Gravitation with superposed Gauss-Bonnet terms in higher dimensions: Black hole metrics and maximal extensions

A. Chakrabarti*
Centre de Physique Théorique, Ecole Polytechnique, F-91128 Palaiseau, France

D. H. Tchrakian†
Department of Mathematical Physics, National University of Ireland Maynooth, Maynooth, Ireland and School of Theoretical Physics–DIAS, 10 Burlington Road, Dublin 4, Ireland

(Received 25 April 2001; published 26 December 2001)

Our starting point is an iterative construction suited to combinatorics in arbitarary dimensions d, of totally anisymmetrized p-Riemann $2p$ forms $(2p \leqslant d)$ generalizing the $(1-)$Riemann curvature 2-forms. The superposition of p-Ricci scalars obtained from the p-Riemann forms defines the maximally Gauss-Bonnet extended gravitational Lagrangian. Metrics, spherically symmetric in $(d-1)$ space dimensions, are constructed for the general case. The problem is directly reduced to solving polynomial equations. For some black-hole type metrics the horizons are obtained by solving polynomial equations. Corresponding Kruskal-type maximal extensions are obtained explicitly in complete generality, as is also the periodicity of time for the Euclidean signature. We show how to include a cosmological constant and a point charge. Possible further developments and applications are indicated.

DOI: 10.1103/PhysRevD.65.024029 PACS number(s): 04.50.+h

I. INTRODUCTION

Higher-order terms in the Riemann tensor appear in the gravitational sector of string theory [1]. Here, we choose to consider only the Gauss-Bonnet (GB) terms [2] of all orders which assure that the Lagrangian contains only quadratic powers of the velocity fields (i.e., the derivatives of the metric or the vielbein). For the second-order (in the Riemann curvature) terms, which will be labeled by $p=2$ in the following, it was indeed shown [3] that the corresponding terms arising in the string theoretic context can be reduced to the GB form by suitably redefining the fields. However, already at the level of cubic curvatures, in addition to the $p=3$ GB term there occurs [4,5] the additional term

$$R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\tau\lambda}R_{\tau\lambda}{}^{\mu\nu}$$

in the case of the bosonic string. The inclusion of this term causes the appearance of the cubic (i.e., higher than quadratic) power of the velocity fields. In this paper we have excluded all such terms and restrict our analysis to GB terms only.

GB extended Einstein equations in higher dimensions have been studied by various authors for a long time in various contexts [6–9], namely, that of cosmological solutions [8], gravitational instantons [6,7] and black holes [8,9]. Of these, the work of [9] is the closest to our present work in that all possible GB terms are taken into account.

Recently we have studied black-hole solutions of generalized gravitational systems consisting of single Gauss-Bonnet terms, considered as members of a hierarchy of generalized gravitational systems, each labeled with an integer p corresponding to the $2p$-form Riemann curvature defining it. Each member is the p-Ricci scalar $R_{(p)}$ formed by the contraction of the indices on the $2p$-form Riemann tensor. These described black-hole vacuum metrics [10] and metrics with point charge [11] generalizing the Reissner-Nordström solutions. In the present work we present a particularly convenient and systematic formalism for a Lagrangian consisting of the superposition of these individual p-Ricci scalars, $R_{(p)}$, with constant dimensional coefficients $\kappa_{(p)}$,

$$\mathcal{L}=\sum_{p=1}^{P}\frac{1}{2p}\kappa_{(p)}R_{(p)}. \tag{1.1}$$

The systems considered must be in dimensions $d \geqslant 5$, and due to the antisymmetry of the $2p$-forms consist of P terms such that $2P \leqslant d$.

For spherical symmetry in $d-1$ *space* dimensions with metric

$$ds^2=\mp N(r)dt^2+N(r)^{-1}dr^2+r^2d\Omega_{(d-2)}^2, \tag{1.2}$$

it is shown in Secs. III and IV that the metric pertaining to the system (1.1), generalizing the standard Schwarzschild, Reissner-Nordström, and de Sitter soltions, is obtained by solving the polynomial equation for $[1-N(r)]$

$$\sum_{p=1}^{P}\frac{\kappa_{(p)}}{2^p p!}(d-2)(d-3)\cdots(d-2p)\left(\frac{1-N}{r^2}\right)^p$$
$$=\frac{c}{r^{d-1}}-\frac{b}{r^{2(d-2)}}+\lambda. \tag{1.3}$$

This is our crucial result. For $P>2$, elliptic and hyperelliptic functions (theta functions of suitably higher genus) are needed to construct the solutions explicitly. Relating the pa-

*Email address: chakra@cpht.polytechnique.fr
†Email address: tigran@thphys.may.ie

rameters characterizing these functions to those appearing in Eq. (1.3) is in general a difficult task. However, in principle a complete set of solutions can be obtained in each case, the required prescriptions [12] for which are available. Hence for spherical symmetry, the problem of the construction of the metric pertaining to the system (1.1) can be considered solved.

