

BPS kinks in the Gross-Neveu model

Paul Fendley

Department of Physics, University of Virginia, Charlottesville, Virginia 22904-4714

Hubert Saleur

Department of Physics and Astronomy, University of Southern California, Los Angeles, California 90089

(Received 18 May 2001; published 29 November 2001)

We find the exact spectrum and degeneracies for the Gross-Neveu model in two dimensions. This model describes N interacting Majorana fermions; it is asymptotically free, and has dynamical mass generation and spontaneous chiral symmetry breaking. We show here that the spectrum contains $2^{N/2}$ kinks for any N . The unusual $\sqrt{2}$ in the number of kinks for odd N comes from restrictions on the allowed multi-kink states. These kinks are the Bogomol'nyi-Prasad-Sommerfield states for a generalized supersymmetry where the conserved current is of dimension $N/2$; the $N=3$ case is the $\mathcal{N}=1$ supersymmetric sine-Gordon model, for which the spectrum consists of $2\sqrt{2}$ kinks. We find the exact S matrix for these kinks, and the exact free energy for the model.

DOI: 10.1103/PhysRevD.65.025001

PACS number(s): 11.10.Kk, 11.30.Pb, 11.55.Ds

I. INTRODUCTION

The Gross-Neveu model describes interacting fermions in two dimensions. It has no gauge fields or gauge symmetry, yet it exhibits much of the behavior of gauge theories in four dimensions. The coupling constant is naively dimensionless, but radiative corrections result in a mass scale to the theory. Thus the theory is asymptotically free and strongly interacting in the infrared. A (discrete) chiral symmetry is spontaneously broken, which gives the fermions mass. In condensed-matter language, the interaction is marginally relevant, and there is no nontrivial fixed point.

The Gross-Neveu model consists of N Majorana fermions ψ^i , $\bar{\psi}^i$, with $i=1, \dots, N$. The action is

$$S = \int d^2z [\psi^i \bar{\partial} \psi^i + \bar{\psi}^i \partial \bar{\psi}^i + g(\bar{\psi}^i \psi^i)(\bar{\psi}^j \psi^j)] \quad (1)$$

where repeated indices are summed over. At the critical point $g=0$, the fermions are free, with ψ^i a function only of z and $\bar{\psi}^i$ a function only of \bar{z} . At this critical point in two dimensions, the fermions have left and right dimensions $(1/2, 0)$ for ψ and $(0, 1/2)$ for $\bar{\psi}$, so the coupling g is naively dimensionless. However, the beta function for this interaction is nonvanishing [1]. For $g > 0$, the trivial free-fermion fixed point is unstable, and a mass scale is generated. We denote this mass scale M . In this paper we study $N \geq 3$; for $N=2$ the model reduces to the well-known massless Thirring model (the Luttinger model in the condensed-matter literature), and for $N=1$ it is free. The action (1) is invariant under the symmetry group $O(N)$. In a separate paper, we will discuss generalizations to other symmetries.

Despite a huge number of papers discussing various aspects of this model, there remained an important unanswered question: what is the spectrum of particles? Just because the fields are known does not mean the spectrum is known: there may be kink states or bound states, and in some cases there are no one-particle states corresponding to the fields them-

selves. In particular, it was long known that for even N the spectrum contained $2^{N/2}$ kink states in the two spinor representations of the $SO(N)$ symmetry algebra [2]. The arguments which led to the existence of kinks for even N apply equally well to odd N , but it was not clear how many such kinks there were when N is odd. It was shown in [3] that there is no consistent S matrix for a single spinor representation, and by utilizing a generalized supersymmetry, Witten showed that the kinks are in a reducible representation of $SO(N)$ when $N=2P+1$ is odd. He went on to conjecture that they are in two copies of the 2^P -dimensional spinor representation.

In this paper we will answer this question and at long last complete the computation of the mass spectrum. We show that there are $\sqrt{2}$ copies of the spinor representation when N is odd: the number of kinks for any N is $2^{N/2}$. A noninteger number of kinks means that there are restrictions on the allowed multi-kink states; we give a precise definition below. These kinks are in Bogomol'nyi-Prasad-Sommerfield (BPS) representations of the generalized supersymmetry. For $N=3$, the model is equivalent to the $\mathcal{N}=1$ supersymmetric sine-Gordon model, so our result shows that these unusual BPS kinks appear even for ordinary supersymmetry. To confirm our claims we compute the exact S matrix and free energy of this model, and find agreement with the known results in the ultraviolet limit.

II. THE SYMMETRIES

To determine the spectrum and S matrices of the Gross-Neveu model, we need to understand the symmetry structure of the model in depth. There are actually four different symmetries which we utilize in this paper. These are all discussed in [2]; this section is basically a review of these results.

A. $O(N)$ symmetry

The action has a global $O(N)$ symmetry $\psi^i \rightarrow U^{ij} \psi^j$ and $\bar{\psi}^i \rightarrow U^{ijT} \bar{\psi}^j$. The matrix U must be an element of $O(N)$

because the Majorana fermions are real. The existence of this global symmetry means that the particles of the model must transform in representations of this symmetry.

B. Spontaneously-broken chiral symmetry

The action (1) has a \mathbf{Z}_2 chiral symmetry $\psi \rightarrow -\psi$, $\bar{\psi} \rightarrow \bar{\psi}$. However, for $g > 0$, this symmetry is spontaneously broken, because the fermion bilinear $\sigma \equiv \bar{\psi}^i \psi^i$ gets an expectation value. This expectation value results in a mass for the fermions. Note that this expectation value does not break the $O(N)$ symmetry; a continuous symmetry cannot be spontaneously broken in two spacetime dimensions. Equivalently, the discrete parity symmetry $\psi(z, \bar{z}) \rightarrow \bar{\psi}(\bar{z}, z)$ is spontaneously broken.

C. Local conserved charges

When a model possesses an infinite number of local conserved currents transforming non-trivially under the Lorentz group, it is said to be integrable. The integrability results in powerful constraints on the S matrix, which will be discussed below. It was shown in [2] that the action (1) has at least one of these conserved currents. When $g=0$, the energy-momentum tensor $T = \psi^i \partial \psi^i$ obeys $\bar{\partial} T^n = 0$ for any integer n . This no longer holds for $g \neq 0$. However, $\bar{\partial} T^2$ must have dimension 5 and Lorentz spin 3. All operators of this dimension and spin must be a total derivative:

$$\bar{\partial} T^2 = \bar{\partial} A + \partial B.$$

Thus there is a conserved current with components $(T^2 - A, B)$ in the Gross-Neveu model. This already requires that the scattering be factorizable, and it seems very likely that there is a conserved current of dimension $2n$ (and hence a charge of dimension $2n-1$) for all integer n .

D. Generalized supersymmetry

A very striking feature of the Gross-Neveu model shown in [2] is that it possesses an extra conserved current of spin $N/2$. This conserved current is

$$J = \epsilon_{i_1 i_2 \dots i_N} \psi^{i_1} \psi^{i_2} \dots \psi^{i_N}. \quad (2)$$

Another conserved current \bar{J} of spin $-N/2$ follows from replacing ψ with $\bar{\psi}$. To prove that this is a conserved current even for $g \neq 0$, one shows that all possible contributions to $\bar{\partial} J$ are themselves total derivatives. In an equation,

$$\bar{\partial} J = \partial \bar{J}.$$

For $N=2$, \bar{J} vanishes and this conserved current is the axial symmetry of the massless Thirring model (Luttinger model). For $N=4$, the Gross-Neveu model decouples into two sine-Gordon models, and the spin-2 current J is the difference of the energy-momentum tensors of the two theories.

For odd $N=2P+1$, the results are much more surprising. For $N=3$, the current is of dimension $3/2$, and generates

supersymmetry transformations. This seems odd in a theory of fermions, but by bosonizing two of the three fermions, one indeed obtains the $\mathcal{N}=1$ supersymmetric sine-Gordon model [2]. For general odd N , this results in a generalized supersymmetry. The full current algebra of these currents seems quite tricky. It seems very likely that at $g=0$ it is the WB_P algebra studied in [4], which indeed involves a current of spin $P+1/2$ along with those of even integer spin. This was explicitly worked out for the $P=2$ case (when $g=0$) in [5]. This current algebra involves the spin-4 local current as well. These currents remain conserved when $g \neq 0$, just like the supersymmetry currents do when $P=1$.

Luckily, to determine the S matrix we do not need the full current algebra: we need only to understand how the conserved charges act on the particles. The conserved charge Q of dimension $N/2-1$ is defined by $Q = \int dz J + \int d\bar{z} \bar{J}$, and likewise for \bar{Q} . The operator Q^2 must be of dimension and Lorentz spin $N-2$. For example, for $N=3$, using the explicit form of the charge gives $Q^2 \propto P_L$ and $\bar{Q}^2 = P_R$, with P_L and P_R the left and right momenta [6]. For general N , this means that when Q^2 acts on a particle with energy E and momentum p ,

$$Q^2 \propto (E+p)^{N-2},$$

since this is the only combination of energy and momentum with the correct Lorentz properties. It is convenient to define the rapidity θ so that a particle of mass m has energy $E = m \cosh \theta$ and momentum $p = m \sinh \theta$. The symmetry algebra acting on a particle for odd N can then be written

$$\begin{aligned} \{Q, Q\} &= 2m^{N-2} e^{(N-2)\theta} & \{\bar{Q}, \bar{Q}\} &\propto 2m^{N-2} e^{-(N-2)\theta} \\ \{Q, \bar{Q}\} &= 2Z \end{aligned} \quad (3)$$

The central term Z acts on the states as $Z|\alpha\rangle = m^{N-2} z_\alpha |\alpha\rangle$. The (real) numbers z_α vanish at $g=0$, but do not otherwise. As we will discuss, non-zero z_α occur because of the non-trivial boundary conditions on the kink states [7].

