Meson mass differences in two dimensional gauged four-Fermi models

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We quantitatively analyze the meson mass inequality relations of two dimensional gauged four-Fermi models in the large N limit. The class of models we study includes the 't Hooft model and the chiral and nonchiral Gross-Neveu models as special points in the space of field theories. Cases in which the chiral symmetry is spontaneously or explicitly broken are both studied. We study the meson mass differences quantitatively and define a susceptibility which allows us to systematically analyze these quantities. In the generalized Gross-Neveu model limit, we derive an analytic expression for this susceptibility. Even though no analytic proof of the validity of the classic mass inequality exists for the generic case, the mass difference is found to have the standard sign throughout most of the parameter space. We point out that the difference might have the opposite sign in certain cases.

I. INTRODUCTION

Determining the properties of composite particles in relativistic field theories, such as mesons or baryons in QCD, is an essentially nonperturbative problem and is quite difficult to address from first principles. While impressive progress has been made in numerical approaches to the problem [1], it is highly desirable to also have a more analytic understanding of the properties of the composite particles. Important relations that can be applied to composite particles in vector-like theories, such as massive QCD, are the mass inequality relations [2]. While these elegant relations are quite useful and have therefore been well studied (for a recent review, see [3]), there seems to be little understanding regarding their quantitative behavior. Furthermore, while the proof of inequalities does not apply to chiral, explicitly left-right asymmetric theories or theories with Yukawa couplings, no examples of theories wherein the mass difference has been shown to be negative exists among relativistic quantum-field-theories. The mass differences provide nontrivial insight into the dynamics of the interacting model. An understanding of the behavior of the inequalities will undoubtedly further our understanding of the spectrum of bound states in relativistic quantum-field-theories.

Whether the mass differences might or might not be positive in chiral models is of import, in particular to supersymmetric theories. In supersymmetric theories, one often "solves" the gauge hierarchy problem by invoking the chiral symmetry of the fermions, which is related in turn to the mass of the scalar bosons by supersymmetry. While some nonperturbative aspects of supersymmetric gauge theories are recently being clarified [4,5], relations analogous to the mass inequality relations seems not to be known. Such relations, if they exist, should shed light on the properties of the

spectrum on supersymmetric theories and on the possibility of spontaneous breaking of supersymmetry. In attempting to extend the mass inequality relations to supersymmetric theories, we believe a deeper understanding of its properties to be crucial. Of course, nonsupersymmetric theories are of interest on its own, one of the reasons being that the low-energy world is not supersymmetric.

While, needless to say, computing the mass differences in full QCD would be of import, not to mention very interesting, this is a daunting task. One approach is to compute the mass differences and develop an understanding of their behavior in analytically solvable relativistic quantum-field-theory models, such as the classic 't Hooft model [6] and the Gross-Neveu models [7]. This is the approach we shall adopt in this work. These models are tractable yet nontrivial and have proven to be quite instructive by providing physics insight into the nonperturbative aspects of field theories [10]. The dynamics of gauge theories is certainly of import especially since it is an integral part of the standard model. Also, theories of four-Fermi interactions have been serving an important role in particle physics and other fields of physics [11,12].

In this work, we shall analyze the properties of mass differences of gauged four-Fermi models in $(1+1)$-dimensions, using the large N limit. In these models, the properties of the "meson" states can be reduced analytically to the problem of solving mathematical equations [13,16]. This will allow us to study the problem analytically, even if the final equations need to be solved numerically [14]. The parameters in this family of models which we can arbitrarily control are the scalar and the pseudoscalar four-Fermi couplings, the gauge coupling, and the "quark" masses. The class of models we study contains the 't Hooft model [6] and the chiral and nonchiral Gross-Neveu models [7] for particular choices of the parameters. It includes cases where the chiral symmetry is spontaneously or explicitly broken. This is of particular interest, since the Gross-Neveu model is known to be equivalent to a Yukawa model where

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the scalar field develops an expectation value [7]. This is precisely the kind of situation where one might suspect that the standard mass inequality may be violated [8,9].

In order to study the mass differences quantitatively and systematically, we need a measure for the size of the mass difference. We shall find such a measure in a parameter which we shall call the "meson mass susceptibility" to be explained below. This parameter will allow us to compare the mass differences of various field theories in a natural manner.

The work is organized as follows. In Sec. II, we shall define and explain the meson mass susceptibility. In Sec. III, we discuss mass differences in quantum mechanics. In Sec. IV, we first treat the simpler case of generalized Gross-Neveu models and analyze the mass differences there. In particular, an analytic form of the mass susceptibility will be explicitly presented. In Sec. V, we work with the general gauged four-Fermi models. We first summarize the class of models we study and explain how the meson spectrum can be computed. Particular care has been paid to presenting the methods we use explicitly for further possible use. We then compute and study the properties of the mass differences. We end with discussions of the results in Sec. VI. A short Appendix on some of the technical aspects of the computation is included.

II. THE MESON MASS SUSCEPTIBILITY

In studying the mass differences quantitatively, we need a quantitative measure of how "large" the mass difference is, in order to compare within the field-theory space. This encodes information, intuitively speaking, on how strong the attractive interactions in the theory are. The mass difference we consider is

$$\delta\mu_{ab} \equiv \mu_{ab} - \frac{(\mu_{aa} + \mu_{bb})}{2} \qquad (1)$$

which is known to be positive for vectorlike gauge theories [2]. Here, denoting the constituents as q's, μ_{ab} is the mass of the lightest meson that overlaps with the $\overline{q_a}q_b$ state, and so on.

This quantity is dimensionful and depends not only on the model, but also on the difference of the masses of the constituents. The quantity may be made dimensionless trivially by taking the ratio of the mass difference to a meson mass. However, this difference may become large just because the constituent mass difference is large, so it is not the most appropriate parameter for investigating the intrinsic dynamics of the theory. In fact, if we consider the field theory space to be parametrized by the couplings of the model which include the masses, the mass difference is *not* a local quantity in the parameter space. It is more natural to define a local parameter in the field-theory space. Let us define the mass squared of the constituents to be

$$M_a^2 = M^2(1+\Delta), \qquad M_b^2 = M^2(1-\Delta). \qquad (2)$$

The meson mass difference $\delta\mu_{ab}$ is even under the interchange of M_a and M_b and is, therefore, an even function of Δ.

We shall characterize the mass difference by a parameter we refer to as the "meson mass susceptibility." This quantity is defined by

$$\mathcal{R} \equiv \lim_{\Delta \to 0} \frac{\delta\mu_{ab}}{\Delta^2 \mu_{ab}} = \lim_{\Delta \to 0} \frac{2\mu_{ab} - (\mu_{aa} + \mu_{bb})}{\Delta^2 2\mu_{ab}}, \qquad (3)$$

which is a function of the couplings and the mass of the constituent quark. The susceptibility we defined above is also useful from a practical point of view: Since the mass difference is an expansion in the constituent mass difference squared, the susceptibility together with the meson mass for equal mass quarks, distill the meson mass information when the quark mass difference is not too large.

Perhaps we should discuss here the relation of this susceptibility to the global properties of the mass difference, namely, when the mass differences are arbitrary. A natural question is whether the positivity of the susceptibility in a parameter region guarantees the positivity of the mass difference when the mass differences are large. In quantum mechanics, the situation is quite simple; if the susceptibility is positive everywhere, the mass difference is positive for arbitrary mass differences. This can be derived from the convexity of the meson mass with respect to the reduced mass of the two quarks. In quantum-field-theory, however, no such argument exists in general, since the meson mass needs not and will not depend only on the reduced mass of the two quarks. To make an analogous argument in quantum-field-theory, we need further information regarding the relation between the meson mass and the quark masses. While it seems quite natural to assume that relations exist such that the positivity of the local susceptibility guarantees the validity of the standard mass inequality globally, we do not know if this is in fact true. In practice, we have found no counterexamples to this statement.