We will study the properties of some relatively simple cases in the following sections. We will assume that in general, in Eq. (1.3),

$$\kappa_{(1)} \geqslant \kappa_{(p)}, \quad p>1.$$

Hence, having obtained the solution of Eq. (1.3) for the usual Einstein-Hilbert case, with $\kappa_{(p)}=0$ for $p>1$, one can consider systems consisting of a series of terms with coefficients $\kappa_{(2)}, \kappa_{(3)}, \ldots$, with $\kappa_{(2)} \geqslant \kappa_{(3)}, \ldots$. In this sense, such systems could be considered as perturbative series, and successive terms are expected to become appreciable with increasing energies. For a fixed number of $\kappa_{(p)}$ our results per se are, in principle, exact. Let us consider an example of interest, namely the horizons, $r=r_H$. These are by definition obtained setting $N(r_H)=0$ in Eq. (1.3), and then solving for r_H. In constructing exact solutions of polynomial equations for horizons in Sec. V, out of the full set of solutions one can select the real positive one by comparing with perturbative (in $\kappa_{(p)}$, $p>1$) solutions. Consider for simplicity the Schwarzschild-like case, with $b=0=\lambda$, with moreover $\kappa_{(p)}=0$ for $p>2$. One can first, setting $\kappa_{(2)}=0$, obtain the real positive horizon $r_H^{(0)}$, and then for small nonzero $\kappa_{(2)}$ obtain

$$r_H = r_H^{(0)} + o(\kappa_{(2)}) + o(\kappa_{(2)}^2).$$

Consistency with this perturabtive series solution will select out the real positive horizon r_H from the exact solutions of the relevant polynomials (1.3). Examples will be given in Sec. V.

In contrast to our considerations in Sec. V, where we truncated the values of p to lower than the maximum possible value P (consistent with $2P<d$), we emphasize that our formalism yields some exact results (for systems featuring all $\kappa_{(p)}$ up to $\kappa_{(P)}$). One such case concerns maximal extensions in Sec. VI. We assume $r=r_H$ to be an exact solution of Eq. (1.3) with $N(r_H)=0$ and then for

$$r=r_H+\rho, \quad \rho \ll 1$$

we set

$$N(\rho)=2\delta\rho+o(\rho^2).$$

We obtain an exact solution for δ, viz. Eq. (6.3), as a function of r_H, the *space-time* dimension d, and the $\kappa_{(p)}$. The parameter δ plays a crucial role concerning the near-horizon geometry, the periodicity of the Euclidean time, and the Hawking temperature. The exact general solution (6.3) is hence of considerable interest.

In Sec. VII, we have indicated how our results of Sec. V can be extended to include metrics pertaining to systems with Maxwell and cosmological terms added to Eq. (1.1), i.e., the construction of Reissner-Nordström and de Sitter metrics.

Possibilities of generalizations and applications of our present study are discussed in our conclusions in Sec. VIII.

II. GENERALIZED LAGRANGIAN AND EINSTEIN TENSOR

Let e^a be the tangent frame vector 1-forms

$$e^a = e^a_\mu dx^\mu, \tag{2.1}$$

where as usual $a,b,\ldots$ denote frame indices and $\mu,\nu,\ldots$ space-time ones, and ω^{ab} denotes the antisymmetric Levi-Civita spin-connection 1-forms

$$\omega^{ab} = \omega^{ab}_\mu dx^\mu = -\omega^{ba}, \tag{2.2}$$

satisfying

$$de^a + \omega^{ab} \wedge e_b = 0. \tag{2.3}$$

The curvature 2-forms are then

$$\begin{aligned} R^{ab} &= d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} = -R^{ba}, \\ &= R^{ab}_{\mu\nu} dx^\mu \wedge dx^\nu = R^{ab}_{a'b'} e^{a'} \wedge e^{b'}. \end{aligned} \tag{2.4}$$

We will often use the last form involving only tangent plane indices.

Starting with R^{ab}, higher-order p-form terms, totally antisymmetrized in the indices $a,b,c,\ldots$, are defined iteratively as follows:

$$R^{abcd} = R^{ab} \wedge R^{cd} + R^{ad} \wedge R^{bc} + R^{ac} \wedge R^{db}, \tag{2.5}$$

$$R^{a_1 a_2 \cdots a_{2p}} = R^{a_1 a_2} \wedge R^{a_3 a_4 \cdots a_{2p}} + \text{cyclic permutations of } (a_2, a_3, \ldots, a_{2p}), \tag{2.6}$$

for the $p=2$ and the generic p cases, respectively. The antisymmetric $2p$-form curvature (2.6) consists of $3.5\ldots(2p-3)(2p-1)$ terms of the type

$$R^{a_1 a_2} \wedge R^{a_3 a_4} \wedge \cdots \wedge R^{a_{2p-1} a_{2p}}.$$

For $p=1$, Eq. (2.6) coincides with Eq. (2.4), the usual curvature, in any dimension d. For $d<2p$, Eq. (2.6) vanishes due to the antisymmetry. For $d=2p$, Eq. (2.6) becomes the topological Euler density in those dimensions. For odd dimensions, $p=\frac{1}{2}(d-1)$ in Eq. (2.6) leads to features analogous to that in $d=3$ for Eq. (2.4).

One can express Eq. (2.6) in a *vielbein* basis, generalizing Eq. (2.4), as

$$R^{a_1 a_2 \cdots a_{2p}} = R^{a_1 a_2 \cdots a_{2p}}_{b_1 b_2 \cdots b_{2p}} e^{b_1} \wedge e^{b_2} \cdots \wedge e^{b_{2p}}. \tag{2.7}$$

The p-Ricci tensor is then defined as

$$R_{(p)}{}^{a_1}_{b_1} = \sum_{(a_2,\ldots,a_{2p})} R^{a_1a_2\cdots a_{2p}}_{b_1a_2\cdots a_{2p}}, \tag{2.8}$$

and the p-Ricci scalar as

$$R_{(p)} = \sum_a R_{(p)}{}^a_a. \tag{2.9}$$

The generalized Einstein-Hilbert Lagrangian is now defined to be

$$\mathcal{L} = \sum_{p=1}^{P} \frac{1}{2p}\kappa_{(p)} R_{(p)}, \tag{2.10}$$

where $2P=d$ and $d-1$, respectively, for even and odd d. Each coupling constant $\kappa_{(p)}$ is taken to be positive and has dimension (length)^{2p}, rendering the Lagrangian dimensionless. For

$$\kappa_{(2)} = \kappa_{(3)} = \cdots = \kappa_{(P)} = 0$$

one recovers the usual Einstein-Hilbert Lagrangian in dimension d. We will always set

$$\kappa_{(1)} > 0.$$

Some or all of the others, $\kappa_{(p)}$, $p=2,3,\ldots,P$, can then be chosen to be nonzero. We will choose at least $\kappa_{(2)}>0$ so as to illustrate higher-order effects, and $\kappa_{(1)}$ will be taken to be much larger than $\kappa_{(p)}$, $p=2,3,\ldots,P$, so that the latter terms can be considered to play a perturbative role.