III. THE KINK SPECTRUM OF THE GROSS-NEVEU MODEL

One interesting feature of the Gross-Neveu model is that the spectrum is far more intricate than a glance at the action (1) would suggest. For any value of N , the spectrum includes kink states, as pointed out in [8,2]. In fact, for $N=3$ and $N=4$, the spectrum does not contain anything but the kink states. The spectrum includes kink states because of the spontaneously broken \mathbf{Z}_2 chiral symmetry. The kink interpolates between the two degenerate vacua; in two-dimensional spacetime there are domain walls between regions of the two vacua. The main result of this paper is to answer the question: how many kink states are there?

The result requires first finding how the kink states transform under the symmetries discussed in the last section. We will find the minimal number of particles required to transform under these symmetries non-trivially. In Sec. IV, we will work out the S matrix of these particles, and then in Sec.

V we will show that these particles and S matrices give the correct free energy.

Again, we follow the analysis of [2] and study the kinks semiclassically. The fermion bilinear $\sigma \equiv \bar{\psi}^i \psi^i$ gets an expectation value σ_0 ; the kink is a configuration of σ interpolating between $+\sigma_0$ and $-\sigma_0$. In the semi-classical approach, one treats σ as a classical background field, and quantizes the fermions in this background. The fermions interact with the background by the interaction

$$g \sigma \bar{\psi}^i \psi^i.$$

Thus in the semiclassical limit, this problem is equivalent to quantizing N Majorana fermions in a kink background. A famous result of Jackiw and Rebbi shows that the Dirac equation with this background possesses a single normalizable zero-energy mode f_0, \bar{f}_0 [9]. The zero-energy solution is real, while the finite-energy solutions are complex, so the zero-energy state is created/annihilated by a real operator b^i , while the finite-energy states are annihilated and created by a_n^i and $a_n^{i\dagger}$ respectively. Thus the semiclassical expansion for ψ^i and $\bar{\psi}^i$ is

$$\begin{aligned} \psi^i &= f_0 b^i + \sum_{n=0}^{\infty} [f_n a_n^i + f_n^* a_n^{i\dagger}] \\ \bar{\psi}^i &= \bar{f}_0 b^i + \sum_{n=0}^{\infty} [\bar{f}_n a_n^i + \bar{f}_n^* a_n^{i\dagger}]. \end{aligned} \quad (4)$$

The canonical commutation relations for ψ^i require that

$$\{b^i, b^j\} = 2 \delta^{ij}. \quad (5)$$

This is called a Clifford algebra; the combinations $M^{ij} = [b^i, b^j]$ generate the $SO(N)$ algebra. In this paper, when we use $O(N)$ we generally mean the symmetry group, while $SO(N)$ refers to the resulting symmetry algebra.

The presence of these zero modes means that there must be more than one kink state, because one can always act with any of the b^i and get a different state. In other words, denoting the kink states by $|\alpha\rangle$, some matrix elements

$$\langle \beta | b^i | \alpha \rangle \quad (6)$$

are non-vanishing. The kink states therefore form a representation of the Clifford algebra. The simplest possibility is that the b^i act like the gamma matrices of $SO(N)$. A gamma matrix $(\gamma^i)_{rs}$ obeys the Clifford algebra (5), and is an $SO(N)$ invariant where $i=1 \dots N$ transforms in the vector representation of $SO(N)$, while $r, s=1 \dots 2^{\text{int}(N/2)}$ transform in the spinor representation(s). The kinks must therefore be in the spinor representation(s) of $SO(N)$. This is effectively charge fractionalization, because the vector representation appears in the tensor product of two spinor representations. Thus the original fermions ψ^i in the vector representation “split” into kinks in the spinor representation(s).

The properties of the spinor representation(s) of $SO(N)$ are different for even and odd N . For even N , there is an operator $\gamma^{N+1} \equiv i b^1 b^2 \dots b^N$ which commutes with all the $SO(N)$ generators M^{ij} but anticommutes with the Clifford algebra generators. Therefore the $2^{N/2}$ -dimensional represen-

tation of the Clifford algebra is not irreducible under the $SO(N)$ when N is even. It decomposes into irreducible representations with $\gamma^{N+1} = \pm 1$. These representations are each $2^{N/2-1}$ -dimensional, and are known as the spinor representations of the algebra $SO(N)$. The entire kink spectrum for even N consists of $2^{N/2}$ kinks in the two spinor representations. It was shown in [10] that including these kinks along with the other particles gives the correct free energy. This effectively proves that these are all the kink states.

For odd $N=2P+1$, the spinor representation of dimension 2^P is irreducible. There is only one such spinor representation of $SO(2P+1)$. Thus it might seem that the simplest possibility is that there are precisely 2^P kinks in the Gross-Neveu model for odd N . However, Witten shows that there must be more [2]. The key is to look at the discrete symmetry $\psi^i \rightarrow -\psi^i, \bar{\psi}^i \rightarrow -\bar{\psi}^i$. One can think of this symmetry as being $(-1)^F$, with the caveat that the fermion number F is not defined for N odd: only $(-1)^F$ is. This symmetry is part of the $O(N)$ group, but not part of the connected $SO(N)$ subgroup when N is odd. Because of the form (4), $b^i \rightarrow -b^i$ under this symmetry. Thus there is a symmetry operator $\Lambda = (-1)^F$ which obeys $\Lambda^2 = 1$ and $\Lambda b^i \Lambda = -b^i$. In other words, Λ anticommutes with all the Clifford algebra generators, and commutes with the $SO(2P+1)$ generators. Thus the kink states in an irreducible representation of $SO(2P+1)$ must all have the same eigenvalue of Λ . The fact that the matrix element (6) does not vanish means that some kink states have eigenvalue $+1$, while some have eigenvalue -1 . Thus there *must* be more kink states than those in the single irreducible representation of $SO(2P+1)$. Note that for even N , we can identify Λ with γ^{N+1} , and the multiple representations with the two different spinor representations. For odd N , there is only one kind of spinor representation, so it must appear at least twice.

So what happens for odd N ? The simplest possibility is that there are kinks in two spinor representations, each of dimension 2^P [2,11]. We will see that this is essentially correct, but there is a major subtlety. The key to the answer is in the generalized supersymmetry.

It is very useful to first examine the case $N=3$, where the answer is known [12–15]. Here the Gross-Neveu model is equivalent to the $\mathcal{N}=1$ supersymmetric sine-Gordon model [2] at a fixed value of the sine-Gordon coupling β_{SG} where the model has an extra $SO(3)$ symmetry. One might expect that the kink spectrum would generalize the kink and antikink of the ordinary sine-Gordon model to account for the supersymmetry. For the $\mathcal{N}=1$ supersymmetric sine-Gordon model, it seems plausible that the kink and antikink each are a boson/fermion doublet, making four particles in all [16]. However, this is not the correct spectrum. The reason is the central charge Z appearing in the supersymmetry algebra [(3) with $N=3$]. For the supersymmetric sine-Gordon model, Z does not vanish [7]: it is the integral of a total derivative. Z acting on the kink states is non-zero because here the boundary conditions on the field at positive spatial infinity and negative spatial infinity are different.

The consequences of a non-vanishing Z are familiar in supersymmetric field theories. Representations where the mass $m = |Z|$ are called BPS states [17]. BPS representations

usually have a smaller number of states than those with $Z = 0$ [7] and so are often referred to as “reduced multiplets.” When the BPS condition $m = |Z|$ holds, one can find a combination of the generators which annihilates all BPS states. For the generalized supersymmetry, the condition is

$$\left(Q - \frac{m^{N-2}}{Z} e^{(N-2)\theta} \bar{Q} \right) |\alpha\rangle = 0 \quad (7)$$

for all BPS states $|\alpha\rangle$.

We need now to introduce a precise definition of the “number of kink states” K . We say there are K kink states if the number of n -particle kink states at large n depends on n as K^n . Put another way, the entropy per kink of a gas of non-interacting kinks is $\ln K$. K is usually an integer in field theory: e.g., if there are two one-particle states, there are four two-particle states, eight three-particle states, and so on. The kink structure need not be so simple, though. For example, consider kinks in a potential $V(\phi) = (\phi^2 - 1)^2$. The potential has minima at $\phi = \pm 1$, so there are two one-kink states, having $\phi(\infty) = \pm 1$ and $\phi(-\infty) = \mp 1$. However, there are only two two-kink states. In fact, there are only two kink states for any n . Thus the number of kink states in this example is $K=1$. The example of most importance for this paper is kinks in a potential like $V(\phi) = \phi^2(\phi^2 - 1)^2$. Here the minima are $\phi = 0, \pm 1$. If kinks are allowed to only interpolate between adjacent minima (i.e., only between 0 and ± 1), then the number of n -kink states doubles every time n is increased by 2. Therefore, $K = \sqrt{2}$ in this triple-well potential.

For the BPS representations of the generalized supersymmetry (3), $K = \sqrt{2}$. This was already known for $\mathcal{N}=1$ supersymmetry in two dimensions [12,14,11], the $N=3$ case here. The BPS representations were initially given for the thermal $(\phi_{1,3})$ perturbation of the tricritical Ising model, which has $\mathcal{N}=1$ supersymmetry as well as a Landau-Ginzburg description in terms of a field ϕ with three degenerate minima [12]. To reconcile these two facts, Zamolodchikov showed how supersymmetry acts non-locally on the $\sqrt{2}$ kinks in this triple-well potential. Somewhat more surprisingly, there are $\sqrt{2}$ kinks in the $\mathcal{N}=1$ sine-Gordon model as well.

