu\nu}^T \mathcal{R}_k h^{T\mu\nu} + \int \xi_\mu \mathcal{R}_k \xi^\mu + \cdots$.

For $\alpha = 1$ the new β -function for λ_k , β_{λ} , agrees perfectly with the result in [6], whereas the coefficients B_1 , B_2 in the β -function for g_k , β_g , do not coincide with the corresponding results derived there. However, with both cutoffs these coefficients are of the form

$$B_{1}(\lambda_{k}; \alpha = 1, d) = \frac{1}{3} (4\pi)^{1-d/2} k^{d-2} \{ e_{1}(d) \Phi_{d/2-1}^{1}(-2\lambda_{k}) + e_{2}(d) \Phi_{d/2}^{2}(-2\lambda_{k}) + e_{3}(d) \Phi_{d/2-1}^{1}(0) + e_{4}(d) \Phi_{d/2}^{2}(0) \},$$

$$B_{2}(\lambda_{k}; \alpha = 1, d) = -\frac{1}{6} (4\pi)^{1-d/2} k^{d-2} \{ e_{1}(d) \}$$

$$\times \tilde{\Phi}^{1}_{d/2-1}(-2\lambda_{k}) + e_{2}(d)\tilde{\Phi}^{2}_{d/2}(-2\lambda_{k})\}.$$
(4.34)

In the present paper (cutoff B) the coefficients e_i are obtained as

$$e_{1}(d) = \frac{d^{4} - 13d^{2} - 24d + 12}{d(d-1)},$$

$$e_{2}(d) = -6\frac{d^{4} - 2d^{3} - d^{2} - 4d + 2}{d(d-1)},$$

$$e_{3}(d) = -4\frac{d^{2} - 6}{d}, \quad e_{4}(d) = -24\frac{d+1}{d} \quad (4.35)$$

while in [6] (cutoff A) they are given by

$$e_1(d) = d(d+1), \quad e_2(d) = -6d(d-1),$$

 $e_3(d) = -4d, \quad e_4(d) = -24.$ (4.36)

Upon subtracting the coefficients in Eq. (4.35) from those in Eq. (4.36) we obtain $\Delta e_1 = -\Delta e_2 = 12(d^2 + 2d - 1)/(d(d - 1))$ and $\Delta e_3 = -\Delta e_4 = -24/d$. Quite remarkably, the sum of the deviations Δe_i vanishes not only in total but also separately for the gravitational contributions, involving e_1 and e_2 , and the contributions from the ghosts, which contain e_3 and e_4 . Obviously, this amounts to a shift from the gravitational (p=1,n=d/2-1) sector to the (p=2,n=d/2) sector as well as to a shift between the corresponding ghost sectors. The simplicity of this result is somewhat mysterious.

V. THE FIXED POINTS

A. Fixed points, critical exponents, and nonperturbative renormalizability

Because of its complexity it is impossible to solve the system of flow equations for g_k and λ_k , Eqs. (4.28) and (4.30), exactly. Even a numerical solution would be a formidable task. However, it is possible to gain important information about the general structure of the RG flow by looking at its fixed point structure.

Given a set of β -functions corresponding to an arbitrary set of dimensionless essential couplings $g_i(k)$, it is often possible to predict the scale dependence of the couplings for very small and/or very large scales k by investigating their fixed points. The fixed points are those points in the space spanned by the g_i where all β -functions vanish. (The essential couplings are those combinations of the couplings appearing in the action functional that are invariant under point transformations of the fields.) Fixed points are characterized by their stability properties. A given eigendirection of the linearized flow is said to be UV or IR attractive (or stable) if, for $k \rightarrow \infty$ or $k \rightarrow 0$, respectively, the trajectories are attracted towards the fixed point along this direction. The UV critical hypersurface in the space of all couplings is defined to consist of all trajectories that run into the fixed point for $k \rightarrow \infty$.

In quantum field theory, fixed points play an important role in the modern approach to renormalization theory [3]. At a UV fixed point the infinite cutoff limit can be taken in a controlled way. As for gravity, Weinberg [16] argued that a theory described by a trajectory lying on a finite-dimensional UV critical hypersurface of some fixed point is presumably free from unphysical singularities. It is predictive since it depends only on a *finite* number of free (essential) parameters. In Weinberg's words, such a theory is asymptotically safe. Asymptotic safety has to be regarded as a generalized, nonperturbative version of renormalizability. It covers the class of perturbatively renormalizable theories, whose infinite cutoff limit is taken at the Gaussian fixed point $g_{*i}=0$, as well as those perturbatively nonrenormalizable theories which are described by a RG trajectory on a finitedimensional UV critical hypersurface of a non-Gaussian fixed point $g_{*i} \neq 0$ and are nonperturbatively renormalizable therefore [16].

Let us now consider the system of differential equations

$$k \,\partial_k \mathbf{g}_i(k) = \boldsymbol{\beta}_i(\mathbf{g}) \tag{5.1}$$

for a set of dimensionless essential couplings $g(k) \equiv \{g_1(k), \ldots, g_n(k)\}$. We assume that g_* is a fixed point of Eq. (5.1), i.e. $\beta_i(g_*) = 0$ for all $i = 1, \ldots, n$. We linearize the RG flow about g_* which leads to

$$k \,\partial_k \mathbf{g}_i(k) = \sum_{j=1}^n B_{ij}(\mathbf{g}_j(k) - \mathbf{g}_{*j})$$
(5.2)

where $B_{ij} \equiv \partial_j \boldsymbol{\beta}_i(\mathbf{g}_*)$ are the entries of the stability matrix $\mathbf{B} = (B_{ij})$. Diagonalizing **B** according to $S^{-1}\mathbf{B}S = -\text{diag}(\theta_1, \ldots, \theta_n), S = (V^1, \ldots, V^n)$, where V^I is the right eigenvector of **B** with eigenvalue $-\theta_I$ we have

$$\sum_{j=1}^{n} B_{ij} V_{j}^{I} = -\theta_{I} V_{i}^{I}, \quad I = 1, \dots, n.$$
 (5.3)

The general solution to Eq. (5.2) may be written as

$$\mathbf{g}_{i}(k) = \mathbf{g}_{*i} + \sum_{I=1}^{n} C_{I} V_{i}^{I} \left(\frac{k_{0}}{k} \right)^{\theta_{I}}.$$
 (5.4)

Here

$$C_{I} \equiv \sum_{j=1}^{n} (S^{-1})_{Ij} g_{j}(k_{0})$$
(5.5)

are arbitrary real parameters and k_0 is a reference scale.

Obviously the fixed point g_* is UV attractive (i.e. attractive for $k \rightarrow \infty$) only if all C_I corresponding to negative $\theta_I < 0$ are set to zero. Therefore the dimensionality of the UV critical hypersurface equals the number of positive $\theta_I > 0$. Conversely, setting to zero all C_I corresponding to positive θ_I , g_* becomes an IR attractive fixed point (approached in the limit $k \rightarrow 0$) with an IR critical hypersurface whose dimensionality equals the number of negative θ_I .

In a slight abuse of language we shall refer to the θ_I 's as the *critical exponents*.

Strictly speaking, the solution (5.4) and its above interpretation is valid only in such cases where all eigenvalues $-\theta_I$ are real, which is not guaranteed since the matrix **B** is not symmetric in general. If complex eigenvalues occur one has to consider complex C_I 's and to take the real part of Eq. (5.4), see below. Then the real parts of the critical exponents determine which directions in coupling constant space are attractive or repulsive.

At this point it is necessary to discuss the impact a change of the cutoff scheme has on the scaling behavior. Since the path integral for Γ_k depends on the cutoff scheme, i.e. on the $\Delta_k S$ chosen, it is clear that the couplings and their fixed point values are scheme dependent. Hence a variation of the cutoff scheme, i.e. of \mathcal{R}_k , induces a change in the corresponding B-matrix. So one might naively expect that also its eigenvalues, the critical exponents, are scheme dependent. In fact, this is not the case. According to the general theory of critical phenomena and a recent reanalysis in the framework of the exact RG equations [18] any variation of the cutoff scheme can be generated by a specific coordinate transformation in the space of couplings with the cutoff held fixed. Such transformations leave the eigenvalues of the B-matrix invariant, so that the critical behavior near the corresponding fixed point is universal. The positions of fixed points are scheme dependent but their (non)existence and the qualitative structure of the RG flow are universal features. Therefore a truncation can be considered reliable only if it predicts the same fixed point structure for all admissible choices of \mathcal{R}_k .

In the context of the Einstein-Hilbert truncation the space of couplings is parametrized by $g_1 = \lambda$ and $g_2 = g$. The β -functions occurring in the two flow equations PHYSICAL REVIEW D 65 025013

are given in Eqs. (4.28) and (4.30), respectively. As we shall see in subsection B, they have a trivial zero at $\lambda_* = g_* = 0$, referred to as the *Gaussian fixed point*. The analysis of subsection C reveals that there exists also a *non-Gaussian fixed point* at $\lambda_* \neq 0$, $g_* \neq 0$. In subsection C we study its cutoff dependence and the cutoff dependence of the associated critical exponents employing the above β -functions of type B as well as those of Refs. [22,19] based on the cutoff type A, with the families of shape functions (3.8) or (3.9) inserted.

B. The Gaussian fixed point

In this subsection we discuss the features of the Gaussian fixed point $(\lambda_*, g_*) = (0,0)$. In order to investigate the RG flow in its vicinity we expand the β -functions in powers of λ and g according to Eq. (H5) of Appendix H and read off the **B**-matrix. It takes the form

$$\mathbf{B} = \begin{pmatrix} -2 & \nu_d d \\ 0 & d-2 \end{pmatrix}. \tag{5.7}$$

Here ν_d is a *d*-dependent parameter defined as

$$\nu_d \equiv (d-3)(4\pi)^{1-d/2} \Phi^1_{d/2}(0).$$
(5.8)

Diagonalizing the matrix (5.7) yields the (obviously universal) critical exponents $\theta_1 = 2$ and $\theta_2 = 2 - d$ which are associated with the eigenvectors $V^1 = (1,0)^T$ and $V^2 = (\nu_d, 1)^T$. Hence, for the linearized system obtained from Eq. (H5) the solution (5.4) assumes the following form:

$$\lambda_{k} = (\lambda_{k_{0}} - \nu_{d}g_{k_{0}}) \left(\frac{k_{0}}{k}\right)^{2} + \nu_{d}g_{k_{0}} \left(\frac{k}{k_{0}}\right)^{d-2},$$

$$g_{k} = g_{k_{0}} \left(\frac{k}{k_{0}}\right)^{d-2}.$$
(5.9)

Since the expanded β -function β_g of Eq. (H5) is λ_k -independent up to terms of third order in the couplings we can easily calculate also the next-to-leading approximation for g_k near the fixed point. In terms of the dimensionful quantity G_k this improved solution reads

$$G_k = G_{k_0} [1 - \omega_d G_{k_0} (k_0^{d-2} - k^{d-2})]^{-1}$$
(5.10)

with

$$\omega_{d} \equiv -\frac{1}{d-2} B_{1}(0;\alpha,d)$$

$$= -2(4\pi)^{1-d/2} \left\{ \frac{(d+2)(d^{3}-6d^{2}+3d-6)}{6d(d-1)(d-2)} \Phi^{1}_{d/2-1}(0) - \left(\frac{d^{4}-4d^{3}+9d^{2}-8d-2}{d(d-1)(d-2)} + 2\alpha\right) \Phi^{2}_{d/2}(0) \right\}$$
(5.11)

a *d*- and α -dependent parameter. For $k \ll |\omega_d G_{k_0}|^{-1/(d-2)}$ and with the reference scale $k_0 = 0$ (which is admissible only for trajectories lying on the IR critical hypersurface of the fixed point) Eq. (5.10) yields

$$\partial_t \lambda_k = \boldsymbol{\beta}_{\lambda}(\lambda_k, g_k), \quad \partial_t g_k = \boldsymbol{\beta}_g(\lambda_k, g_k)$$
 (5.6)

$$G_k = G_0 [1 - \omega_d G_0 k^{d-2} + \mathcal{O}(G_0^2 k^{2(d-2)})].$$
 (5.12)

For the dimensionful cosmological constant we obtain, from Eq. (5.9),

$$\bar{\lambda}_{k} = \bar{\lambda}_{k_{0}} + \nu_{d} G_{k_{0}}(k^{d} - k_{0}^{d}).$$
(5.13)

Apart from the different expression of ω_d due to the new cutoff, the solutions (5.12) and (5.13) coincide in d=4 dimensions with those derived in Ref. [6] by using a similar approximation scheme, see Eqs. (5.18) and (5.25) of this reference.

Let us now analyze the scaling behavior near $(\lambda_*, g_*) = (0,0)$. Since $\theta_1 = 2 > 0$ the V^1 -eigendirection, which coincides with the λ direction, is IR repulsive (and thus UV attractive). For d < 2, θ_2 is positive which implies that the Gaussian fixed point is UV attractive for any direction in the 2-dimensional parameter space.

For d>2 we have $\theta_2 < 0$ so that the V^2 eigendirection is IR attractive (and UV repulsive). Hence, in this case both the UV and the IR critical hypersurface of the Gaussian fixed point are one-dimensional, i.e. they consist of a single trajectory. For the IR critical trajectory that hits the fixed point in the limit $k \rightarrow 0$ we have

$$\bar{\lambda}_k = \nu_d G_k k^d \Leftrightarrow \lambda_k = \nu_d g_k \tag{5.14}$$

for sufficiently small values of k [31], with G_k given by Eq. (5.12). Since $\Phi_{d/2}^1(0)$ depends on the shape function $R^{(0)}$, ν_d is not a universal quantity. Therefore the slope of the distinguished trajectory (5.14) is not fixed in a universal manner. This is in accordance with the general expectation that the eigenvalues of **B** should be universal, but not its eigenvectors.

For $d \neq 2$ the parameters $\Phi^1_{d/2-1}(0)$ and $\Phi^2_{d/2}(0)$ appearing in ω_d are scheme dependent as well. Furthermore, ω_d is a function of the gauge parameter α . Hence ω_d is a nonuniversal quantity, too. In the most interesting case of d=4 dimensions it takes the form

$$\omega_4 = \frac{1}{24\pi} [13\Phi_1^1(0) + (55 + 24\alpha)\Phi_2^2(0)]. \quad (5.15)$$

Since $\Phi_1^1(0)$ and $\Phi_2^2(0)$ are positive for any admissible shape function we can infer from Eq. (5.15) that ω_4 is positive for all $\alpha > \alpha_0$ and negative for all $\alpha < \alpha_0$. Here α_0 $\equiv [-13\Phi_1^1(0)/\Phi_2^2(0)-55]/24$ is a negative number of order unity. Thus, if we identify Einstein gravity with the theory described by the IR critical trajectory of the Gaussian fixed point, Eq. (5.12) implies that Einstein gravity is antiscreening for all $\alpha > \alpha_0$, i.e. G_k decreases as k increases. On the other hand if $\alpha < \alpha_0$ gravity would exhibit a screening behavior. As argued in Sec. IV A, the gauge parameter should be regarded as a scale dependent parameter in an exact treatment where it is expected to approach the fixed point value $\alpha_*=0$. Setting $\alpha=0$ from the outset we may conclude that the physical G_k displays the antiscreening behavior found for $\alpha > \alpha_0$. In Ref. [22] a similar result was obtained with a cutoff of type A, while the above calculation employed the cutoff B. The only difference between our result for the behavior of G_k and the one obtained in [22] lies in the slightly differing value of α_0 which is a scheme dependent parameter. For the cutoff type A, $\omega_4 \equiv \omega_4^{(A)}$ is given by (see [22])

$$\omega_4^{(A)} = \frac{1}{6\pi} [(18 + 6\alpha)\Phi_2^2(0) - \Phi_1^1(0)] \qquad (5.16)$$

so that $\alpha_0 = [\Phi_1^1(0)/\Phi_2^2(0) - 18]/6$ in this case, while ν_4 is the same with both cutoffs. For $\alpha = 1$, Eq. (5.16) boils down to $\omega_4^{(A)} = [24\Phi_2^2(0) - \Phi_1^1(0)]/(6\pi)$ which equals the result obtained in the original paper [6]. This is because for $\alpha = 1$ the cutoff type A coincides with the one used in Ref. [6]. For a comparison of this result with the one for the cutoff type B we insert $\alpha = 1$ into Eq. (5.15) which yields $\omega_4^{(B)}$ $= [13\Phi_1^1(0) + 79\Phi_2^2(0)]/(24\pi)$. Using the exponential shape function $R^{(0)}$ with s = 1 we have $\Phi_1^1(0) = \pi^2/6$, $\Phi_2^2(0) = 1$ so that $\omega_4^{(B)} \approx 1.33$ for the cutoff type B, which lies rather close to the value $\omega_4^{(A)} \approx 1.19$ obtained in [6,22] for the cutoff type A. Furthermore, we have $\Phi_2^1(0) = 2\zeta(3)$ where ζ denotes the zeta function, and thus $\nu_4 \approx 0.19$ with both cutoffs.

C. The non-Gaussian fixed point

Now we turn to the nontrivial zeros of the set of $\boldsymbol{\beta}$ -functions { $\boldsymbol{\beta}_{\lambda}, \boldsymbol{\beta}_{g}$ } given by Eqs. (4.28), (4.30). Such non-Gaussian fixed points (λ_{*}, g_{*}) \neq (0,0) satisfy the condition

$$\eta_{N*} = 2 - d$$
 (5.17)

which follows immediately from Eq. (4.30).

