

Doubly covariant action principle of singular hypersurfaces in general relativity and scalar-tensor theories

Shinji Mukohyama

Department of Physics, Harvard University, Cambridge, Massachusetts 02138

*Canadian Institute for Theoretical Astrophysics, University of Toronto, Toronto, Ontario, Canada M5S 3H8
and Department of Physics and Astronomy, University of Victoria, Victoria, British Columbia, Canada V8W 3P6*

(Received 21 August 2001; published 26 December 2001)

An action principle of singular hypersurfaces in general relativity and scalar-tensor type theories of gravity in the Einstein frame is presented without assuming any symmetry. The action principle is manifestly doubly covariant in the sense that coordinate systems on and off a hypersurface are disentangled and can be independently specified. It is shown that, including variation of the metric, the position of the hypersurface, and the matter fields, the variational principle gives the correct set of equations of motion: the Einstein equation off the hypersurface, Israel's junction condition in a doubly covariant form, and the equations of motion of the matter fields including the scalar fields. The position of the hypersurface measured from one side of the hypersurface and that measured from the other side can be independently varied as required by the double covariance.

DOI: 10.1103/PhysRevD.65.024028

PACS number(s): 04.50.+h, 98.80.Cq

I. INTRODUCTION

Spatially extended objects such as branes, membranes, shells, and walls have played many important roles in the recent progress in theoretical physics including string theory [1], particle phenomenology [2–4], the theory of black holes [5–7], cosmology [8,9], and so on. Hence, it is important to investigate the dynamics of such extended objects. In particular, the so-called brane-world scenario is based on the idea that our four-dimensional universe may be the world volume of a brane in a higher dimensional spacetime [2–4,10]. Thus, in the brane-world scenario the dynamics of the brane is the dynamics of our universe itself and is of the utmost physical importance.

It is well known and the most commonly adopted picture that the dynamics of extended objects is elegantly described as a geometrical embedding of world-volume surfaces into spacetime in a certain limit. In particular, in the case of codimension 1, or when the world-volume surface is a hypersurface, the geometrical description becomes simpler than in other cases with higher codimension. In fact, in general relativity or other theories of gravity in the Einstein frame, the classical dynamics of a hypersurface is perfectly described by Israel's junction condition [11].

One of the main advantages of the junction condition is that it is manifestly doubly covariant in the sense that coordinate systems on and off a hypersurface are disentangled and can be independently specified. More precisely, there are three independent coordinate systems: that on the hypersurface and those in the two regions separated by the hypersurface. From the brane world point of view, double covariance is important since it allows us to separate the coordinate system in our world from that in the higher dimensional spacetime.

Once the classical dynamics is understood, one would usually like to understand quantum mechanical dynamics [12–21]. For this purpose, we would like to obtain the action principle for a system including a hypersurface.

The easiest way to obtain the action may be to adopt the Gaussian normal coordinate system based on the hypersurface and to consider the Einstein-Hilbert action with a delta function source. The action obtained in this way gives the correct set of equations in the coordinate system, provided that the position of the hypersurface and coordinates in the neighborhood of the hypersurface are fixed by the Gaussian normal coordinate condition. However, in this method we lose the double covariance: the coordinates on the hypersurface are part of the coordinates off the hypersurface so that the coordinates satisfy the Gaussian normal coordinate condition. The loss of the double covariance is regrettable.

Actually, as far as the author knows, a doubly covariant action principle has not yet been obtained in the literature. One of the main difficulties seems to be due to the fact that the spacetime metric on one side of the hypersurface and that on the other side are independent variables in the variational principle. Hence, the question arises: How can we ensure the regularity of the intrinsic geometry of the hypersurface without entangling the coordinate systems on and off the hypersurface? This question will be answered in this paper as a manifestly covariant action principle will be presented.

Another difficulty is due to the fact that the double covariance requires inclusion of the position of the hypersurface as a dynamical variable in the action principle. In fact, in the doubly covariant formulation of the junction condition, it is easy to see that variables specifying the position are not invariant under coordinate transformation and should not be fixed [22]. More about why we need to include the position of the hypersurface will be explained from the brane world point of view in Sec. V. Here, we mention that, since the coordinate systems on the two sides of the hypersurface are independent, it should be possible to independently vary the position of the hypersurface measured from one side and that measured from the other side in the variational principle.