III. MASS DIFFERENCES IN QUANTUM MECHANICS

In this section, we briefly discuss mass differences in quantum mechanics (see also [3]). While the discussion is not necessary for computing mass differences in relativistic field theories, we feel that it is nonetheless quite instructive and provides a broader perspective on mass differences in quantum theories. Also, the mass differences in relativistic field theories should reduce to that of quantum mechanics in the nonrelativistic limit. As such, some of the results here will be later compared to those from the full quantum-field-theory below. It should be noted, however, that phenomena such as symmetry breaking which plays a large role in (1+1)-dimensional gauge theories studied in this work are essentially quantum-field-theoretical so that quantum-mechanical behavior is not sufficient for understanding the full relativistic behavior, even qualitatively.

In quantum mechanics, the problem of two-body bound states under a local potential reduces to a model with the Hamiltonian

$$H = \frac{p^2}{2M_{12}} + V(x), \qquad (4)$$

where M_{12} denotes the reduced mass $1/M_{12}=1/M_1+1/M_2$. We will analyze one dimensional models, but similar analysis can be applied to higher dimensional models.

A. Infinitely deep square-well potential

The potential of the model is

$$V(x)=\begin{cases} 0 & (0\leqslant x\leqslant L), \\ \infty & (x<0,x>L). \end{cases} \tag{5}$$

The spectrum of the bound states is known to be $E_{12,n}=\hbar^2\pi^2n^2/(2M_{12}L^2)$, $(n=1,2,\dots)$. This is somewhat trivial but an interesting case. The susceptibility $\mathcal{R}=0$ and we can understand this as the signature of the model being free within the well.

B. Delta function potential

The delta function potential

$$V(x)=-V_0\delta(x), \quad (V_0>0) \tag{6}$$

has a bound state with the binding energy $-M_{12}V_0^2/(2\hbar^2)$

$$\delta\mu_{ab}=E_{ab}-\frac{E_{aa}+E_{bb}}{2}=\frac{V_0^2}{8\hbar^2}\frac{(M_a-M_b)^2}{(M_a+M_b)}. \tag{7}$$

This leads to the susceptibility

$$\mathcal{R}=\frac{V_0^2}{32(1-V_0^2/8)}>0. \tag{8}$$

The susceptibility increases with larger V_0, as expected. In the nonrelativistic limit, $V_0\ll1$.

C. Monomial potentials

Let us also discuss potentials whose behavior is governed by a monomial

$$V(x)=Ax^\gamma, \quad A\gamma>0, \tag{9}$$

where γ needs not be an integer but $\gamma>-2$ needs to be satisfied for sensible physics behavior. $A\gamma>0$ needs to be imposed for the existence of bound states. $\gamma=2$ and -1 corresponds to the harmonic oscillator and the three dimensional Coulomb case, respectively.

We can use the uncertainty principle to crudely estimate the bound-state energy as

$$E_{12}\simeq\left(\frac{\gamma}{2}+1\right)A\left(\frac{\hbar^2}{\gamma AM_{12}}\right)^{\gamma/(\gamma+2)}. \tag{10}$$

We can obtain the susceptibility from this energy as

$$\mathcal{R}\simeq\frac{\hbar^{2\gamma/(\gamma+2)}}{8(\gamma+2)c^2}\left(\frac{\gamma A}{2}\right)^{2/(\gamma+2)}M^{-2(\gamma+1)/(\gamma+2)}>0 \tag{11}$$

in the nonrelativistic limit. While the derivation is not rigorous, in the harmonic oscillator and the Coulomb cases, the susceptibilities agree with those obtained from exact methods.

IV. GENERALIZED GROSS-NEVEU MODELS

In this section, we analyze the mass differences in the generalized Gross-Neveu models, described by the Lagrangian

$$\mathcal{L}=\sum_{f=1}^{N_F}\bar{\psi}_f(i\partial\!\!\!/-m_f)\psi_f+\frac{a^2}{2}\sum_{f,f'=1}^{N_F}(\bar{\psi}_{f'}\psi_f)(\bar{\psi}_f\psi_{f'})$$
$$-\frac{a_5^2}{2}\sum_{f,f'=1}^{N_F}(\bar{\psi}_{f'}\gamma_5\psi_f)(\bar{\psi}_f\gamma_5\psi_{f'}). \tag{12}$$

In addition to the flavor indices f,f' denoted explicitly in the above formula, the fermions carry an additional internal space index, the "color" index $(1,2,\dots,N)$ which has been suppressed in the notation. This index should not be confused with the flavor index. We take the large N limit while keeping a^2N,a_5^2N fixed. When $m_f=0,a_5^2=0$, the model reduces to the original Gross-Neveu model and when $m_f=0,a^2=a_5^2$, the model reduces to the chiral Gross-Neveu model with continuous chiral symmetry. When $m_f=0$ and the couplings are not equal, we are left with discrete chiral symmetry in the model. We need to consider multiple flavors for the analysis of the mass differences.

This class of models is included in the gauged four-Fermi models we deal with below and the analytic methods discussed there can be applied here also. However, the generalized Gross-Neveu models can be solved completely analytically using different methods than the gauged four-Fermi model case, so we shall discuss it separately. Here, we shall need the spectrum in the general case when two flavors have different masses, $m_1^2\neq m_2^2$, and $a^2\neq a_5^2$, which was not solved explicitly in [16]. We shall present the spectrum and analyze the mass differences.

Let us consider a meson bound state of constituents with masses M_1,M_2. These constituent masses are physical fermion masses that include the effects of spontaneous chiral symmetry breaking that occurs dynamically in the Gross-Neveu model. We dispense with the derivation here, but the Bethe-Salpeter equation for the meson state can be solved algebraically to obtain the meson "wave function," $\varphi(x)$ as

$$\varphi(x)=\varphi^{(0)}+\varphi^{(1)}(1-2x)+\hat{\varphi}(x), \quad (0\leqslant x\leqslant 1), \tag{13}$$

$$\hat{\varphi}(x)=\frac{\mu_{12}^2[\varphi^{(0)}+\varphi^{(1)}(1-2x)]+2(M_1^2-M_2^2)\varphi^{(1)}}{-\mu_{12}^2+\frac{M_1^2}{x}+\frac{M_2^2}{1-x}}, \tag{14}$$

where $\varphi^{(0)},\varphi^{(1)}$ are constants and $\hat{\varphi}(x)/[x(1-x)]$ is integrable at $x=0,1$. The meson wave function satisfies the following boundary conditions:

$$\begin{pmatrix} b_+ & (1+4G_5)b_- \\ b_- & (1+4G)b_+ \end{pmatrix}\begin{pmatrix} \varphi^{(0)} \\ \varphi^{(1)} \end{pmatrix} = \int_0^1 dx \frac{\hat{\varphi}(x)}{x(1-x)}\begin{pmatrix} G_5 & 0 \\ 0 & G \end{pmatrix}$$

$$\times \begin{pmatrix} b_+ & b_- \\ b_- & b_+ \end{pmatrix}\begin{pmatrix} 1 \\ 1-2x \end{pmatrix}.$$

(15)

Here, we used the notation $G \equiv a^2 N/(4\pi)$, $G_5 \equiv a_5^2 N/(4\pi)$ for the *renormalized* couplings and defined $b_\pm \equiv (M_1 \pm M_2)/2$. When the coupling constants are equal, $a^2 = a_5^2$, or when the masses are equal, $M_1 = M_2$, the two boundary condition equations simply decouple, but do *not* in the general case.