For each p, the p-Einstein tensor is defined as

$$G_{(p)}{}^a_b = R_{(p)}{}^a_b - \frac{1}{2p}\eta^a_b R_{(p)}, \tag{2.11}$$

and for the system (2.10) it is

$$G^a_b = \sum_{p=1}^{P} \frac{1}{2p}\kappa_{(p)} G_{(p)}{}^a_b. \tag{2.12}$$

III. SPHERICAL SYMMETRY

We impose spherical symmetry in the $d-1$ space-dimensions by requiring the diagonal metric, for Lorentz and Euclidean signatures, respectively,

$$ds^2 = \mp N(r)dt^2 + N(r)^{-1}dr^2 + r^2 d\Omega^2_{(d-2)}, \tag{3.1}$$

where

$$d\Omega^2_{(d-2)} = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \cdots + \left(\prod_{n=1}^{d-3}\sin\theta_n\right)^2 d\theta^2_{d-2}.$$

We shall henceforth use the following notation[1] for the frame indices:

$$x^a = (t,r,\theta_1,\theta_2,\ldots,\theta_{d-2}) = (t,r,1,2,\ldots,d-2),$$

so that for diagonal metrics one can set (with no summation)

$$e^a = \sqrt{|g_{aa}|}dx^a. \tag{3.2}$$

The consequent simplifying properties [10] of the spin connections lead finally to, using labeling with $i,j=1,2,\ldots,d-2$, and the notation

$$N(r) = 1 - L(r), \quad \text{with} \quad L' = \frac{dL}{dr}, \tag{3.3}$$

$$R^{tr} = \frac{1}{2}L'' e^t \wedge e^r, \quad R^{ti} = \frac{1}{2r}L' e^t \wedge e^i,$$

$$R^{ri} = \frac{1}{2r}L' e^r \wedge e^i, \quad R^{ij} = \frac{1}{r^2}L\, e^i \wedge e^j. \tag{3.4}$$

IV. METRICS

From the results of the previous two sections, namely by applying Eqs. (3.4) to Eq. (2.11), we arrive at the remarkably compact expressions for the nonvanishing components of G^a_b

$$G^t_t = G^r_r = -\left(r\frac{d}{dr} + (d-1)\right)V(r) \tag{4.1}$$

$$G^i_i = -\frac{1}{d-2}\left(r\frac{d}{dr} + (d-2)\right) \times \left(r\frac{d}{dr} + (d-1)\right)V(r), \tag{4.2}$$

with $i=1,2,\ldots,d-2$, and where

$$V(r) = \sum_{p=1}^{P} \kappa_{(p)} \frac{(d-2)(d-3)\cdots(d-2p)}{2^p p!}\left(\frac{L}{r^2}\right)^p. \tag{4.3}$$

It is clear that the term with $d=2p$ vanishes, so that the summation in Eq. (4.3) runs up to $2P<d$.

All the p dependence is contained in $V(r)$, *which is a polynomial in* L/r^2. Hence the constraints on $V(r)$ itself, namely the variational equations, are independent of p and are the same as for the $p=1$ case. Once these dynamical equations are solved, the next step is to solve a polynomial equation in L/r^2. This result is crucial.

[1] We do not distinguish the tangent plane indices from the coordinate ones $(t,r,1,2,\ldots,d-2)$, for example by introducing yet another notation $(\hat{t},\hat{r},\hat{1},\hat{2},\ldots)$. For diagonal metrics, this simplifying notation does not cause ambiguities. As tangent plane indices they are raised and lowered using $\eta^a{}_b$ rather than $g^\mu{}_\nu$.

Consistently with spherical symmetry one can set

$$V(r)=\frac{c}{r^{d-1}}-\frac{b}{r^{2(d-2)}}+\lambda \tag{4.4}$$

when

$$G_t^t=G_r^r=-\frac{(d-3)b}{r^{2(d-2)}}-(d-1)\lambda, \tag{4.5}$$

$$G_i^i=\frac{(d-3)b}{r^{2(d-2)}}-(d-1)\lambda. \tag{4.6}$$

The first term in Eq. (4.4), namely $cr^{-(d-1)}$, represents a vacuum solution leading to the generalized Schwarzschild-type black hole [10]. This is annihilated by the operator $[r(d/dr)+(d-1)]$ present in each G_a^a in Eqs. (4.1) and (4.2).

Adding the second term in Eq. (4.4), $-br^{-2(d-2)}$, leads to generalized Reissner-Nordström type [11] solutions in the presence of a point charge in d dimensions. Finally adding λ in Eq. (4.4) can include the presence of a cosmological constant.