It may be worth reviewing the present status in the literature regarding the second difficulty. However, the author knows of only a few papers referring to this point. Here, we only quote a sentence from one of them: “The variational

equations that arose from the unreduced Hamiltonian action were not strictly consistent in a distributional sense, but we were able to localize the ambiguity into the single equation that arises by varying the action with respect to the shell position. [20]” One might think that another paper [21] obtained the correct set of equations, but in that paper the position of the hypersurface measured from one side and that measured from the other side cannot be varied independently. In fact, if we simply varied the position of the hypersurface measured from one side and that measured from the other side independently, then the variational principle presented in Ref. [21] would give wrong equations. Moreover, in both of these papers, the hypersurface represents only a dust shell and the coordinate systems on and off the hypersurface are not independent. One of them [20] assumes spherical symmetry, too.

The purpose of this paper is to provide a manifestly doubly covariant action principle of singular hypersurfaces in general relativity and scalar-tensor type theories of gravity in the Einstein frame without assuming any symmetry. In addition to the scalar fields included in the scalar-tensor type theories, any kind of matter Lagrangian density on the hypersurface, which may also depend on the pullback of the scalar fields, can be included. It is shown that, including variation of the metric, the position of the hypersurface, and the matter fields, the variational principle gives the correct set of equations of motion: the Einstein equation off the hypersurface, Israel’s junction condition in a doubly covariant form, and the equations of motion of the matter fields including the scalar fields. As required by the double covariance, the position of the hypersurface measured from one side of the hypersurface and that measured from the other side can be independently varied.

This paper is organized as follows. In Sec. II a doubly covariant action of a singular hypersurface is derived from the standard Einstein-Hilbert action. In Sec. III variation of the action is calculated for the variations of the metric and the position of the hypersurface, and the corresponding equations are obtained. In Sec. IV the variation of the action corresponding to the variations of scalar fields is evaluated. Section V is devoted to a summary of this paper and some discussion.

II. ACTION OF SINGULAR HYPERSURFACE

Let us consider a D -dimensional spacetime (\mathcal{M}, g_{MN}) and a timelike or spacelike hypersurface Σ which separates \mathcal{M} into two regions \mathcal{M}_+ and \mathcal{M}_- . Since we would like to consider Σ as a physical object (e.g., the world volume of a brane or the world volume of a bubble wall in a first-order phase transition) or a physical event (e.g., an instantaneous global phase transition [23]), we assume that the $(D-1)$ -dimensional intrinsic geometry on Σ is regular. On the other hand, the D -dimensional geometry is not necessarily regular on Σ .

In the following arguments we shall estimate the action for the system including the singular hypersurface Σ . We

assume that the system is described by the action¹

$$I_{tot} = I_{EH} + I_{matter}, \quad (1)$$

where I_{EH} is the Einstein-Hilbert action with a cosmological constant

$$I_{EH} = \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^D x \sqrt{|g|} (R - 2\Lambda), \quad (2)$$

and I_{matter} is the matter action of the form

$$I_{matter} = \int_{\mathcal{M}_+} d^D x_+ \mathcal{L}_+ + \int_{\mathcal{M}_-} d^D x_- \mathcal{L}_- + \int_{\Sigma} d^{D-1} y \mathcal{L}_0. \quad (3)$$

Here, $\{x_{\pm}^M\}$ are D -dimensional coordinate systems in \mathcal{M}_{\pm} , respectively, and $\{y^{\mu}\}$ is a $(D-1)$ -dimensional coordinate system in Σ . The three coordinate systems can be independent of each other.