1. In $2+\varepsilon$ dimensions

As a warm up we consider the case of $d=2+\varepsilon$ dimensions with $0 < |\varepsilon| \le 1$ which can be dealt with analytically and for which the existence of a non-Gaussian fixed point has already been shown [16,6,32–34]. In this case the condition (5.17) takes the form $\eta_{N*} = -\varepsilon$ with

$$\eta_{N*} = \frac{g_{*}(\varepsilon)B_{1}(\lambda_{*}(\varepsilon);\alpha,2+\varepsilon)}{1 - g_{*}(\varepsilon)B_{2}(\lambda_{*}(\varepsilon);\alpha,2+\varepsilon)}.$$
 (5.18)

Solving Eq. (5.18) for $g_*(\varepsilon)$ and expanding the result with respect to ε leads to

$$g_{*}(\varepsilon) = -[B_{1}(\lambda_{*}(0);\alpha,2)]^{-1}\varepsilon + \mathcal{O}(\varepsilon^{2}). \quad (5.19)$$

Furthermore, expanding also $\beta_{\lambda}(\lambda_{*}(\varepsilon), g_{*}(\varepsilon); \alpha, 2+\varepsilon)$ with respect to ε and equating equal powers of ε yields

$$\lambda_*(\varepsilon) = [B_1(0;\alpha,2)]^{-1} \Phi_1^1(0)\varepsilon + \mathcal{O}(\varepsilon^2). \quad (5.20)$$

In particular we obtain $\lambda_*(0) = 0$ which implies $g_*(\varepsilon) = -[B_1(0;\alpha,2)]^{-1}\varepsilon + \mathcal{O}(\varepsilon^2)$.

The parameters $B_i(0; \alpha, 2)$, i = 1, 2, may be obtained from the zeroth order terms of the expansions $B_i(\lambda_k; \alpha, 2+\varepsilon)$ $= B_i^{(0)}(\lambda_k; \alpha) + B_i^{(1)}(\lambda_k; \alpha)\varepsilon + \mathcal{O}(\varepsilon^2)$ which take the form

$$B_{1}^{(0)}(\lambda_{k};\alpha) = -\frac{34}{3}(1-2\lambda_{k})^{-1} - \frac{2}{3}(1-2\alpha\lambda_{k})^{-1} + 4\Phi_{1}^{2}(-2\lambda_{k}) + 6\Phi_{1}^{2}(-2\alpha\lambda_{k}) - \frac{32}{3},$$

$$B_{2}^{(0)}(\lambda_{k};\alpha) = \frac{17}{3}(1-2\lambda_{k})^{-1} + \frac{1}{3}(1-2\alpha\lambda_{k})^{-1} - 2\tilde{\Phi}_{1}^{2}(-2\lambda_{k}) - 3\tilde{\Phi}_{1}^{2}(-2\alpha\lambda_{k}). \quad (5.21)$$

Inserting $\lambda_k = 0$ into Eq. (5.21) and using that $\Phi_1^2(0) = 1$ is scheme independent [6] yields

$$B_1(0;\alpha,2) = B_1^{(0)}(0;\alpha) = -\frac{38}{3},$$

$$B_2(0;\alpha,2) = B_2^{(0)}(0;\alpha) = 6 - 5\tilde{\Phi}_1^2(0).$$
(5.22)

)

In contrast to the universal quantity $B_1^{(0)}(0;\alpha)$, $B_2^{(0)}(0;\alpha)$ depends on the shape of $R^{(0)}$ via $\tilde{\Phi}_1^2(0)$. However, $B_2^{(0)}(0;\alpha)$ does not enter the leading order term of $\lambda_*(\varepsilon)$ and $g_*(\varepsilon)$, which may now be written as

$$\lambda_{*}(\varepsilon) = -\frac{3}{38} \Phi_{1}^{1}(0)\varepsilon + \mathcal{O}(\varepsilon^{2}),$$

$$g_{*}(\varepsilon) = \frac{3}{38}\varepsilon + \mathcal{O}(\varepsilon^{2}).$$
(5.23)

The leading order term of $\lambda_*(\varepsilon)$ is nonuniversal since it contains the scheme dependent parameter $\Phi_1^1(0)$. This is not the case for $g_*(\varepsilon)$ whose leading order contribution has a universal meaning.

Let us now analyze the scaling behavior near the non-Gaussian fixed point (5.23). One finds that the associated **B**-matrix is of the form

$$\mathbf{B} = \begin{pmatrix} -2 + \frac{12\alpha - 13}{19}\varepsilon + \mathcal{O}(\varepsilon^2) & -2\Phi_1^1(0) + \mathcal{O}(\varepsilon) \\ \mathcal{O}(\varepsilon^2) & -\varepsilon + \mathcal{O}(\varepsilon^2) \end{pmatrix}.$$
(5.24)

From Eq. (5.24) we obtain the critical exponents $\theta_1 = 2 - [(12\alpha - 13)/19]\varepsilon + \mathcal{O}(\varepsilon^2)$ and $\theta_2 = \varepsilon + \mathcal{O}(\varepsilon^2)$. θ_1 and θ_2 are scheme independent up to terms of $\mathcal{O}(\varepsilon^2)$. θ_1 depends on the gauge parameter. For $\alpha = 1$ the critical exponents co-incide with those following from the β -functions of Ref. [6], where the cutoff type A is used. These findings nicely confirm that, to lowest order, *the critical exponents are the same for the cutoffs A and B and are independent of R*⁽⁰⁾.

For $\varepsilon > 0$ both critical exponents are positive. Hence the non-Gaussian fixed point (5.23) is UV attractive for all trajectories so that the condition for the asymptotic safety scenario is met. It is interesting to investigate whether this result stabilizes in the sense that more general truncations including higher powers of the curvature tensor reproduce this

fixed point and lead to a finite-dimensional UV critical hypersurface. In [24] we will discuss this point in detail.

2. Location of the fixed point (d=4)

In d=4 dimensions, and for the cutoff A, the non-Gaussian fixed point of the Einstein-Hilbert truncation was first discussed in [17], and in Ref. [19] the α and $R^{(0)}$ dependence of its projection $(0,g_*)$ onto the g direction has been investigated. However, since for $\alpha \neq 1$ the cutoff of type A is introduced by an *ad hoc* modification of the standard one-loop determinants it is not clear whether it may be expressed in terms of an action $\Delta_k S$, except for the case α = 1 [6]. Since a specification of $\Delta_k S$ is indispensable for the construction of Γ_k , the status of the results derived in [19] is somewhat unclear. In the following we determine the fixed point properties using different cutoffs of type B, for which a $\Delta_k S$ is known to exist, and compare them to the analogous results for the cutoff A.

In a first attempt to determine the non-Gaussian fixed point we neglect the cosmological constant and set $\lambda_k = \lambda_* = 0$, thereby projecting the renormalization group flow onto the one-dimensional space parametrized by g. In this case the non-Gaussian fixed point is obtained as the nontrivial solution of $\beta_g(0,g_*;\alpha,d)=0$. It is determined in Appendix H with the result given by Eq. (H2). In order to get a first impression of the position of g_* we insert the exponential shape function with s=1 into Eq. (H2) and set d=4, α = 1. We obtain $g_*\approx 0.590$.

Assuming that for the combined λ -g system both g_* and λ_* are of the same order of magnitude as g_* above we expand the β -functions about $(\lambda_k, g_k) = (0,0)$ and neglect terms of higher orders in the couplings. Again in Appendix H we determine the non-Gaussian fixed point for the corresponding system of differential equations. Inserting the shape function (3.7) and setting d=4, $\alpha=1$, we find $(\lambda_*, g_*) \approx (0.287, 0.751)$.

In order to determine the *exact* position of the non-Gaussian fixed point (λ_*, g_*) we have to resort to numerical methods. Given a starting value for the fixed point, e.g. one of the approximate solutions above, the program we use determines a numerical solution which is exact up to an arbitrary degree of accuracy. Under the same conditions as above, i.e. s=1, d=4, we obtain

$$(\lambda_*, g_*) = \begin{cases} (0.348, 0.272) & \text{for } \alpha = 1, \\ (0.339, 0.344) & \text{for } \alpha = 0. \end{cases}$$
(5.25)

Next we study the gauge and scheme dependence of the non-Gaussian fixed point. The scheme dependence is investigated by looking at the *s* dependence introduced via the family of exponential shape functions (3.8) where *s* parametrizes the profile of $R^{(0)}$.

Figure 1 shows $g_*(\alpha, s)$ obtained from the approximation $\lambda_k = \lambda_* = 0$, while Figs. 2 and 3 display the (exact) functions $g_*(\alpha, s)$ and $\lambda_*(\alpha, s)$ resulting from the combined λ -g system. In each of these figures the plot on the LHS [i.e. Figs.



FIG. 1. g_* as a function of *s* and α from the approximation $\lambda_k = \lambda_* = 0$, using (a) the cutoff type A, and (b) the cutoff type B, with the family of exponential shape functions (3.8) inserted.

1,2,3(a)] is obtained from the cutoff type A and the one on the RHS [i.e. Figs. 1,2,3(b)] is obtained from the cutoff type B used in the present paper.

Our results establish the existence of the non-Gaussian fixed point in a wide range of α and *s* values. As expected, the position of the fixed point turns out to be *s*, i.e. scheme, dependent, but the crucial point is that it exists for any of the cutoffs employed. This is one of the important results of our analysis because it gives a first hint at the reliability of the Einstein-Hilbert truncation.

As for the α dependence, $\alpha = 0$ is, in principle, the only relevant case since according to Sec. IV A, $\alpha = 0$ is assumed to be the physical value of the gauge parameter. In practical calculations $\alpha = 1$ is often used instead, because this simplifies the evaluation of the flow equation considerably. In [24], for instance, all calculations are performed with $\alpha = 1$ for this reason. Therefore it is necessary to compare the gauges $\alpha = 0$ and $\alpha = 1$ in order to judge whether the results obtained by using $\alpha = 1$ are a sensible approximation to the physical case $\alpha = 0$. Here we see that this is indeed the case.

As for comparing different types of cutoffs, we recognize from Fig. 1 that, in the approximation $\lambda_k = \lambda_* = 0$, the *s* dependence of g_* is much weaker for type B than for type A. Contrary to this, both cutoffs yield nearly the same results for g_* and λ_* if we consider the combined λ -*g* system, see Figs. 2 and 3. Furthermore, the scheme dependence of g_* in Fig. 2 is stronger than in Fig. 1(b), but much weaker than in Fig. 1(a). Figure 1(a) reproduces the result of Ref. [19] obtained from the cutoff A, see Fig. 2 of this reference.

It should be noted that we are forced to restrict our considerations to shape functions (3.8) with $s \ge 1$. This is because for s < 1 the numerical integrations are plagued by



FIG. 2. The exact g_* as a function of *s* and α from the combined λ -*g* system, using (a) the cutoff type A, and (b) the cutoff type B, with the family of exponential shape functions (3.8) inserted.

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FIG. 3. The exact λ_* as a function of s and α from the combined λ -g system, using (a) the cutoff type A, and (b) the cutoff type B, with the family of exponential shape functions (3.8) inserted.

convergence problems which is due to the fact that in d=4 dimensions the threshold functions in β_{λ} and β_{g} diverge in the limit $s \rightarrow 0$, see also [19].

Because the scale k enters the flow equation via \mathcal{R}_k as a purely mathematical device it is clear that the functions $k \mapsto \lambda_k, g_k$ and their UV limits λ_*, g_* are scheme dependent and not directly observable therefore. It can be argued that the product $g_*\lambda_*$ must be scheme independent, however. While k and, at a fixed value of k, G_k and $\overline{\lambda}_k$ cannot be measured separately, we may invert the function $k \mapsto G_k$ and insert the result k = k(G) into $\overline{\lambda}_k$. This leads to a relationship between Newton's constant and the cosmological constant which, at least in principle, could be tested experimentally: $\overline{\lambda} = \overline{\lambda}(G)$. In general this relation depends on the RG trajectory chosen (specified by its IR values $\overline{\lambda}_0$ and G_0 , for instance), but in the fixed point regime all trajectories approach $\overline{\lambda}_k = \lambda_* k^2$ and $G_k = g_* / k^2$ which gives rise to

$$\bar{\lambda}(G) = \frac{g_* \lambda_*}{G}.$$
(5.26)

Equation (5.26) is valid if $\overline{\lambda} \gg m_{\rm Pl}^2$ and $G \ll m_{\rm Pl}^{-2}$. (We define the Planck mass in terms of the IR limit of G_k , $m_{\rm Pl} \equiv G_0^{-1/2}$.) Assuming that $\overline{\lambda}$ and *G* have the status of observable quantities, Eq. (5.26) shows that $g_*\lambda_*$ must be observable, and hence scheme independent, too. (For a related discussion see [33].) Below the Planck regime the function $\overline{\lambda}(G)$ becomes much more complicated than $\overline{\lambda} \propto 1/G$ (which follows already from dimensional analysis) because the dimensionful quantities $\overline{\lambda}_0$ and G_0 enter explicitly there.

As for the universality of $g_*\lambda_*$, it is also interesting to note that, for any k, the product $g_k\lambda_k = G_k\bar{\lambda}_k$ is essentially the inverse of the on shell value of Γ_k . The stationary points of Eq. (1.2) with $\bar{g}_{\mu\nu} = g_{\mu\nu}$ satisfy Einstein's equation $G_{\mu\nu}$ $= -\bar{\lambda}_k g_{\mu\nu}$. Hence $R = 4\bar{\lambda}_k$, so that from Eq. (1.2) in four dimensions,

$$\Gamma_k[\text{ on shell}] = -\frac{v}{8\pi G_k \bar{\lambda}_k} = -\frac{v}{8\pi g_k \lambda_k}.$$
 (5.27)

Here we used that, for dimensional reasons, $\int d^4x \sqrt{g} = v/\overline{\lambda}_k^2$ where *v* is a finite, positive constant for any solution with a finite four-volume.

Quite remarkably, the universality of the product $g_*\lambda_*$ is confirmed by our results in a rather impressive manner, as is illustrated in Figs. 4–7. Figure 4(a) contains several parametric plots of $[\lambda_*(s), g_*(s)]$ for various values of α , obtained from the β -functions (4.28) and (4.30) which are based on the cutoff type B. The hyperbolic shape of these plots is a first hint at the s independence of the product $g_*\lambda_*$. Its direct confirmation is supplied by Fig. 5 which shows g_* , λ_* , and $g_*\lambda_*$ as functions of s for $\alpha = 0$ [Figs. 5(a),(b)] and $\alpha = 1$ [Figs. 5(c),(d)], again using the cutoff type B. In Figs. 5(a),(c) these functions are plotted in the range of values 1 $\leq s \leq 30$ while Figs. 5(b),(d) contain the sector corresponding to $1 \le s \le 5$ where the largest changes in λ_* and g_* occur. In any of these figures the product of λ_* and g_* is almost constant for the whole range of s values considered. Its universal value is

$$g_* \lambda_* \approx \begin{cases} 0.12 & \text{for } \alpha = 1, \\ 0.14 & \text{for } \alpha = 0. \end{cases}$$
 (5.28)

Obviously the difference between the physical case $\alpha = 0$ and the case preferred for technical reasons, $\alpha = 1$, is rather small.

It is reassuring to see that employing the β -functions of Refs. [22,19] which are based on the cutoff A we obtain almost identical results. They are illustrated by means of Fig. 6 which shows g_* , λ_* , and $g_*\lambda_*$ as functions of *s*, $1 \le s \le 5$, for $\alpha = 0$ [Fig. 6(a)] and $\alpha = 1$ [Fig. 6(b)].

It is also interesting to compare the above results with those obtained from a different shape function. Figure 7 displays the behavior of g_* , λ_* , and their product resulting from the β -functions (4.28) and (4.30) of the cutoff B, with



FIG. 4. (a) *s*-parametric plot of $[\lambda_*(s), g_*(s)]$ in the range $1 \le s \le 50$ for various values of α . Each curve starts on the left at s = 1, and ends on the right at s = 50. (b) α -parametric plot of $[\lambda_*(\alpha), g_*(\alpha)]$ for various values of *s*. In both (a) and (b) the cutoff type B is used, with the family of exponential shape functions (3.8) inserted.

the family of shape functions $R^{(0)}(y;b)$ with compact support, Eq. (3.9), inserted. Here $b \in [0,1.5)$ parametrizes the profile of these shape functions. In Fig. 7(a) we present a parametric plot of $[\lambda_*(b), g_*(b)]$ for $\alpha = 0$ starting at b = 0 and ending at b = 1.5. Furthermore, Fig. 7(b) shows the product $g_*\lambda_*$ as a function of *b* for $\alpha = 0$ and $\alpha = 1$. For $b \leq 1.2$, the parametric plot in Fig. 7(a) exhibits an approximately linear behavior which leads to a $g_*\lambda_*$ plateau in Fig.

7(b) where $g_*\lambda_*$ is nearly constant. Remarkably, the position of these plateaus coincides quite precisely with those of the corresponding plateaus in Figs. 5 and 6 obtained with the other cutoffs. As for the quality of the Einstein-Hilbert truncation this result is rather encouraging.

For b > 1.2 the curves in Fig. 7 have a rather strong and erratic *b* dependence. This is because $R^{(0)}(y;b)$ approaches a sharp cutoff as $b \rightarrow 1.5$, which introduces discontinuities into



FIG. 5. g_* , λ_* , and $g_*\lambda_*$ as functions of s for (a) $\alpha=0$, $1 \le s \le 30$, (b) $\alpha=0$, $1 \le s \le 5$, (c) $\alpha=1$, $1 \le s \le 30$, and (d) $\alpha=1$, $1 \le s \le 5$, using the cutoff type B with the family of exponential shape functions (3.8) inserted.



FIG. 6. g_* , λ_* , and $g_*\lambda_*$ as functions of s for (a) $\alpha = 0$, and (b) $\alpha = 1$, using the cutoff type A with the family of exponential shape functions (3.8) inserted.

the integrands of the threshold functions Φ_n^p and $\tilde{\Phi}_n^p$. Already for $b \ge 1.2$ the β -functions start to "feel" the sharp cutoff limit so that the results cannot be trusted beyond this point.

As for the α dependence of the fixed point, in Fig. 4(b) we present parametric plots of $[\lambda_*(\alpha), g_*(\alpha)]$ for various fixed values of *s*. Here we used the cutoff type B with the shape function (3.8) inserted. These plots start at positions in the λ -*g* plane which correspond to $\alpha = 0$ and which are different for the distinct *s* values. As $\alpha \rightarrow \infty$ all curves run into the Gaussian fixed point.

To summarize: (a) In 4 dimensions, the Einstein-Hilbert truncation leads to a non-Gaussian fixed point with positive values of λ_* and g_* for all admissible cutoffs, both of type A and type B. The scheme independence of this prediction is a nontrivial result. (To emphasize this point we mention that in higher dimensions, where the Einstein-Hilbert truncation is less reliable, the fixed point exists or does not exist depending on the cutoff chosen [31].)