Note the effect of the extra factor

$$\frac{1}{d-2}\left(r\frac{d}{dr}+(d-2)\right) \tag{4.7}$$

in Eq. (4.2) as compared to its absence in Eq. (4.1). In Eq. (4.6) this induces just a change of sign as compared to Eq. (4.5). This compensates precisely for the corresponding sign of the angular components T_i^i of the stress-energy tensor of a point charge q in d dimensions: namely,

$$T_t^t=T_r^r=-T_i^i=-\frac{(d-3)q^2}{2r^{2(d-2)}}. \tag{4.8}$$

Acting on the third term λ of Eq. (4.4), the action of the operator (4.7) yields *unity*. The constants b and λ are to be fixed, finally, after choosing suitable units, by inserting Eqs. (4.5) and (4.6) in

$$G^a{}_b=\text{const}\times T^a{}_b+\Lambda\ \delta^a{}_b\,. \tag{4.9}$$

In the following sections we will give examples of explicit solutions for $L(r)$ using Eqs. (4.3) and (4.4). We will start with Schwarzschild-type black holes with $b=\lambda=0$ and will study their properties.

V. GENERALIZED SCHWARZSCHILD-TYPE BLACK HOLES

In the case of a vanishing stress-energy tensor, Eqs. (4.5) and (4.6) are solved by $V(r)=cr^{-(d-1)}$, with $b=0=\lambda$, yielding

$$\frac{\kappa_{(1)}}{2}(d-2)\left(\frac{L}{r^2}\right)+\frac{\kappa_{(2)}}{8}(d-2)(d-3)(d-4)\left(\frac{L}{r^2}\right)^2+\cdots+\frac{\kappa_{(P)}}{2^P P!}(d-2)(d-3)\cdots(d-2P)\left(\frac{L}{r^2}\right)^P=\frac{c}{r^{d-1}} \tag{5.1}$$

with $2P<d$. With a single nonvanishing p, the solution of Eq. (5.1) reduces to the function $L(r)$ desribing the p-Schwarzschild metric of [10], and with $p=1$ to the d-dimensional Schwarzschild metric of [13].

For a horizon, denoted by $r=r_H$, by definition

$$N(r_H)=1-L(r_H)=0 \quad\mapsto\quad L(r_H)=1. \tag{5.2}$$

Hence

$$\frac{\kappa_{(1)}}{2}(d-2)(r_H)^{-2}+\frac{\kappa_{(2)}}{8}(d-2)(d-3)(d-4)(r_H)^{-4}+\cdots+\frac{\kappa_{(P)}}{2^P P!}(d-2)(d-3)\cdots(d-2P)(r_H)^{-2P}=c(r_H)^{-(d-1)}. \tag{5.3}$$

We will look for *positive real roots* only.

Let us now look at particular cases to better understand the possibilities. In practice we will restrict to the case $\kappa_{(3)}=\kappa_{(4)}=\cdots=\kappa_{(P)}=0$, keeping only $\kappa_{(1)}$ and $\kappa_{(2)}$. One obtains, for $d>4$,

$$L(r)=\left(\frac{2\kappa_{(1)}}{(d-3)(d-4)\kappa_{(2)}}\right)\times\left[\left(1+\frac{2c(d-3)(d-4)\kappa_{(2)}}{(d-2)\kappa_{(1)}^2 r^{(d-1)}}\right)^{1/2}-1\right]r^2. \tag{5.4}$$

For $\kappa_{(2)}>0$ this is real and positive. For $\kappa_{(2)}=0$ it reduces to the usual Schwarzschild solution in d dimensions [13]. In all cases one obtains asymptotically flat solutions.

For the horizon, one has

$$\frac{\kappa_{(1)}}{2}(d-2)(r_H)^{-2}+\frac{\kappa_{(2)}}{8}(d-2)(d-3)(d-4)(r_H)^{-4}=c(r_H)^{-(d-1)}. \tag{5.5}$$

(i) For dimension $d=5$,

$$r_H^2=\frac{2}{3\kappa_{(1)}}\left(c-\frac{3}{4}\kappa_{(2)}\right). \tag{5.6}$$

Hence for $c>\frac{3}{4}\kappa_{(2)}$, there is a single real horizon at

$$r_H=\left(\frac{2}{3\kappa_{(1)}}\right)^{1/2}\left(c-\frac{3}{4}\kappa_{(2)}\right)^{1/2}. \tag{5.7}$$

Compare this with the case $p=2$ in [10] when $\kappa_{(1)}=0$.

(ii) For dimension $d=6$,

$$r_H^3+\frac{3}{2}\frac{\kappa_{(2)}}{\kappa_{(1)}}r_H-\frac{c}{2\kappa_{(1)}}=0. \tag{5.8}$$

Hence

$$r_H=(\alpha+\beta),\quad (\alpha e^{i(2\pi/3)}+\beta e^{-i(2\pi/3)}),$$
$$(\alpha e^{-i(2\pi/3)}+\beta e^{i(2\pi/3)}) \tag{5.9}$$

with

$$\alpha=\left\{\frac{c}{4\kappa_{(1)}}+\left[\left(\frac{c}{4\kappa_{(1)}}\right)^2+\left(\frac{\kappa_{(2)}}{2\kappa_{(1)}}\right)^3\right]^{1/2}\right\}^{1/3},$$

$$\beta=\left\{\frac{c}{4\kappa_{(1)}}-\left[\left(\frac{c}{4\kappa_{(1)}}\right)^2+\left(\frac{\kappa_{(2)}}{2\kappa_{(1)}}\right)^3\right]^{1/2}\right\}^{1/3},$$

which are real cube roots. Thus, for sufficiently small $\kappa_{(2)} \ll \kappa_{(1)}$, there is only one real horizon at

$$r_H=(\alpha+\beta), \tag{5.10}$$

consistently with

$$r_H=\left(\frac{c}{2\kappa_{(1)}}\right)^{1/3}-\frac{1}{2}\frac{\kappa_{(2)}}{\kappa_{(1)}}\left(\frac{c}{2\kappa_{(1)}}\right)^{-1/3}+o(\kappa_{(2)}^2).$$