In order to evaluate the gravitational part of the action, we first regularize the D -dimensional geometry in the neighborhood of Σ by introducing the finite thickness δ of the object corresponding to Σ . Of course, in the final step below, we shall take the limit $\delta \rightarrow +0$, where the hypersurface becomes singular again. That is we consider the prescription

$$I_{EH} = \lim_{\delta \rightarrow +0} (I_+^{\delta} + I_0^{\delta} + I_-^{\delta}),$$

$$I_0^{\delta} = \frac{1}{2\kappa^2} \left[\int_{\mathcal{M}_0^{\delta}} d^D x \sqrt{|g|} (R - 2\Lambda) + 2\epsilon \int_{\mathcal{B}_+^{\delta}} d^{D-1} y \sqrt{|q|} K - 2\epsilon \int_{\mathcal{B}_-^{\delta}} d^{D-1} y \sqrt{|q|} K \right],$$

$$I_{\pm}^{\delta} = \frac{1}{2\kappa^2} \left[\int_{\mathcal{M}_{\pm}^{\delta}} d^D x \sqrt{|g|} (R - 2\Lambda) \mp 2\epsilon \int_{\mathcal{B}_{\pm}^{\delta}} d^{D-1} y \sqrt{|q|} K \right], \quad (4)$$

where \mathcal{M}_0^{δ} is a spacetime neighborhood of Σ representing the regularized object, $\mathcal{M}_{\pm}^{\delta}$ are the two regions separated by \mathcal{M}_0^{δ} so that

$$\mathcal{M}_0^{\delta} \supset \Sigma, \mathcal{M}_{\pm}^{\delta} \subset \mathcal{M}_{\pm}, \lim_{\delta \rightarrow +0} \mathcal{M}_{\pm}^{\delta} = \mathcal{M}_{\pm}, \quad (5)$$

and $\mathcal{B}_{\pm}^{\delta}$ is the boundary between \mathcal{M}_0^{δ} and $\mathcal{M}_{\pm}^{\delta}$, respectively. Note that surface terms have been included in $I_{0,\pm}^{\delta}$ for later convenience but that these exactly cancel each other on common boundaries $\mathcal{B}_{\pm}^{\delta}$. Each surface term is defined as an integral over the $(D-1)$ -dimensional intrinsic coordinates y^{μ} on $\mathcal{B}_{\pm}^{\delta}$, q is the determinant of the induced metric, K is

¹For simplicity we do not consider the boundary of \mathcal{M} , but it is easy to take it into account by imposing suitable boundary conditions and introducing boundary terms appropriate for the boundary condition.

the trace of the extrinsic curvature associated with the unit normal n^M directed from \mathcal{M}_0^δ to \mathcal{M}_+^δ or from \mathcal{M}_-^δ to \mathcal{M}_0^δ , and $\epsilon = g_{MN}n^M n^N = \pm 1$.

Next, in order to estimate I_0^δ , we foliate \mathcal{M}_0^δ by such a one-parameter family of hypersurfaces Σ_τ that Σ_0 coincides with Σ and that $\Sigma_{\pm 1}$ coincides with the boundary \mathcal{B}_\pm^δ , respectively. Hence, we can decompose the D -dimensional Ricci scalar R as

$$R = R^{(D-1)} + \epsilon K^2 - \epsilon K^{\mu\nu} K_{\mu\nu} - 2\epsilon (Kn^M - n^M_{;N}n^N)_{;M}, \quad (6)$$

where $R^{(D-1)}$ is the Ricci scalar of the $(D-1)$ -dimensional induced metric on Σ_τ , the semicolon represents the covariant derivative compatible with g_{MN} , n^M is the unit normal to Σ_τ directed toward \mathcal{B}_+^δ , $\epsilon = g_{MN}n^M n^N = \pm 1$, $K_{\mu\nu}$ is the extrinsic curvature associated with n^M , the indices $\{\mu, \nu\}$ are raised by the inverse of the induced metric, and $K = K^\mu_\nu$. By integrating over \mathcal{M}_0^δ and taking the limit $\delta \rightarrow +0$, we obtain

$$I_0^\delta = \frac{1}{2\kappa^2} \int_{\mathcal{M}_0^\delta} d^D x \sqrt{|g|} (R^{(D-1)} + \epsilon K^2 - \epsilon K^{\mu\nu} K_{\mu\nu} - 2\Lambda) \rightarrow 0 (\delta \rightarrow +0). \quad (7)$$

Here, we have used the assumption that the intrinsic geometry on Σ is regular even in the limit $\delta \rightarrow +0$. We have also assumed that the extrinsic curvature remains finite.