The boundary conditions lead to a secular equation

$$\det\begin{pmatrix} J_{12} - \frac{1}{2}\left(\frac{1}{G} + \frac{1}{G_5}\right) - \left(\frac{1}{G_5} - \frac{1}{G}\right)\frac{M_1^2 + M_2^2}{4M_1 M_2} & \frac{M_1^2 - M_2^2}{\mu^2}J_{12} + \ln\frac{M_1^2}{M_2^2} - \left(\frac{1}{G_5} - \frac{1}{G}\right)\frac{M_1^2 - M_2^2}{4M_1 M_2} \\ -\frac{M_1^2 - M_2^2}{\mu^2}J_{12} + \ln\frac{M_1^2}{M_2^2} + \left(\frac{1}{G_5} - \frac{1}{G}\right)\frac{M_1^2 - M_2^2}{4M_1 M_2} & \left(1 - 2\frac{M_1^2 + M_2^2}{\mu^2}\right)J_{12} - \frac{1}{2}\left(\frac{1}{G} + \frac{1}{G_5}\right) + \left(\frac{1}{G_5} - \frac{1}{G}\right)\frac{M_1^2 + M_2^2}{4M_1 M_2} \end{pmatrix} = 0.$$

(16)

Here, we defined

$$J_{12} \equiv \int_0^1 \frac{\mu^2\, dx}{-\mu^2 x(1-x) + M_1^2(1-x) + M_2^2 x}. \quad (17)$$

It should be noted that since the couplings G, G_5 are dimensionless, the overall mass scale M can always be scaled out of the problem and only the relative masses have a physical meaning. The physical parameters of this quantum-field-theory are the two dimensionless renormalized couplings G, G_5 and the mass ratio M_1/M_2.

Before we analyze the behavior of the mass differences, we first need to understand the behavior of the spectrum when the masses of the constituents are the same. In this case, the secular equation (16) splits into two independent equations for the pseudoscalar and scalar bound states, χ and σ:

$$\chi: \quad \frac{1}{G_5} = \int_0^1 dx \frac{(\mu_\chi/M)^2}{1 - (\mu_\chi/M)^2 x(1-x)}$$

$$= \frac{4}{\sqrt{4(M/\mu_\chi)^2 - 1}}\tan^{-1}\left(\frac{1}{\sqrt{4(M/\mu_\chi)^2 - 1}}\right),$$

(18)

$$\sigma: \quad \frac{1}{G} = \int_0^1 dx \frac{(\mu_\sigma/M)^2 - 4}{1 - (\mu_\sigma/M)^2 x(1-x)}. \quad (19)$$

It should be noted that only G_5 (G) appears in the equation for χ (σ).

The state σ exists as a nontachyonic bound state only for $G < -1/4$. It is not clear whether the theory is unitary for negative G and we shall consider the region $G \geq 0$, so we shall not have much more to say on σ. The original Gross-Neveu model corresponds to $G \to -\infty$ in our scheme and in this limit, $\mu_\sigma^2 \to 4M^2$.

χ exists as a bound state for any $G_5 \geq 0$ and $0 \leq \mu_\chi^2 \leq 4M^2$. The dependence of the bound-state mass on the coupling is plotted in Fig. 1. This is the only bound state in the model for $G, G_5 > 0$ and corresponds to the Nambu-Goldstone-like particle when the constituent masses are zero [7,17], as in the chiral Gross-Neveu model. It is the dependence of this meson state on the constituent masses that we shall investigate. As a side note, in a region we shall not investigate, there is an intriguing possibility when $G_5 \geq 0$ and $G < -1/4$, in some cases, the χ mass can be larger than the σ mass. We do not know whether this can be achieved in a physically consistent situation. Another comment is perhaps appropriate; in the literature, the Gross-Neveu model ($G \neq 0, G_5 = 0$) is often used as a prototypical simple model with a bound state. However, the original Gross-Neveu model has no binding energy for the meson and has barely a bound state. It seems to us that in fact, the simplest theory that may be considered in this family that is useful in analyzing bound-state dynamics is $G_5 \neq 0, G = 0$ case. In this case, we have a bound state whose mass depends on the coupling as in Fig. 1.

The meson mass susceptibility may be obtained by perturbing the Eq. (16) in the mass difference parameter Δ in Eq. (2). After some computation, we derive
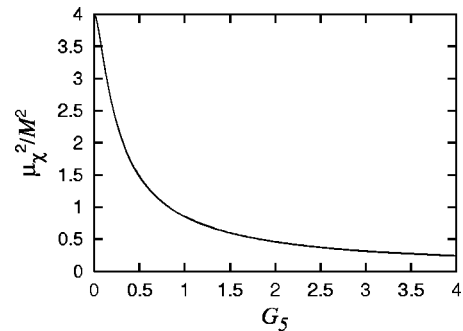

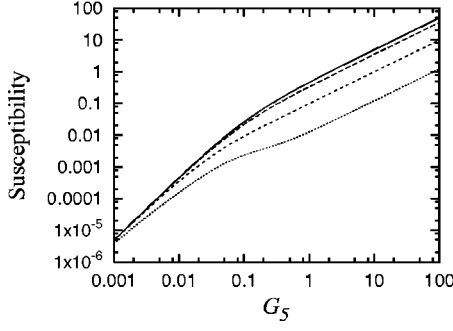FIG. 1. The behavior of the χ meson mass μ_χ^2/M^2 with respect to the coupling G_5.

FIG. 2. The behavior of the meson mass susceptibility $\mathcal{R}$ with respect to the coupling G_5 for the generalized Gross-Neveu models. The lines represent, from top to bottom, $\mathcal{R}$ for $G = 0, 0.1, 1, 10$, respectively.

$$\mathcal{R} = \left(\zeta - \frac{1}{4} \right) \left\{ \frac{1}{2} - \frac{1}{\left(1 + \frac{\zeta}{G_5} \right) \left[\left(\zeta - \frac{1}{4} \right) \frac{1}{G_5} + \frac{1}{4G} \right]} \right\}. \tag{20}$$

We defined $\zeta \equiv M^2/\mu_0^2$, where μ_0^2 is the mass of the meson in the unperturbed case, when $\Delta = 0$. The susceptibility is independent of the overall mass scale M, since it can be scaled out of the problem. The susceptibility may be shown analytically to be positive for any $G, G_5 > 0$. Since G, G_5 are scalar and pseudoscalar couplings that can take on arbitrary values, the standard proof of the mass inequality [2,3] does *not* apply to the models we study, except at special points. We believe that an analytic expression has not been previously derived for the mass difference in any relativistic quantum-field-theory.

It is interesting to check the asymptotic behavior of the susceptibility for small and large couplings. For small G_5 couplings,

$$\mathcal{R} = \frac{\pi^2 G_5^2}{2} [1 - 8 G_5 (1 + 4G) + \mathcal{O}(G_5^2)]. \tag{21}$$

This behavior is consistent with that for the δ function problem discussed in Sec. II. For large G_5 couplings,

$$\mathcal{R} = \frac{G_5}{2(4G+1)} - \frac{1}{24(4G+1)^2} + \mathcal{O}(G_5^{-1}). \tag{22}$$

The behavior of the susceptibility with respect to G_5 is shown for $G = 0, 0.1, 1, 10$ in Fig. 2. The dependence on G is not strong; this is because the properties of the bound state χ are governed mostly by the pseudoscalar coupling G_5. The crossover from G_5^2 behavior to G_5 behavior in the susceptibility can be clearly seen in the plot.