(iii) For dimension $d=7$,

$$r_H^4+3\frac{\kappa_{(2)}}{\kappa_{(1)}}r_H^2-\frac{2c}{5\kappa_{(1)}}=0, \tag{5.11}$$

giving the real positive value for the horizon

$$r_H=\left(\frac{3\kappa_{(2)}}{2\kappa_{(1)}}\right)^{1/2}\left[\left(1+\frac{8c}{45}\frac{\kappa_{(1)}}{\kappa_{(2)}^2}\right)^{1/2}-1\right]^{1/2}. \tag{5.12}$$

(iv) For dimension $d=8$,

$$r_H^5+5\frac{\kappa_{(2)}}{\kappa_{(1)}}r_H^3-\frac{c}{3\kappa_{(1)}}=0. \tag{5.13}$$

This is a quintic equation whose solution [12] can be expressed in terms of *elliptic functions*. Setting

$$r_H=\frac{a}{z},$$

Eq. (5.13) transforms into

$$z^5-\left(\frac{5\kappa_{(2)}}{a^2\kappa_{(1)}}\right)\left(\frac{3a^5\kappa_{(1)}}{c}\right)z^2-\left(\frac{3a^5\kappa_{(1)}}{c}\right)=0. \tag{5.14}$$

Equation (5.14) is already a *principal quintic* [12] with in addition the linear term absent (vanishing coefficient of z). Moreover, by choosing a suitably one can obtain a conveniently simple value for one of the two coefficients in Eq. (5.14) or for their ratio. These features simplify the task of explicitly constructing the solutions [12].

However since considerable more work is needed to realize these explicit solutions, we will not pursue them further here. We just add that the determination of the real positive root must be consistent with

$$r_H=\left(\frac{c}{3\kappa_{(1)}}\right)^{1/5}-\frac{\kappa_{(2)}}{\kappa_{(1)}}\left(\frac{c}{3\kappa_{(1)}}\right)^{-1/5}+o(\kappa_{(2)}^2). \tag{5.15}$$

(v) For dimension $d=11$, there is a special simplification, namely that Eq. (5.5) reduces to a *quartic* in r_H^2, which permits an elementary solution. Thus Eq. (5.5) here is

$$r_H^8+\frac{14\kappa_{(2)}}{\kappa_{(1)}}r_H^6-\frac{2c}{9\kappa_{(1)}}=0. \tag{5.16}$$

This can first be solved as a quartic in r_H^2 and then the positive square root taken consistently with

$$r_H=\left(\frac{2c}{9\kappa_{(1)}}\right)^{1/8}-\frac{7}{4}\frac{\kappa_{(2)}}{\kappa_{(1)}}\left(\frac{2c}{9\kappa_{(1)}}\right)^{-1/8}+o(\kappa_{(2)}^2). \tag{5.17}$$

(vi) For arbitrary dimension d, one can solve, in principle, polynomial equations of any degree in terms of *theta functions* of suitably high genus [12]. This applies both to Eqs. (5.1) and (5.3). Hence, in principle, exact solutions can be constructed though it would be a very complicated task in practice.

One may note certain qualitative features easy to observe. Thus, for example, the qualitative features described by Eqs. (5.15) and (5.17) hold more generally. For the conditions concerning $\kappa_{(p)}$, stated after (2.10), the single real positive r_H tends to shrink due to $\kappa_{(2)}$ and $\kappa_{(p)}$ with $p>2$, the black hole becoming smaller in radius.

In the preceding examples we have retained only $\kappa_{(1)}$ and $\kappa_{(2)}$ to illustrate basic features. For $d\geqslant 7$ one can include $\kappa_{(3)}$, for $d\geqslant 9$, $\kappa_{(4)}$, and so on. The equations become more difficult to solve but the general features appear already in our examples above.

In the illustrative examples considered in this section, we were mostly concerned with Eq. (5.3) to find the horizon r_H. Concerning the evaluation of the function $L(r)$, on the other hand, one may note that in Eq. (5.1) for L, up to $d=10$ one has a quartic or an equation of lower degree for L. For $d=11$ one has, retaining all possible nonzero contributions, for the first time, a quintic for L.

VI. MAXIMAL EXTENSIONS AND PERIODICITY FOR EUCLIDEAN SIGNATURE

We start by deriving a crucial ingredient in this context, determining both the maximal extension and the near-horizon geometry. For details we refer to Secs. III and V of [10] and references cited therein.

Set

$$r=r_H+\rho, \quad \rho \ll 1 \quad . \tag{6.1}$$

Then, near the horizon, we define δ through

$$N(\rho)=1-L(\rho)=2\delta\rho+o(\rho^2). \tag{6.2}$$

It can be shown that

$$2\delta=\left(\frac{d-3}{r_H}\right)\left[\frac{\kappa_{(1)}+\frac{1}{4}(d-4)(d-5)\kappa_{(2)}r_H^{-2}+\cdots+\frac{1}{2^{n-1}n!}(d-4)\cdots(d-2n-1)\kappa_{(n)}r_H^{-2(n-1)}}{\kappa_{(1)}+\frac{1}{2}(d-3)(d-4)\kappa_{(2)}r_H^{-2}+\cdots+\frac{1}{2^{n-1}(n-1)!}(d-3)\cdots(d-2n)\kappa_{(n)}r_H^{-2(n-1)}}\right] \tag{6.3}$$

with r_H satisfying Eq. (5.3), where we have assumed that the first n $\kappa_{(p)}$, $p=1,2,\ldots,n$ are nonzero. The next step would be to substitute for r_H an explicit solution such as Eq. (5.7), Eq. (5.10), and so on. But the general expression (6.3) is particularly suitable for our present purpose.