To give explicitly the action of the generalized supersymmetry for odd N , we denote a kink with $\phi(x = -\infty) = r$ and $\phi(x = +\infty) = s$ as $|rs\rangle$. In the triple-well potential, $r, s = 0, \pm 1$, and $r - s = \pm 1$. The central charge Z acting on a kink of mass m in a BPS representation is $m^{N-2}(r^2 - s^2)$. The BPS representation of the generalized supersymmetry for odd N is non-local: it changes the kink configuration all the way to spatial $-\infty$. On a single-particle state of rapidity θ ,

$$Q|r0\rangle = ir(me^\theta)^{(N-2)/2}|-r0\rangle$$

$$Q|0r\rangle = r(me^\theta)^{(N-2)/2}|0r\rangle$$

$$\bar{Q}|r0\rangle = ir(me^{-\theta})^{(N-2)/2}|-r0\rangle$$

$$\bar{Q}|0r\rangle = -r(me^{-\theta})^{(N-2)/2}|0r\rangle.$$

One can indeed verify that Q and \bar{Q} acting on this representation satisfy the algebra (3). Moreover, the discrete symmetry Λ acts non-diagonally as

$$\Lambda|rs\rangle = |-r-s\rangle.$$

Λ indeed anticommutes with the generalized supersymmetry generators. With the interpretation of Λ as $(-1)^F$, we see that the kinks are neither bosons nor fermions. Since the action of Q and \bar{Q} is non-local, we must be careful to define their action on multiparticle states as well. For example on a three-kink state, Q acts as

$$\begin{aligned} Q(|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle) &= (Q|\alpha\rangle) \otimes |\beta\rangle \otimes |\gamma\rangle + (\Lambda|\alpha\rangle) \otimes (Q|\beta\rangle) \\ &\otimes |\gamma\rangle + (\Lambda|\alpha\rangle) \otimes (\Lambda|\beta\rangle) \otimes (Q|\gamma\rangle) \end{aligned}$$

and likewise for \bar{Q} . Thus when Q acts on a kink, it flips all kinks to the left of it. For $N=3$, this reduces to the action of supersymmetry discussed in [12,14].

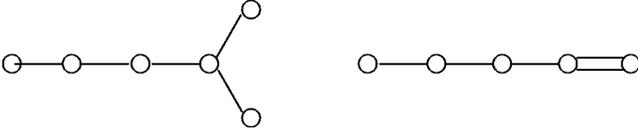
Of course, the kinks in the Gross-Neveu model must still transform as a spinor of $SO(N)$. Thus we see that for odd N , the simplest non-trivial kink spectrum consistent with all the symmetries of the theory is for the kinks to be spinors of $SO(2P+1)$ and interpolate between the three degenerate wells of this potential. Each kink is labeled by a spinor index $1 \dots 2^P$ and by a pair of vacua. The number of kinks is $\sqrt{2} \times 2^P = 2^{N/2}$. We emphasize that the generalized supersymmetry and the discrete symmetry Λ require that there be more than the spinor: $2^{N/2}$ is the minimum number of kinks satisfying these constraints. In subsequent sections, we will find the S matrix for these kinks, and show that it gives the correct free energy, effectively confirming the presence of these BPS kinks.

IV. THE S MATRICES

In this section we work out the S matrix for the kinks in the Gross-Neveu model for odd $N = 2P + 1$.

This S matrix must obey a variety of constraints. These are easiest to write in terms of rapidity variables, defined so that the rapidity θ_1 of a particle is related to its energy and momentum by $E = m \cosh \theta_1$, $p = m \sinh \theta_1$. Lorentz invariance requires that the two-particle S matrix depend only on the difference $\theta = \theta_1 - \theta_2$. Any S matrix arising from a unitary field theory must be unitary and crossing-symmetric. Moreover, the integrability of the Gross-Neveu model means that the multi-particle S matrix must factorize into the product of two-particle ones. The resulting constraint is called the Yang-Baxter equation. Finally, the S matrix must obey the bootstrap equations. These mean that the S matrix elements of a bound state can be expressed in terms of the S matrix of the constituents. This constraint is explained in detail in [3], for example. Bound states show up as poles in the S matrix in the “physical strip” $0 < \text{Im}(\theta) < \pi$. If there is such a pole at $\theta = \theta_j$ in the kink-kink S matrix, then the kinks of mass m have a bound state at mass

$$m_j = 2m \cosh(\theta_j/2). \quad (8)$$


 FIG. 1. The $SO(N)$ Dynkin diagrams for even N and odd N .

When $N=3$ and $N=4$, the kinks are the only states in the spectrum [8]. For $N=3$, this was confirmed by the computation of the exact free energy of the $\mathcal{N}=1$ supersymmetric sine-Gordon model [15]. For the latter, it follows because the $N=4$ Gross-Neveu model can be mapped to two decoupled ordinary sine-Gordon models, each at the $SU(2)$ -invariant coupling ($\beta_{SG}^2 \rightarrow 8\pi$ in the conventional normalization) [2]. This special decoupling happens because the algebra $SO(4)$ is equivalent to $SU(2) \times SU(2)$. At the $SU(2)$ invariant coupling of sine-Gordon, there are no bound states in the spectrum: the only particles are the kinks in doublets of each $SU(2)$. In the $SO(4)$ language, these are the spinor representations.

For $N > 4$, there are states other than the kinks. They have masses [18]

$$m_j = 2m \sin\left(\frac{\pi j}{N-2}\right) \quad (9)$$

where m is the mass of the kink and $j = 1 \dots \text{int}(N-3)/2$. The first of these states corresponds to the particle created by ψ^j ; it is in the N -dimensional vector representation of $SO(N)$. The other states are bound states of the fermions. One useful fact to note is that each type of particle corresponds to a node on the Dynkin diagram for $SO(N)$, as displayed in Fig. 1. For even N , the kinks correspond to the two nodes on the right, while for odd N , the kink is the node on the right. The vector representation is the node on the left.

The exact S matrix of the vector particles for any N was worked out in [18]. Using the bootstrap gives the S matrix for fermion bound states, but not the kinks. The explicit expression for the S matrix of particles with mass m_2 [which are in the antisymmetric and singlet representations of $SO(N)$] can be found in [19]. The S matrix for the kinks for even N is worked out in [20,21] (see also [3]). Thus to complete this picture we need to work out the scattering of the kinks for odd N , including the fact that they have the additional BPS structure discussed in the last section.

The BPS kinks in the Gross-Neveu model are in the 2^P -dimensional spinor representation of $SO(2P+1)$, and they also form a multiplet transforming under the generalized supersymmetry. Since the two symmetries commute with each other, the simplest two-particle S matrix invariant under these symmetries is of the tensor-product form

$$S(\theta) = S_{spinor}(\theta) \otimes S_{BPS}(\theta). \quad (10)$$

The matrices $S_{spinor}(\theta)$ and $S_{BPS}(\theta)$ are respectively the S matrices for particles in the spinor representation of $SO(2P+1)$, and for kinks in the triple-well potential.

The spinor part of the S matrix was found in [3]. The fact that it is invariant under $SO(2P+1)$ means that it can be

written in terms of projection operators. A projection operator \mathcal{P}_a maps the tensor product of two spinor representations onto an irreducible representation labeled by a . Here $a = 0 \dots P$, where $a=0$ labels the identity representation, $a=1$ the vector representation, $a=2$ the antisymmetric tensor, and so on up to $a=P-1$. The representation with $a=P$ is the representation with highest weight $2\mu_P$, where μ_P is the highest weight of the spinor representation. We do not give the explicit expressions for the projectors because we will not need them; they can be written explicitly in terms of the gamma matrices for the algebra $SO(2P+1)$, which in turn can be written as tensor products of Pauli matrices. The spinor S matrix is

$$S_{spinor}(\theta) = \sum_{a=0}^P f_a(\theta) \mathcal{P}_a(\theta). \quad (11)$$

The functions $f_a(\theta)$ are not constrained by the $SO(2P+1)$ symmetry, but all can be related to $f_P(\theta)$ by using the Yang-Baxter equation. The result is [3]

$$f_{P-1} = \frac{\theta + i\pi\Delta}{\theta - i\pi\Delta} f_P$$

$$f_{P-a-1} = \frac{\theta + i\pi\Delta(2a+1)}{\theta - i\pi\Delta(2a+1)} f_{P-a+1} \quad (12)$$

where the Yang-Baxter equation does not determine Δ .

The function $f_P(\theta)$ must satisfy the unitarity and crossing relations. These do not determine the function uniquely, because one can obtain a new solution of these equations by multiplying any given solution by a function $F(\theta)$ obeying $F(\theta) = 1$ and $F(\theta) = F(i\pi - \theta)$. This is known as the CDD ambiguity. The minimal solution of the unitarity and crossing relations is a solution without any poles in the physical strip. The minimal spinor S matrix of $SO(2P+1)$ has

$$\Delta = \frac{1}{N-2} = \frac{1}{2P-1}$$

and [3]

$$f_P(\theta) = \prod_{b=0}^{P-1} \frac{\Gamma\left(1 - b\Delta - \frac{\theta}{2\pi i}\right) \Gamma\left(\left(b + \frac{1}{2}\right)\Delta + \frac{\theta}{2\pi i}\right)}{\Gamma\left(1 - b\Delta + \frac{\theta}{2\pi i}\right) \Gamma\left(\left(b + \frac{1}{2}\right)\Delta - \frac{\theta}{2\pi i}\right)}. \quad (13)$$

Notice that this has zero at $\theta = i\pi\Delta(2b+1)$ for all $b = 0 \dots P-1$. These cancel poles arising from Eqs. (12), and ensure that there are no poles in the physical strip for any of the f_a . The following integral representation for f_P will be useful in the future:

$$f_P(\theta) = \exp\left[\int_{-\infty}^{\infty} \frac{d\omega}{\omega} e^{i(2P-1)\omega\theta/\pi}\right] \times \frac{e^{-|\omega|/2} \sinh(P\omega)}{2 \cosh[(2P-1)\omega/2] \sinh(\omega)}. \quad (14)$$

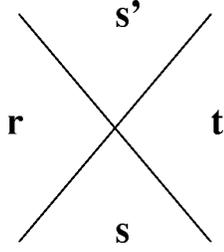


FIG. 2. Representing kink scattering by four vacua.