V. GAUGED FOUR-FERMI MODELS

A. The model

Let us now discuss the most general gauged four-Fermi model described by the Lagrangian:

$$\mathcal{L} = -\frac{1}{2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) + \sum_{f=1}^{N_F} \bar{\psi}_f (i\slashed{D} - m_f) \psi_f$$

$$+ \frac{a^2}{2} \sum_{f,f'=1}^{N_F} (\bar{\psi}'_f \psi_f)(\bar{\psi}'_f \psi'_f) - \frac{a_5^2}{2} \sum_{f,f'=1}^{N_F} (\bar{\psi}'_f \gamma_5 \psi_f)$$

$$\times (\bar{\psi}_f \gamma_5 \psi'_f). \tag{23}$$

We have gauged the internal index in the generalized Gross-Neveu model (12) so that when we set the gauge coupling to zero, we recover the generalized Gross-Neveu model discussed in the previous section. When we set $G = G_5 = 0$, we recover the 't Hooft model. We take the large-N limit in a manner similar to that of the previous section but also for the gauge coupling; namely, we keep $g^2 N, a^2 N$, and $a_5^2 N$ fixed while we take N to infinity.

We again split the meson wave function as in Eq. (13). Then the Bethe-Salpeter equations for the meson bound states with the fermion constituents with masses $M_{1,2}$ may be obtained in a simple closed form [16]:

$$\mu^2 \varphi(x) = H\varphi(x) = \left(\frac{\beta_1 - 1}{x} + \frac{\beta_2 - 1}{1-x} \right) \hat{\varphi}(x)$$

$$- P \int_0^1 dy \frac{\hat{\varphi}(y)}{(y-x)^2} + 2 \varphi^{(1)} \left(-\beta_1 + \beta_2 + \ln \frac{1-x}{x} \right) \tag{24}$$

satisfying the boundary conditions (15). Since the gauge coupling g has the dimensions of the mass in $(1+1)$-dimensions, we introduced the dimensionless mass parameters $\beta_{1,2} \equiv \pi M_{1,2}^2/(g^2 N)$. The renormalized couplings G, G_5 are defined in the same way as in the previous section. We should point out that all the parameters in this equation are finite renormalized parameters, so that the problem has been reduced to solving a somewhat complicated integral equation. The physical parameters in this theory are the three dimensionless renormalized parameters $\beta, G, G_5 > 0$.

For later purposes, we also derive the matrix elements of the "Hamiltonian," H, in the most general case, when the couplings and the masses are arbitrary:

$$(\varphi', H\varphi) = \left[\frac{\beta_1 + \beta_2}{4} \left(\frac{1}{G} + \frac{1}{G_5} \right) + \frac{1}{2} \sqrt{\beta_1 \beta_2} \left(\frac{1}{G_5} - \frac{1}{G} \right) \right]$$

$$\times \overline{\varphi^{(0)'}} \varphi^{(0)} + \frac{\beta_1 - \beta_2}{4} \left(\frac{1}{G} + \frac{1}{G_5} \right) (\overline{\varphi^{(0)'}} \varphi^{(1)}$$

$$+ \overline{\varphi^{(1)'}} \varphi^{(0)}) + \left[\frac{\beta_1 + \beta_2}{4} \left(\frac{1}{G} + \frac{1}{G_5} + 8 \right) + 2 \right.$$

$$\left. + \frac{1}{2} \sqrt{\beta_1 \beta_2} \left(\frac{1}{G} - \frac{1}{G_5} \right) \right] \overline{\varphi^{(1)'}} \varphi^{(1)}$$

$$+ \int_0^1 dx \left(\frac{\beta_1 - 1}{x} + \frac{\beta_2 - 1}{1-x} \right) \overline{\hat{\varphi}'(x)} \hat{\varphi}(x)$$

$$-P\int_0^1\int_0^1\frac{dx\,dy}{(x-y)^2}\overline{\hat{\varphi}'(x)}\hat{\varphi}(y)$$

$$+\int_0^1dx2\left(-\beta_1+\beta_2+\ln\frac{1-x}{x}\right)\overline{(\varphi^{(1)\prime}}\hat{\varphi}(x)$$

$$+\overline{\hat{\varphi}'(x)}\varphi^{(1)}).\tag{25}$$

B. Methods for obtaining the spectrum

In generalized Gross-Neveu models, the spectrum could be obtained by just solving an ordinary equation, albeit a transcendental one. In contrast, for the gauged four-Fermi model, we need to solve an integral equation which is technically more involved. Of course, this is to be expected, since the usual 't Hooft model, which is a simpler model, is solved in terms of an integral equation. To solve the integral equation (24), we employ two methods to be explained in this subsection, generalizing the methods used previously in the 't Hooft model [18,19,15,16]. With either of the two methods, we can solve for the spectrum and the wave functions of any of the meson states for arbitrary combinations of masses and couplings in the gauged four-Fermi models. By using the two different methods simultaneously, we are able to obtain a better control over the error in the results which inevitably arise when we solve the integral equation numeri-

cally, in addition to checking the internal consistency. We will be succinct and summarize the results. Even though the basic ideas are the same as those in [16], the results are substantially more complicated since we need to treat the most general case which was previously not necessary.

1. Variational method

One method for solving the Bethe-Salpeter Eq. (24), familiar from solving the Schrödinger equation, is the variational method. We choose the basis functions $\{\varphi_j|\ j=2,3,\dots\}$ as

$$\varphi_{2k}(x)=c_{11}+c_{21}(1-2x)+\frac{[x(1-x)]^k}{B(k,k)},$$

$$\varphi_{2k+1}(x)=c_{12}+c_{22}(1-2x)$$

$$+\frac{(2k+1)(1-2x)[x(1-x)]^k}{B(k,k)}$$

$$(k=1,2\dots).\tag{26}$$

The c_{ij}'s need to be determined to satisfy the boundary conditions (15) as

$$\begin{pmatrix}c_{11}&c_{12}\\c_{21}&c_{22}\end{pmatrix}=\begin{pmatrix}b_+&(1+4G_5)b_-\\b_-&(1+4G)b_+\end{pmatrix}^{-1}\begin{pmatrix}G_5&0\\0&G\end{pmatrix}\begin{pmatrix}b_+&b_-\\b_-&b_+\end{pmatrix}\tag{27}$$

$$=\frac{1}{d}\begin{pmatrix}-\frac{1}{4}(G-G_5)(\beta_1+\beta_2)+\left(\frac{G_5+G}{2}+4G_5G\right)\sqrt{\beta_1\beta_2}&-\frac{1}{4}(G-G_5)(\beta_1-\beta_2)\\\frac{1}{4}(G-G_5)(\beta_1-\beta_2)&\frac{1}{4}(G-G_5)(\beta_1+\beta_2)+\frac{1}{2}(G_5+G)\sqrt{\beta_1\beta_2}\end{pmatrix}$$

where

$$d\equiv(1+4G)b_+^2-(1+4G_5)b_-^2=\{(G-G_5)(\beta_1+\beta_2)$$

$$+[1+2(G_5+G)]\sqrt{\beta_1\beta_2}\}.\tag{28}$$

In the variational method, the problem of obtaining the meson states is reduced to solving an eigenvalue problem:

$$(\mu^2N_{kl}-H_{kl})w_l=0,\quad H_{kl}\equiv(\varphi_k,H\varphi_l),\quad N_{kl}\equiv(\varphi_k,\varphi_l),$$

$$k,l=2,3,4\dots.\tag{29}$$

We will approximate the solution by using basis elements up to a certain number and check the convergence by varying

the dimension of this basis space. With some work, the matrix elements can be computed to be