Therefore, we obtain the following form of the Einstein-Hilbert action for the system including the singular hypersurface Σ :

$$I_{EH} = \frac{1}{2\kappa^2} \left[\int_{\mathcal{M}_+} d^D x_+ \sqrt{|g_+|} (R_+ - 2\Lambda_+) + \int_{\mathcal{M}_-} d^D x_- \sqrt{|g_-|} (R_- - 2\Lambda_-) - 2\epsilon \int_{\Sigma} d^{D-1} y (\sqrt{|q_+|} K_+ - \sqrt{|q_-|} K_-) + \epsilon \int_{\Sigma} d^{D-1} y \lambda^{\mu\nu} (q_{+\mu\nu} - q_{-\mu\nu}) \right], \quad (8)$$

where $q_{\pm\mu\nu}$ is the induced metric, q_\pm is the determinant of $q_{\pm\mu\nu}$, $K_\pm = q_\pm^{\mu\nu} K_{\pm\mu\nu}$ is the trace of the extrinsic curvature $K_{\pm\mu\nu}$, and $q_\pm^{\mu\nu}$ is the inverse of $q_{\pm\mu\nu}$. In the expression (8) we have distinguished geometrical quantities in \mathcal{M}_+ and \mathcal{M}_- by introducing the subscript \pm , and have allowed the cosmological constant to have different values in these two regions. We have introduced the Lagrange multiplier field $\lambda^{\mu\nu}(y)$ to ensure the regularity of the intrinsic geometry of Σ . When we regularized the system and decomposed I_{EH} into I_0^δ and I_\pm^δ as in Eq. (4), we implicitly assumed that the induced metric and the extrinsic curvature are continuous across the boundaries \mathcal{B}_\pm^δ . After taking the limit $\delta \rightarrow +0$, the extrinsic curvature remains finite but can be discontinuous across Σ . On the other hand, the induced metric should re-

main continuous across Σ even after taking the limit $\delta \rightarrow +0$ because of the finiteness of the extrinsic curvature. Provided that the hypersurface Σ is specified as the boundary of \mathcal{M}_\pm by the parametric equation

$$x_\pm^M = Z_\pm^M(y^\mu), \quad (9)$$

the induced metric and the extrinsic curvature are given by

$$q_{\pm\mu\nu}(y) = e_{\pm\mu}^M(y) e_{\pm\nu}^N(y) g_{MN}|_{x_\pm = Z_\pm(y)},$$

$$K_{\pm\mu\nu}(y) = \frac{1}{2} e_{\pm\mu}^M(y) e_{\pm\nu}^N(y) \mathcal{L}_{n_\pm} g_{\pm MN}|_{x_\pm = Z_\pm(y)}, \quad (10)$$

where $e_{\pm\nu}^M$ are vectors tangent to Σ defined by

$$e_{\pm\mu}^M(y) = \frac{\partial Z_\pm^M}{\partial y^\mu}, \quad (11)$$

and n_\pm^M is the unit normal to Σ directed from \mathcal{M}_- to \mathcal{M}_+ . To be precise, n_+^M is the inward-directed unit normal to Σ as the boundary of \mathcal{M}_+ and n_-^M is the outward-directed unit normal to Σ as the boundary of \mathcal{M}_- .

Finally, the total action of the system is given by Eq. (1), where I_{EH} and I_{matter} are given by Eq. (8) and (3), respectively.

III. VARIATIONAL PRINCIPLE

In this section we derive equations of motion from the variational principle based on the action obtained in the previous section; namely, we shall extremize the action I_{tot} with respect to the variation

$$g_{\pm MN}(x) \rightarrow g_{\pm MN}(x) + \delta g_{\pm MN}(x),$$

$$Z_\pm^M(y) \rightarrow Z_\pm^M(y) + \delta Z_\pm^M(y). \quad (12)$$

In the following we omit the subscript \pm unless there is a possibility of confusion.