(b) Universal quantities are strictly cutoff independent only in an exact treatment. Any truncation leads to a scheme dependence of these quantities. The extent of this scheme dependence is a measure for the reliability of the truncation. The product $g_*\lambda_*$ is an example of a universal quantity. While we find a considerable scheme dependence of g_* and λ_* separately, their product is scheme independent at a quite amazing level of accuracy, see Fig. 5. As for the reliability of the Einstein-Hilbert truncation, we consider this result a highly nontrivial confirmation of our assumption that the region of parameter space where the fixed point occurs is well described by this truncation ansatz so that the fixed point also exists in the *exact* theory and is not a truncation artifact.

3. Higher and lower dimensions

Before continuing our analysis of the d=4 dimensional case we study the *d*-dependence of the non-Gaussian fixed point. This is done by means of the parametric plots in Fig. 8 which are obtained from the β -functions of type B, Eqs. (4.28) and (4.30), with the shape function (3.8) with s=1inserted. Figure 8(b) shows $[\lambda_*(d), g_*(d)]$ in $2 \le d \le 4$ for $\alpha = 1$. Remarkably, this plot is almost identical with that in Fig. 4 of Ref. [17] which was derived from the β -functions based on the cutoff type A.

In order to gain information about the behavior of the fixed point in d>4 we also plotted $[\lambda_*(d), g_*(d)]$ in $2 \le d \le 6$, but this time for $\alpha=0$, see Fig. 8(a). For *d* beyond $d\approx 5$, the value of g_* increases significantly, whereas λ_* seems to approach a constant value.



FIG. 7. (a) *b*-parametric plot of $[\lambda_*(b), g_*(b)]$ for $\alpha = 0$, and (b) $g_*\lambda_*$ as a function of *b* for $\alpha = 0$ and $\alpha = 1$, using the cutoff type B with the family of shape functions with compact support (3.9) inserted.



FIG. 8. *d*-parametric plot of $[\lambda_*(d), g_*(d)]$ for (a) $\alpha = 0, 2 \le d \le 6$, and (b) $\alpha = 1, 2 \le d \le 4$, using the cutoff type B with the exponential shape function (3.8) with s = 1 inserted.

However, this result might be a truncation artifact since it is plausible to assume that in higher dimensions the Einstein-Hilbert truncation becomes less reliable. This is because with increasing dimensionality d the number of terms which are relevant at the non-Gaussian fixed point and which are neglected in the Einstein-Hilbert truncation increases most probably. At the non-Gaussian fixed point the scaling dimensions of local invariants such as $\int \sqrt{g}$, $\int \sqrt{gR}$, $\int \sqrt{gR^2}$, $\int \sqrt{g} R_{\mu\nu} R^{\mu\nu}$, etc. are not known a priori. We only know that with respect to the Gaussian fixed point all local monomials R, R^2, \ldots are relevant or marginal if their canonical mass dimension does not exceed d. A sensible truncation should retain at least the relevant terms, whence it is clear that the number of terms needed increases with the dimensionality d. [In 4 dimensions, $\int \sqrt{g}$ and $\int \sqrt{g}R$ are relevant and the (curvature)² invariants are marginal.] By analogy we expect that the description of the non-Gaussian fixed point, too, requires increasingly high powers of the curvature when d is increased [31].

4. The critical exponents (d=4)

Let us now return to the 4-dimensional case and analyze the critical behavior near the non-Gaussian fixed point. In order to get a first impression of its features we restrict our considerations to the cutoff type B with the exponential shape function (3.8) with s = 1 and to the gauge $\alpha = 1$. In this case we have $(\lambda_*, g_*) = (0.348, 0.272)$, see above. The corresponding **B**-matrix assumes the form

$$\mathbf{B} = \begin{pmatrix} -0.187 & 5.129 \\ -3.228 & -2.907 \end{pmatrix}.$$
 (5.29)

It leads to a pair of *complex* critical exponents $\theta_1 \equiv \theta' + i\theta''$ and $\theta_2 = \theta_1^* \equiv \theta' - i\theta''$. For the real quantities θ' and θ'' we find $\theta' = 1.547$ and $\theta'' = 3.835$. (In general we define θ_1 as the critical exponent with the positive imaginary part so that $\theta'' > 0$.) The behavior of λ_k and g_k near the fixed point is described by the real part of Eq. (5.4) in this case. Using that $V^2 = (V^1)^*$, and setting $V^1 \equiv V$ and $C_1 \equiv C$, the general solution to the linearized flow equation may then be written as

$$\begin{pmatrix} \lambda_k \\ g_k \end{pmatrix} = \begin{pmatrix} \lambda_* \\ g_* \end{pmatrix} + 2\{[\operatorname{Re} C \cos(\theta'' t) + \operatorname{Im} C \sin(\theta'' t)] \operatorname{Re} V \\ + [\operatorname{Re} C \sin(\theta'' t) - \operatorname{Im} C \cos(\theta'' t)] \operatorname{Im} V\} e^{-\theta' t}.$$
(5.30)

Here $t \equiv \ln(k/k_0)$. Obviously the non-Gaussian fixed point is UV (IR) attractive if $\theta' \equiv \operatorname{Re} \theta_1 = \operatorname{Re} \theta_2 > 0(<0)$. The imaginary parts $\pm \theta''$ of the critical exponents do not influence the stability of the fixed point. They only give rise to a rotation of the vector $(\lambda_k - \lambda_*, g_k - g_*)^{\mathrm{T}}$ about the fixed point.

In the case under consideration we have $\theta' > 0$ which implies that the non-Gaussian fixed point is UV attractive in both directions of (λ, g) space. All RG trajectories which reach its basin of attraction spiral into the fixed point for $k \rightarrow \infty$. Thus, the Einstein-Hilbert truncation predicts all the ingredients which are necessary for the asymptotic safety scenario and the nonperturbative renormalizability of 4-dimensional quantum gravity. Clearly the dimensionality of the UV critical hypersurface cannot be determined within the present approach. We shall come back to this question in the framework of a more general truncation including higher derivative terms [24].

As discussed in Sec. V A the critical exponents are universal in an exact treatment, in contrast to g_* and λ_* . However, in a truncated parameter space a scheme dependence is expected to occur as an artifact of the truncation. Therefore we may use this scheme dependence of the critical exponents to judge the quality of our truncation. Also in this respect the Einstein-Hilbert truncation yields satisfactory results, which we display in Figs. 9-11. First of all it should be noted that the critical exponents obtained from our numerical analysis have a nonzero imaginary part in any of the cases considered. Figures 9–11(a) show their real part θ' while their imaginary part θ'' is depicted in Figs. 9–11(b). Figures 9 and 10 are derived from the β -functions (4.28) and (4.30) based on the cutoff B of the present paper, with the family of shape functions (3.8) (in Fig. 9) and (3.9) (in Fig. 10) inserted. For comparison, Fig. 11 is obtained from the β -functions of Refs. [22,19] derived from the cutoff type A, with the family



FIG. 9. (a) $\theta' = \text{Re } \theta_1$, and (b) $\theta'' = \text{Im } \theta_1$ as functions of *s* for various values of α , using the cutoff type B with the family of exponential shape functions (3.8) inserted.

of shape functions (3.8) inserted. The figures contain various plots for distinct values of α , which describe the *s* or *b* dependence of θ' and θ'' .

The θ' plots in Figs. 9(a) and 11(a) for the cutoffs B and A, respectively, exhibit a similar *s* dependence, and the same holds true for the θ'' plots in Figs. 9(b) and 11(b). Moreover, in the range $1 \le s \le 5$ one recognizes a total variation of both θ' and θ'' which is approximately of the order of magnitude of θ' and θ'' , respectively. For s > 5 there remains only a rather weak dependence on *s*, such that the functions $\theta'(s)$ and $\theta''(s)$ develop a plateau-like shape. The *s* dependence is slightly stronger for the cutoff type B than for type A, and the positions of the "plateaus" are different for both cutoff types. For $\alpha = 1$, for instance, we have $\theta'(s = 30) \approx 1.56$ and $\theta''(s = 30) \approx 3.06$ using type A, while employing type B yields $\theta'(s = 30) \approx 1.75$ and $\theta''(s = 30) \approx 2.31$. These differences have to be interpreted as truncation artifacts.

The above results may now be compared with those for the cutoff B with the shape functions with compact support, Eq. (3.9), inserted. Our results are shown in Fig. 10. Remarkably, θ' is almost constant in the range $0 \le b \le 1.2$. Furthermore, the change in θ'' is only very weak in this region and occurs mainly for $b \ge 0.4$. For $b \le 0.4$ θ'' is almost constant as well. In the region $1.2 \le b \le 1.5$ both θ' and θ'' strongly vary with *b* which is caused by the sharp cutoff limit as $b \rightarrow 1.5$, see above. The position of the plateaus is nearly identical with those obtained above by using the same cutoff B, but the shape function (3.8). For $\alpha = 1$ we have $\theta'(b=0) \approx 1.69$ and $\theta''(b=0) \approx 2.56$.

As discussed above, $\alpha = 0$ is assumed to be the physical value of the gauge parameter while in practical applications $\alpha = 1$ is often preferred. For a further justification of this approximation we compare the critical exponents obtained from $\alpha = 1$ with those obtained from the (physical) gauge $\alpha = 0$. The results are qualitatively the same, but quantitatively we find a relative deviation of about 10 to 20 percent. This has to be kept in mind when calculations are performed with $\alpha = 1$.

To summarize: For all admissible cutoffs, both of type A and type B, the non-Gaussian fixed point is UV attractive in both directions of parameter space. It is characterized by a pair of complex conjugate critical exponents which leads to spiral-type trajectories in its vicinity. The exact critical exponents are universal. Those obtained from the Einstein-Hilbert truncation are approximately scheme independent, but their scheme dependence is stronger than that of $g_*\lambda_*$. This might be related to the fact that the $\int \sqrt{gR^2}$ term neglected by the Einstein-Hilbert truncation is (very weakly) relevant at



FIG. 10. (a) $\theta' = \text{Re } \theta_1$, and (b) $\theta'' = \text{Im } \theta_1$ as functions of b for $\alpha = 0$ and $\alpha = 1$, using the cutoff type B with the family of shape functions with compact support (3.9) inserted.

the non-Gaussian fixed point. We shall come back to this point in Ref. [24].

VI. THE GRAVITON PROPAGATOR AT LARGE MOMENTA

The UV fixed point is characterized by an anomalous dimension $\eta \equiv \eta_N(g_*, \lambda_*) = -2$. We can use this information in order to determine the effective momentum dependence of the dressed graviton propagator for momenta in the fixed point regime, i.e. for $p^2 \ge m_{\text{Pl}}^2$. Expanding the truncated Γ_k about flat space and omitting the standard tensor structures we find the inverse propagator $\tilde{\mathcal{G}}_k(p)^{-1} \propto Z_N(k)p^2$. The conventional dressed propagator $\tilde{\mathcal{G}}(p)$ contained in $\Gamma \equiv \Gamma_{k=0}$ obtains from $\tilde{\mathcal{G}}_k$ in the limit $k \rightarrow 0$. Assuming that the actual physical cutoff scale is the momentum p^2 itself (which is indeed true for $p^2 > k^2 \ge m_{\text{Pl}}^2$), the *k* evolution of $\tilde{\mathcal{G}}_k(p)$ stops at the threshold $k = \sqrt{p^2}$. Therefore

$$\tilde{\mathcal{G}}(p)^{-1} \propto Z_N(k = \sqrt{p^2}) p^{2} \propto (p^2)^{1 - \eta/2}$$
(6.1)

because $Z_N(k) \propto k^{-\eta}$ when $\eta \equiv -\partial_t \ln Z_N$ is (approximately) constant. In *d* dimensions, and for $\eta \neq 2-d$, the Fourier transform of $\tilde{\mathcal{G}}(p) \propto 1/(p^2)^{1-\eta/2}$ yields the following Euclidean propagator in position space:

$$\mathcal{G}(x;y) \propto \frac{1}{|x-y|^{d-2+\eta}}.$$
(6.2)

This is a standard result well known from the theory of critical phenomena, for instance. In the latter case it applies to large distances, while in quantum gravity we are interested in the extreme short distance regime governed by the UV fixed point. However, as it stands Eq. (6.2) is not valid for the case of interest, d=4 and $\eta=-2$. For $\eta=-2$ the dressed propagator is $\tilde{\mathcal{G}}(p)=1/p^4$ which, in d=4 dimensions, has the following representation in position space:

$$\mathcal{G}(x;y) = -\frac{1}{8\pi^2} \ln(\mu |x-y|).$$
(6.3)

Here μ is an arbitrary constant with the dimension of a mass. Usually in ordinary matter field theories on flat space a $1/p^4$ propagator is considered problematic because it is incompatible with a positive spectral density. It is by no means clear, however, that the familiar notions of positivity, analyticity, and causality which are appropriate for Minkowski space are of relevance to the situation under consideration. For $k \rightarrow \infty$ the "on shell" spacetimes of Γ_k have a large curvature $\propto k^2$, after all.

So let us assume that the improvement (6.1) is indeed correct and that in the fixed point regime, i.e. for distances much smaller than the Planck length, the effective graviton propagator has a logarithmic dependence on the distance. This result is quite remarkable because it implies a kind of dimensional reduction from 4 to 2 dimensions. In fact, Eq. (6.3) is precisely what one obtains from a standard $1/p^2$ propagator in d=2 dimensions. This means that, in a certain sense, spacetime appears to be two-dimensional when it is probed by a very high energetic graviton.

Since, symbolically, $R = \partial \partial h$, the propagator (6.3) yields the curvature-curvature correlation function

$$\langle \mathbf{R}(x)\mathbf{R}(y)\rangle \propto \frac{1}{(x-y)^4}.$$
 (6.4)

Its short-distance singularity has to be contrasted with the $1/(x-y)^6$ behavior one finds at tree level. Here **R** stands for the curvature scalar or for any component of the Riemann or Ricci tensor.

Switching for a moment to spacetimes with a Lorentzian signature, it is interesting to look at the linearized gravitational field produced very close to a *static* source. Decomposing $x \equiv (x^0, \mathbf{x})$, the relevant Green's function for static problems reads

$$\mathcal{G}_{\text{stat}}(\mathbf{x};\mathbf{y}) \equiv \int_{-\infty}^{\infty} dy^0 \mathcal{G}(x^0,\mathbf{x};y^0,\mathbf{y}).$$
(6.5)

In our case this is the 3-dimensional Fourier transform of $1/|\mathbf{p}|^4$, i.e.

$$\mathcal{G}_{\text{stat}}(\mathbf{x};\mathbf{y}) = \langle \mathbf{x} | (\nabla^2 \nabla^2)^{-1} | \mathbf{y} \rangle = -\frac{1}{8\pi} | \mathbf{x} - \mathbf{y} | \qquad (6.6)$$

provided $|\mathbf{x}-\mathbf{y}| \leq m_{\text{Pl}}^{-1}$. In an, admittedly somewhat naive, Newtonian language this result would mean that a point mass located at $\mathbf{y}=0$ creates a gravitational potential which behaves as $\Phi(\mathbf{x}) \propto |\mathbf{x}|$ as long as $|\mathbf{x}|$ is much smaller than the Planck length. Probably this linear potential is related to a similar phenomenon shown by the renormalization group improved Schwarzschild black hole which has been constructed recently [13]. The radial geodesics in this spacetime, in a fully relativistic treatment, experience a linear⁶ repulsive "potential" close to the core of the black hole.

VII. SUMMARY AND CONCLUSION

On the formal side, the main result of this paper is the construction of a new exact RG equation for the gravitational effective average action. It is formulated in terms of the component fields appearing in the TT decomposition of the metric. It is defined on a sufficiently large class of background spacetimes so as to facilitate the projection of the RG flow onto very general truncated parameter spaces. It also helps in finding admissible IR cutoffs. A formalism of this kind is mandatory for truncations including higher-derivative invariants, matter fields, or a running gauge fixing, for instance. In a forthcoming paper [24] we shall use this technology in

⁶In the notation of Ref. [13] the potential is linear if $\gamma = 0$. This corresponds to the case $k \propto 1/r$ which ignores the impact the background curvature has on the cutoff identification, see Eq. (4.35) of [13]. This is consistent with the fact that Eq. (6.3) is valid for a flat background.



FIG. 11. (a) $\theta' = \operatorname{Re} \theta_1$, and (b) $\theta'' = \operatorname{Im} \theta_1$ as functions of *s* for $\alpha = 0$ and $\alpha = 1$, using the cutoff type A with the family of exponential shape functions (3.8) inserted.

order to explore a more general truncation with an R^2 term. In the present paper we analyzed the Einstein-Hilbert truncation with the new equation and a new cutoff.

After deriving the exact functional equation in Sec. II, we discussed in Sec. III a general strategy for constructing an appropriate cutoff operator in the context of a class of truncations which is still very general. In Sec. IV we specialized for the Einstein-Hilbert truncation and derived the nonperturbative β -functions which govern the RG evolution of g_k and λ_k .

On the applied side, most of our results concern the non-Gaussian fixed point of the λ -*g* system in 4 dimensions which we analyzed in Sec. V. If this fixed point is actually present in the exact theory, its importance can hardly be overestimated. Its existence would imply that in spite of its notorious perturbative nonrenormalizability quantum Einstein gravity is most probably renormalizable at the nonperturbative level and thus qualifies as a fundamental (microscopic) quantum theory of gravity. Clearly the crucial question is whether the fixed point of the Einstein-Hilbert truncation is indeed genuine or merely a truncation artifact.

In order to get an impression of the reliability of the Einstein-Hilbert truncation we investigated how its predictions change when we vary the cutoff which is built into the RG equations. We used both the original cutoff of type A, formulated in terms of $h_{\mu\nu}$, and the new cutoff of type B which is natural in the TT approach; the cutoffs were equipped with two different one-parameter families of shape functions.

In 4 dimensions, we found that the Einstein-Hilbert truncation leads to a non-Gaussian fixed point for all admissible cutoffs, both of type A and B. The robustness of this prediction is a nontrivial result since in higher dimensions, for instance, where this truncation is less reliable, the fixed point is present for some cutoffs but absent for others.