$$N_{2k,2l}=c_{11}^2+\frac{c_{21}^2}{3}+\frac{c_{11}}{2}\left(\frac{k}{2k+1}+\frac{l}{2l+1}\right)$$

$$+\frac{k+l}{2(2k+2l+1)}\frac{B(k+l,k+l)}{B(k,k)B(l,l)},$$

$$N_{2k+1,2l+1}=c_{12}^2+\frac{c_{22}^2}{3}+\frac{c_{22}}{2}\left(\frac{k}{2k+3}+\frac{l}{2l+3}\right)$$

$$+\frac{(k+l)(2k+1)(2l+1)}{2(2k+2l+1)(2k+2l+3)}\frac{B(k+l,k+l)}{B(k,k)B(l,l)},$$

$$\tag{30}$$

$$N_{2k,2l+1}=N_{2l+1,2k}=c_{11}c_{12}+\frac{1}{3}c_{21}c_{22}$$

$$+\frac{k}{2(2k+1)}c_{12}+\frac{l}{2(2l+3)}c_{21},$$

$$H_{2k,2l}=\frac{1}{d}\sqrt{\beta_1\beta_2}\left[\frac{1}{4}(G+G_5+8GG_5)(\beta_1+\beta_2)\right.$$

$$-\frac{1}{2}(G-G_5)\sqrt{\beta_1\beta_2}\bigg]+2c_{12}^2+(\beta_1-\beta_2)$$

$$\times c_{12}\left(2c_{11}+\frac{k}{2k+1}+\frac{l}{2l+1}\right)$$

$$+\left(\frac{\beta_1+\beta_2}{2}-1\right)\frac{B(k+l,k+l)}{B(k,k)B(l,l)}+\frac{kl}{2(k+l)},$$

$$H_{2k+1,2l+1}=\frac{1}{d}\sqrt{\beta_1\beta_2}\left[\frac{1}{4}(G+G_5)(\beta_1+\beta_2)\right.$$

$$+\frac{1}{2}(G-G_5)\sqrt{\beta_1\beta_2}\bigg]$$

$$+2c_{22}[c_{22}-(\beta_1-\beta_2)c_{12}]$$

$$+c_{22}\left(\frac{k}{k+1}+\frac{l}{l+1}\right)+\left(\frac{\beta_1+\beta_2}{2}-1\right)$$

$$\times\frac{(2k+1)(2l+1)}{2k+2l+1}\frac{B(k+l,k+l)}{B(k,k)B(l,l)}$$

$$+\frac{kl(2k+1)(2l+1)}{2(k+l)(k+l+1)}, \tag{31}$$

$$H_{2k,2l+1}=H_{2l+1,2k}=\frac{1}{4d}\sqrt{\beta_1\beta_2}(\beta_1-\beta_2)(G+G_5)$$

$$-2c_{12}[c_{22}-(\beta_1-\beta_2)c_{12}]$$

$$-c_{12}\frac{l}{l+1}-c_{22}(\beta_1-\beta_2)\frac{k}{2k+1}+\frac{1}{2}(\beta_1-\beta_2)$$

$$\times\frac{(2l+1)B(k+l,k+l)}{(2k+2l+1)B(k,k)B(l,l)}.$$

When $\beta_1=\beta_2$, the even and the odd sectors completely decouple.

2. Multhopp's method

Rather than using a variational method, we can expand the meson wave function and solve the eigenvalue problem directly [19]. Defining $x\equiv(1+\cos\theta)/2$, the wave function can be expanded in a manner consistent with the boundary conditions as

$$\varphi(x)=2\pi\left(c_{11}\sum_{n:\text{odd}}^{K}v_n-c_{12}\sum_{n:\text{even}}^{K}v_n\right)-2\pi\left(c_{21}\sum_{n:\text{odd}}^{K}v_n\right.$$

$$\left.-c_{22}\sum_{n:\text{even}}^{K}v_n\right)\cos\theta+\sum_{n=1}^{K}v_n\sin n\theta, \tag{32}$$

where c_{ij}'s were defined in Eq. (27). This reduces the Bethe-Salpeter equation (24) to

$$\sum_{n=1}^{K}[\mu^2\hat{P}_n(\theta)-\hat{M}_n(\theta)]v_n=0, \tag{33}$$

where

$$\hat{P}_n(\theta)\equiv\sin n\theta+2\pi\begin{cases}c_{11}-c_{21}\cos\theta & n:\text{ odd}\\ -c_{12}+c_{22}\cos\theta & n:\text{ even,}\end{cases} \tag{34}$$

$$\hat{M}_n(\theta)\equiv2\left(\frac{\beta_1-1}{1+\cos\theta}+\frac{\beta_2-1}{1-\cos\theta}\right)\sin n\theta+2\pi\frac{n\sin n\theta}{\sin\theta}$$

$$+4\pi\left(\beta_1-\beta_2+\ln\frac{1+\cos\theta}{1-\cos\theta}\right)$$

$$\times\begin{cases}-c_{21} & n:\text{ odd}\\ c_{22} & n:\text{ even.}\end{cases}$$

The above Eq. (33) is still a functional equation, with the dependence on the parameter θ.

This can be further reduced to a generalized matrix eigenvalue problem

$$(\mu^2P-M)v=0. \tag{35}$$

The matrices are defined as

$$P_{mn}\equiv\sum_{l=1}^{K}g_m(\theta_l)\hat{P}_n(\theta_l),\quad M_{mn}\equiv\sum_{l=1}^{K}g_m(\theta_l)\hat{M}_n(\theta_l),$$

$$\theta_j\equiv\pi\frac{j}{K+1}. \tag{36}$$

The function $g_m(\theta)$ is arbitrary, but using functions with the property $g_m(\theta_l)=(-1)^{m+1}g_m(\theta_{K+1-l})$ simplifies the matrix elements. With this condition, the matrix elements are

$$P_{mn}=\sum_{l=1}^{K}g_m(\theta_l)\sin\theta_{nl}+2\pi\sum_{l=1}^{K}g_m(\theta_l)$$

$$\times\begin{cases}c_{11} & (m,n:\text{ odd})\\ c_{22}\cos\theta_l & (m,n:\text{ even}),\end{cases}$$

$$P_{mn}=-2\pi\sum_{l=1}^{K}g_m(\theta_l)$$

$$\times\begin{cases}c_{12} & (m:\text{ odd},n:\text{ even})\\ c_{21}\cos\theta_l & (m:\text{ even},n:\text{ odd}),\end{cases} \tag{37}$$
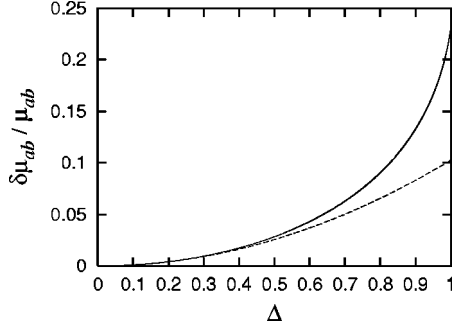
FIG. 3. The normalized meson mass difference as a function of Δ (solid). The mass difference expected from the susceptibility (dashes) is also shown.

$$M_{mn} = \sum_{l=1}^{K} g_m(\theta_l) \left[2 \frac{(\beta_1 + \beta_2 - 2)}{\sin^2 \theta_l} + \frac{2\pi n}{\sin \theta_l} \right] \sin \theta_{nl}$$

$$+ 4\pi \sum_{l=1}^{K} g_m(\theta_l) \begin{cases} (-c_{21})(\beta_1 - \beta_2) & (m,n: \text{ odd}) \\ c_{22} \ln \dfrac{1 + \cos \theta_l}{1 - \cos \theta_l} & (m,n: \text{ even}), \end{cases}$$

$$M_{mn} = -2(\beta_1 - \beta_2) \sum_{l=1}^{K} g_m(\theta_l) \frac{\cos \theta_l \sin \theta_{nl}}{\sin^2 \theta_l}$$

$$+ 4\pi \sum_{l=1}^{K} g_m(\theta_l)$$

$$\times \begin{cases} c_{22}(\beta_1 - \beta_2) & (m: \text{ odd}, n: \text{ even}) \\ (-c_{21}) \ln \dfrac{1 + \cos \theta_l}{1 - \cos \theta_l} & (m: \text{ even}, n: \text{ odd}). \end{cases}$$

(38)

In what follows, we adopt $g_m(\theta) = 2 \sin[m\theta/(K+1)]$ as was done so for the 't Hooft model [19].