If $\kappa_{(n)}=0$ for $n>1$, one obtains

$$2\delta=\left(\frac{d-3}{r_H}\right)=(d-3)\left(\frac{\kappa_{(1)}}{2c}\right)^{1/(d-3)}, \tag{6.4}$$

and usually, for $d=4$, $(\kappa_{(1)}/2c)$ is defined as $(2M)^{-1}$.

Now we proceed to construct the Kruskal-type maximal extension and the periodicity of the Euclidean metric (3.1) with Euclidean signature, namely

$$ds^2=Ndt^2+N^{-1}dr^2+r^2d\Omega^2_{(d-2)}. \tag{6.5}$$

We follow the standard procedure, which was generalized to one (single) member p of the hierarchy in Sec. 3 of [10]. Using Eqs. (6.1) and (6.2), we set

$$r^\star=\int\frac{dr}{N}=\frac{1}{2\delta}\ln\rho+h(\rho), \tag{6.6}$$

in which the function $h(\rho)$ is not relevant for the singularity at the horizon.

We also introduce the coordinates (η,ζ) by

$$e^{2\delta r^\star}=\frac{1}{4}(\eta^2+\zeta^2), \quad e^{i\delta t}=\left(\frac{\eta-i\zeta}{\eta+\zeta}\right)^{1/2}, \tag{6.7}$$

δ being given by Eq. (6.3). Here (η,ζ) provide the generalization of Kruskal coordinates.

One obtains from Eqs. (6.5) and (6.7)

$$ds^2=(4\delta^2e^{2\delta r^\star})^{-1}N(\eta,\zeta)(d\zeta^2+d\eta^2)+r^2(\eta,\zeta)d\Omega^2_{(d-2)}, \tag{6.8}$$

the factor of r^2 in the last term being implicitly a function of (η,ζ). The factor $(e^{2\delta r^\star})^{-1}N$ tends to unity as $\rho\to 0$, assuring maximal extension, there being neither a divergence nor a zero at the horizon.

One obtains from the second member of Eq. (6.7), for the period of t,

$$P=\frac{2\pi}{|\delta|}, \tag{6.9}$$

where δ is given by Eq. (6.3). For $n=2$, i.e., only $\kappa_{(1)}$ and $\kappa_{(2)}$ nonzero, one obtains

$$P=\frac{4\pi r_H}{d-3}\left[1+\frac{(d-1)(d-4)\kappa_{(2)}}{4\pi\kappa_{(1)}r_H^2+(d-4)(d-5)\kappa_{(2)}}\right]. \tag{6.10}$$

The period P is inversely proportional to the Hawking temperature of the black hole. Substituting for r_H in Eq. (6.10) one obtains the full modification due to $\kappa_{(2)}$.

VII. COSMOLOGICAL CONSTANT AND POINT CHARGE

So far we restricted our attention to vacuum metrics. Let us now consider the more general case, keeping Eqs. (4.8) and (4.9) in mind, namely Eq. (4.4),

$$V(r)=\frac{c}{r^{d-1}}-\frac{b}{r^{2(d-2)}}+\lambda.$$

We illustrate some basic features by setting

$$\kappa_{(3)}=\kappa_{(4)}=\cdots=\kappa_{(P)}=0$$

in Eq. (4.3), whence (for $d>4$)

$$\frac{\kappa_{(1)}}{2}(d-2)\left(\frac{L}{r^2}\right)+\frac{\kappa_{(2)}}{8}(d-2)(d-3)(d-4)\left(\frac{L}{r^2}\right)^2=\frac{c}{r^{d-1}}-\frac{b}{r^{2(d-2)}}+\lambda. \tag{7.1}$$

Setting

$$L=\hat{L}+\eta r^2, \tag{7.2}$$

the left-hand side of Eq. (7.1) becomes

$$\beta_1\left(\frac{\hat{L}}{r^2}\right)+\beta_2\left(\frac{\hat{L}}{r^2}\right)^2+\beta_3 \tag{7.3}$$

with

$$\beta_1=\frac{1}{2}(d-2)\kappa_{(1)}+(d-2)(d-3)(d-4)\eta\kappa_{(2)}, \tag{7.4}$$

$$\beta_2=\frac{1}{8}(d-2)(d-3)(d-4)\kappa_{(2)}, \tag{7.5}$$

$$\beta_3=\frac{1}{2}(d-2)\eta\kappa_{(1)}+\frac{1}{8}(d-2)(d-3)(d-4)\eta^2\kappa_{(2)}. \tag{7.6}$$

Setting

$$\beta_3=\lambda \tag{7.7}$$

determines η. Then one solves

$$\beta_2\left(\frac{\hat{L}}{r^2}\right)^2+\beta_1\left(\frac{\hat{L}}{r^2}\right)-\left(\frac{c}{r^{d-1}}-\frac{b}{r^{2(d-2)}}\right)=0. \tag{7.8}$$

One obtains, with sign conventions consistent with Eq. (5.4),

$$\hat{L}(r)=\left(\frac{\beta_1}{2\beta_2}\right)\left\{\left[1+\frac{4\beta_2}{\beta_1^2}\left(\frac{c}{r^{d-1}}-\frac{b}{r^{2(d-2)}}\right)\right]^{1/2}-1\right\}r^2, \tag{7.9}$$

so that $L(r)$ is given by Eq. (7.2).