Another consistency test successfully passed by the truncation is that all cutoffs agree on *positive* values of g_* and λ_* . A negative g_* would probably be unacceptable for stability reasons, but there is no mechanism in the flow equation which would exclude it on general grounds. In fact, $g_* < 0$ is realized for d < 2. For all cutoffs of type A and B the non-Gaussian fixed point is found to be UV attractive in both directions of the λ -g plane. The linearized flow in its vicinity is always characterized by a pair of complex conjugate critical exponents leading to spiral-type trajectories which hit the fixed point for $k \rightarrow \infty$. This is precisely the stability property needed for asymptotic safety.

By definition, universal quantities are scheme-, or cutoffindependent in an exact calculation. Truncations lead to a scheme dependence, however. We can use the degree of the scheme dependence as a measure for the reliability of the truncation. The critical exponents and, as we argued, the product $g_*\lambda_*$ are universal quantities. The existence of fixed points is a universal feature of the RG flow, but not their precise location in parameter space.

The critical exponents were indeed found to be reasonably constant for a wide range of shape parameters. The universality properties of $g_*\lambda_*$ are much more impressive though. While we found a considerable scheme dependence of g_* and λ_* separately, their product is scheme independent at a rather amazing level of accuracy.

We believe that these results hardly can be a mathematical accident, and we consider them a very nontrivial confirmation of the hypothesis that the region of parameter space where the non-Gaussian fixed point is situated is well described by the Einstein-Hilbert truncation. As a consequence, the fixed point should exist in the exact theory, too.

Apart from the renormalizability issue the nontrivial fixed point is also intriguing from a "phenomenological" point of view. Its relevance for the structure of black holes [13] and the cosmology of the Planck era [14] has been pointed out already. Moreover, we saw in Sec. VI that the RG improvement of the graviton propagator suggests a kind of dimensional reduction from 4 to 2 dimensions when spacetime is probed at sub-Planckian length scales.

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APPENDIX A: THE TT DECOMPOSITION

1. Pseudo-projectors for the TT decomposition

In Sec. II B we introduced the TT decomposition

$$h_{\mu\nu} = h_{\mu\nu}^{T} + h_{\mu\nu}^{L} + h_{\mu\nu}^{Tr}$$
(A1)

valid for arbitrary symmetric rank 2 tensors defined on either closed or open, asymptotically flat Riemannian *d* spaces. Here $h_{\mu\nu}^T$, $h_{\mu\nu}^L$ and $h_{\mu\nu}^{Tr}$ represent the transverse traceless, longitudinal traceless and pure trace part, respectively, which are mutually orthogonal. According to Eq. (2.9) these parts may be expressed in terms of pure spin-2, spin-1 and spin-0 component fields $h_{\mu\nu}^T$, $\hat{\xi}_{\mu}$, $\hat{\sigma}$, ϕ . In this subsection we show that the component fields can be obtained by applying certain operators Π to the full field $h_{\mu\nu}$. In the following these operators will be termed pseudo-projectors.

As a first step we express the longitudinal-transverse and the pure trace part as

$$h_{\mu\nu}^{L} = (L\varepsilon)_{\mu\nu} \equiv \bar{D}_{\mu}\varepsilon_{\nu} + \bar{D}_{\nu}\varepsilon_{\mu} - \frac{2}{d}\bar{g}_{\mu\nu}\bar{D}_{\lambda}\varepsilon^{\lambda},$$
$$h_{\mu\nu}^{Tr} = \frac{1}{d}\bar{g}_{\mu\nu}\phi, \phi \equiv \bar{g}^{\mu\nu}h_{\mu\nu}^{Tr}.$$
(A2)

Here the operator L maps vectors onto longitudinal traceless tensors. Given a tensor $h_{\mu\nu}^L$, the equation $(L\varepsilon)_{\mu\nu} = h_{\mu\nu}^L$ is solved by

$$\varepsilon_{\mu} \equiv \hat{\xi}_{\mu} + \frac{1}{2} \bar{D}_{\mu} \hat{\sigma} \tag{A3}$$

where $\hat{\xi}_{\mu}$ is a transverse vector and $\hat{\sigma}$ a scalar. This solution is unique up to the addition of conformal Killing vectors (CKV's), as discussed in Sec. II B. We recover Eq. (2.9) by inserting Eqs. (A2), (A3) into Eq. (A1). Contrary to ε_{μ} , the scalar ϕ is uniquely determined by $h_{\mu\nu}$.

Now taking the covariant divergence of Eq. (A1) with Eq. (A2) inserted, and using the transversality requirement $\bar{D}^{\mu}h_{\mu\nu}^{T}=0$ leads to

$$(\mathcal{D}\varepsilon)_{\mu} = -\bar{D}^{\nu} \left(h_{\mu\nu} - \frac{1}{d} \bar{g}_{\mu\nu} \phi \right) \tag{A4}$$

with the operator \mathcal{D} defined by

$$(\mathcal{D}\varepsilon)_{\mu} \equiv -\bar{D}^{\nu} (L\varepsilon)_{\mu\nu}. \tag{A5}$$

As shown in Ref. [20], \mathcal{D} is a positive definite, Hermitian operator mapping vectors onto vectors. Moreover, the equation $(\mathcal{D}\varepsilon)_{\mu} = u_{\mu}$ with an arbitrary given vector u_{μ} always possesses solutions ε_{μ} which are unique up to CKV's. However, even if these CKV's exist they cause no problems in solving Eq. (A4) for ε_{μ} , see Ref. [20] for details. In order to determine this solution we have to invert \mathcal{D} . For this purpose we assume that \mathcal{D} (and any additional operator that needs to be inverted in the course of this discussion) has a complete set of orthogonal eigenfunctions and that the corresponding eigenvalues do not have zero as an accumulation point. Then \mathcal{D}^{-1} exists and the solution to Eq. (A4) may be written as

$$\varepsilon_{\mu} = (\mathcal{D}^{-1} \widetilde{\delta} h)_{\mu}. \tag{A6}$$

Here the operator $\tilde{\delta}$ maps tensors onto vectors according to

$$(\tilde{\delta}h)_{\mu} = -\bar{D}^{\nu}\Lambda_{\mu\nu}{}^{\alpha\beta}h_{\alpha\beta} \tag{A7}$$

with

$$\Lambda_{\mu\nu}{}^{\alpha\beta} \equiv \frac{1}{2} \left(\delta^{\alpha}_{\mu} \delta^{\beta}_{\nu} + \delta^{\beta}_{\mu} \delta^{\alpha}_{\nu} \right) - \frac{1}{d} \overline{g}_{\mu\nu} \overline{g}^{\alpha\beta} \tag{A8}$$

being the operator that projects symmetric tensors onto their traceless part:

$$\Lambda_{\mu\nu}^{\ \alpha\beta}h_{\alpha\beta} = h_{\mu\nu} - \frac{1}{d}\overline{g}_{\mu\nu}\phi.$$
(A9)

Taking the covariant divergence of the solution (A6), inserting Eq. (A3), and using $\bar{D}_{\mu}\hat{\xi}^{\mu}=0$ yields

$$\bar{D}^{\mu}\varepsilon_{\mu} = \frac{1}{2}\bar{D}^{2}\hat{\sigma} = \bar{D}^{\mu}(\mathcal{D}^{-1}\tilde{\delta}h)_{\mu}.$$
 (A10)

This leads to the final result for $\hat{\xi}_{\mu}$ and $\hat{\sigma}$ in terms of $h_{\mu\nu}$:

$$\hat{\sigma} = 2(\bar{D}^2)^{-1}\bar{D}^{\mu}(\mathcal{D}^{-1}\tilde{\delta}h)_{\mu} \equiv \Xi h,$$

$$\hat{\xi}_{\mu} = (\mathcal{D}^{-1}\tilde{\delta}h)_{\mu} - \bar{D}_{\mu}(\bar{D}^2)^{-1}\bar{D}^{\nu}(\mathcal{D}^{-1}\tilde{\delta}h)_{\nu} \equiv (\Omega h)_{\mu}.$$
(A11)

Hence the pseudo-projectors Π which map $h_{\mu\nu}$ onto the individual component fields are obtained as

$$h_{\mu\nu}^{T} = (\Pi_{TT}h)_{\mu\nu} \equiv (\Lambda h)_{\mu\nu} - (L\mathcal{D}^{-1}\tilde{\delta}h)_{\mu\nu},$$

$$\xi_{\mu} = (\Pi_{LT}h)_{\mu} \equiv (\sqrt{-\bar{D}^{2} - \mathrm{Ric}} \Omega h)_{\mu},$$

$$\sigma = \Pi_{LL}h \equiv \sqrt{(\bar{D}^{2})^{2} + \frac{d}{d-1}\bar{D}_{\mu}\bar{R}^{\mu\nu}\bar{D}_{\nu}}\Xi h,$$

$$\phi = \Pi_{Tr}h \equiv \bar{g}^{\mu\nu}h_{\mu\nu}.$$
 (A12)

Here we defined Π_{LT} and Π_{LL} in terms of the redefined fields ξ_{μ} and σ which are related to $\hat{\xi}_{\mu}$ and $\hat{\sigma}$ by Eq. (2.21).

Furthermore, the pseudo-projectors for the transverse decomposition of arbitrary vector fields can be inferred from Eq. (A11). For $C^{\mu} = C^{T\mu} + \bar{D}^{\mu} (-\bar{D}^2)^{-1/2} \eta$ with $\bar{D}_{\mu} C^{T\mu} = 0$ they are determined by

$$\eta = \Pi_L C \equiv (-\bar{D}^2)^{-1/2} \bar{D}_{\mu} C^{\mu},$$

$$C^{T\mu} = (\Pi_T C)^{\mu} \equiv C^{\mu} - \bar{D}^{\mu} (-\bar{D}^2)^{-1} \bar{D}_{\nu} C^{\nu}.$$
 (A13)

Obviously Π_{LT} maps tensors onto vectors, Π_{LL} and Π_{Tr} map tensors onto scalars, and Π_L maps vectors onto scalars. Hence these operators cannot be projection operators in the usual sense. However, projection operators **P** mapping arbitrary $h_{\mu\nu}$ onto $h_{\mu\nu}^{Tr}$ or $h_{\mu\nu}^{Tr}$, or arbitrary ε_{μ} onto their longitudinal component can be constructed from them [20]. Since the **II**'s map vectors and symmetric tensors onto their component fields they generate a kind of projection in a wider sense of the word. Therefore we call the **II**'s pseudoprojectors. Contrary to $\mathbf{\Pi}_{LT}$, $\mathbf{\Pi}_{LL}$, $\mathbf{\Pi}_{Tr}$ and $\mathbf{\Pi}_{L}$, the operators $\mathbf{\Pi}_{TT}$ and $\mathbf{\Pi}_{T}$ are genuine projection operators mapping symmetric tensors and vectors onto their ST^2 and T component, respectively.

2. Construction of the cutoff and the source terms

In the present paper we need a formulation for Γ_k which allows for a description in terms of the fundamental as well as the component fields. The translation between the two descriptions can be achieved by using the pseudo-projectors for the construction of the cutoff and for an appropriate decomposition of the source terms.

Starting from the definition of the cutoff in terms of the fundamental fields, Eq. (2.25), we choose the cutoff operators $\mathcal{R}_{k}^{\text{grav}}$ and $\mathcal{R}_{k}^{\text{gh}}$ as

$$\mathcal{R}_{k}^{\text{grav}} = \sum_{\zeta_{1}, \zeta_{2} \in \{h^{T}, \xi, \sigma, \phi\}} \Pi_{\zeta_{1}}^{\dagger}(\mathcal{R}_{k})_{\zeta_{1}\zeta_{2}} \Pi_{\zeta_{2}},$$
$$\mathcal{R}_{k}^{\text{gh}} = \sum_{\bar{\vartheta}_{1} \in \{\bar{C}^{T}, \bar{\eta}\}} \sum_{\vartheta_{2} \in \{C^{T}, \eta\}} \Pi_{\bar{\vartheta}_{1}}^{\dagger}(\mathcal{R}_{k})_{\bar{\vartheta}_{1}\vartheta_{2}} \Pi_{\vartheta_{2}}.$$
(A14)

Here $((\mathcal{R}_k)_{\bar{\vartheta}_1\vartheta_2})_{\bar{\vartheta}_1 \in \{\bar{C}^T, \bar{\eta}\}, \vartheta_2 \in \{C^T, \eta\}}$ represents a block of the more general matrix operator $((\mathcal{R}_k)_{\psi_1\psi_2})_{\psi_1,\psi_2 \in \{\bar{C}^T, \bar{\eta}, C^T, \eta\}}$ for which $(\mathcal{R}_k)_{\psi_1\psi_2} \equiv 0$ if both $\psi_1 \in \{C^T, \eta\}$ and $\psi_2 \in \{C^T, \eta\}$, or if both $\psi_1 \in \{\bar{C}^T, \bar{\eta}\}$ and $\psi_2 \in \{\bar{C}^T, \bar{\eta}\}$. The operators $(\mathcal{R}_k)_{\zeta_1\zeta_2}$ and $(\mathcal{R}_k)_{\psi_1\psi_2}$ are required to satisfy the Hermiticity conditions $(\mathcal{R}_k)_{\zeta_2\zeta_1} = (\mathcal{R}_k)_{\zeta_1\zeta_2}^{\dagger}$ and $(\mathcal{R}_k)_{\psi_2\psi_1} = -(\mathcal{R}_k)_{\psi_1\psi_2}^{\dagger}$. Furthermore, we set $\Pi_h \tau \equiv \Pi_{TT}$, $\Pi_{\xi} \equiv \Pi_{LT}$, $\Pi_{\sigma} \equiv \Pi_{LL}$, $\Pi_{\phi} \equiv \Pi_{Tr}$, $\Pi_{\bar{C}} \tau \equiv \Pi_C \tau \equiv \Pi_T$, $\Pi_{\bar{\eta}} \equiv \Pi_{\eta} \equiv \Pi_L$, and Π_{ζ}^{\dagger} , Π_{ψ}^{\dagger} denote the Hermitian conjugates of Π_{ζ} , Π_{ψ} . Hence the operators II appearing in Eq. (A14) can be inferred from Eqs. (A12), (A13).

Inserting Eq. (A14) into Eq. (2.25) leads to

$$\Delta_{k}S[h,C,\bar{C};\bar{g}]$$

$$=\frac{1}{2}\sum_{\zeta_{1},\zeta_{2}\in\{h^{T},\xi,\sigma,\phi\}} \langle \Pi_{\zeta_{1}}h,(\mathcal{R}_{k})_{\zeta_{1}\zeta_{2}}\Pi_{\zeta_{2}}h\rangle$$

$$+\sum_{\bar{\vartheta}_{1}\in\{\bar{C}^{T},\bar{\eta}\}}\sum_{\vartheta_{2}\in\{C^{T},\eta\}} \langle \Pi_{\bar{\vartheta}_{1}}\bar{C},(\mathcal{R}_{k})_{\bar{\vartheta}_{1}\vartheta_{2}}\Pi_{\vartheta_{2}}C\rangle \quad (A15)$$

which can be rewritten in terms of the component fields. In fact, applying Eqs. (A12) and (A13) and using the relation between the matrix operators $(\mathcal{R}_k)_{\bar{\vartheta}_1\vartheta_2}$ and $(\mathcal{R}_k)_{\psi_1\psi_2}$ stated above, we end up with Eq. (2.26).

Let us now consider the source terms in Eq. (2.27). Decomposing the sources for the ghost fields according to

$$K^{\mu} = K^{\mu}_{\bar{C}^{T}} + \bar{D}^{\mu} (-\bar{D}^{2})^{-1/2} K_{\bar{\eta}},$$

$$\bar{K}^{\mu} = \bar{K}^{\mu}_{C^{T}} + \bar{D}^{\mu} (-\bar{D}^{2})^{-1/2} \bar{K}_{\eta}$$
(A16)

with $\bar{D}_{\mu}K^{\mu}_{\bar{C}^{T}} = \bar{D}_{\mu}\bar{K}^{\mu}_{C^{T}} = 0$ yields

$$\langle \bar{K}, C \rangle + \langle K, \bar{C} \rangle = \sum_{\psi \in \{C^T, \eta\}} \langle \bar{K}_{\psi}, \psi \rangle + \sum_{\psi \in \{\bar{C}^T, \bar{\eta}\}} \langle K_{\psi}, \psi \rangle.$$
(A17)

The decomposition of $\langle J,h \rangle$ is more involved. In analogy with $h_{\mu\nu}$ in Eq. (A1) we decompose the source $J_{\mu\nu}$ into its orthogonal parts:

$$J_{\mu\nu} = J_{\mu\nu}^{T} + J_{\mu\nu}^{L} + J_{\mu\nu}^{Tr}.$$
 (A18)

According to Eqs. (A2), (A3) we write $J_{\mu\nu}^{L}$ as

$$J^{L}_{\mu\nu} = J^{LT}_{\mu\nu} + J^{LL}_{\mu\nu}$$
(A19)

with

$$J_{\mu\nu}^{LT} = (L\Theta)_{\mu\nu} = \bar{D}_{\mu}\Theta_{\nu} + \bar{D}_{\nu}\Theta_{\mu},$$

$$J_{\mu\nu}^{LL} = \frac{1}{2}(L\bar{D}\nu)_{\mu\nu} = \bar{D}_{\mu}\bar{D}_{\nu}\nu - \frac{1}{d}\bar{g}_{\mu\nu}\bar{D}^{2}\nu.$$
 (A20)

Analogous to Eq. (A2), the vector $\Theta_{\mu} + \bar{D}_{\mu} v/2$ with $\bar{D}_{\mu} \Theta^{\mu} = 0$ is unique up to CKV's.