C. Mass differences

Since we have at hand the methods for obtaining the physical properties of meson states, we are in a position to compute the mass differences. For investigating the mass differences, we use the properties of the lightest meson state in each channel. In Fig. 3 we first plot the behavior of the mass difference $\delta\mu_{ab}/\mu_{ab}$ defined in Eq. (1) for finite mass differences for a typical case of $G = G_5 = 1$, $\beta = 1$. The relative quark mass difference parameter $|\Delta| \leq 1$ by definition and the mass difference is symmetric with respect to the interchange $\Delta \leftrightarrow -\Delta$. At the same time, we also plot the behavior expected from the susceptibility $\mathcal{R}\Delta^2$. We see that the susceptibility describes the mass difference quite well unless the quark mass difference is quite large, say $\Delta \gtrsim 0.4$.

Let us move on to the behavior of meson mass susceptibilities. To compute the susceptibilities, we may just use the methods explained in the previous section and obtain the susceptibility as the limiting case of small mass differences going to zero. While this is logically fine, it incurs unnecessary numerical errors during the process. Therefore, we can
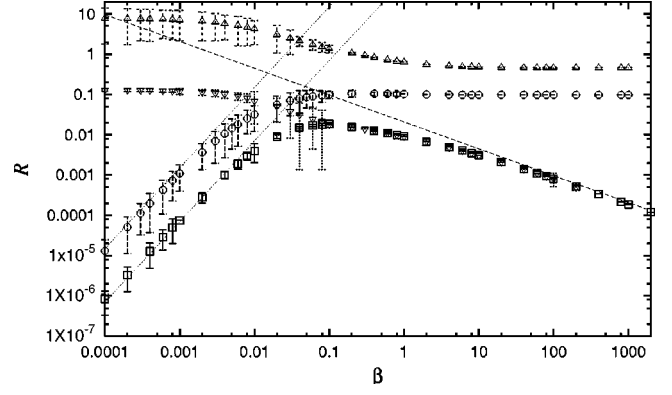

FIG. 4. The behavior of $\mathcal{R}$ against β for $(G, G_5) = (0,0)$ ($\square$), $(1,1)$ ($\bigcirc$), $(0,1)$ ($\triangle$), and $(1,0)$ ($\triangledown$). Dashes indicate $\beta^{-2/3}$ behavior and dots indicate β^2 behavior.

refine the method by perturbing in the mass difference *analytically* and then obtaining the mass susceptibilities directly. However, the standard perturbation formulas are *not* applicable to either of the two methods explained in the previous section since we are dealing with a perturbation that changes the boundary conditions, as we can see from Eq. (15). While the formulas that need to be derived should be of use in further studies, since this is technical and somewhat involved, we have chosen to describe the methods concretely in the Appendix. We have computed the susceptibility using both methods and have checked that the results agree.

The parameters of the gauged four-Fermi models are β, G, G_5 and the ratios of constituent masses. We expect G to play a not so dominant role in determining the properties of the lightest meson state. G_5 is the pseudoscalar coupling that strongly affects the lightest meson. β is effectively the inverse of the strength of the gauge coupling.

We first investigate the behavior of $\mathcal{R}$ with respect to β as in Fig. 4. When $G_5 \neq 0$, for large β, the susceptibilities approach those of the generalized Gross-Neveu model, which is quite natural since the gauge coupling is effectively weak. This behavior is quite visible for $(G, G_5) = (0,1), (1,1)$ cases in Fig. 4 and the approach already occurs for moderate β values, $\beta \gtrsim 0.1$. When $G_5 = 0$, as we can see from Eq. (21), $\mathcal{R} = 0$ in the generalized Gross-Neveu model. In the gauged four-Fermi model, $\mathcal{R}$ behaves as $\sim \beta^{-2/3}$, when $G_5 = 0$ and large β as we can see for $(G, G_5) = (0,0), (1,0)$ cases in Fig. 4. This is consistent with the expectation from the quantum-mechanics calculation in Eq. (10) for the linear confining potential. For $G = G_5$ and $\beta = 0$, it can be shown that $\mathcal{R} \to 0$ as β^2. This behavior is indeed seen in Fig. 4 for the $(G, G_5) = (0,0), (1,1)$ cases.

Let us now analyze how $\mathcal{R}$ behaves with respect to G_5 as in Fig. 5. It can be seen that for fixed β, $\mathcal{R}$ approaches the generalized Gross-Neveu model value as we increase G or G_5. Qualitatively, this can be understood as the gauge coupling, becoming relatively less important when the other couplings are strong. For small G_5, the behavior is governed by the gauge coupling and we see in Fig. 5 that the susceptibilities for the same β value approach each other. While these behaviors can be understood from the physics of the
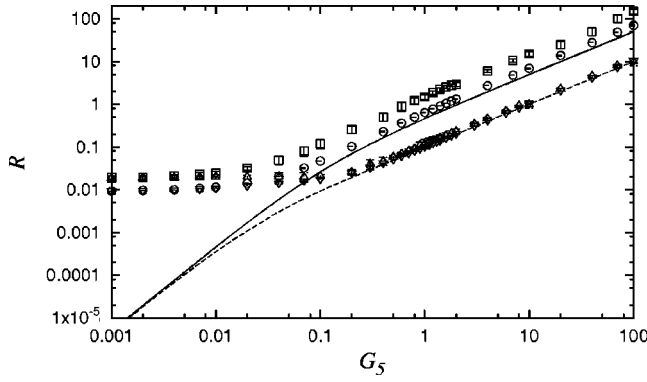
FIG. 5. The behavior of $\mathcal{R}$ against G_5 for (G,β) $=(0,0.1)$ ($\square$), $(0,1)$ ($\bigcirc$), $(1,0.1)$ ($\triangle$), and $(1,1)$ ($\bigtriangledown$). For comparison, $\mathcal{R}$ for the generalized Gross-Neveu model is also displayed for $G=0$ (solid) and $G=1$ (dashes).
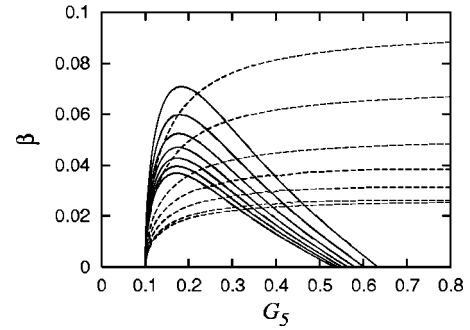

FIG. 6. The zeros of the mass susceptibility parameter in the $G_5-\beta$ plane when $G=0.1$, computed using finite-dimensional basis spaces. The solid curves, from top to bottom, correspond to the zeros in the variational method for the basis space dimensions of 8, 10, 12, 14, 16, 18, and 20. The dashed curves, from top to bottom, correspond to the zeros of the susceptibility in the Multhopp's method for the basis space dimensions of 20, 40, 100, 200, 400, 800, and 1000. In the small regions below the respective curves, the susceptibility is negative. The negative region becomes smaller with the increase in the size of the basis space in both methods.

model as we did here, it is quite nontrivial to derive them analytically.