First, it is easy to show that the integrand of the volume term in I_{EH} changes as follows:

$$\sqrt{|g|} (R - 2\Lambda) \rightarrow \sqrt{|g|} [(R - 2\Lambda) - (G^{MN} + \Lambda g^{MN}) \delta g_{MN} + (\delta g^M{}_{;N} - \delta g^M)_{;M} + O(\delta^2)], \quad (13)$$

where the semicolon represents the covariant derivative compatible with the background metric g_{MN} (not with the perturbed metric $g_{MN} + \delta g_{MN}$), the indices M, N, \dots are lowered and raised by the background metric g_{MN} and its inverse g^{MN} , and δg is defined by $\delta g = \delta g^M_M$. Hence,

$$\begin{aligned}
& \delta \int_{\mathcal{M}_{\pm}} d^D x_{\pm} \sqrt{|g|} (R - 2\Lambda) \\
&= - \int_{\mathcal{M}_{\pm}} d^D x_{\pm} \sqrt{|g|} (G^{MN} + \Lambda g^{MN}) \delta g_{MN} \\
&\mp \epsilon \int_{\Sigma} d^{D-1} y \sqrt{|q|} n^M (\delta g_{M;N} - \delta g_{;M}) \Big|_{x_{\pm}=Z_{\pm}} \\
&\mp \epsilon \int_{\Sigma} d^{D-1} y \sqrt{|q|} n_M \delta Z_{\pm}^M (R - 2\Lambda) \Big|_{x_{\pm}=Z_{\pm}}. \tag{14}
\end{aligned}$$

The second term on the right hand side came from the total derivative in Eq. (13) and the last term is due to the change of the region to be integrated over.

Next, let us consider the surface term in I_{EH} . As shown in Ref. [22] the variations of the induced metric and the extrinsic curvature are given by

$$\begin{aligned}
\delta q_{\mu\nu} &= e_{\mu}^M e_{\nu}^N (\delta g_{MN} + \delta Z_{M;N} + \delta Z_{N;M}), \\
\delta K_{\mu\nu} &= \frac{\epsilon}{2} n^M n^N (\delta g_{MN} + 2\delta Z_{M;N}) K_{\mu\nu} \\
&\quad - \frac{1}{2} n^L e_{\mu}^M e_{\nu}^N [2\delta\Gamma_{LMN} + \delta Z_{L;MN} \\
&\quad + \delta Z_{L;NM} + (R_{L'MLN} + R_{L'NLM}) \delta Z^{L'}], \tag{15}
\end{aligned}$$

where the right hand side is evaluated at $x_{\pm}^M = Z_{\pm}^M(y)$ and

$$\delta\Gamma_{LMN} = \frac{1}{2} (\delta g_{LM;N} + \delta g_{LN;M} - \delta g_{MN;L}). \tag{16}$$

In order to make the covariant derivatives of δZ^M well-defined, we have to extend δZ^M off Σ . The expressions (15) are independent of the method of the extension. For details, see Ref. [22]. Hence,

$$\sqrt{|q|} K \rightarrow \sqrt{|q|} K + \delta(\sqrt{|q|} K) + O(\delta^2), \tag{17}$$

where

$$\begin{aligned}
\delta(\sqrt{|q|} K) / \sqrt{|q|} &= - \left(K^{\mu\nu} - \frac{1}{2} K q^{\mu\nu} \right) \delta q_{\mu\nu} + \frac{\epsilon}{2} n^M n^N (\delta g_{MN} \\
&\quad + 2\delta Z_{M;N}) K - n^L q^{\mu\nu} e_{\mu}^M e_{\nu}^N (\delta\Gamma_{LMN} \\
&\quad + \delta Z_{L;MN} + R_{L'MLN} \delta Z^{L'}). \tag{18}
\end{aligned}$$