According to Eq. (2.14) the inner product $\langle J,h \rangle$ may now be decomposed as follows:

$$\langle J,h\rangle = \langle J^{T},h^{T}\rangle + \langle J^{L},h^{LT}\rangle + \langle J^{L},h^{LL}\rangle + \langle J^{Tr},h^{Tr}\rangle.$$
(A21)

It is important to note that, for arbitrary nonvanishing $h_{\mu\nu}^{T}$, $h_{\mu\nu}^{LT}$, $h_{\mu\nu}^{LL}$, $h_{\mu\nu}^{T}$, we have

$$(\Pi_{TT}h^{T})_{\mu\nu} \neq 0, \ (\Pi_{LT}h^{LT})_{\mu} \neq 0, \ \Pi_{LL}h^{LL} \neq 0, \ \Pi_{Tr}h^{Tr} \neq 0.$$
(A22)

This implies that the operators Π_{TT} , Π_{LT} , Π_{LL} , Π_{Tr} are invertible if their action is restricted to fields of the type $h_{\mu\nu}^{T}$, $h_{\mu\nu}^{LT}$, $h_{\mu\nu}^{LL}$, $h_{\mu\nu}^{Tr}$, respectively. Therefore the inner product of Eq. (A21) may be written as

$$\langle J,h \rangle = \langle J^{T}, \Pi_{TT}^{-1} \Pi_{TT} h^{T} \rangle + \langle J^{L}, \Pi_{LT}^{-1} \Pi_{LT} h^{LT} \rangle$$

$$+ \langle J^{L}, \Pi_{LL}^{-1} \Pi_{LL} h^{LL} \rangle + \langle J^{Tr}, \Pi_{Tr}^{-1} \Pi_{Tr} h^{Tr} \rangle$$

$$= \sum_{\zeta \in \{h^{T}, \xi, \sigma, \phi\}} \langle J_{\zeta}, \zeta \rangle.$$
(A23)

Here we used Eq. (A12) and introduced the source components

$$\begin{split} J_{hT}^{\mu\nu} &= ((\Pi_{TT}^{-1})^{\dagger} J^{T})^{\mu\nu} = J^{T\mu\nu}, \\ J_{\xi}^{\mu} &= ((\Pi_{LT}^{-1})^{\dagger} J^{L})^{\mu} = 2\sqrt{-\bar{D}^{2} - \bar{\operatorname{Ric}}} \ \theta^{\mu} \\ &+ 2(-\bar{D}^{2} - \bar{\operatorname{Ric}})^{-1/2} \bigg[\bigg(\frac{1}{d} \bar{g}^{\mu\nu} \bar{R} - \bar{R}^{\mu\nu} \bigg) \bar{D}_{\nu} + \frac{1}{d} (\bar{D}^{\mu} \bar{R}) \bigg] \nu, \\ J_{\sigma} &= (\Pi_{LL}^{-1})^{\dagger} J^{L} = 2 \bigg((\bar{D}^{2})^{2} + \frac{d}{d-1} \bar{D}_{\mu} \bar{R}^{\mu\nu} \bar{D}_{\nu} \bigg)^{-1/2} \\ &\times \bar{D}_{\alpha} \bar{R}^{\alpha\beta} \theta_{\beta} + \frac{d-1}{d} \\ &\times \sqrt{(\bar{D}^{2})^{2} + \frac{d}{d-1}} \bar{D}_{\mu} \bar{R}^{\mu\nu} \bar{D}_{\nu} \nu, \\ J_{\phi} &= (\Pi_{Tr}^{-1})^{\dagger} J^{Tr} = \frac{1}{d} \bar{g}^{\mu\nu} J_{\mu\nu}^{Tr}. \end{split}$$
(A24)

Then combining the results of Eqs. (A17) and (A23) we eventually arrive at Eq. (2.28).

APPENDIX B: EVALUATING THE RHS OF THE TRUNCATED FLOW EQUATION

In this section we present several rather lengthy calculations needed for the discussion of the Einstein-Hilbert truncation in Sec. IV. In the following, all calculations are performed with $g_{\mu\nu} = \overline{g}_{\mu\nu}$ where $\overline{g}_{\mu\nu}$ is assumed to correspond to a spherical background and the bars are omitted from the metric, the curvature and the operators.

1. Computation of the inverse operators $(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}$ and $(S_{\rm gh}^{(2)} + \mathcal{R}_k)^{-1}$

In Sec. IV we derived explicit expressions for the kinetic operators $\tilde{\Gamma}_{k}^{(2)} \equiv \Gamma_{k}^{(2)} + \mathcal{R}_{k}$ and $\tilde{S}_{\rm gh}^{(2)} \equiv S_{\rm gh}^{(2)} + \mathcal{R}_{k}$. They may be represented as matrix differential operators acting on the column vectors $(\bar{h}^{T}, \bar{\xi}, \bar{\phi}_{0}, \bar{\sigma}, \bar{\phi}_{1})^{\rm T}$ and $(\bar{v}^{T}, v^{T}, \bar{\varrho}, \varrho)^{\rm T}$, respectively. In this representation they take the form

$$\tilde{\Gamma}_{k}^{(2)}[g,g] = \begin{pmatrix} (\tilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{h}}\tau_{\bar{h}}\bar{\tau} & 0 & 0 & 0_{1\times 2} \\ 0 & (\tilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\xi}\bar{\xi}} & 0 & 0_{1\times 2} \\ 0 & 0 & (\tilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\phi}_{0}}\bar{\phi}_{0} & 0_{1\times 2} \\ 0_{2\times 1} & 0_{2\times 1} & 0_{2\times 1} & \mathcal{Q}_{k} \end{pmatrix}$$
(B1)

and

$$\tilde{S}_{gh}^{(2)}[g,g] = \begin{pmatrix} 0 & (\tilde{S}_{gh}^{(2)}[g,g])_{\bar{v}^T \bar{v}^T} & 0 & 0 \\ (\tilde{S}_{gh}^{(2)}[g,g])_{\bar{v}^T \bar{v}^T} & 0 & 0 & 0 \\ 0 & 0 & 0 & (\tilde{S}_{gh}^{(2)}[g,g])_{\bar{\varrho}\bar{\varrho}} \\ 0 & 0 & 0 & (\tilde{S}_{gh}^{(2)}[g,g])_{\bar{\varrho}\bar{\varrho}} \\ 0 & 0 & 0 & (\tilde{S}_{gh}^{(2)}[g,g])_{\bar{\varrho}\bar{\varrho}} & 0 \end{pmatrix}$$
(B2)

where

$$\mathcal{Q}_{k} \equiv \begin{pmatrix} (\widetilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\sigma}\bar{\sigma}} & (\widetilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\phi}_{1}\bar{\sigma}} \\ (\widetilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\phi}_{1}\bar{\sigma}} & (\widetilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\phi}\bar{\phi}} \end{pmatrix}.$$
(B3)

The entries of these matrices are given in Eq. (4.16). On the RHS of the flow equation (3.3) these operators appear in terms of their inverses, which are determined in the following. At this point it is important to note that, because of the maximally symmetric background, all covariant derivatives contained in the operators (B1) and (B2) appear as covariant Laplacians and that the various entries are *x*-independent otherwise. This implies that these entries are *commuting* differential operators which allows for particularly simple manipulations. Therefore it is not difficult to verify that the inverse operators assume the form

$$(\tilde{\Gamma}_{k}^{(2)}[g,g])^{-1} = \begin{pmatrix} [(\tilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{h}^{T}\bar{h}^{T}}]^{-1} & 0 & 0 & 0_{1\times 2} \\ 0 & [(\tilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\xi}\bar{\xi}}]^{-1} & 0 & 0_{1\times 2} \\ 0 & 0 & [(\tilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\phi}_{0}\bar{\phi}_{0}}]^{-1} & 0_{1\times 2} \\ 0_{2\times 1} & 0_{2\times 1} & 0_{2\times 1} & \mathcal{Q}_{k}^{-1} \end{pmatrix}$$
(B4)

and

$$(\tilde{S}_{gh}^{(2)}[g,g])^{-1} = \begin{pmatrix} 0 & [(\tilde{S}_{gh}^{(2)}[g,g])_{\bar{v}^{T}\bar{v}^{T}}]^{-1} & 0 & 0 \\ [(\tilde{S}_{gh}^{(2)}[g,g])_{\bar{v}^{T}\bar{v}^{T}}]^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & [(\tilde{S}_{gh}^{(2)}[g,g])_{\bar{\varrho}\varrho}]^{-1} \\ 0 & 0 & 0 & [(\tilde{S}_{gh}^{(2)}[g,g])_{\varrho\bar{\varrho}}]^{-1} & 0 \end{pmatrix}$$
(B5)

with

$$\mathcal{Q}_{k}^{-1} = \left[(\tilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\sigma}\bar{\sigma}} (\tilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\phi}_{1}\bar{\phi}_{1}} - (\tilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\phi}_{1}\bar{\sigma}}^{2} \right]^{-1} \begin{pmatrix} (\tilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\phi}_{1}\bar{\phi}_{1}} & -(\tilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\phi}_{1}\bar{\sigma}} \\ -(\tilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\phi}_{1}\bar{\sigma}} & (\tilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\sigma}\bar{\sigma}} \end{pmatrix}.$$
(B6)

Inserting these expressions into the RHS of the flow equation (3.3) leads to

$$S_{k}(R) = \frac{1}{2} \operatorname{Tr}' \left[\sum_{\zeta \in \{\bar{h}^{T}, \bar{\xi}, \bar{\phi}_{0}\}} \left[(\Gamma_{k}^{(2)}[g,g] + \mathcal{R}_{k}[g])_{\zeta\zeta} \right]^{-1} \partial_{t} (\mathcal{R}_{k}[g])_{\zeta\zeta} \right] + \frac{1}{2} \operatorname{Tr}' \left[\{ (\tilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\sigma}\bar{\sigma}} (\tilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\phi}_{1}\bar{\phi}_{1}} - (\tilde{\Gamma}_{k}^{(2)}[g,g])_{\bar{\phi}_{1}\bar{\sigma}} \right]^{-1} \{ (\Gamma_{k}^{(2)}[g,g] + \mathcal{R}_{k}[g])_{\bar{\sigma}\bar{\sigma}} \partial_{t} (\mathcal{R}_{k}[g])_{\bar{\phi}_{1}\bar{\phi}_{1}} + (\Gamma_{k}^{(2)}[g,g] + \mathcal{R}_{k}[g])_{\bar{\phi}_{1}\bar{\phi}_{1}} \partial_{t} (\mathcal{R}_{k}[g])_{\bar{\sigma}\bar{\sigma}} - 2(\Gamma_{k}^{(2)}[g,g] + \mathcal{R}_{k}[g])_{\bar{\phi}_{1}\bar{\sigma}} \partial_{t} (\mathcal{R}_{k}[g])_{\bar{\phi}_{1}\bar{\sigma}} \} - \operatorname{Tr}' \left[\sum_{\psi \in \{v^{T}, \varrho\}} \left[(S_{gh}^{(2)}[g,g] + \mathcal{R}_{k}[g])_{\bar{\psi}\psi} \right]^{-1} \partial_{t} (\mathcal{R}_{k}[g])_{\bar{\psi}\psi} \right]$$
(B7)

where we used the relations

$$((S_{\rm gh}^{(2)})_{\bar{v}}\tau_{v}\tau)^{\mu x}{}_{\nu y} = -((S_{\rm gh}^{(2)})_{v}\tau_{\bar{v}}\tau)_{\nu y}{}^{\mu x} = \frac{1}{\sqrt{g(y)}}\frac{\delta}{\delta v}^{T\nu(y)}\frac{1}{\sqrt{g(x)}}\frac{\delta S_{\rm gh}}{\delta \bar{v}_{\mu}^{T}(x)}$$
$$((S_{\rm gh}^{(2)})_{\bar{\varrho}\bar{\varrho}})^{x}{}_{y} = -((S_{\rm gh}^{(2)})_{\varrho\bar{\varrho}})^{x}{}_{y} = \frac{1}{\sqrt{g(y)}}\frac{\delta}{\delta \varrho(y)}\frac{1}{\sqrt{g(x)}}\frac{\delta S_{\rm gh}}{\delta \bar{\varrho}(x)}.$$
(B8)

The trace of the ϕ_0 term appearing in Eq. (B7) may be easily evaluated since only the scalar eigenmodes T^{01} and T^{1m} contribute. We obtain

$$\frac{1}{2} \operatorname{Tr}' \left(\left[\left(\Gamma_{k}^{(2)} \left[g, g \right] + \mathcal{R}_{k} \left[g \right] \right)_{\phi_{0} \phi_{0}} \right]^{-1} \partial_{t} \left(\mathcal{R}_{k} \left[g \right] \right)_{\phi_{0} \phi_{0}} \right) \\
= \frac{1}{2Z_{Nk}} \sum_{l=0}^{1} \sum_{m=1}^{D_{l}(d,0)} \int d^{d}x \sqrt{g(x)} T^{lm}(x) \left[P_{k} + A_{S1}(d,\alpha)R + B_{S1}(d,\alpha)\bar{\lambda}_{k} \right]^{-1} \partial_{t} \left[Z_{Nk} k^{2} R^{(0)}(-D^{2}/k^{2}) \right] T^{lm}(x) \\
= \frac{1}{2Z_{Nk}} \sum_{l=0}^{1} \left[D_{l}(d,0) \frac{\partial_{t} \left[Z_{Nk} k^{2} R^{(0)}(\Lambda_{l}(d,0)/k^{2}) \right]}{\Lambda_{l}(d,0) + k^{2} R^{(0)}(\Lambda_{l}(d,0)/k^{2}) + A_{S1}(d,\alpha)R + B_{S1}(d,\alpha)\bar{\lambda}_{k}} \right].$$
(B9)

Here $\Lambda_l(d,0)$ is the eigenvalue with respect to $-D^2$ corresponding to T^{lm} . Inserting also the remaining operators given in Eq. (4.16) into Eq. (B7) finally leads to Eq. (4.17).

Γ

2. Evaluation of the traces

In this part of the appendix we evaluate the traces appearing in Eq. (4.20) by applying the asymptotic heat kernel expansion. In its original form it has often been used to compute traces of operators acting on unconstrained fields, see e.g. Refs. [35]. For our purposes we need the corresponding expansions for operators acting on constrained fields, i.e. fields satisfying appropriate transversality conditions. In Appendix E these expansions are derived in detail for Laplacians D^2 acting on symmetric transverse traceless tensors, on transverse vectors and on scalars, with the following results:

$$Tr_{(2ST^{2})}[e^{-(is-\varepsilon)D^{2}}] = \left(\frac{i}{4\pi(s+i\varepsilon)}\right)^{d/2} \int d^{d}x \sqrt{g} \left\{\frac{1}{2}(d-2)(d+1) - \frac{(d+1)(d+2)(d-5+3\delta_{d,2})}{12(d-1)}(is-\varepsilon)R + \mathcal{O}(R^{2})\right\},$$
(B10)

$$\operatorname{Tr}_{(1T)}\left[e^{-(\mathrm{i}s-\varepsilon)D^{2}}\right]$$

$$=\left(\frac{\mathrm{i}}{4\pi(s+\mathrm{i}\varepsilon)}\right)^{d/2}\int d^{d}x\sqrt{g}$$

$$\times\left\{d-1-\frac{(d+2)(d-3)+6\,\delta_{d,2}}{6d}(\mathrm{i}s-\varepsilon)R+\mathcal{O}(R^{2})\right\},$$
(B11)

$$\operatorname{Tr}_{(0)}\left[e^{-(\mathrm{i}s-\varepsilon)D^{2}}\right] = \left(\frac{\mathrm{i}}{4\pi(s+\mathrm{i}\varepsilon)}\right)^{d/2} \int d^{d}x \sqrt{g} \left\{1 - \frac{1}{6}(\mathrm{i}s-\varepsilon)R + \mathcal{O}(R^{2})\right\}.$$
(B12)

The next step is to consider an arbitrary function W(z) with a Fourier transform $\tilde{W}(s)$. For such functions W, we may express the trace of the operator $W(-D^2)$ that results from replacing the argument of W with $-D^2$ in terms of $\tilde{W}(s)$:

$$\operatorname{Tr}[W(-D^{2})] = \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} ds \ \tilde{W}(s) \operatorname{Tr}[e^{-(\mathrm{i}s-\varepsilon)D^{2}}].$$
(B13)

We obtain the asymptotic expansion of $\text{Tr}[W(-D^2)]$ by inserting the heat kernel expansion for $\text{Tr}[e^{-(is-\varepsilon)D^2}]$ into Eq. (B13). For Laplacians acting on the constrained fields considered here they read as follows:

$$\begin{aligned} \operatorname{Tr}_{(2ST^2)}[W(-D^2)] \\ &= (4\pi)^{-d/2} \bigg\{ \frac{1}{2} (d-2)(d+1) Q_{d/2}[W] \int d^d x \sqrt{g} \\ &+ \frac{(d+1)(d+2)(d-5+3\delta_{d,2})}{12(d-1)} Q_{d/2-1}[W] \\ &\times \int d^d x \sqrt{g} R + \mathcal{O}(r^{< d-2}) \bigg\}, \end{aligned} \tag{B14}$$

$$Tr_{(1T)}[W(-D^{2})] = (4\pi)^{-d/2} \left\{ (d-1)Q_{d/2}[W] \int d^{d}x \sqrt{g} + \frac{(d+2)(d-3) + 6\delta_{d,2}}{6d} Q_{d/2-1}[W] \times \int d^{d}x \sqrt{g}R + \mathcal{O}(r^{\leq d-2}) \right\},$$
(B15)

$$Tr_{(0)}[W(-D^{2})] = (4\pi)^{-d/2} \left\{ Q_{d/2}[W] \int d^{d}x \sqrt{g} + \frac{1}{6} Q_{d/2-1}[W] \int d^{d}x \sqrt{g}R + \mathcal{O}(r^{
(B16)$$

Here the set of functionals $Q_n[W]$ is defined as

$$Q_n[W] \equiv \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} ds (-is + \varepsilon)^{-n} \widetilde{W}(s).$$
 (B17)

By virtue of the Mellin transformation we may now reexpress Q_n in terms of W so that

$$Q_{n}[W] = \frac{1}{\Gamma(n)} \int_{0}^{\infty} dz \, z^{n-1} W(z), \quad n > 0;$$
$$Q_{n}[W] = \frac{(-1)^{m}}{\Gamma(m+n)} \int_{0}^{\infty} dz \, z^{m+n-1} \frac{d^{m} W(z)}{dz^{m}},$$
$$n \leq 0, \quad m > -n, \quad m \in |\mathbb{N} \text{ arbitrary. (B18)}$$

In particular we obtain $Q_0[W] = W(0)$.