We have investigated the susceptibility extensively within the parameter space of the theory and found that it is positive, except for a relatively small region, which we now discuss. In most regions in the parameter space, the numerical convergence of the susceptibility is quite rapid at least in one of the methods and both methods yield consistent results. In all these regions, the susceptibility parameter satisfies $\mathcal{R} > 0$. However, for small β, the convergence is rather slow. Particularly intriguing is the region $G \gtrsim G_5, 0 \leq \beta \ll 1$. A simple argument shows why the behavior in this region can be subtle: In general, the finite-dimensional numerical results are analytic with respect to the parameters G, G_5, and β. From the behavior $\mathcal{R} \to 0$ as $\beta \to 0$, $G=G_5$, we know that unless $\mathcal{R}$ vanishes as $(G-G_5)^2$ or some higher even power for $\beta=0$, there will be a region where $\mathcal{R}$ is negative. Indeed, we find in the numerical results that $\mathcal{R}$ is negative in the region $G \gtrsim G_5, 0 \leq \beta \ll 1$ using both the variational and the Multhopp's method. Even the extrapolated values, in some cases, are negative. The meson mass squared is always positive even in these cases and the physics of the system seems to be quite consistent. Naively, we would claim that the susceptibility and hence also the standard inequality does not hold in this regime. This, to our knowledge, would *not* conflict with any general arguments regarding mass inequality relations. However, to conclude, this would be somewhat premature, since if we study the negative region in the parameter space, we find as in Fig. 6 that it shrinks when the basis space is enlarged and the negative region is quite small compared to $\mathcal{O}(1)$ which is the natural scale in the problem. It should also be noted that even if $\mathcal{R}=a_1(G-G_5)+\mathcal{O}[(G-G_5)^2]$ for $\beta=0$ when the basis space is finite, it is still possible that the coefficient a_1 approaches 0 in the full basis space, so that the positive susceptibility is compatible with a negative one in the truncated basis space. On the other hand, the regions of negative susceptibility have a common region with respect to both methods so it is also possible that a finite region of negative susceptibility remains even when the basis space is complete. We therefore conclude that while the susceptibility may be negative in the regime $G \gtrsim G_5, 0 \leq \beta \ll 1$,

further investigation is necessary to clarify this point. An analytic computation determining the sign of the mass difference would be ideal. If this is not possible, a set of basis optimized for the gauged four-Fermi models in this particular parameter regime, in either variational or Multhopp's method, should settle this issue.

VI. SUMMARY AND DISCUSSIONS

We have systematically and quantitatively studied the mass differences in gauged four-Fermi models in $(1+1)$-dimensions. Even in the cases where the mass inequality relations have been shown to be valid from general arguments, the size of the difference is unknown unless an explicit computation is made. We believe that the mass difference is an interesting dynamical quantity characterizing the spectrum of relativistic quantum-field-theories. To analyze the differences quantitatively, we adopted a natural susceptibility parameter to compare the size of the mass differences throughout the field-theory space. We found that the parameter captures the essence of the mass difference when the constituent mass differences are not too large. In the family of generalized Gross-Neveu models, we were able to derive an analytic expression for the meson mass susceptibility. In the more general case of gauged four-Fermi models, we have developed methods for obtaining the mass differences systematically and have computed them. Since not much seems to be known about the quantitative behavior of mass differences, we think that it is significant to have a class of relativistic field-theory models where it has been studied explicitly. While the results are interesting, there remain further questions which should be answered.

An important question is whether the positivity of the mass difference is much more general than the cases where it has been shown to hold [2]. In particular, an intriguing problem is whether there is a relativistic quantum-field-theory wherein the mass difference is negative yet its physics be-

havior is consistent. We have found that the meson mass susceptibility is positive for most of the parameter space in the gauged four-Fermi models and have explained some of the behavior analytically. The models we studied here include the celebrated models of 't Hooft and of Gross and Neveu for special choices of the parameters. For the 't Hooft model, the standard arguments [2] *do* apply and we may show analytically that the difference is positive. However, in general, no such arguments can be applied to gauged four-Fermi models. Furthermore, the Gross-Neveu models are known to be equivalent to models with Yukawa couplings and also display dynamical chiral symmetry-breaking behavior. These are exactly the kind of situations in which we might doubt whether the mass difference is positive [8,9]. It is interesting that even in these cases, the mass difference is positive, so that in fact, the property holds in much more general than those situations where it has been proven. It would be interesting to find an analytic proof for this property if possible. It is important to understand why the difference is positive for the gauged four-Fermi models and clarify if this can be extended to other theories, such as supersymmetric theories with bound states. There still remains a small region within the field-theory space wherein the sign of the susceptibility, hence also the mass difference, remains uncertain and further investigation is necessary to establish its sign. While $(1+1)$-dimensional theories such as the 't Hooft model or the Gross-Neveu model have physical behavior resembling those of higher dimensions, we should mention the possibility that in higher dimensions, the behavior of the susceptibility might be quite different. Also, even in $(1+1)$ dimensions, the mass differences might behave qualitatively differently for other classes of models.

We have seen in Sec. III that the mass difference is positive in a large class of quantum-mechanics models [3]. This leads us to suspect that the mass inequality relations are valid for a large class of relativistic quantum-field-theories also. This is certainly consistent with our findings here. However, it should be noted that spontaneous breaking of symmetries is essentially a field-theoretical behavior, which is also quite relevant to the theory we studied. Therefore, we believe that it would be worthwhile to perform further research and in particular, clarify whether the mass difference can become negative in relativistic quantum-field-theories. In another direction, large N limit of field theories, such as the class of models we study, are presumably described by some kind of string theories [6]—an idea, which has recently been made more concrete [4]. It would be interesting to find out what kind of string theories our models correspond to and to elucidate how mass inequalities fit into the string picture.

APPENDIX: PERTURBATION THEORY IN THE MASS DIFFERENCES FOR THE SPECTRUM

Here, we shall briefly outline how to perform perturbation with respect to the relative constituent mass difference, Δ, in the methods explained in Sec. V B for obtaining the spectrum. The standard perturbation methods cannot be applied here. One major reason is that the boundary conditions (15) depend on the masses of the constituents so that they need to

be perturbed also. There are additional complications for both the methods used in Sec. V B, as we shall describe below.

In both cases, we perturb in the relative mass difference Δ and obtain an expansion for the meson mass in terms of Δ, for the cases $(M_1^2,M_2^2)=(M_a^2,M_a^2),(M_b^2,M_b^2)$, and (M_a^2,M_b^2)

$$\mu^2 \equiv \mu_0^2 + \Delta\mu_1^2 + \Delta^2\mu_2^2 + \mathcal{O}(\Delta^3). \quad (A1)$$

In the first two cases, the first-order term exists and is of the same size but of opposite sign, and in the last case the first-order term is absent. Therefore, the leading-order term in the mass difference $\delta\mu_{ab}$ will be of order Δ^2, as it should be.

1. Variational method

In the variational method, we need to consider a generalized eigenvalue problem with the normalization matrix not being the identity matrix. In theory, we can just orthonormalize the basis vectors, but in practice, this is not numerically equivalent since the normalization matrices can become almost singular even though we have tried to normalize the matrix elements to be of order one. Furthermore, since the boundary conditions also are perturbed, the normalization matrices will also have a nontrivial expansion in Δ.