The minimal S matrix for BPS kinks in a triple-well potential can also be determined by imposing the same criteria. In addition, we can utilize the generalized supersymmetry. Because the generalized supersymmetry operators Q and \bar{Q} commute with the Hamiltonian, they must also commute with the S matrix. Since we know how they act on multiparticle states, this is simple to implement. In fact, some of the work has already been done for us. The S matrix for the tricritical Ising model describes the scattering of kinks in a triple well, and is invariant under ordinary supersymmetry [12]. The generalized supersymmetry algebra acting on the states (3) is related to the ordinary supersymmetry algebra by making the substitution $\theta \rightarrow \pm(N-2)\theta$. Thus in order to commute with the generalized supersymmetry,

$$S_{BPS}(\theta) \propto S_{TCI}((N-2)\theta)$$

$$\text{or } S_{BPS}(\theta) \propto S_{TCI}(-(N-2)\theta).$$

Such an S matrix will automatically satisfy the Yang-Baxter equation as well (any solution remains a solution under the scaling $\theta \rightarrow \lambda\theta$).

It is easiest to label the S matrix elements for kinks by their vacua. A two-kink configuration can be labeled by three vacua. As shown in Fig. 2, a two-particle S -matrix element can be labeled by four vacua because only the middle vacuum can change in a collision. Thus this S matrix element $S_{ss'}^{(rt)}$ describes scattering the initial state rst to the final state $rs't$. For the triple well, the labels r, s, s', t take the values $0, \pm 1$. The elements of S_{BPS} are then

$$\begin{aligned} S_{rr}^{(r-1, r+1)}(\theta) &= B(\theta) \left(\frac{\beta_r}{\beta_{r+1}^{1/2} \beta_{r-1}^{1/2}} \right)^{i(\theta/\pi)} i \sinh \left[\lambda \theta - \frac{i\pi}{4} \right] \\ S_{\pm 1, \mp 1}^{(rr)}(\theta) &= B(\theta) \left(\frac{\beta_{r+1}^{1/2} \beta_{r-1}^{1/2}}{\beta_r} \right)^{1+i(\theta/\pi)} (-1)^P i \sinh[\lambda \theta] \\ S_{r+1, r+1}^{(rr)}(\theta) &= B(\theta) \left(\frac{\beta_{r+1}}{\beta_r} \right)^{i(\theta/\pi)} \frac{\beta_1}{\beta_r} \cosh \left[\frac{i\pi}{4} r + \lambda \theta \right] \\ S_{r-1, r-1}^{(rr)}(\theta) &= B(\theta) \left(\frac{\beta_{r-1}}{\beta_r} \right)^{i(\theta/\pi)} \frac{\beta_1}{\beta_r} \cosh \left[\frac{i\pi}{4} r - \lambda \theta \right] \end{aligned} \quad (15)$$

where

$$\beta_r = \cos \left(\frac{\pi}{4} r \right).$$

The tricritical Ising model S matrix has $\lambda = 1/4$, so in general λ must be $\pm(N-2)/4$. Crossing symmetry fixes the sign, requiring that

$$\lambda = (-1)^{P+1} \frac{N-2}{4} = (-1)^{P+1} \frac{1}{4\Delta}.$$

We should note that some extra minus signs appear in the crossing relations because the kinks are neither bosons nor fermions [21]. This solution of the Yang-Baxter equation was originally found in an integrable lattice model, the hard-hexagon model, where the kink vacua correspond to the heights in the lattice model. This lattice model is part of the ‘‘restricted solid-on-solid’’ (RSOS) hierarchy [22]. More general RSOS models let the heights or vacua run over more values; these arise in the multi-flavor and symplectic Gross-Neveu models to be discussed in [35].

The last step is to find the function $B(\theta)$. Crossing symmetry requires that

$$B(\theta) = B(i\pi - \theta)$$

while unitarity requires that

$$B(\theta)B(-\theta) = \frac{1}{\sinh \left[\lambda \theta - \frac{i\pi}{4} \right] \sinh \left[\frac{i\pi}{4} + \lambda \theta \right]}.$$

One can easily find the minimal solution of these relations, where $B(\theta)$ has no poles in the physical strip $0 < \text{Im } \theta < \pi$. For the $N=3$ case, the $\mathcal{N}=1$ supersymmetric sine-Gordon model, there are no bound states: the BPS kinks make up the entire spectrum [2]. There are no poles in the physical strip for $N=3$. This result is confirmed by calculating the free energy [15], a computation we will repeat in the next section.

In general, we need a $B(\theta)$ with poles, because the Gross-Neveu model has bound states. For $P > 1$, there are particles other than the kink in the spectrum. These have masses given by Eq. (9), with $j = 1 \dots P-1$. The fermion ψ_j is the $j=1$ state, and higher j corresponds to bound states of the fermion [18]. For odd $N=2P+1$ all fundamental representations except for the spinor appear in the tensor product of two spinor representations. Moreover, the poles in Eqs. (12) correspond to bound-state particles in the correct representations of $SO(2P+1)$. We thus expect all states (other than the kink itself) to appear as bound states of two kinks. It follows from Eqs. (8) and (9) that the bound states appear as poles in $B(\theta)$ at $\theta_j = i\pi(1-2j\Delta)$ for $j = 1 \dots P-1$. Crossing symmetry means that there must be poles at $\theta_j = i\pi 2j\Delta$ as well.

We find that

$$B(\theta) = \frac{i}{\sinh\left[\lambda\theta + \frac{i\pi}{4}\right]} \prod_{b=1}^{\text{int}\{(P-1)/2\}} \frac{\sinh\left[\frac{1}{2}(\theta + i\pi - i4b\pi\Delta)\right] \sinh\left[\frac{1}{2}(\theta + i4b\pi\Delta)\right]}{\sinh\left[\frac{1}{2}(\theta - i\pi + i4b\pi\Delta)\right] \sinh\left[\frac{1}{2}(\theta - i4b\pi\Delta)\right]} \\ \times \prod_{l=1}^{\infty} \frac{\Gamma\left[\frac{1}{2} + (l-1)\frac{1}{2\Delta} - \frac{\theta}{4\pi i\Delta}\right] \Gamma^2\left[\frac{1}{2} + (2l-1)\frac{1}{4\Delta} + \frac{\theta}{4\pi i\Delta}\right] \Gamma\left[\frac{1}{2} + l\frac{1}{2\Delta} - \frac{\theta}{4\pi i\Delta}\right]}{\Gamma\left[\frac{1}{2} + (l-1)\frac{1}{2\Delta} + \frac{\theta}{4\pi i\Delta}\right] \Gamma^2\left[\frac{1}{2} + (2l-1)\frac{1}{4\Delta} - \frac{\theta}{4\pi i\Delta}\right] \Gamma\left[\frac{1}{2} + l\frac{1}{2\Delta} + \frac{\theta}{4\pi i\Delta}\right]}.$$

The $1/\sinh(\lambda\theta + i\pi/4)$ gives poles corresponding to bound states with odd j . The finite product of sinh functions gives the appropriate poles for even j . The infinite product of gamma functions assures that crossing symmetry is obeyed. Another way of writing $B(\theta)$ is as

$$B(\theta) = \frac{i}{\sinh\left[\frac{i\pi}{4} - \lambda\theta\right]} \prod_{b=1}^{\text{int}\{P/2\}} \frac{\sinh\left[\frac{1}{2}[\theta + i\pi - i(4b-2)\pi\Delta]\right] \sinh\left[\frac{1}{2}[\theta + i(4b-2)\pi\Delta]\right]}{\sinh\left[\frac{1}{2}[\theta - i\pi + i(4b-2)\pi\Delta]\right] \sinh\left[\frac{1}{2}[\theta - i(4b-2)\pi\Delta]\right]} \\ \times \prod_{l=1}^{\infty} \frac{\Gamma\left[1 + (l-1)\frac{1}{2\Delta} - \frac{\theta}{4\pi i\Delta}\right] \Gamma\left[(2l-1)\frac{1}{4\Delta} + \frac{\theta}{4\pi i\Delta}\right] \Gamma\left[1 + (2l-1)\frac{1}{4\Delta} + \frac{\theta}{4\pi i\Delta}\right] \Gamma\left[l\frac{1}{2\Delta} - \frac{\theta}{4\pi i\Delta}\right]}{\Gamma\left[1 + (l-1)\frac{1}{2\Delta} + \frac{\theta}{4\pi i\Delta}\right] \Gamma\left[(2l-1)\frac{1}{4\Delta} - \frac{\theta}{4\pi i\Delta}\right] \Gamma\left[1 + (2l-1)\frac{1}{4\Delta} - \frac{\theta}{4\pi i\Delta}\right] \Gamma\left[l\frac{1}{2\Delta} + \frac{\theta}{4\pi i\Delta}\right]}.$$

Here the $1/\sinh(\lambda\theta - i\pi/4)$ gives poles corresponding to bound states with even j , while the finite product of sinh functions gives the appropriate poles for odd j . With a little tedium one can show that the two expressions for $B(\theta)$ are in fact equal. The following integral representation will be useful:

$$B(\theta) = \frac{\sqrt{2}}{\sqrt{\cosh 2\lambda\theta}} \exp\left[-\int_{-\infty}^{\infty} \frac{d\omega}{\omega} e^{i(2P-1)\omega\theta/\pi}\right] \\ \times \frac{2 \sinh[(P-3/2)\omega] + \sinh[(P-1/2)\omega]}{4 \sinh[\omega] \cosh[(P-1/2)\omega]}.$$

(16)

It is most easily obtained by multiplying the two expressions for $B(\theta)$, a standard rewriting of the logarithm of the gamma functions in terms of an integral, and then taking the square root.