Let us now consider the case where isolated eigenvalues have to be excluded from Tr[$W(-D^2)$]. According to Appendix E such traces can be expressed as the difference between the complete trace Tr[$W(-D^2)$] and a term of the form $\Sigma_{l \in \{l_1, \ldots, l_n\}} D_l(d,s) W(\Lambda_l(d,s))$. Here l_1, \ldots, l_n refer to the modes to be omitted and $\Lambda_l(d,s)$ and $D_l(d,s)$ denote the corresponding eigenvalues of $-D^2$ and their degrees of degeneracy, respectively. Since $\Lambda_l(d,s) \propto R$ we may view $W(\Lambda_l(d,s))$ as a function of R. As outlined in Sec. IV such a function contributes to the evolution of Z_{Nk} and $\overline{\lambda}_k$ only for d=2, with the contribution given by W(0). Using the explicit expressions for $D_l(d,s)$ (see Table I in Appendix D) and applying Eq. (4.7) we therefore obtain for the traces relevant to the flow equation

$$\operatorname{Tr}_{(1T)}^{\prime}[W(-D^{2})] = \operatorname{Tr}_{(1T)}[W(-D^{2})] - \frac{3\delta_{d,2}}{8\pi}W(0)\int d^{2}x\sqrt{g}R + \mathcal{O}(r^{< d-2}),$$

$$\operatorname{Tr}_{(0)}^{\prime\prime}[W(-D^{2})] = \operatorname{Tr}_{(0)}[W(-D^{2})] - \frac{\delta_{d,2}}{2\pi}W(0)\int d^{2}x\sqrt{g}R + \mathcal{O}(r^{< d-2}),$$

$$\operatorname{Tr}_{(0)}^{\prime}[W(-D^{2})] = \operatorname{Tr}_{(0)}[W(-D^{2})] - \frac{\delta_{d,2}}{8\pi}W(0)\int d^{2}x\sqrt{g}R + \mathcal{O}(r^{< d-2}), \tag{B19}$$

where the primes have to be interpreted as in Sec. IV.

The next step is to insert the expansions of the traces into $S_k(R)$, Eq. (4.20), and to compare the coefficients of the operators $\int d^d x \sqrt{g} R$ with those on the LHS, Eq. (4.5). This leads to the following differential equations:

$$\partial_t (Z_{Nk}\bar{\lambda}_k) = (4\kappa^2)^{-1} (4\pi)^{-d/2} \left\{ \frac{1}{2} d(d-1) Q_{d/2} [(\mathcal{A}_0 - 2\bar{\lambda}_k)^{-1} \mathcal{N}] + d Q_{d/2} [(\mathcal{A}_0 - 2\alpha\bar{\lambda}_k)^{-1} \mathcal{N}] - 2d Q_{d/2} [\mathcal{A}_0^{-1} \mathcal{N}_0] \right\},$$
(B20)

$$\partial_{t} Z_{Nk} = -(2\kappa^{2})^{-1} (4\pi)^{-d/2} \Biggl\{ c_{1}(d) Q_{d/2-1} [(\mathcal{A}_{0} - 2\bar{\lambda}_{k})^{-1} \mathcal{N}] + c_{2}(d) Q_{d/2-1} [(\mathcal{A}_{0} - 2\alpha\bar{\lambda}_{k})^{-1} \mathcal{N}] + c_{3}(d) Q_{d/2} [(\mathcal{A}_{0} - 2\bar{\lambda}_{k})^{-2} \mathcal{N}] + c_{4}(d, \alpha) Q_{d/2} [(\mathcal{A}_{0} - 2\alpha\bar{\lambda}_{k})^{-2} \mathcal{N}] - 2c_{2}(d) Q_{d/2-1} [\mathcal{A}_{0}^{-1} \mathcal{N}_{0}] + c_{5}(d) Q_{d/2} [\mathcal{A}_{0}^{-2} \mathcal{N}_{0}] + 3 \delta_{d,2} \Biggl[\frac{\partial_{t}(Z_{Nk}k^{2})}{2Z_{Nk}(k^{2} - 2\bar{\lambda}_{k})} - \frac{\partial_{t}(Z_{Nk}k^{2})}{2Z_{Nk}(k^{2} - 2\alpha\bar{\lambda}_{k})} \Biggr] \Biggr\}.$$
(B21)

Here the coefficients c_i are defined as in Eq. (4.24).

In Eqs. (B20), (B21), the various Q_n may now be expressed in terms of the cutoff-dependent threshold functions Φ_n^p and Φ_n^p introduced in Eq. (4.23). Using the relations

$$Q_{n}[(\mathcal{A}_{0}+c)^{-p}\mathcal{N}] = k^{2(n-p+1)}\Phi_{n}^{p}(c/k^{2}) - \frac{1}{2}\eta_{N}(k)k^{2(n-p+1)}\tilde{\Phi}_{n}^{p}(c/k^{2})$$

$$Q_{n}[(\mathcal{A}_{0}+c)^{-p}\mathcal{N}_{0}] = k^{2(n-p+1)}\Phi_{n}^{p}(c/k^{2})$$
(B22)

we arrive at the differential equations (4.21) and (4.22).

At this stage the following point should be mentioned. In order to achieve that the integrals in Eq. (B17) actually converge we have to demand that $R^{(0)}(y)$ rapidly decreases as $y \to \pm \infty$. However, since from now on its form for y < 0 does not play a role any more we identify $R^{(0)}(y)$ with its part for nonnegative arguments and assume that $R^{(0)}(y)$ is a smooth function defined only for $y \ge 0$ and endowed with the properties stated in Sec. III B.

APPENDIX C: FLOW EQUATIONS IN FOUR DIMENSIONS

In this section we compare our results to those derived in Ref. [21] for the d=4 dimensional case. Inserting d=4 into Eq. (4.17) we obtain for the RHS S_k of the evolution equation

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$$S_{k}(R) = \frac{1}{2} \operatorname{Tr}_{(2ST^{2})} \left[\frac{\partial_{l}P_{k}}{P_{k} + \frac{2}{3}R - 2\bar{\lambda}_{k}} \right] + \frac{1}{2} \operatorname{Tr}_{(1T)}' \left[\frac{\partial_{l}P_{k}}{P_{k} + \frac{2\alpha - 1}{4}R - 2\alpha\bar{\lambda}_{k}} \right] + \frac{1}{2} \operatorname{Tr}_{(0)}'' \left[\frac{\partial_{l}P_{k}}{P_{k} + \frac{\alpha - 1}{2}R - 2\alpha\bar{\lambda}_{k}} \right] + \frac{1}{2} \operatorname{Tr}_{(0)}' \left[\frac{\partial_{l}P_{k}}{P_{k} - 2\bar{\lambda}_{k}} \right] \right] \\ + \frac{1}{2} \sum_{l=0}^{1} \left[D_{l}(4,0) \frac{\partial_{l}P_{k}(\Lambda_{l}(4,0))}{P_{k}(\Lambda_{l}(4,0)) - \frac{4\alpha}{3\alpha - 1}\bar{\lambda}_{k}} \right] - \operatorname{Tr}_{(1T)}' \left[\frac{\partial_{l}P_{k}}{P_{k} - \frac{R}{4}} \right] - \operatorname{Tr}_{(0)}' \left[\frac{\partial_{l}P_{k}}{P_{k} - \frac{R}{2}} \right] \\ + \frac{\partial_{l}Z_{Nk}}{Z_{Nk}} \left\{ \frac{1}{2} \operatorname{Tr}_{(2ST^{2})} \left[\frac{P_{k} + D^{2}}{P_{k} + \frac{2}{3}R - 2\bar{\lambda}_{k}} \right] + \frac{1}{2} \operatorname{Tr}_{(1T)}' \left[\frac{P_{k} + D^{2}}{P_{k} + \frac{2\alpha - 1}{4}R - 2\alpha\bar{\lambda}_{k}} \right] \right] \\ + \frac{1}{2} \sum_{l=0}^{1} \left[D_{l}(4,0) \frac{P_{k}(\Lambda_{l}(4,0)) - \Lambda_{l}(4,0)}{P_{k}(\Lambda_{l}(4,0)) - \frac{4\alpha}{3\alpha - 1}\bar{\lambda}_{k}} \right] - \frac{1}{4\alpha} \operatorname{Tr}_{(0)}' \left[\left[\left[P_{k} - 2\bar{\lambda}_{k} \right] \left[P_{k} + \frac{\alpha - 1}{2}R - 2\alpha\bar{\lambda}_{k} \right] \right]^{-1} \\ \times \left\{ \left(\left(1 - 3\alpha \right) \left[(3 - \alpha)P_{k} + \frac{\alpha - 1}{2}R \right] + 4\alpha(\alpha + 1)\bar{\lambda}_{k} \right) (P_{k} + D^{2}) - 3(1 - \alpha)^{2}\sqrt{P_{k}}\sqrt{P_{k} - \frac{R}{3}} \right] \right\} \right] \right\}$$
(C1)

with $P_k(\Lambda_l(4,0)) \equiv \Lambda_l(4,0) + k^2 R^{(0)}(\Lambda_l(4,0)/k^2)$. Our result (C1) agrees with Eq. (3.22) of Ref. [21], up to a few (typographical) errors occurring in Eq. (3.22). To be more precise, the prime at the Tr'_(1T)-term in lines 1 and 4 of Eq. (C1), the factor $D_l(4,0)$ appearing in the first term of lines 3 and 5, and the factor 1/2 contained in the term $\propto R$ in line 7 are mistakenly left out in the corresponding equation of Ref. [21].

Expanding the flow equations for $Z_{Nk}\bar{\lambda}_k$ and Z_{Nk} , Eqs. (4.21) and (4.22), with respect to $\bar{\lambda}_k$, using the relation $\partial_t \bar{\lambda}_k = Z_{Nk}^{-1} \partial_t (Z_{Nk} \bar{\lambda}_k) + \bar{\lambda}_k \eta_N(k)$ and setting, as in [21],

$$\kappa_{k} \equiv 2\kappa^{2}Z_{Nk}, \quad Z_{k} \equiv Z_{Nk}, \quad \eta_{k} \equiv \partial_{t} \ln Z_{k} \equiv -\eta_{N}(k),$$
$$q_{n}^{p} \equiv 2\Phi_{n}^{p}(0), \quad \widetilde{q}_{n}^{p} \equiv \widetilde{\Phi}_{n}^{p}(0) \tag{C2}$$

leads to

$$\partial_t \kappa_k = (4\pi)^{-2} k^2 \left\{ \frac{13}{24} q_1^1 + \left(\frac{55}{24} + \alpha \right) q_2^2 + \eta_k \left[\frac{1}{8} \tilde{q}_1^1 + \left(\frac{25}{24} + \alpha \right) \tilde{q}_2^2 \right] \right\} + \mathcal{O}(\bar{\lambda}_k), \quad (C3)$$

$$\partial_t \bar{\lambda}_k = \kappa_k^{-1} (4\pi)^{-2} \left\{ k^4 \left[\frac{1}{2} q_2^1 + \eta_k \frac{5}{2} \tilde{q}_2^1 \right] \right. \\ \left. + \bar{\lambda}_k k^2 \left[-\frac{13}{24} q_1^1 + \left(\frac{17}{24} + \alpha \right) q_2^2 \right. \\ \left. + \eta_k \left(-\frac{1}{8} \tilde{q}_1^1 + \left(\frac{47}{24} + \alpha \right) \tilde{q}_2^2 \right) \right] \right\} + \mathcal{O}(\bar{\lambda}_k^2).$$
(C4)

In this form the flow equations for the couplings are suitable for a comparison with the corresponding results in Eqs. (4.6)-(4.9) of Ref. [21]. Apart from the contributions from matter fields, which are also considered there, the results differ by a factor 60/24 in front of the q_2^2 terms. Presumably, this deviation can be explained by a wrong sign introduced in [21] for a certain term. This term is a contribution from the ghosts produced by heat kernel expanding the last two terms in line 3 of Eq. (C1), and it carries the prefactor 30/24. We may conclude that, apart from these corrections, our results agree with those in [21], which is as it should be since the same cutoff is used.

APPENDIX D: TENSOR SPHERICAL HARMONICS

In this section we introduce the spherical harmonics $T^{lm}_{\mu\nu}$, T^{lm}_{μ} and T^{lm} for symmetric transverse traceless (ST^2) tensors $h^T_{\mu\nu}$, transverse (T) vectors ξ_{μ} , and scalars ϕ on a

Eigenfunction	Spin s	Eigenvalue $\Lambda_l(d,s)$	Degeneracy $D_l(d,s)$	
$T^{lm}_{\mu\nu}(x)$	2	$\frac{l(l+d-1)-2}{d(d-1)}\overline{R}$	$\frac{(d+1)(d-2)(l+d)(l-1)(2l+d-1)(l+d-3)!}{2(d-1)!(l+1)!}$	$l = 2, 3, \ldots$
$T^{lm}_{\mu}(x)$	1	$\frac{l(l+d-1)-1}{d(d-1)}\overline{R}$	$\frac{l(l+d-1)(2l+d-1)(l+d-3)!}{(d-2)!(l+1)!}$	$l = 1, 2, \ldots$
$T^{lm}(x)$	0	$\frac{l(l+d-1)}{d(d-1)}\overline{R}$	$\frac{(2l+d-1)(l+d-2)!}{l!(d-1)!}$	$l = 0, 1, \ldots$

TABLE I. Eigenvalues of $-\overline{D}^2$ and their degeneracy on the *d*-sphere.

d-dimensional spherical background S^d . These harmonics form complete sets of orthogonal eigenfunctions with respect to the covariant Laplacians acting on ST^2 tensors, *T* vectors and scalars, i.e. they satisfy

$$\begin{split} &-\bar{D}^{2}T^{lm}_{\mu\nu}(x) = \Lambda_{l}(d,2)T^{lm}_{\mu\nu}(x), \\ &-\bar{D}^{2}T^{lm}_{\mu}(x) = \Lambda_{l}(d,1)T^{lm}_{\mu}(x), \\ &-\bar{D}^{2}T^{lm}(x) = \Lambda_{l}(d,0)T^{lm}(x) \end{split}$$
(D1)

and, after proper normalization,

$$\delta^{lk} \delta^{mn} = \int d^d x \sqrt{\overline{g}} (\mathbf{1}_{(2ST^2)})^{\mu\nu\rho\sigma} T^{lm}_{\mu\nu} T^{kn}_{\rho\sigma}$$
$$= \int d^d x \sqrt{\overline{g}} (\mathbf{1}_{(1T)})^{\mu\nu} T^{lm}_{\mu} T^{kn}_{\nu}$$
$$= \int d^d x \sqrt{\overline{g}} T^{lm} T^{kn}.$$
(D2)

Here $(\mathbf{1}_{(2ST^2)})^{\mu\nu\rho\sigma} = (d-2)/(2d)(\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma} + \bar{g}^{\mu\sigma}\bar{g}^{\nu\rho})$ and $(\mathbf{1}_{(1T)})^{\mu\nu} = (d-1)/d \bar{g}^{\mu\nu}$ are the unit matrices in the spaces of ST^2 tensors and transverse vectors, respectively. In Eq. (D1) the $\Lambda_l(d,s)$'s denote the eigenvalues with respect to $-\overline{D}^2$ where s refers to the spin of the field under consideration and *l* takes the values $\overline{s, s+1, s+2, \ldots}$. Furthermore, the second upper index at $T^{lm}_{\mu\nu}$, T^{lm}_{μ} and T^{lm} , *m*, takes the degeneracy of the eigenvalues into account. It assumes values from one to $D_1(d,s)$ with $D_1(d,s)$ the degree of degeneracy. In Ref. [36] explicit expressions for $\Lambda_l(d,s)$ and $D_{I}(d,s)$ are derived which can be found in Table I. The eigenvalues are expressed in terms of the curvature scalar $\overline{R} = d(d-1)/r^2$ of the sphere with radius r. In Ref. [36] it is also shown that the spherical harmonics $T^{lm}_{\mu\nu}$, T^{lm}_{μ} and T^{lm} span the spaces of ST^2 tensors, T vectors and scalars so that we may expand arbitrary functions $h_{\mu\nu}^T$, ξ_{μ} and ϕ according to

$$h_{\mu\nu}^{T}(x) = \sum_{l=2}^{\infty} \sum_{m=1}^{D_{l}(d,2)} h_{lm}^{T} T_{\mu\nu}^{lm}(x),$$

$$\xi_{\mu}(x) = \sum_{l=1}^{\infty} \sum_{m=1}^{D_{l}(d,1)} \xi_{lm} T_{\mu}^{lm}(x),$$

$$\phi(x) = \sum_{l=0}^{\infty} \sum_{m=1}^{D_l(d,0)} \phi_{lm} T^{lm}(x).$$
(D3)

Here the coefficients $\{h_{lm}^T\}$, $\{\xi_{lm}\}$ and $\{\phi_{lm}\}$ are countably infinite sets of constants that are uniquely determined by $h_{\mu\nu}^T$, ξ_{μ} and ϕ . Equation (D3) may now be used to expand also any symmetric non- T^2 tensor and nontransverse vector in terms of spherical harmonics since they may be expressed in terms of ST^2 tensors, T vectors and scalars by using the decompositions (2.9), (2.18), see e.g. [36–39].

In this context it is important to note that the $D_1(d,1) = d(d+1)/2$ modes $T^{1,m}_{\mu}$ and the $D_1(d,0) = d+1$ modes $T^{1,m}$ satisfy the Killing equation (2.11) and the scalar equation (2.12), respectively, and that $T^{0,1} = \text{const.}$ As discussed in Sec. II B, arbitrary symmetric rank 2 tensors receive no contribution from these modes. In the case of arbitrary vectors it is the constant scalar mode that does not contribute. Such modes have no physical meaning and have to be omitted therefore.

APPENDIX E: HEAT KERNEL COEFFICIENTS FOR DIFFERENTIALLY CONSTRAINED FIELDS

In this part of the appendix we supply the tools necessary for the evaluation of functional traces and we derive the heat kernel expansions for Laplacians acting on differentially constrained fields.