Combining this with the second term in Eq. (14), we obtain

$$\begin{aligned}
& n^M (\delta g_{M;N}^N - \delta g_{;M}) + 2\delta(\sqrt{|q|} K) / \sqrt{|q|} \\
&= - (K^{\mu\nu} - K q^{\mu\nu}) \delta q_{\mu\nu} - 2n^M R_{MN} \delta Z^N \\
&\quad - \frac{1}{\sqrt{|q|}} [\sqrt{|q|} q^{\mu\nu} n^M e_{\nu}^N (\delta g_{MN} + 2\delta Z_{M;N})]_{,L} e_{\mu}^L \tag{19}
\end{aligned}$$

where we have used the equations

$$[e_{\mu}, e_{\nu}]^M = 0,$$

$$e_{\mu}^M (e_{\nu}^N n_N)_{;M} = e_{\mu}^M (n^N n_N)_{;M} = 0,$$

$$q^{\mu\nu} e_{\mu}^M e_{\nu}^N + \epsilon n^M n^N = g^{MN}. \tag{20}$$

Thus, the variation of the Einstein-Hilbert action I_{EH} is given by

$$\begin{aligned}
2\kappa^2 \delta I_{EH} &= - \int_{\mathcal{M}_+} d^D x_+ \sqrt{|g_+|} (G_+^{MN} + \Lambda_+ g_+^{MN}) \delta g_{+MN} \\
&\quad - \int_{\mathcal{M}_-} d^D x_- \sqrt{|g_-|} (G_-^{MN} + \Lambda_- g_-^{MN}) \delta g_{-MN} \\
&\quad + \epsilon \int_{\Sigma} d^{D-1} y \{ [\sqrt{|q_+|} (K_+^{\mu\nu} - K_+ q_+^{\mu\nu}) \\
&\quad + \lambda^{\mu\nu}] \delta q_{+\mu\nu} - [\sqrt{|q_-|} (K_-^{\mu\nu} - K_- q_-^{\mu\nu}) \\
&\quad + \lambda^{\mu\nu}] \delta q_{-\mu\nu} + 2n_+^M (G_{+MN} \\
&\quad + \Lambda_+ g_{+MN}) \Big|_{x_+=Z_+} \delta Z_+^N - 2n_-^M (G_{-MN} \\
&\quad + \Lambda_- g_{-MN}) \Big|_{x_-=Z_-} \delta Z_-^N + (q_{+\mu\nu} - q_{-\mu\nu}) \delta \lambda^{\mu\nu} \}. \tag{21}
\end{aligned}$$

Now let us consider the variation of I_{matter} :

$$\begin{aligned}
2\delta I_{matter} &= \int_{\mathcal{M}_+} d^D x_+ \sqrt{|g_+|} T_+^{MN} \delta g_{+MN} \\
&\quad + \int_{\mathcal{M}_-} d^D x_- \sqrt{|g_-|} T_-^{MN} \delta g_{-MN} \\
&\quad + \epsilon \int_{\Sigma} d^{D-1} y [\sqrt{|q|} S^{\mu\nu} \delta q_{\mu\nu} + 2F_{+M} \delta Z_+^M \\
&\quad + 2F_{-M} \delta Z_-^M], \tag{22}
\end{aligned}$$

where $q_{\mu\nu}$ is either $q_{+\mu\nu}$ or $q_{-\mu\nu}$, and

$$\sqrt{|g_{\pm}|} T_{\pm}^{MN}(x_{\pm}) = 2 \frac{\delta}{\delta g_{\pm MN}(x_{\pm})} \int_{\mathcal{M}_{\pm}} d^D x'_{\pm} \mathcal{L}_{\pm},$$

$$\sqrt{|q|} S^{\mu\nu}(y) = 2\epsilon \frac{\delta}{\delta q_{\mu\nu}(y)} \int_{\Sigma} d^{D-1} y' \mathcal{L}_0 |_{\delta Z^M=0},$$

$$\sqrt{|q|}F_{\pm M}(y) = \mp n_{\pm M} \mathcal{L}_{\pm} \Big|_{x_{\pm}=Z_{\pm}(y)} + \epsilon \frac{\delta}{\delta Z_{\pm}^M(y)} \int_{\Sigma} d^{D-1}y' \mathcal{L}_0 \Big|_{\delta q_{\mu\nu}=0} \quad (23)$$