We have thus determined the S matrix (10) for the BPS kinks in the Gross-Neveu model.

By using the bootstrap, one obtains the S matrix for all the particles, for example reproducing the S matrix of [18] for the fermions ψ_i . To apply the bootstrap, we need to utilize a number of results from integrable lattice models. In this context, the process of obtaining a new solution of the Yang-Baxter equation from an existing one is called fusion [23,24]. For example, in a model with $SU(2)$ symmetry, one can fuse the S matrices for two spin-1/2 particles to get that for a spin-1 particle. In the simplest cases, the fusion procedure utilizes

the fact that at a pole $\theta = \theta_j$ of the S matrix, the residue $(\theta - \theta_j)S(\theta_j)$ becomes a projection operator. This happens only at imaginary values of the rapidity, so this does not violate unitarity. For example, from Eqs. (12), it follows that at $\theta = \theta_1 \equiv i\pi(1-2\Delta)$, all the $f_j(\theta_1) = 0$ except for $j=1$. Thus S_{spinor} becomes a projection operator \mathcal{P}_1 onto the $j=1$ bound states. This means that one can think of the fermion of rapidity θ as a bound state of two kinks, one with rapidity $\theta + \theta_1/2$ and the other with $\theta - \theta_1/2$. In general, consider a case where the S matrix of two particles α and β has a pole at $\theta = \theta_j$. The residue of the S matrix at the pole is the sum of projection operators

$$\lim_{\theta \rightarrow \theta_j} [(\theta - \theta_j)S^{(\alpha\beta)}(\theta)] = \sum_a R_a \mathcal{P}_a.$$

The S matrix of the bound state j from another particle γ related to the S matrices of the constituents by the formula [23,3]

$$S^{(j\gamma)} = \left(\sum_a \sqrt{|R_a|} \mathcal{P}_a \right) S^{(\alpha\gamma)}(\theta + \theta_j/2) S^{(\beta\gamma)}(\theta - \theta_j/2) \\ \times \left(\sum_a \frac{1}{\sqrt{|R_a|}} \mathcal{P}_a \right).$$

(17)

Note that the matrices in Eq. (17) are not all acting on the same spaces, so this relation is to be understood as multiplying the appropriate elements. The upshot is that one can

think of the particle with mass m_j and rapidity θ as the bound state of kinks with rapidities $\theta + \theta_j/2$ and $\theta - \theta_j/2$.

The f_a given by Eqs. (12) determine in which representation of $SO(2P+1)$ the particles for a given j transform. For example, the particles with mass m_1 transform in the vector representation, so there are $2P+1$ of them. These are the original fermions. The particles with mass m_2 transform in the antisymmetric representation and the singlet, so there are $2P^2+P+1$ of these. In general, the particles with mass m_j are in the a -index antisymmetric tensor representations with $a = j, j-2, \dots$. The reason for the multiple representations of particles at a given mass is likely that they form an irreducible representation of the Yangian algebra for $SO(2P+1)$ [19]. To compute the explicit S matrix for all the bound states is a formidable task; for some results, see [19]. However, by using fusion we can compute the ‘‘prefactor’’ of the S matrix, which is necessary for the computation of the free energy in the next section. The prefactor of the S matrix for scattering a particle in representation with highest weight μ_a from a particle in representation μ_b is defined as the S matrix element multiplying the projector on the representation with highest weight $\mu_a + \mu_b$. We denote the prefactor as $S_{ab}(\theta)$, and for example, for kink-kink scattering the prefactor is

$$S_{PP}(\theta) = B(\theta)f_P(\theta).$$

An important check on this S matrix is that the scattering closes. This means that all the poles in the physical strip correspond to particles in the spectrum. In fact, zeros coming from the BPS part of the S matrix cancel spurious poles coming from the spinor part: without the BPS part, the bootstrap does not close. This was noted in [3], where it was termed a ‘‘violation’’ of the bootstrap for spinor particles [3]. Solving this problem gives another way of seeing that the extra BPS structure must be present. To find the zeros coming from fusing S_{BPS} , note that θ_j for odd j obeys $\sinh(\lambda\theta_j + i\pi/4) = 0$, while for even j it obeys $\sinh(\lambda\theta_j - i\pi/4) = 0$. Thus at $\lambda\theta = -i\pi/4$, the S matrix in Eqs. (15) projects onto the states

$$|-11\rangle, |1-1\rangle, |00\rangle \quad j \text{ odd}$$

while at $\lambda\theta = i\pi/4$, it projects onto

$$|-1,-1\rangle, |00\rangle, |11\rangle \quad j \text{ even.}$$

Using the formula (17) yields the bound-state S matrix. One finds that fusing S_{BPS} gives an S matrix effectively diagonal in the kink labels, and moreover, it is the same for all labels. Thus it contributes only an overall factor to the S matrix of a bound state j from a kink γ , namely

$$S^{(j\gamma)} \propto \sinh\left[\lambda\theta + \frac{i\pi}{8}\right] \sinh\left[\lambda\theta - \frac{i3\pi}{8}\right] \quad j \text{ odd}$$

$$S^{(j\gamma)} \propto \sinh\left[\lambda\theta - \frac{i\pi}{8}\right] \sinh\left[\lambda\theta + \frac{i3\pi}{8}\right] \quad j \text{ even.}$$

This overall factor is the only effect of the restricted-kink structure on the bound states $j = 1 \dots P-1$. The non-integer

number of kink states and the funny rules do not affect the bound states. However, these extra factors do result in zeros which cancel spurious poles occurring in the fusion of S_{spinor} for odd N .

There is an algebraic explanation for the simplicity of the kink structure in the bound states. Conceptually, it is similar to the cases with a Lie-algebra symmetry, but here it is because S_{BPS} has a quantum-group symmetry. In our case of interest, the quantum-group algebra is $U_q(SU(2))$, with the deformation parameter $q = e^{2\pi i/3}$. This algebra is a deformation of $SU(2)$, and many of its representations have properties similar to ordinary $SU(2)$. In particular, the kinks interpolating between adjacent minima are the spin-1/2 representations of $U_q(SU(2))$. However, when q is a root of unity, various special things happen. In our case of a triple well, what happens is that the spin-1 representation is very simple. A spin-1 kink goes from either the vacuum 1 to -1 , -1 to 1, or from 0 to 0. Note that whichever vacua one is in, there is only one vacua to go to. Hence the number of spin-1 kinks is $K=1$. The triple-well structure can be ignored once one fuses the kinks: effectively the BPS structure goes away for the bound states.

To conclude this section, we note that kinks transforming under the generalized supersymmetry have appeared before, in the ordinary sine-Gordon model at coupling $\beta_{SG}^2 = 16\pi/(2P+1)$ [25]. In fact, the Gross-Neveu model and the sine-Gordon model at these couplings are closely related. The masses m_j (but not the degeneracies) in the spectrum are identical, with the soliton and antisoliton of the sine-Gordon model having the mass m like the kinks here [18]. Even more strikingly, the soliton and antisoliton are representations of the generalized supersymmetry (3), as shown in [25]. The reason for the similarity is that both are related to a certain integrable perturbation [by the (1,1;adjoint) operator] of the coset conformal field theories

$$\frac{SO(N)_k \times SO(N)_1}{SO(N)_{k+1}}.$$

The $k=1$ case is the sine-Gordon model at $\beta_{SG}^2 = 16\pi/N$, while the $k \rightarrow \infty$ case is the Gross-Neveu model [26]. The models for all k should be integrable, and for odd N the S matrix should have the form

$$S = S_q \otimes S_{BPS}$$

where q is related to k . The S matrix S_q is the S matrix associated with the spinor representation of the quantum group $U_q(SO(N))$, which as far as we know has never been worked out explicitly in general. In the $k=1$ sine-Gordon case, the generalized supersymmetry is extended to an $\mathcal{N}=2$ version, with generators Q_{\pm}, \bar{Q}_{\pm} . This means that for $k=1$, S_q is in fact $\propto S_{BPS}$. For $k \rightarrow \infty$, $q \rightarrow 1$ and the quantum-group algebra reduces to the usual Lie algebra $SO(N)$. Thus $\lim_{q \rightarrow 1} S_q = S_{spinor}$.

V. THE FREE ENERGY

In this section we compute the free energy of the Gross-Neveu model at finite temperature using a technique called

the thermodynamic Bethe ansatz (TBA). This gives a non-trivial check on the exact S matrix. In the limit of all masses going to zero, the theorem of [27] says that the free energy per unit length must behave as

$$\lim_{m \rightarrow 0} F = -\frac{\pi T^2}{6} c_{UV} \quad (18)$$

where c_{UV} is the central charge of the conformal field theory describing this ultraviolet limit. The number c_{UV} can usually be calculated analytically from the TBA, because in this limit the free energy can be expressed as a sum of dilogarithms. The c_{UV} computed from the TBA must of course match the c_{UV} from the field theory. This provides a very useful check, because the spectrum is essentially an infrared property. Finding the central charge exactly in the ultraviolet limit requires knowing not only the exact spectrum but the exact S matrix as well: the gas of particles at high temperature is strongly interacting. All particles contribute to the free energy, so if some piece of the spectrum is missing, or if an S matrix is wrong, the TBA will not give the correct c_{UV} .