Let us expand the matrices as

$$H \equiv H_0 + \Delta H_1 + \Delta^2 H_2 + \mathcal{O}(\Delta^3),$$

$$N \equiv N_0 + \Delta N_1 + \Delta^2 N_2 + \mathcal{O}(\Delta^3). \quad (A2)$$

Assume that we have the complete eigen-system for the 0th order problem:

$$H_0 w_{0n} = \mu_{0n}^2 N_0 w_{0n}, \quad (w_{0m}, N_0 w_{0n}) = \delta_{mn}. \quad (A3)$$

Then, we obtain the expansion for mass squared of the meson state labeled by n

$$\mu_{1n}^2 = (w_{0n}, (H_1 - \mu_{0n}^2 N_1) w_{0n}),$$

$$\mu_{2n}^2 = (w_{0n}, (H_2 - \mu_{0n}^2 N_2) w_{0n})$$

$$\quad - (w_{0n}, N_1 w_{0n})(w_{0n}, (H_1 - \mu_{0n}^2 N_1) w_{0n})$$

$$\quad + \sum_m \frac{1}{\mu_{0n}^2 - \mu_{0m}^2} |(w_{0n}, (H_1 - \mu_{0n}^2 N_1) w_{0m})|^2.$$

$$(A4)$$

We need the expansions of the matrices H,N in terms of Δ for the three cases, $(M_1^2,M_2^2)=(M_a^2,M_a^2)$, (M_b^2,M_b^2), and (M_a^2,M_b^2), to obtain the final results. Since this expansion is cumbersome but logically straight forward, it will not be explicitly presented here to save space.

2. Multhopp's method

In Multhopp's method, the matrices are not Hermitian so that we need to perform the perturbation theory with some care. Furthermore, due to the perturbation in the boundary

conditions, the matrix P will also be perturbed. To perform the expansion, we will reduce the equation to a mathematically equivalent problem,

$$(\mu^2 - P^{-1}M)v = 0. \tag{A5}$$

Ideally, it is better not to invert matrices numerically, but it is a substantially more complicated numerical task to solve a generalized nonsymmetric eigenvalue problem and also, in this case, the matrix P turns out to be quite robust against inversion even for moderately large basis spaces with dimensions of order 10^3.

We expand the matrices as

$$P = P_0 + \Delta P_1 + \Delta^2 P_2,$$

$$M = M_0 + \Delta M_1 + \Delta^2 M_2,$$

$$(P^{-1}M) = (P^{-1}M)_0 + \Delta(P^{-1}M)_1$$
$$+ \Delta^2(P^{-1}M)_2 + \mathcal{O}(\Delta^3), \tag{A6}$$

where

$$(P^{-1}M)_0 = P_0^{-1}M_0, \quad (P^{-1}M)_1 = P_0^{-1}(M_1 - P_1 P_0^{-1}M_0),$$

$$(P^{-1}M)_2 = P_0^{-1}(M_2 - P_2 P_0^{-1}M_0 - P_1 P_0^{-1}M_1$$
$$+ P_1 P_0^{-1}P_1 P_0^{-1}M_0). \tag{A7}$$

We need to first solve the 0th order problem for the left and right eigenvectors, $\{u_n\}$ and $\{v_n\}$[1]

$$u_{0m}(P^{-1}M)_0 = \mu_{0m}^2 u_{0m}, \quad (P^{-1}M)_0 v_{0n} = \mu_{0n}^2 v_{0n},$$

$$(u_{0m}, v_{0n}) = \delta_{mn}. \tag{A8}$$

Then, we may obtain the expansion for the meson mass squared of the meson state labeled by n as

$$\mu_{1n}^2 = (u_{0n}, (P^{-1}M)_1 v_{0n}), \tag{A9}$$

$$\mu_{2n}^2 = (u_{0n}, (P^{-1}M)_2 v_{0n})$$
$$+ \sum_{k \neq n} \frac{(u_{0n}, (P^{-1}M)_1 v_{0k})(u_{0k}, (P^{-1}M)_1 v_{0n})}{\mu_{0n}^2 - \mu_{0k}^2}. \tag{A10}$$

The rest proceeds as in the variational method case. As in the case of the variational method, the explicit expressions for the matrices are not shown here due to space considerations.

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[1] From a mathematical point of view, additional complications can arise in general; namely, the eigenvalues may be degenerate so that the matrix is not diagonalizable, or the eigenvalues may be complex. However, we need to keep in mind that we do not have to solve the problem for general dimensions of the basis space, but only for a sequence of spaces that will allow us to obtain the susceptibility. In practice, these complications do not hinder our computations in the cases we have studied.

[1] For instance, see the contributions in Nucl. Phys. B (Proc. Suppl.) **94**, (2001).

[2] D. Weingarten, Phys. Rev. Lett. **51**, 1830 (1983); E. Witten, *ibid.* **51**, 2351 (1983); S. Nussinov, *ibid.* **51**, 2081 (1983).

[3] S. Nussinov and M.A. Lampert, hep-ph/9911532.

[4] N. Seiberg and E. Witten, Nucl. Phys. **B426**, 19 (1994); **B431**, 484 (1994); K. Intriligator and N. Seiberg, Nucl. Phys. B (Proc. Suppl.) **45BC**, 1 (1996); J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, Phys. Lett. B **428**, 105 (1998); E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998); O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).

[5] H. Nishino, Phys. Rev. D **61**, 025008 (2000).

[6] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974); **B75**, 461 (1974).

[7] D.J. Gross and A. Neveu, Phys. Rev. D **10**, 3235 (1974).

[8] C. Vafa and E. Witten, Nucl. Phys. **B234**, 173 (1984).

[9] M. Cvetic, Nucl. Phys. **B279**, 593 (1987); U. Maryland preprint MDDP-PP-85-023, 1984.

[10] S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, England, 1985); B. Rosenstein, B.J. Warr, and S.H. Park, Phys. Rep. **205**, 59 (1991).

[11] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); for recent reviews, see for instance, J.L. Rosner, Comments Mod. Phys., Part C **1**, 11 (1999); R.S. Chivukula, hep-ph/9903500.

[12] Y. Nambu, in *New Theories in Physics*, edited by Z. Ajduk, S. Pokorski, and A. Trautman (World Scientific, Singapore, 1989); and in *New Trends in Strong Coupling Gauge Theories*, edited by M. Bando, T. Muta, and K. Yamawaki (World Scientific, Singapore, 1989); V.A. Miransky, M. Tanabashi, and K. Yamawaki, Mod. Phys. Lett. A **4**, 1043 (1989); Phys. Lett. B **221**, 177 (1989); W.A. Bardeen, C.T. Hill, and M. Lindner, Phys. Rev. D **41**, 1647 (1990).

[13] M. Burkardt, Phys. Rev. D **56**, 7105 (1997).

[14] K. Itakura, Ph.D. thesis, Tokyo University, 1996.

[15] K. Aoki and T. Ichihara, Phys. Rev. D **52**, 6435 (1995).

[16] K. Aoki and K. Ito, Phys. Rev. D **60**, 096004 (1999).

[17] S. Coleman, Commun. Math. Phys. **31**, 259 (1973).

[18] W.A. Bardeen, R.B. Pearson, and E. Rabinovici, Phys. Rev. D **21**, 1037 (1980).

[19] A.J. Hanson, R.D. Peccei, and M.K. Prasad, Nucl. Phys. **B121**, 477 (1977); R.C. Brower, W.L. Spence, and J.H. Weis, Phys. Rev. D **19**, 3024 (1979); S. Huang, J.W. Negele, and J. Polonyi, Nucl. Phys. **B307**, 669 (1988); R.L. Jaffe and P.F. Mende, *ibid.* **B369**, 182 (1992).