As a first step we consider a functional trace of the form

$$\operatorname{Tr}_{(s[C])}[f(-D^{2})] = \int d^{d}x \sqrt{g} \langle x|f(-D^{2})|x\rangle_{\mu_{1}\dots\mu_{s}}^{\mu_{1}\dots\mu_{s}}$$
$$= \int d^{d}x [(f(-D^{2}))^{\mu_{1}\dots\mu_{s}\nu_{1}\dots\nu_{s}}_{\chi(\mathbf{1}_{(s[C])})_{\nu_{1}\dots\nu_{s}\mu_{1}\dots\mu_{s}} \delta^{d}(x-y)]_{x=y}.$$
(E1)

Here *f* is an arbitrary smooth function whose argument is replaced with the covariant Laplacian defined on the space of spin-*s* fields with a possible symmetry and/or transversality constraint *C*, as indicated by the subscript (s[C]) at the trace. Note that *f* inherits the matrix structure from the corresponding Laplacian. Furthermore, $\mathbf{1}_{(s[C])}$ denotes the unit matrix in the space of independent field components. Given a closed⁷ Riemannian manifold (\mathcal{M},g) we now assume that $\{U_{\mu_1 \dots \mu_s}^k(x)\}$ is a complete set of orthonormal functions on (\mathcal{M},g) spanning the space of fields under consideration. Then by making use of the completeness relation

$$(\mathbf{1}_{(s[C])})_{\nu_{1}...\nu_{s}\mu_{1}...\mu_{s}} \frac{\delta^{d}(x-y)}{\sqrt{g(x)}}$$
$$= \frac{1}{2} \sum_{k} (U^{k}_{\nu_{1}...\nu_{s}}(x)U^{k}_{\mu_{1}...\mu_{s}}(y)$$
$$+ U^{k}_{\mu_{1}...\mu_{s}}(x)U^{k}_{\nu_{1}...\nu_{s}}(y))$$
(E2)

Eq. (E1) can be written as

$$Tr_{(s[C])}[f(-D^{2})] = \int d^{d}x \sqrt{g} \sum_{k} U^{k}_{\mu_{1}...\mu_{s}}(x) \times (f(-D^{2}))^{\mu_{1}...\mu_{s}\nu_{1}...\nu_{s}} U^{k}_{\nu_{1}...\nu_{s}}(x).$$
(E3)

Clearly if $\{U_{\mu_1...\mu_s}^k(x)\}$ is taken to be a complete set of orthonormal eigenfunctions with respect to the covariant Laplacian such that $-D^2 U_{\mu_1...\mu_s}^k(x) = \tilde{\Lambda}_k U_{\mu_1...\mu_s}^k(x)$ Eq. (E3) boils down to

$$\operatorname{Tr}_{(s[C])}[f(-D^2)] = \sum_{k} f(\widetilde{\Lambda}_k).$$
(E4)

In general the evaluation of such traces is a formidable task and one has to resort to approximations. The most familiar such approximation is the "early time" expansion of the diagonal heat kernel $\langle x|e^{-itD^2}|x\rangle$ for $|t| \rightarrow 0$. [We write $t \equiv s + i\varepsilon$ with Im(t) = $\varepsilon > 0$.] It has been discussed in many references [35]:

$$\langle x | e^{-it(D^2 + Q)} | x \rangle = \left(\frac{i}{4\pi t} \right)^{d/2} \{ a_0(x;Q) - it \, a_2(x;Q) - t^2 \, a_4(x;Q) + \mathcal{O}(t^3) \}.$$
(E5)

Here Q is an arbitrary smooth matrix potential and the a_n 's are tensor polynomials proportional to the n/2-th power of the curvature and endowed with the same matrix structure as $D^2 + Q$. They depend on the space of fields under consideration. For operators $D^2 + Q$ acting on unconstrained fields, i.e. fields with independent components, the first three coefficients take the form⁸

$$a_{0}(x;Q) = a_{0} = \mathbf{I},$$

$$a_{2}(x;Q) = P,$$

$$a_{4}(x;Q) = \frac{1}{180} (R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - R_{\mu\nu}R^{\mu\nu} + D^{2}R)\mathbf{1}$$

$$+ \frac{1}{2}P^{2} + \frac{1}{12}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \frac{1}{6}D^{2}P.$$
(E6)

Here **1** is the unit matrix in the space of field components, and

$$P \equiv Q + \frac{1}{6}R$$
 1. (E7)

Furthermore, $\mathcal{R}_{\mu\nu}$ is the curvature operator defined as the commutator

$$R_{\mu\nu} = [D_{\mu}, D_{\nu}] \equiv D_{\mu} D_{\nu} - D_{\nu} D_{\mu}.$$
(E8)

Inserting the asymptotic expansion (E5) into Eq. (E1) we obtain for the functional trace of the heat kernel

$$\operatorname{Tr}[e^{-\operatorname{it}(D^{2}+Q)}] = \left(\frac{\mathrm{i}}{4\pi \mathrm{t}}\right)^{d/2} \int d^{d}x \,\sqrt{g} \{\operatorname{tr} a_{0} - \operatorname{it} \operatorname{tr} a_{2}(x;Q) - \mathrm{t}^{2} \operatorname{tr} a_{4}(x;Q) + \mathcal{O}(\mathrm{t}^{3})\}$$
(E9)

where tr is the matrix trace with respect to the tensor or spinor indices.

Let us have a closer look at the coefficients a_n . For arbitrary scalars ϕ , $\mathcal{R}_{\mu\nu}$ vanishes since $D_{\mu}D_{\nu}\phi = D_{\nu}D_{\mu}\phi$. In the case of arbitrary spin-*s* fields $F_{\mu_1...\mu_s}(x)$ with integer spin $s \ge 1$ we obtain

$$\mathcal{R}_{\alpha\beta}F_{\mu_{1}\dots\mu_{s}}(x) = \sum_{i=1}^{s} R_{\alpha\beta\mu_{i}}{}^{\mu}F_{\mu_{1}\dots\mu_{i-1}\mu\mu_{i+1}\dots\mu_{s}}(x).$$
(E10)

Hence, for scalars, a_4 receives no contribution from $\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta}$, while for fields with nonzero spin it amounts to a nonvanishing contribution with a $(d^s \times d^s)$ -matrix structure. For arbitrary vectors and rank-2 tensors we find, respectively,

$$(\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta})_{\mu\nu} = -R_{\alpha\beta\gamma\mu}R^{\alpha\beta\gamma}{}_{\nu},$$

$$(\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta})_{\mu\nu\rho\sigma} = -R_{\alpha\beta\gamma\mu}R^{\alpha\beta\gamma}{}_{\rho}g_{\nu\sigma}$$

$$-R_{\alpha\beta\gamma\nu}R^{\alpha\beta\gamma}{}_{\sigma}g_{\mu\rho} + 2R_{\alpha\beta\mu\rho}R^{\alpha\beta}{}_{\nu\sigma}.$$
(E11)

From now on we restrict our considerations to matrix potentials Q of the form $Q = qR\mathbf{1}_{(s)}$ with q a real constant, and we assume that the metric corresponds to a maximally symmetric background S^d . Setting $a_n(x;qR\mathbf{1}_{(s)}) \equiv a_n(q)$ we obtain $a_0(q) \equiv a_0 = \mathbf{1}_{(s)}$ and $a_2(q) = (1+6q)/6R\mathbf{1}_{(s)}$, independently of the spin s of the field. Here the dependence on s is

⁷The restriction to closed Riemannian manifolds is done for the sake of notational simplicity only. In principle, the results extend to noncompact asymptotically flat Riemannian manifolds.

⁸In the context of the Einstein-Hilbert truncation performed in the present paper we only need the coefficients $a_0 \propto r^0$ and $a_2 \propto r^{-2}$. However, for truncations containing invariants quadratic in the curvature we shall also need $a_4 \propto r^{-4}$, see [24].

totally encrypted in the unit matrix $\mathbf{1}_{(s)}$. This is not the case for the coefficient a_4 . It is given by

$$a_4(q) = \frac{1}{360} \left[\frac{5d^2 - 7d + 6}{d(d-1)} + 60q + 180q^2 \right] \mathbf{1}_{(0)} R^2,$$
(E12)

$$[a_4(q)]_{\mu\nu} = \frac{1}{360} \left[\frac{5d^3 - 7d^2 + 6d - 60}{d^2(d-1)} + 60q + 180q^2 \right] \times (\mathbf{1}_{(1)})_{\mu\nu} R^2,$$
(E13)

$$[a_{4}(q)]_{\mu\nu\rho\sigma} = \frac{1}{360} \left[\frac{5d^{3} - 7d^{2} + 6d - 120}{d^{2}(d-1)} + 60q + 180q^{2} \right]$$
$$\times (\mathbf{1}_{(2)})_{\mu\nu\rho\sigma} R^{2} + \frac{1}{3d^{2}(d-1)^{2}}$$
$$\times (g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\nu\rho})R^{2}$$
(E14)

for scalars, vectors and rank-2 tensors, respectively, with $\mathbf{1}_{(0)} = 1$, $(\mathbf{1}_{(1)})_{\mu\nu} = g_{\mu\nu}$ and $(\mathbf{1}_{(2)})_{\mu\nu\rho\sigma} = g_{\mu\rho}g_{\nu\sigma}$.

Up to this point we considered only unconstrained fields. For fields subject to constraints like transverse vectors ξ_{μ} and ST^2 tensors $h_{\mu\nu}^T$ we cannot directly apply Eqs. (E6) or (E13), (E14). However, the heat kernel coefficients for ξ_{μ} and $h_{\mu\nu}^T$ can be computed from those of the unconstrained fields using the decompositions (2.18), (2.9) for arbitrary vectors ε_{μ} and arbitrary symmetric tensors $h_{\mu\nu}$.

From Appendix D we can infer that the sets of orthonormal $-D^2$ eigenfunctions

$$\{T^{lm}_{\mu}|m \in \{1, \dots, D_{l}(d, 1)\}, \ l = 1, 2, \dots\}$$
$$\cup \{(\Lambda_{l}(d, 0))^{-1/2} D_{\mu} T^{lm}|m$$
$$\in \{1, \dots, D_{l}(d, 0)\}, l = 1, 2, \dots\}$$
(E15)

and

$$\{T_{\mu\nu}^{lm}|m\in\{1,\ldots,D_l(d,2)\}, l=2,3,\ldots\}$$

$$\cup \left\{ \left(2 \left(\Lambda_{l}(d,1) - \frac{R}{d} \right) \right)^{-1/2} (D_{\mu} T_{\nu}^{lm} + D_{\nu} T_{\mu}^{lm}) \middle| m \in \{1, \dots, D_{l}(d,1)\}, l = 2, 3, \dots \right\}$$

$$\cup \left\{ \left(\Lambda_{l}(d,0) \left(\frac{d-1}{d} \Lambda_{l}(d,0) - \frac{R}{d} \right) \right)^{-1/2} \left(D_{\mu} D_{\nu} - \frac{1}{d} g_{\mu\nu} D^{2} \right) T^{lm} \middle| m \in \{1, \dots, D_{l}(d,0)\}, l = 2, 3, \dots \right\}$$

$$\cup \left\{ \frac{1}{\sqrt{d}} g_{\mu\nu} T^{lm} \middle| m \in \{1, \dots, D_{l}(d,0)\}, l = 0, 1, \dots \right\}$$

$$(E16)$$

span the spaces of *all* vectors ε_{μ} and of *all* symmetric tensors $h_{\mu\nu}$, respectively. Here the T^{lm} 's are the normalized spherical harmonics of Eq. (D2).

Now we insert these eigenfunctions into the trace formula (E3) with f taken to be an exponential. Then we use the commutation relations of Appendix G in order to pull the D_{μ} 's from the TT decomposition through the exponentials and to combine them to Laplacians. This leads to the following decomposition for the traced heat kernels of the unconstrained vectors and symmetric tensors in terms of the heat kernels for the differentially constrained fields:

$$\operatorname{Tr}_{(1)}[e^{-\operatorname{it}(D^{2}+qR)}]$$

$$=\operatorname{Tr}_{(1T)}[e^{-\operatorname{it}(D^{2}+qR)}]+\operatorname{Tr}_{(0)}[e^{-\operatorname{it}(D^{2}+[(dq+1)/d]R)}]$$

$$-e^{-\operatorname{it}[(dq+1)/d]R},$$
(E17)

 $\mathrm{Tr}_{(2S)}[e^{-\mathrm{it}(D^2+qR)}]$

$$= \operatorname{Tr}_{(2ST^{2})} \left[e^{-\operatorname{it}(D^{2} + qR)} \right] + \operatorname{Tr}_{(1T)} \left[e^{-\operatorname{it}(D^{2} + ((d+1)/[d(d-1)] + q)R)} \right] + \operatorname{Tr}_{(0)} \left[e^{-\operatorname{it}(D^{2} + [2/(d-1)] + q]R)} \right] + \operatorname{Tr}_{(0)} \left[e^{-\operatorname{it}(D^{2} + qR)} \right] - e^{-\operatorname{it}(2/(d-1)) + q)R} - (d+1) e^{-\operatorname{it}(1/(d-1)) + q)R} - \frac{d(d+1)}{2} e^{-\operatorname{it}(2/[d(d-1)]] + q)R}.$$
(E18)

The last term of Eq. (E17) and the last three terms of Eq. (E18) arise from those spherical harmonics T^{lm} and T^{lm}_{μ} which are not contained in the sets of eigenfunctions (E15) and (E16). To be more precise, the last term in Eq. (E17) comes from the constant eigenmode $T^{0,1}$ of the operator D^2

+(dq+1)/dR. Furthermore, the last but second and the last but first term in Eq. (E18) take account of the eigenmodes $T^{0,1}$ = const and $T^{1,m}$ of the operator $D^2 + (2/(d-1)+q)R$, respectively. As discussed in Sec. II B the $T^{1,m}$'s satisfy the scalar equation (2.12) and are therefore in a one-to-one correspondence with the PCKV's of S^d . The last term in Eq. (E18) comes from the eigenmodes $T^{1,m}_{\mu}$ of the operator D^2 +((d+1)/(d(d-1))+q)R, which are the KV's of S^d .

These subtraction terms compensate for the corresponding unphysical contributions contained in the *complete* traces for the constrained fields on the RHS of Eqs. (E17), (E18). This can be seen as follows. Consider the functional trace of Eq. (E3). Omitting the contributions from the modes U_{k_1}, \ldots, U_{k_n} , we denote the functional trace involving only the remaining modes with $\operatorname{Tr}'_{(s[C])}[f(-D^2)]$. Then Eq. (E4) implies the following relation between $\operatorname{Tr}'_{(s[C])}[f(-D^2)]$ and the *complete* trace $\operatorname{Tr}_{(s[C])}[f(-D^2)]$:

$$\operatorname{Tr}_{(s[C])}'[f(-D^{2})] = \operatorname{Tr}_{(s[C])}[f(-D^{2})] - \sum_{k \in \{k_{1}, \dots, k_{n}\}} f(\tilde{\Lambda}_{k}).$$
(E19)

This rule indeed yields the last term in Eq. (E17) and the last three terms in Eq. (E18).

As the next step we insert the asymptotic expansion (E9) into both sides of Eqs. (E17) and (E18) and compare the coefficients of R. This leads to the following Seeley coefficients for the constrained fields:

$$\operatorname{tr} a_{0}|_{(1T)} = \operatorname{tr} a_{0}|_{(1T)} - \operatorname{tr} a_{0}|_{(0)},$$

$$\operatorname{tr} a_{2}(q)|_{(1T)} = \operatorname{tr} a_{2}(q)|_{(1)} - \operatorname{tr} a_{2}\left(\frac{dq+1}{d}\right)\Big|_{(0)} + \frac{1}{2}\,\delta_{d,2}\,R,$$

$$\operatorname{tr} a_{4}(q)|_{(1T)} = \operatorname{tr} a_{4}(q)|_{(1)} - \operatorname{tr} a_{4}\left(\frac{dq+1}{d}\right)\Big|_{(0)} + \frac{1}{4}\,\delta_{d,2}(1+2q)\,R^{2} + \frac{1}{24}\,\delta_{d,4}\,R^{2},$$

(E20)

$$\operatorname{tr} a_0|_{(2ST^2)} = \operatorname{tr} a_0|_{(2S)} - \operatorname{tr} a_0|_{(1T)} - 2\operatorname{tr} a_0|_{(0)},$$

$$\operatorname{tr} a_{2}(q)|_{(2ST^{2})} = \operatorname{tr} a_{2}(q)|_{(2S)} - \operatorname{tr} a_{2}\left(\frac{d+1}{d(d-1)} + q\right)\Big|_{(1T)}$$
$$-\operatorname{tr} a_{2}\left(\frac{2}{d-1} + q\right)\Big|_{(0)} - \operatorname{tr} a_{2}(q)|_{(0)}$$
$$+ \frac{7}{2}\delta_{d,2}R,$$

$$\operatorname{tr} a_{4}(q)|_{(2ST^{2})} = \operatorname{tr} a_{4}(q)|_{(2S)} - \operatorname{tr} a_{4} \left(\frac{d+1}{d(d-1)} + q \right) \Big|_{(1T)} - \operatorname{tr} a_{4} \left(\frac{2}{d-1} + q \right) \Big|_{(0)} - \operatorname{tr} a_{4}(q)|_{(0)} + \delta_{d,2} \left(4 + \frac{7}{2}q \right) R^{2} + \frac{2}{3} \delta_{d,4} R^{2}.$$
(E21)

The terms proportional to the δ 's originate from the subtraction terms on the RHS of Eqs. (E17), (E18) which are due to the unphysical eigenmodes. These terms have an expansion of the form $\sum_{m=0}^{\infty} b_{2m} r^{-2m}$, while the terms of the heat kernel expansion are of the form $\int d^d x \sqrt{g} \operatorname{tr} a_n \propto r^{d-n}$. Comparing powers of $R \propto 1/r^2$, only under the condition -2m=d-n a given term $b_{2m}r^{-2m}$ contributes to $\int d^d x \sqrt{g} \operatorname{tr} a_n$. Hence for n, m fixed, the Seeley coefficients tr a_n for the differentially constrained fields receive a contribution from a term of the form $b_{2m}r^{-2m}/(\int d^d x \sqrt{g})$ at most for one specific value of the dimensionality d. In particular, the subtraction terms in Eqs. (E17), (E18) do not contribute to tr a_0 , while tr a_2 and tr a_4 on the LHS of Eqs. (E20), (E21) receive contributions from terms of the form $\delta_{d,2}b_0/r^2$ and $\delta_{d,2}b_2/r^4$, $\delta_{d,4}b_0/r^4$, respectively.