The free energy is computed from the TBA by two steps. First one finds how the momenta are quantized when periodic boundary conditions are imposed. Precisely, one demands that the multi-particle wave function $\psi(x_1, x_2, \dots)$ remains the same when any coordinate x_i is shifted to $x_i + L$. In an interacting theory, the quantization involves the S matrix, because as one particle is brought around the periodic world, it scatters through the other particles. In the continuum, this leads to a constraining relation between the densities of states and the actual particle densities. The free energy at temperature T is found in the second step by minimizing it subject to the constraint. The detailed procedure for this computation has been discussed in many places, so we will not repeat these explanations. Several papers closely related to the current computation are [29,15,10]. The TBA computation here is technically complicated because the scattering is not diagonal. This means that, as a particle is going around the periodic interval of length L , it can change states as it scatters through the other particles. The way to proceed then is well known. First, one has to set up the system of auxiliary Bethe equations to diagonalize the transfer matrix, then use its results to determine the allowed rapidities. This computation amounts to a standard Bethe ansatz computation. The end result can be written conveniently by introducing extra zero-mass ‘‘pseudoparticles’’ or ‘‘magnons’’ to the constraining relations. Then one minimizes the free energy to find out the equilibrium distributions at temperature T , and thus the thermal properties of the (1+1)-dimensional quantum field theory.

Two kinds of additional complications occur here. The first difficulty is that, as a striking consequence of the bootstrap analysis, particles appear in all j -index antisymmetric tensor representations Π_j , but in general there is not a single mass associated with a given representation. Instead, particles with mass m_a appear in representations $j = a, a - 2, \dots$, a mixture we call ρ_a . On the other hand, little is known about the diagonalization of transfer matrices acting on products of $SO(2P + 1)$ representations. To proceed, we

have to assume that the usual Bethe equations based on the Dynkin diagram of the underlying Lie algebra [28] apply, in the $SO(2P + 1)$ case, not to the irreducible representations Π_j , but precisely to the mixtures ρ_a . This is a key technical assumption, which is checked *a posteriori*, because the number of particles of each mass can be read off from the TBA equations.

The second is that the kinks in the Gross Neveu model are not only in the 2^P -dimensional spinor representation, they also form a multiplet under the generalized supersymmetry transformations. This means that instead of one, one gets two auxiliary problems, one to diagonalize the transfer matrices acting on the degrees of freedom transforming in the spinor representation, the other to diagonalize the transfer matrices acting on the BPS degrees of freedom. A similar but simpler problem has been solved in the TBA calculation of the $\mathcal{N} = 1$ supersymmetric sine-Gordon model [15].

We first solve the auxiliary problem for the BPS S matrix. This was already done in [29], because S_{BPS} is the same as that in the tricritical Ising model, up to a rescaling of θ . The result is that the TBA equations in the tricritical Ising model require introducing one pseudoparticle with density $\tau(\theta)$, and hole density $\tilde{\tau}(\theta)$. Then the density of states for the particle in the tricritical Ising model is

$$2\pi P_{TCI}(\theta) = m \cosh \theta + \frac{1}{2} \frac{\xi \star \xi}{2\pi} \star \rho_{TCI} + \frac{1}{2} \xi \star (\tau - \tilde{\tau})$$

where we have defined convolution as

$$a \star b(\theta) \equiv \int_{-\infty}^{\infty} a(\theta - \theta') b(\theta') d\theta'.$$

The kernel $\xi(\theta)$ is defined as

$$\xi(\theta) = \frac{2P - 1}{\cosh[(2P - 1)\theta]} \quad (19)$$

in general, with $P = 1$ for the tricritical Ising model. The density of real particles is denoted by ρ_{TCI} . It is most convenient to give most kernels in terms of Fourier transforms, defined as

$$\hat{f}(\omega) \equiv \int \frac{d\theta}{2\pi} e^{i(2P-1)\omega\theta/\pi} f(\theta)$$

so that

$$\hat{\xi}(\omega) = \frac{1}{2 \cosh\left(\frac{\omega}{2}\right)}.$$

The density of states for the pseudoparticles is then related to the density of real particles ρ_{TCI} by

$$2\pi(\tau + \tilde{\tau}) = \xi \star \rho_{TCI}.$$

The pseudoparticles have zero energy, but they contribute entropy to the free energy. The equilibrium values ρ_{TCI} and τ

are found by minimizing the free energy subject to the above constraints. Notice that simple manipulations allow us to re-express the density of states for the particle as

$$2\pi P_{TCI}(\theta) = m \cosh \theta + \frac{\xi \star \xi}{2\pi} \star \rho_{TCI} - \xi \star \tilde{\tau} = m \cosh \theta + \xi \star \tau.$$

The second form will be the most convenient in what follows.

Before giving the full answer for the odd- N Gross-Neveu model, let us first also review the solution for the $P=1$ case, the $\mathcal{N}=1$ supersymmetric sine-Gordon model at its $SO(3)$ symmetric point. As discussed in the last section, the S matrix is

$$S_{SO(3)} = S_{TCI} \otimes S_{\beta_{SG}^2 \rightarrow 8\pi}.$$

The latter piece is the S matrix for the ordinary sine-Gordon model at an $SU(2)$ -invariant point ($\beta_{SG}^2 \rightarrow 8\pi$ in the usual normalization). The auxiliary problem for this piece is equivalent to diagonalizing the Heisenberg spin chain, a problem solved by Bethe 70 years ago. One must introduce an infinite number of pseudoparticles with densities $\rho_j(\theta)$, in addition to the pseudoparticle density $\tau(\theta)$ required for the BPS S matrix. The density of states P_0 for the kinks of mass m is then related to the particle density ρ_0 given by

$$2\pi P_0(\theta) = m \cosh \theta + \mathcal{Y} \star \rho_0(\theta) - \sum_{j=1}^{\infty} \sigma_j^{(\infty)} \star \tilde{\rho}_j(\theta) - \xi \star \tilde{\tau}(\theta). \quad (20)$$

The kernels $\sigma_j^{(\infty)}$ follow from the Bethe ansatz analysis, and are

$$\hat{\sigma}_j^{(\infty)} = e^{-j|\omega|}.$$

The kernel \mathcal{Y} comes from two places. There is a contribution from the prefactor of the S matrix, and for kink-kink scattering there is an extra piece arising from the Bethe ansatz analysis [29]. The formula for any P is

$$\hat{\mathcal{Y}}_{PP}^{(P+1/2)} = \frac{d}{d\theta} \text{Im} \ln B + \frac{d}{d\theta} \text{Im} \ln f_P + \frac{1}{2} (\hat{\xi})^2 \quad (21)$$

so that

$$\hat{\mathcal{Y}} = \hat{\mathcal{Y}}_{PP}^{(P+1/2)}|_{P=1} = 1 - \frac{e^{|\omega|}}{4 \cosh^2(\omega/2)}.$$

The other Bethe equations relate the densities of states for the pseudoparticles to particle and pseudoparticle densities. They are

$$2\pi \rho_j(\theta) = \sigma_j^{(\infty)} \star \rho_0(\theta) - \sum_{l=1}^{\infty} A_{jl}^{(\infty)} \star \tilde{\rho}_l(\theta) \quad (22)$$

where the density of string states P_j is

$$P_j = \tilde{\rho}_j + \rho_j.$$

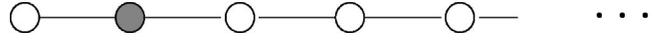


FIG. 3. The TBA for the $O(3)$ Gross-Neveu model (the supersymmetric sine-Gordon model).

The new kernel is given by

$$\hat{A}_{jl}^{(\infty)} = 2 \coth(\omega) e^{-\max(j,l)|\omega|} \sinh(\min(j,l)\omega).$$

These equations can be simplified greatly by inverting the matrix A_{jl} (for details see for example [10]), giving

$$2\pi P_j(\theta) = \delta_{j0} m \cosh \theta + \xi \star (\rho_{j-1} + \rho_{j-1}). \quad (23)$$

Even the pseudoparticle τ coming from the BPS piece is included in this equation, by defining $\rho_{-1}(\theta) \equiv \tau(\theta)$ and $P_{-1} = \tau + \tilde{\tau}$. Minimizing the free energy then yields the TBA equations

$$\epsilon_j(\theta) = \delta_{j0} m \cosh \theta - \xi \star [\ln(1 + e^{-\epsilon_{j-1}/T}) + \ln(1 + e^{-\epsilon_{j+1}/T})]$$

with $j = -1, 0, 1, 2, \dots$, and $\epsilon_{-2} \equiv \infty$. These TBA equations are conveniently encoded in the diagram in Fig. 3. The circles represent the functions ϵ_a ; the filled node represents the fact that the equation for ϵ_0 has a mass term.

The free energy per unit length F is

$$F(m, T) = -Tm \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh \theta \ln(1 + e^{-\epsilon_0(\theta)/T}). \quad (24)$$

One can check that in the limit $m \rightarrow 0$, this free energy yields the correct central charge $3/2$ [15]. This confirms the presence of the BPS kinks in the spectrum.

The TBA for the full $O(2P+1)$ Gross-Neveu model is conceptually similar to the $O(3)$ case, but is much more involved technically. To complicate matters further, there are particles with masses m_a , $a=1 \dots P$. Luckily, many of the technical complications have already been solved. The diagonalization of the BPS part is the same as in the tricritical Ising model [29]. This requires introducing the pseudoparticle density $\tau(\theta)$, as described above. The diagonalization of the auxiliary problem for the $SO(2P+1)$ spinor part is done using the standard string hypothesis based on the Dynkin diagram [30]. For the TBA for the $SU(N)$ and $O(2P)$ Gross-Neveu models discussed in [10], the TBA equations are related to the $SU(N)$ and $SO(2P)$ Dynkin diagrams. However, $SO(2P+1)$ is not simply laced, so this case is somewhat more complicated. Nevertheless, the appropriate computation has been done already in [30], utilizing the Yangian structure of the S matrix. The densities of particles with mass m_a are defined as $\rho_{a,0}$, while the auxiliary problem requires introducing pseudoparticle densities $\rho_{a,j}$ with $j=1 \dots \infty$, as well as τ . In our notation, we try to remain consistent with [10], where the TBA for the $O(2P)$ Gross-Neveu model is discussed.