For $b=0$ and $\lambda\neq 0$, Eq. (7.9) for $\hat{L}$ has the same structure as for L in Eq. (5.4), with the constant coefficients now depending on λ, as given by Eq. (7.4)–(7.7). *It is easy to see that this feature will hold quite generally for all* $\kappa_{(p)}$ (*up to* $p=P$). Even in the presence of a point charge (with $b\neq 0$), the effect of λ can always be taken into account in this way.

When b is nonzero, it is seen from Eq. (4.8) to be positive for a real charge q. Hence for sufficiently small r, it can be seen from Eq. (7.9) that $\hat{L}$ (and hence L) becomes *complex*.

In a previous study [11] of the gravitational systems characterized by $\lambda=0=\eta$, but $b\neq 0$, and with a *single* nonvanishing $\kappa_{(p)}$, it was shown that such a point of transition from a real to a complex metric was situated *inside* the horizon(s) r_H of the generalized p–Reissner-Nordström type solutions. Hence in the *exterior* region the metric was always real. Also in [11], the possibility of a compatible real metric for the *interior* region was indicated. In the present work we will not undertake a parallel detailed study of Eq. (7.9).

VIII. DISCUSSION

We have studied black-hole-type solutions to generalized gravity in d dimensions ($d\geqslant 5$). These systems consist of superpositions of successive higher-order Gauss-Bonnet terms labeled by p ($2p\leqslant d$) which occur, among other gravitational terms, in the superstring inspired [1,3] gravitational system. In the present work, we have omitted the effects of all other fields, e.g. the dilaton, which also occur in superstring inspired systems [1,3]. So far, apart from introducing a cosmological constant and the Maxwell system (i.e., a point charge) in Sec. VII, we have studied only the gravitational field with higher-order terms. The scope can be broadened by including other fields relevant to string theory and by pursuing certain applications.

In the context of string theory, the fields to be added on to the gravitational systems are the dilaton, the Yang-Mills, and the (Abelian) antisymmetric tensor fields.

Exact solutions including dilatons in this context were studied in [14] and [15], but without higher-order gravitational terms. The application of the efficient formalism given in the present work would enable the extension of these results [14,15] to the case of gravitational systems including higher GB terms.

Concerning the interaction of (the usual) Yang-Mills fields interacting with Einstein-Hilbert gravity in 4-dimensions, this has been intensely studied recently, and extensive references to it can be found in the review [16]. The extension of these considerations (with and without dilaton), involving generalized Yang-Mills systems interacting with generalized Einstein-Hilberts fields in higher (than 4) dimensions, would be a very natural and efficient use of our present results. Indeed, we have already considered generalized Yang-Mills fields on *fixed* generalized gravitational *backgrounds* in higher dimensions [17].

Since in the gravitational sector of string theory there appear in addition the dilaton and the antisymmetric tensor fields, it is in principle desirable to include these, as, for example in [18,19], respectively. However, an adequate study with dilatons, antisymmetric tensors, and possible supplementary terms from the cubic order onwards [18,19] is beyond the scope of this work.

Another interesting result of the inclusion of the dilaton and the antisymmetric tensor fields to the usual gravity in the context of the low-energy effective action of supergravity is the construction of solitonic solutions of supergravity [20]. As another application of our results, it would be very interesting to find out what the effect of adding higher-order gravitational terms would be on these solutions.

Finally, we mention some further possible applications of our results in the wake of earlier work in the literature where gravitational Lagrangians with higher terms were studied. These include applications to the elimination of ghosts [3], to the vanishing of the cosmological constant as a stable fixed point [21,22], and to the construction of gravitational instantons [6,7], as well as to cosmological solutions [23,24]. More recent work involving the first GB term [25] pertains to cosmological solutions [26] and to theories with noncompact extra dimensions [27–30] of gravity. The extension of these results [26–29] to include several higher-order GB terms, with and without the inclusion of the dilaton field, would constitute natural applications of our results.

1. Note added

After completion of this manuscript, many previous sources were brought to our attention. To take account of these, we add the following supplementary references and explanations.

Most helpful is the extensive list of references in the review of Myers [31]. Without trying to be complete, we mention the pioneering sources quoted there [32,33,3].

Maximal extensions have been studied systematically in [34]. Our results pertaining to maximal extensions are specifically concerned with the consequences of the supplementary $(d-2)$ dimensions and the higher curvature GB terms implemented through Eq. (6.3) in the standard Kruskal prescription in the r-t plane.

Particularly relevant for us is [35]. The metric function (5.4) and the period (6.10) obtained by us as particular cases by setting $\kappa_{(p)}=0$ for $p>2$, match with Eqs. (3) and (9) of [35], respectively, for

$$\hat{\lambda}=\frac{\kappa_{(2)}}{2\kappa_{(1)}}(d-3)(d-4). \tag{8.1}$$

For comparison with the *total Euclidean action*, Eq. (12) in [35], we briefly present the corresponding general result for our case:

Total Euclidean Action: = period × area of unit sphere in $(d-2)$ dimensions × radial integral,

Period $=2\pi/|\delta|$; area of unit sphere in $(d-2)$ dimensions,

$$A_{(d-2)}=\frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}.$$

The radial integral is

$$\sum_p \frac{\kappa_{(p)}}{2p}\int_{r_H}^{\infty}R_{(p)}dr =\sum_{p=1}^{P}\kappa_{(p)}\frac{(d-2)(d-3)\cdots(d-2p+1)}{2^p p!}\int_{r_H}^{\infty}I_{(p)}dr, \tag{8.2}$$

where

$$\int_{r_H}^{\infty}I_{(p)}dr=\int_{r_H}^{\infty}dr\, r^{d-2}\left(r\frac{d}{dr}+2\right)\left(r\frac{d}{dr}+1\right)\left[r^{d-2}\left(\frac{L}{r^2}\right)^p\right] =\left\{\frac{d}{dr}\left[r^d\left(\frac{L}{r^2}\right)^p\right]\right\}_{r_H}^{\infty} =\frac{2c}{(d-2)\kappa_{(1)}}\delta_{p,1}+\left((2\delta)p-\frac{(d-2p)}{r_H}\right)r_H^{d-2p} \tag{8.3}$$

with

$$I_{(1)}=\frac{2c}{(d-2)\kappa_{(1)}}+2\delta r_H^{d-2}-(d-2)r_H^{d-3}, \tag{8.4}$$

$$I_{(2)}=4\,\delta r_H^{d-4}-(d-4)r_H^{d-5}, \tag{8.5}$$

and so forth.