The matrix traces on the RHS of Eqs. (E20), (E21) can now be evaluated by using the heat kernel coefficients for the (differentially) unconstrained fields. For scalars we have tr $a_n(q) = a_n(q)(\mathbf{1}_{(0)} = 1)$. For vectors the traces are evaluated according to tr $a_n(q) = g^{\mu\nu}[a_n(q)]_{\mu\nu}$ so that we obtain from Eqs. (E6), (E13)

$$\operatorname{tr} a_0|_{(1)} = d$$

$$\operatorname{tr} a_2(q)|_{(1)} = \frac{1+6q}{6} dR$$

$$\operatorname{tr} a_4(q)|_{(1)} = \frac{1}{360} \left[\frac{5d^3 - 7d^2 + 6d - 60}{d(d-1)} + 60dq + 180dq^2 \right] R^2.$$
(E22)

In order to determine tr a_n for symmetric tensor fields we have to symmetrize the heat kernel for unconstrained rank-2 tensors according to

$$\langle x | e^{-is(D^{2}+qR)} | x \rangle^{\mu\nu\rho\sigma} |_{(2S)} = \frac{1}{4} \{ \langle x | e^{-is(D^{2}+qR)} | x \rangle^{\mu\nu\rho\sigma} |_{(2)}$$

$$+ \langle x | e^{-is(D^{2}+qR)} | x \rangle^{\nu\mu\rho\sigma} |_{(2)}$$

$$+ \langle x | e^{-is(D^{2}+qR)} | x \rangle^{\mu\nu\sigma\rho} |_{(2)}$$

$$+ \langle x | e^{-is(D^{2}+qR)} | x \rangle^{\nu\mu\sigma\rho} |_{(2)} \}$$
(E23)

before we can apply Eq. (E9) with Eqs. (E6) and (E14). This leads to

Field	tr a ₀	tr $a_2(q)$	tr $a_4(q)$
<i>ST</i> ² tensor	$\frac{(d-2)(d+1)}{2}$	$\frac{(d+1)(d+2)(d-5+3 \delta_{d,2})}{12(d-1)}R \\ +q \frac{(d-2)(d+1)}{2}R$	$\frac{(d+1)(5d^4 - 22d^3 - 83d^2 - 392d - 228 + 1440 \delta_{d,2} + 3240 \delta_{d,4})}{720d(d-1)^2} R^2 + q \frac{(d+1)(d+2)(d-5+3 \delta_{d,2})}{12(d-1)} R^2 + q^2 \frac{(d-2)(d+1)}{4} R^2$
<i>T</i> vector	d - 1	$\frac{(d+2)(d-3)+6 \delta_{d,2}}{6d}R \\ +q(d-1)R$	$\frac{5d^4 - 12d^3 - 47d^2 - 186d + 180 + 360 \delta_{d,2} + 720 \delta_{d,4}}{360d^2(d-1)} R^2 + q \frac{(d+2)(d-3) + 6 \delta_{d,2}}{6d} R^2 + q^2 \frac{d-1}{2} R^2$
scalar	1	$\frac{1+6q}{6}R$	$\frac{5d^2 - 7d + 6}{360d(d-1)}R^2 + \frac{1}{6}q R^2 + \frac{1}{2}q^2 R^2$

TABLE II. Heat kernel coefficients.

$$[a_{n}(q)]_{\mu\nu\rho\sigma}|_{(2S)} = \frac{1}{4} ([a_{n}(q)]_{\mu\nu\rho\sigma}|_{(2)} + [a_{n}(q)]_{\nu\mu\rho\sigma}|_{(2)} + [a_{n}(q)]_{\mu\nu\sigma\rho}|_{(2)} + [a_{n}(q)]_{\nu\mu\sigma\rho}|_{(2)})$$
(E24)

and in particular $(\mathbf{1}_{(2S)})_{\mu\nu\rho\sigma} = (g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho})/2$. For tensors the matrix traces are computed according to tr $a_n = g^{\mu\rho}g^{\nu\sigma}[a_n(q)]_{\mu\nu\rho\sigma}$ which yields

$$\operatorname{tr} a_0|_{(2S)} = \frac{1}{2}d(d+1)$$
$$\operatorname{tr} a_2(q)|_{(2S)} = \frac{1+6q}{12}d(d+1)R$$

$$\operatorname{tr} a_4(q)\big|_{(2S)} = \frac{1}{720} \bigg[\frac{5d^4 - 2d^3 - d^2 - 114d - 240}{d(d-1)} + 60q(1+3q)d(d+1) \bigg] R^2.$$
(E25)

Finally we insert the matrix traces Eqs. (E22), (E25) and tr $a_n|_{(0)} = a_n|_{(0)}$ into Eqs. (E20), (E21) and determine the heat kernel coefficients for transverse vectors and ST^2 tensors. The results are summarized in Table II.

Let us add a final remark concerning the applicability of the asymptotic expansion (E5). Since it is valid only in the limit $|t| \rightarrow 0$ it is clear that it cannot be integrated over $\operatorname{Re}(t) = s$ or $\operatorname{Im}(t) = \varepsilon$ term by term, in general. However, this is possible if the heat kernel is integrated against a "test" function which suppresses large values of s or ε . This is indeed the case for our application of the asymptotic expansion presented in Appendix B 2.

APPENDIX F: VARIOUS COEFFICIENTS

In this appendix we define the coefficient functions which appear in Eqs. (4.13)-(4.20) of Sec. IV C and in Eq. (B9) of Appendix B 1:

$$A_{T}(d) \equiv \frac{d(d-3)+4}{d(d-1)}, \quad A_{V}(d,\alpha) \equiv \frac{\alpha(d-2)-1}{d}, \quad A_{S1}(d,\alpha) \equiv \frac{\alpha(d-4)}{2\alpha(d-1)-(d-2)}$$

$$A_{S2}(d,\alpha) \equiv -\frac{\alpha(d-2)-2}{\alpha(d-2)-2(d-1)}, \quad A_{S3}(d) \equiv \frac{d-4}{d}, \quad A_{S4}(d,\alpha) \equiv \frac{\alpha(d-2)-2}{d},$$

$$A_{S5} \equiv \frac{(d-2)(d+2)\alpha^{2} + (d^{2}-10d+8)\alpha + 2(d-2)}{d^{2}\alpha},$$

$$B_{S1}(d,\alpha) \equiv -\frac{2\alpha d}{2\alpha (d-1) - (d-2)}, \ B_{S2}(d,\alpha) \equiv \frac{2\alpha d}{\alpha (d-2) - 2(d-1)},$$

$$C_{S1}(d,\alpha) = -\frac{2\alpha(d-1) - (d-2)}{4(d-1) - 2\alpha(d-2)} \frac{d-2}{d-1}, \quad C_{S2}(d,\alpha) = \frac{d-1}{d^2} \frac{2(d-1) - \alpha(d-2)}{\alpha},$$

$$C_{S3}(d,\alpha) = \frac{(d-2)(\alpha-1)}{\alpha(d-2) - 2(d-1)}, \quad E_S(d,\alpha) = -\frac{d-2}{4\alpha d^2} [2\alpha(d-1) - (d-2)],$$

$$F_{S1}(d,\alpha) = -\frac{2[\alpha(d-2) - 2(d-1)][2\alpha(d-1) - (d-2)]}{d^2\alpha},$$

$$F_{S2}(d,\alpha) = \frac{4(d-2)(d-1)(\alpha-1)^2}{d^2\alpha}.$$
(F1)

APPENDIX G: COMMUTATION RELATIONS FOR A MAXIMALLY SYMMETRIC BACKGROUND

In the following we summarize the commutation relations which were used in order to derive Eq. (4.13) of Sec. IV C, and Eqs. (E17) and (E18) of Appendix E. They are valid for the class of maximally symmetric backgrounds:

$$\bar{D}_{\mu}\bar{D}_{\nu}\hat{\xi}^{\mu} = \frac{R}{d}\hat{\xi}_{\nu} \tag{G1}$$

$$\bar{D}^{2}(\bar{D}_{\mu}\hat{\xi}_{\nu}+\bar{D}_{\nu}\hat{\xi}_{\mu}) = \bar{D}_{\mu}\left(\bar{D}^{2}+\frac{(d+1)\bar{R}}{d(d-1)}\right)\hat{\xi}_{\nu}+\bar{D}_{\nu}\left(\bar{D}^{2}+\frac{(d+1)\bar{R}}{d(d-1)}\right)\hat{\xi}_{\mu}$$
(G2)

$$(\bar{D}_{\mu}\hat{\xi}_{\nu}+\bar{D}_{\nu}\hat{\xi}_{\mu})(\bar{D}^{\mu}\hat{\xi}^{\nu}+\bar{D}^{\nu}\hat{\xi}^{\mu}) = -2\hat{\xi}_{\mu}\left(\bar{D}^{2}+\frac{\bar{R}}{d}\right)\hat{\xi}^{\mu} + \text{cov. divergence}$$
(G3)

$$\bar{D}^2 \bar{D}_{\mu} \hat{\sigma} = \bar{D}^{\nu} \bar{D}_{\mu} \bar{D}_{\nu} \hat{\sigma} = \bar{D}_{\mu} \left(\bar{D}^2 + \frac{\bar{R}}{d} \right) \hat{\sigma}$$
(G4)

$$\bar{D}^2 \left(\bar{D}_{\mu} \bar{D}_{\nu} - \frac{1}{d} \bar{g}_{\mu\nu} \bar{D}^2 \right) \hat{\sigma} = \left(\bar{D}_{\mu} \bar{D}_{\nu} - \frac{1}{d} \bar{g}_{\mu\nu} \bar{D}^2 \right) \left(\bar{D}^2 + \frac{2\bar{R}}{d-1} \right) \hat{\sigma}$$
(G5)

$$\left(\bar{D}_{\mu}\bar{D}_{\nu}\hat{\sigma} - \frac{1}{d}\bar{g}_{\mu\nu}\bar{D}^{2}\hat{\sigma}\right)\left(\bar{D}^{\mu}\bar{D}^{\nu}\hat{\sigma} - \frac{1}{d}\bar{g}^{\mu\nu}\bar{D}^{2}\hat{\sigma}\right) = \frac{d-1}{d}\hat{\sigma}\bar{D}^{2}\left(\bar{D}^{2} + \frac{\bar{R}}{d-1}\right)\hat{\sigma} + \text{cov. divergence}$$
(G6)

$$(\bar{D}_{\mu}\hat{\xi}_{\nu}+\bar{D}_{\nu}\hat{\xi}_{\mu})\exp(\bar{D}^{2})(\bar{D}^{\mu}\hat{\xi}^{\nu}+\bar{D}^{\nu}\hat{\xi}^{\mu}) = -2\hat{\xi}_{\mu}\left(\bar{D}^{2}+\frac{\bar{R}}{d}\right)\exp\left(\bar{D}^{2}+\frac{(d+1)\bar{R}}{d(d-1)}\right)\hat{\xi}^{\mu} + \text{cov. divergence}$$
(G7)

$$(\bar{D}_{\mu}\hat{\sigma})\exp(\bar{D}^{2})\,\bar{D}^{\mu}\hat{\sigma} = -\,\hat{\sigma}\,\bar{D}^{2}\exp\left(\bar{D}^{2} + \frac{\bar{R}}{d}\right)\,\hat{\sigma} + \text{cov. divergence}$$
(G8)

$$\left[\left(\bar{D}_{\mu}\bar{D}_{\nu}-\frac{1}{d}\bar{g}_{\mu\nu}\bar{D}^{2}\right)\hat{\sigma}\right]\exp(\bar{D}^{2})\left(\bar{D}^{\mu}\bar{D}^{\nu}-\frac{1}{d}\bar{g}^{\mu\nu}\bar{D}^{2}\right)\hat{\sigma}=\frac{d-1}{d}\hat{\sigma}\bar{D}^{2}\left(\bar{D}^{2}+\frac{\bar{R}}{d-1}\right)\exp\left(\bar{D}^{2}+\frac{2\bar{R}}{d-1}\right)\hat{\sigma}+\text{cov. divergence.}$$
(G9)

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APPENDIX H: APPROXIMATE SOLUTIONS FOR THE FIXED POINT

determine g_* from the condition $\eta_{N*} = 2 - d$ alone. Solving this equation for g_* leads to

In the following we determine the approximate formula for the position of the non-Gaussian fixed point discussed in Sec. V C. In a first approximation we set $\lambda_k = \lambda_* = 0$ and

$$g_* = \frac{2-d}{B_1(\lambda_*;\alpha,d) - (d-2)B_2(\lambda_*;\alpha,d)}$$
(H1)

which, for $\lambda_* = 0$, boils down to

$$g_{*} = \frac{2-d}{4} (4\pi)^{d/2-1} \left\{ k_{1}(d) \Phi_{d/2-1}^{1}(0) + k_{2}(d) \tilde{\Phi}_{d/2-1}^{1}(0) + k_{3}(d,\alpha) \Phi_{d/2}^{2}(0) + k_{4}(d,\alpha) \tilde{\Phi}_{d/2}^{2}(0) + 3 \delta_{d,2} \left[\frac{1}{1-2\lambda_{k}} - \frac{1}{1-2\alpha\lambda_{k}} \right] \right\}^{-1}.$$
 (H2)

Here k_1, \ldots, k_4 are *d*- and α -dependent coefficients defined as

$$k_{1}(d) = \frac{d^{4} - 4d^{3} - 9d^{2} - 12}{12d(d-1)},$$

$$k_{2}(d) = \frac{(d-2)(d^{4} - 13d^{2} - 24d + 12)}{24d(d-1)},$$

$$k_{3}(d,\alpha) = -\frac{d^{4} - 4d^{3} + 9d^{2} - 8d - 2}{2d(d-1)} - (d$$

$$-2)\alpha,$$

$$k_{4}(d,\alpha) = -\frac{(d-2)(d^{4} - 4d^{3} + 5d^{2} - 8d + 2)}{4d(d-1)}$$

$$d, \alpha) = -\frac{(d-2)^2}{4d(d-1)} -\frac{(d-2)^2}{2}\alpha.$$
 (H3)

Employing the exponential shape function (3.8) with s=1, and setting d=4 and $\alpha=1$, for instance, Eq. (H2) yields $g_* \approx 0.590$. Here we used that for this shape function $\Phi_1^1(0) = \pi^2/6$, $\Phi_2^2(0) = 1$, $\tilde{\Phi}_1^1(0) = 1$, $\tilde{\Phi}_2^2(0) = 1/2$.

As a different approximation scheme, we determine (λ_*, g_*) from a set of Taylor-expanded β -functions. Using

$$\frac{d}{d\lambda_{k}}\Phi_{n}^{p}(-2\alpha\lambda_{k}) = 2\alpha p \Phi_{n}^{p+1}(-2\alpha\lambda_{k}),$$
$$\frac{d}{d\lambda_{k}}\Phi_{n}^{p}(-2\alpha\lambda_{k}) = 2\alpha p \tilde{\Phi}_{n}^{p+1}(-2\alpha\lambda_{k}),$$
(H4)

we expand the β -functions (4.28) and (4.30) about $g_k = \lambda_k$ = 0 and obtain

$$\begin{aligned} \boldsymbol{\beta}_{\lambda}(\lambda_{k},g_{k};\alpha,d) &= -2\lambda_{k} + \nu_{d} d g_{k} + [2d(d-1+2\alpha) \\ &\times (4\pi)^{1-d/2} \Phi_{d/2}^{2}(0) - (d-2) \omega_{d}]\lambda_{k} g_{k} \\ &+ \frac{1}{2} d(d+1)(d-2) \\ &\times (4\pi)^{1-d/2} \omega_{d} \Phi_{d/2}^{1}(0) g_{k}^{2} + \mathcal{O}(g^{3}), \end{aligned}$$
$$\\ \boldsymbol{\beta}_{g}(\lambda_{k},g_{k};\alpha,d) &= (d-2) g_{k} - (d-2) \omega_{d} g_{k}^{2} + \mathcal{O}(g^{3}). \end{aligned} \tag{H5}$$

Here ν_d and ω_d are defined as in Eqs. (5.8) and (5.11), and $\mathcal{O}(g^3)$ stands for terms of third and higher orders in the couplings $g_1(k) = \lambda_k$ and $g_2(k) = g_k$. Now g_* is obtained as the nontrivial solution to $\boldsymbol{\beta}_g = 0$, which reads

$$g_{*} = \omega_{d}^{-1}$$

$$= \frac{2-d}{4} (4\pi)^{d/2-1} \{k_{1}(d) \Phi_{d/2-1}^{1}(0)$$

$$+ k_{3}(d, \alpha) \Phi_{d/2}^{2}(0)\}^{-1}.$$
(H6)

Inserting Eq. (H6) into β_{λ} and neglecting also the terms quadratic in the couplings the condition $\beta_{\lambda} = 0$ leads to

$$\lambda_{*} = \frac{\nu_{d} d}{2\omega_{d}}$$

$$= -\frac{d(d-2)(d-3)}{8} \Phi^{1}_{d/2}(0) \{k_{1}(d) \Phi^{1}_{d/2-1}(0)$$

$$+ k_{3}(d, \alpha) \Phi^{2}_{d/2}(0)\}^{-1}.$$
(H7)

Using the shape function (3.8) with s=1 we obtain from Eqs. (H6) and (H7) in d=4 dimensions

$$g_{*} = \left(\frac{13\pi}{144} + \frac{55}{24\pi} + \frac{\alpha}{\pi}\right)^{-1},$$
$$\lambda_{*} = \zeta(3) \left(\frac{13\pi^{2}}{144} + \frac{55}{24} + \alpha\right)^{-1},$$
(H8)

which yields $(\lambda_*, g_*) = (0.287, 0.751)$ for $\alpha = 1$, for instance.

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