The quantization conditions for the densities of states $P_{a,j} = \rho_{a,j} + \tilde{\rho}_{a,j}$ are then

$$\begin{aligned}
2\pi P_{a,0}(\theta) &= m_a \cosh \theta + \sum_{b=1}^P \mathcal{Y}_{ab}^{(P+1/2)} \star \rho_{b,0}(\theta) \\
&\quad - \sum_{j=1}^{\infty} \sigma_j^{(\infty)} \star \tilde{\rho}_{a,j}(\theta), \\
2\pi P_{P,0} &= m \cosh \theta + \sum_{b=1}^P \mathcal{Y}_{Pb}^{(P+1/2)} \star \rho_{b,0}(\theta) \\
&\quad - \sum_{j=1}^{\infty} \sigma_{j/2}^{(\infty)} \star \tilde{\rho}_{P,j}(\theta) - \xi \star \tilde{\tau}(\theta). \tag{25}
\end{aligned}$$

The last terms in Eqs. (25) are expressions of the logarithmic derivatives of the transfer matrix \mathcal{T} eigenvalues. The kernels \mathcal{Y}_{ab} are simply related with the S matrices described in the previous sections. Except for the case $a=b=P$ given above in Eq. (21), one has $\mathcal{Y}_{ab}^{(P+1/2)} = (1/i)(d/d\theta) \ln S_{ab}$, where S_{ab} is the prefactor for scattering particles belonging to the representations with highest weights μ_a and μ_b . As explained above, these S matrix elements do not probe the kink structure, so the same computation will give these kernels for $O(N)$ for even and odd N . Thus these kernels for odd N can be read off from the N even results in [10], giving

$$\begin{aligned}
\hat{\mathcal{Y}}_{ab}^{(P+1/2)}(\omega) &= \delta_{ab} - e^{|\omega|} \frac{\cosh((P-1/2-\max(a,b))\omega) \sinh(\min(a,b)\omega)}{\cosh((P-1/2)\omega) \sinh(\omega)} \\
\hat{\mathcal{Y}}_{a,P}^{(P+1/2)} &= -e^{|\omega|} \frac{\sinh(a\omega)}{2 \cosh((P-1/2)\omega) \sinh(\omega)} \tag{26} \\
\hat{\mathcal{Y}}_{P,P}^{(P+1/2)} &= 1 - e^{|\omega|} \frac{\sinh(P\omega)}{4 \cosh((P-1/2)\omega) \sinh(\omega) \cosh(\omega/2)}
\end{aligned}$$

for $a, b = 1 \dots P-1$. The latter kernel came from Eq. (21). The pseudoparticle densities are given by the auxiliary Bethe system

$$\begin{aligned}
2\pi \rho_{a,j}(\theta) &= \sigma_j^{(\infty)} \star \rho_{a,0} - \sum_{l=1}^{\infty} \sum_{b=1}^{P-1} A_{jl}^{(\infty)} \star K_{ab} \star \tilde{\rho}_{b,l}(\theta) \\
&\quad - \sum_{l=1}^{\infty} A_{j,l/2}^{(\infty)} \star K_{aP} \star \tilde{\rho}_{P,l}, \\
2\pi \rho_{P,j}(\theta) &= \sigma_{j/2}^{(\infty)} \star \rho_{P,0} - \sum_{l=1}^{\infty} \sum_{b=1}^{P-1} A_{j/2,l}^{(\infty)} \star K_{Pb} \star \tilde{\rho}_{b,l}(\theta) \\
&\quad - \sum_{l=1}^{\infty} A_{j/2,l/2}^{(\infty)} \star K_{PP} \star \tilde{\rho}_{P,l} \\
2\pi(\tau + \tilde{\tau}) &= \xi \star \rho_{P,0} \tag{27}
\end{aligned}$$

where the kernel K in Fourier space is

$$\begin{aligned}
\hat{K}_{a,a\pm 1} &= -\frac{1}{2 \cosh \omega} \quad \hat{K}_{aa} = 1 \\
\hat{K}_{P,P-1} &= \frac{\cosh \omega/2}{\cosh \omega} \quad \hat{K}_{PP} = \frac{\coth \omega/2}{\coth \omega}. \tag{28}
\end{aligned}$$

Equations (27) are the continuum limit of the $SO(2P+1)$ Bethe equations, with source terms associated to all the ρ_a representations. In the preceding equations (25)–(28), the index a takes values $1 \dots P-1$; the equations involving the index P are given explicitly. The key feature of this system is

the appearance of factors of 2 in the terms involving the latter, which in the Dynkin diagram corresponds to the spinor representation. It is directly related to the fact that the P th root is the shortest; it has length 1 while all the others have length 2.

The TBA equations are written in terms of functions $\epsilon_{a,j}$, defined as

$$\frac{\rho_{a,j}}{P_{a,j}} = \frac{1}{1 + e^{\epsilon_{a,j}/T}}, \quad \frac{\tau}{\tau + \tilde{\tau}} = \frac{1}{1 + e^{\epsilon_{P,-1}/T}}$$

for $a = 1 \dots P$. The values that j runs over depend on the value of a : for $a = 1 \dots P-1$, j takes values $0, 1, \dots, \infty$, while for $a = P$, $j = -1, 0, 1 \dots \infty$. The extra function $\epsilon_{P,-1}$ arises from the diagonalization of the BPS S matrix. This problem can now be put in a considerably simpler and rather universal form by inverting the kernels. The resulting TBA equations are

$$\begin{aligned}
\epsilon_{a,j}(\theta) &= -T\phi \star [\ln(1 + e^{-\epsilon_{a,j+1}/T}) + \ln(1 + e^{-\epsilon_{a,j-1}/T})] \\
&\quad + T\phi \star [\ln(1 + e^{\epsilon_{a+1,j}/T}) + \ln(1 + e^{\epsilon_{a-1,j}/T})], \\
a &= 1, \dots, P-2 \tag{29}
\end{aligned}$$

together with

$$\begin{aligned}
\epsilon_{P-1,j}(\theta) &= -T\phi \star [\ln(1 + e^{-\epsilon_{P-1,j+1}/T}) \\
&\quad + \ln(1 + e^{-\epsilon_{P-1,j-1}/T})] + T\phi \star [\ln(1 + e^{\epsilon_{P-2,j}/T}) \\
&\quad + \ln(1 + e^{\epsilon_{P,2j+1}/T}) + \ln(1 + e^{\epsilon_{P,2j-1}/T})] \\
&\quad + T\psi \star \ln(1 + e^{\epsilon_{P,2j}/T}) \tag{30}
\end{aligned}$$

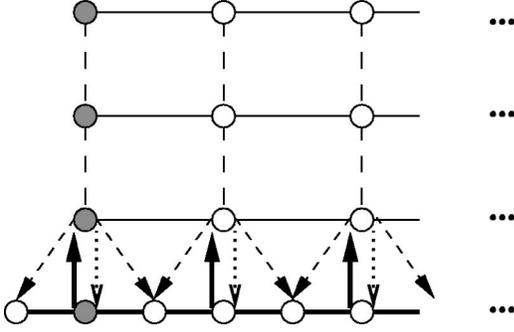


FIG. 4. The TBA for the $O(2P+1)$ Gross-Neveu model (here $P=4$).

and

$$\begin{aligned} \epsilon_{P,j}(\theta) = & -T\xi\star[\ln(1+e^{-\epsilon_{P,j+1}/T})+\ln(1+e^{-\epsilon_{P,j-1}/T})] \\ & +T\xi\star\ln(1+e^{\epsilon_{P-1,j/2}/T}). \end{aligned} \quad (31)$$

In the last equation, the coupling to $\epsilon_{P-1,j/2}$ occurs only when j is even. The kernels are defined by their Fourier transform,

$$\hat{\phi} = \frac{1}{2 \cosh(\omega)}, \quad \hat{\psi} = \frac{\cosh\left(\frac{\omega}{2}\right)}{\cosh(\omega)}, \quad \hat{\xi} = \frac{1}{2 \cosh\left(\frac{\omega}{2}\right)}.$$

As usual, the mass terms disappear from the equations, but are encoded in the asymptotic boundary conditions

$$\epsilon_{a,0} \rightarrow m_a \cosh \theta \quad \epsilon_{P,0} \rightarrow m \cosh \theta, \quad \theta \rightarrow \infty.$$

The TBA equations can be conveniently encoded in the diagram of Fig. 4. This is the non-simply-laced generalization of the diagrams of [10]. One remarkable feature is how the extra pseudoparticle coming from the BPS structure (the node on the bottom left) fits in perfectly with the pseudoparticles coming from the diagonalization of the spinor part of the S matrix. In this sense one could infer the existence of the BPS structure from a careful examination of the structure of the TBA equations.

The entire purpose of this section is to verify that this S matrix does give the correct free energy (18) in the ultraviolet $m \rightarrow 0$. This is of course a major check: it shows that this complicated spectrum and S matrix conspire to give the correct central charge $c = P + 1/2$ in the UV, corresponding to $2P + 1$ Majorana fermions. The free energy per unit length F is given in terms of these dressed energies ϵ_a as

$$F(m, T) = -T \sum_a m_a \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh \theta \ln(1 + e^{-\epsilon_{a,0}(\theta)/T}). \quad (32)$$

By rewriting F as $m \rightarrow 0$ in terms of dilogarithms, we have verified that one does indeed obtain $c = P + 1/2$ as required. Thus we take it as proven that this is the correct S matrix.