Note that the term independent of r_H in Eq. (8.4) comes from

$$\lim_{r\to\infty}\frac{d}{dr}\left[r^d\left(\frac{L}{r^2}\right)\right]=\lim_{r\to\infty}\frac{d}{dr}\left(\frac{2c}{(d-2)\kappa_{(1)}r^{d-1}}\cdot r^d\right) =\frac{2c}{(d-2)\kappa_{(1)}}.$$

The Euclidean action and the near-horizon factor [see Eq. (6.3)] provide basic ingredients for studying thermodynamics and surface deformations of black holes in the context of our formalism. We hope to present such a study elsewhere.

ACKNOWLEDGMENTS

We would like to acknowledge some interesting correspondence with R.B. King. It is a pleasure to thank John Rizos for illuminating discussions. This work was carried out in the framework of the Enterprise-Ireland/CNRS program, under project FR/00/018. Center de Physique Théorique is Laboratoire Popre du CNRS UMR7644.

[1] D.J. Gross and E. Witten, Nucl. Phys. **B277**, 1 (1986).
[2] B. Zumino, Phys. Rep. **137**, 2656 (1985).
[3] B. Zwiebach, Phys. Lett. **156B**, 915 (1985).
[4] R.R. Metsaev and A.A. Tseytlin, Phys. Lett. B **185**, 52 (1987).
[5] M.C. Bento and O. Bertolami, Phys. Lett. B **228**, 348 (1989).
[6] A. Chakrabarti and D.H. Tchrakian, Phys. Lett. **168B**, 187 (1986).
[7] G.M. O'Brien and D.H. Tchrakian, J. Math. Phys. **29**, 1212 (1988).
[8] J.T. Wheeler, Nucl. Phys. **B268**, 737 (1986).
[9] J.T. Wheeler, Nucl. Phys. **B273**, 733 (1986).
[10] A. Chakrabarti and D.H. Tchrakian, Adv. Theor. Math. Phys. **4**, 791 (2000).
[11] A. Chakrabarti and D.H. Tchrakian, "Antisymmetrised $2p$-forms generalising curvature 2-forms II: a p-hierarchy of Reissner-Nordström type metrics in dimensions $d>2p+1$," hep-th/0010011.
[12] See, for example, R. Bruce King, *Beyond the Quartic Equation* (Birkhäuser, Boston, 1996).
[13] R.C. Myers and M.J. Perry, Ann. Phys. (N.Y.) **172**, 304 (1986).
[14] G. Gibbons and K. Maeda, Nucl. Phys. **B298**, 741 (1988).
[15] D. Garfinkle, G.T. Horowitz, and A. Strominger, Phys. Rev. D **43**, 3140 (1991).
[16] M.S. Volkov and D.V. Gal'tsov, Phys. Rep. **319**, 1 (1999).
[17] Y. Brihaye, A. Chakrabarti, and D.H. Tchrakian, J. Math. Phys. **41**, 5490 (2000).
[18] M.C. Bento and O. Bertolami, Phys. Lett. B **368**, 198 (1996).
[19] M.C. Bento and N.E. Mavromatos, Phys. Lett. B **190**, 105

(1987).
[20] See, for example, Th. Mohaupt, Class. Quantum Grav. **17**, 3429 (2000).
[21] J. Madore, Phys. Lett. A **110**, 289 (1985).
[22] N. Deruelle and J. Madore, Phys. Lett. A **114**, 185 (1985).
[23] B. Giorgini and R. Kerner, Class. Quantum Grav. **5**, 339 (1988).
[24] Müller-Hoissen, Phys. Lett. **163B**, 106 (1985).
[25] P. Kanti, J. Rizos, and K. Tamvakis, Phys. Rev. D **59**, 083512 (1999).
[26] See, S. Alexeyev, A. Toporensky, and V. Ustiansky, Phys. Lett. B **509**, 151 (2001).
[27] J.E. Kim, B. Kyae, and H.M. Lee, Phys. Rev. D **62**, 045013 (2000).
[28] I. Low and A. Zee, Nucl. Phys. **B585**, 395 (2000).
[29] N.E. Mavromatos and J. Rizos, Phys. Rev. D **62**, 124004 (2000).
[30] K.A. Meissner and M. Olechowski, Phys. Rev. Lett. **86**, 3708 (2001).
[31] R.C. Myers, ‘‘Black holes in higher curvature gravity,’’ gr-qc/9811042.
[32] D. Lovelock, J. Math. Phys. **12**, 498 (1971); **13**, 874 (1972).
[33] D.G. Boulware and S. Deser, Phys. Rev. Lett. **55**, 2656 (1985).
[34] T. Klösch and T. Srtobl, Class. Quantum Grav. **13**, 965 (1996); **14**, 825(E) (1997); **14**, 1689 (1997).
[35] R.C. Myers and J.Z. Simon, Phys. Rev. D **38**, 2434 (1988).