Another check is that in the IR limit $m_a \rightarrow \infty$, the correct particle multiplicities are obtained. Namely, one should have

$$\lim_{m_a \rightarrow \infty} F(m, T) = -T \sum_a n_a m_a \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh \theta e^{-m_a \cosh(\theta)/T} \quad (33)$$

where n_a is the number of particles with mass m_a . To obtain this by taking the limit of Eq. (32) takes a little bit of work. Define

$$Y_{a,j} \equiv \lim_{\theta \rightarrow \infty} e^{-\epsilon_{a,j}(\theta)}, \quad j \geq 1$$

$$Y_{a,0} \equiv \lim_{\theta \rightarrow \infty} e^{m_a \cosh \theta - \epsilon_{a,0}(\theta)}.$$

It then follows from Eqs. (32),(33) that $n_a = Y_{a,0}$. To find the $Y_{a,j}$ requires taking the $m \rightarrow \infty$ limit of the TBA equations (29),(30),(31). In this limit, one can replace the $\epsilon_{a,j}(\theta)$ by $\ln Y_{a,j}$. Thus the integrals can be done explicitly, giving a set of polynomial equations for the $Y_{a,j}$. For example, for $j = 0$ they are

$$(Y_{a,0})^2 = Y_{a-1,0} Y_{a+1,0} (1 + Y_{a,1}) \quad a = 1, \dots, P-2$$

$$\begin{aligned} (Y_{P-1,0})^2 = & Y_{P-2,0} (Y_{P,0})^2 (1 + Y_{P-1,1}) Y_{P,1} (1 + Y_{P,1})^{-1} \\ & \times Y_{P,-1} (1 + Y_{P,-1})^{-1} \end{aligned}$$

$$(Y_{P,0})^2 = Y_{P-1,0} (1 + Y_{P,-1}) (1 + Y_{P,1})$$

where $Y_{0,0} \equiv 1$. The (not written) equations for $Y_{a,j}$ for $j \geq 1$ do not depend on the $Y_{a,0}$, so these can be solved separately. One finds immediately that $Y_{P,-1} = 1$, but unfortunately, we were not able to derive an explicit closed-form solution for the other $Y_{a,j}$. However, it is easy to find them by solving the polynomial equations numerically. For example, for $P=2$, one finds that $Y_{1,1} = 14/11$, and $Y_{2,1} = 11/5$. Plugging this into the equation for $Y_{a,0}$ gives $Y_{1,0} = n_1 = 5$ and $Y_{2,0} = n_2 = 4\sqrt{2}$. These indeed are the correct multiplicities for $P=2$. For $P=3$, we find $Y_{1,1} = 27/22$, $Y_{2,1} = 95/147$, and $Y_{3,1} = 21/11$. This yields $n_1 = 7$, $n_2 = 22$, and $n_3 = 8\sqrt{2}$. In particular, note that $n_2 = 21 + 1$, the dimensions of the antisymmetric and singlet representations of $O(7)$. This checks that the transfer matrix diagonalization in [30] indeed includes all the particles at each mass, even though they come from more than one representation for $a = 2 \dots P-1$.

As another check, we can easily generalize this TBA calculation to the $SO(2P+1)_k \times SO(2P+1)_1 / SO(2P+1)_{k+1}$ coset models mentioned at the end of the last section. The TBA is almost the same as that above, except that the right hand side is truncated, so the j in $\epsilon_{a,j}$ runs only from $0 \dots k-1$ for $a = 1, \dots, P-1$, while $j = -1, 0, 1 \dots 2k-1$ for the $a = P$ nodes [30]. One can check that for this truncated TBA system

$$c_{UV} = \frac{k(2P+1)(4P-1+k)}{2(k+2P-1)(k+2P)}$$

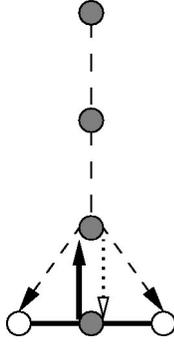


FIG. 5. The TBA for the sine-Gordon model at $\beta_{SG}^2/8\pi = 2/(2P+1)$ (here $P=4$).

as required. For $k=1$, the conformal theory has central charge $c=1$, and the coset perturbation coincides with the sine-Gordon model for $\beta_{SG}^2/8\pi=2/(2P+1)$. This TBA is represented by the diagram in Fig. 5, and was first studied in [25,31].

VI. CONCLUSION

We have completed the solution of the Gross-Neveu model for any number N of fermions. For odd N , generalized

supersymmetry results in the existence of BPS kinks. We found the exact S matrix for these kinks, and used this to compute the exact free energy.

One striking feature is that there is a non-integer number $K=2^{N/2}$ of these kinks, in the sense that the number of n -kink states goes as K^n . In the simplest case, the generalized supersymmetry reduces to $\mathcal{N}=1$ supersymmetry in two dimensions. In fact, the “single-state supermultiplet” of [32] is closely related to the multiplet with the $\sqrt{2}$ particles. Moreover, a $(2+1)$ -dimensional analog is discussed in [33], and unusual particle statistics related to Clifford algebras have also been discussed in [34].

We have generalized these results to a large number of models with four-fermion interactions [35]. These models include Gross-Neveu-like models with $Sp(2N)$ symmetry, and multi-flavor generalizations of the Gross-Neveu model. As in the $O(N)$ case, kinks with non-integer K appear whenever the symmetry algebra is not simply laced.

ACKNOWLEDGMENTS

The work of P.F. is supported by the DOE OJI program, the Alfred Sloan Foundation Fellowship, and by NSF grant DMR-9802813. The work of H.S. is supported by the DOE and the NSF through the NYI program.

-
- [1] D. Gross and A. Neveu, Phys. Rev. D **10**, 3235 (1974).
 - [2] E. Witten, Nucl. Phys. **B142**, 285 (1978).
 - [3] E. Ogievetsky, N. Reshetikhin, and P. Wiegmann, Nucl. Phys. **B280**, 45 (1987).
 - [4] V. A. Fateev and S. L. Lukyanov, Int. J. Mod. Phys. A **7**, 853 (1992); **7**, 1325 (1992).
 - [5] C. Ahn, Int. J. Mod. Phys. A **7**, 6799 (1992).
 - [6] H. Aratyn and P. H. Damgaard, Phys. Lett. **B241**, 253 (1984).
 - [7] E. Witten and D. Olive, Phys. Lett. **78B**, 97 (1978).
 - [8] A. B. Zamolodchikov (unpublished).
 - [9] R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976).
 - [10] P. Fendley, J. High Energy Phys. **05**, 050 (2001).
 - [11] A general discussion related to this issue is in Sec. 12.3 of E. Witten, in *Quantum Fields and Strings: A Course for Mathematicians*, edited by P. Deligne *et al.* (AMS/IAS, Princeton, NJ, 1999), Vol. 2.
 - [12] A. B. Zamolodchikov (unpublished).
 - [13] A. M. Tselik, Sov. J. Nucl. Phys. **47**, 172 (1988).
 - [14] K. Schoutens, Nucl. Phys. **B344**, 665 (1990).
 - [15] P. Fendley and K. Intriligator, Nucl. Phys. **B380**, 265 (1992); **B372**, 533 (1992).
 - [16] R. Shankar and E. Witten, Phys. Rev. D **17**, 2134 (1978).
 - [17] E. B. Bogomolny, Sov. J. Nucl. Phys. **24**, 449 (1976); M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. **35**, 760 (1975).
 - [18] A. B. Zamolodchikov and Al. B. Zamolodchikov, Ann. Phys. (N.Y.) **120**, 253 (1979).
 - [19] N. MacKay, Nucl. Phys. **B356**, 729 (1991).
 - [20] R. Shankar and E. Witten, Nucl. Phys. **B141**, 349 (1978).
 - [21] M. Karowski and H. Thun, Nucl. Phys. **B190**, 61 (1981).
 - [22] G. E. Andrews, R. J. Baxter, and P. J. Forrester, J. Stat. Phys. **35**, 193 (1984).
 - [23] P. Kulish, N. Reshetikhin, and E. Sklyanin, Lett. Math. Phys. **5**, 393 (1981).
 - [24] E. Date, M. Jimbo, T. Miwa, and M. Okado, Lett. Math. Phys. **12**, 209 (1986).
 - [25] H. Itoyama and T. Oota, Nucl. Phys. **B419**, 632 (1994).
 - [26] P. Fendley, Phys. Rev. Lett. **83**, 4468 (1999).
 - [27] H. Blöte, J. Cardy, and M. Nightingale, Phys. Rev. Lett. **56**, 742 (1986); I. Affleck, *ibid.* **56**, 746 (1986).
 - [28] N. Yu. Reshithikin, Lett. Math. Phys. **14**, 235 (1987).
 - [29] Al. B. Zamolodchikov, Nucl. Phys. **B358**, 497 (1991).
 - [30] A. Kuniba, T. Nakanishi, and J. Suzuki, Int. J. Mod. Phys. A **9**, 5215 (1994); **9**, 5267 (1999).
 - [31] R. Tateo, Int. J. Mod. Phys. A **10**, 1357 (1995).
 - [32] A. Losev, M. Shifman, and A. Vainshtein, hep-th/0011027.
 - [33] C. Nayak and F. Wilczek, Nucl. Phys. **B479**, 529 (1996).
 - [34] J. Baugh, D. R. Finkelstein, A. Galiatdinov, and H. Saller, J. Math. Phys. **42**, 1489 (2001); D. R. Finkelstein and A. A. Galiatdinov, hep-th/0005039.
 - [35] P. Fendley and H. Saleur, “Integrable models with four-fermion interactions.”