Therefore, $\delta I_{tot}=0$ is equivalent to the following set of equations:

$$G_{\pm}^{MN} + \Lambda_{\pm} g_{\pm}^{MN} = \kappa^2 T_{\pm}^{MN},$$

$$q_{+\mu\nu} - q_{-\mu\nu} = 0,$$

$$K_{+}^{\mu\nu} - K_{-}^{\mu\nu} = -\kappa^2 \left(S^{\mu\nu} - \frac{1}{D-2} S q^{\mu\nu} \right), \quad (24)$$

$$F_{\pm N} = \mp n_{\pm}^M T_{\pm MN} \Big|_{x_{\pm}=Z_{\pm}}, \quad (25)$$

and

$$\lambda^{\mu\nu} = -\sqrt{|q|} (K_{+}^{\mu\nu} - K_{-}^{\mu\nu}). \quad (26)$$

On the right hand side of the last equation, the subscript $-$ (or $+$) should be taken when \mathcal{L}_0 is written in terms of $q_{+\mu\nu}$ (or $q_{-\mu\nu}$).

Note that Eqs. (24) are the Einstein equation and Israel's junction condition [11]. The last equation is just to determine the Lagrange multiplier field $\lambda^{\mu\nu}$. Although Eq. (25) looks like a new independent equation, it will be shown in the next section for simple examples that the equation is compatible with the equations of motion of the matter fields. Therefore, the action principle gives the correct set of equations: the Einstein equation, Israel's junction condition, and the equations of motion of matter fields.

IV. SIMPLE EXAMPLES

In this section we show that for simple examples Eq. (25) is compatible with the equations of motion of matter fields. The first trivial example is the case in which all matter fields are confined on the hypersurface Σ . This case includes a shell with an arbitrary equation of state in a vacuum and the brane world scenario in a purely gravitational bulk with a bulk cosmological constant and arbitrary matter fields on the brane. In this case, the consistency condition (25) is trivially satisfied since $\mathcal{L}_{\pm}=0$ and \mathcal{L}_0 does not change when Z_{\pm}^M is changed with $q_{\mu\nu}$ fixed.

As the second example, let us consider a simple case in which there is only a scalar field other than those matter fields confined on the hypersurface Σ ; namely, let us consider the following Lagrangian densities:

$$\mathcal{L}_{\pm} = -\sqrt{|g|} \left[\frac{1}{2} g^{MN} \partial_M \Phi \partial_N \Phi + V_{\pm}(\Phi) \right],$$

$$\mathcal{L}_0 = \bar{\mathcal{L}}_0(\phi_{+}) + \lambda_{\phi}(\phi_{+} - \phi_{-}), \quad (27)$$

where $\bar{\mathcal{L}}_0$ is the Lagrangian density for matter confined on Σ , and ϕ_{\pm} is the pullback of Φ on Σ defined by

$$\phi_{\pm}(y) = \Phi \Big|_{x_{\pm}=Z_{\pm}(y)}. \quad (28)$$

The matter Lagrangian density $\bar{\mathcal{L}}_0$ on Σ can depend on ϕ_{+} as well. Note that the Lagrange multiplier field $\lambda_{\phi}(y)$ is necessary in order that the scalar field should have a single value on Σ and that Σ should be regular. For this example we can easily calculate T_{\pm}^{MN} , $S^{\mu\nu}$, and $F_{\pm M}$ as follows.

$$T_{\pm}^{MN} = \partial_M \Phi \partial_N \Phi - g_{MN} \left[\frac{1}{2} g^{M'N'} \partial_{M'} \Phi \partial_{N'} \Phi + V(\Phi) \right],$$

$$S^{\mu\nu} = \frac{2\epsilon}{\sqrt{|q|}} \frac{\delta}{\delta q_{\mu\nu}(y)} \int_{\Sigma} d^{D-1}y' \bar{\mathcal{L}}_0 \Big|_{\delta\phi_{+}=0}, \quad (29)$$

and

$$F_{+M} = \sqrt{|q|} \left[\frac{1}{2} g^{M'N'} \partial_{M'} \Phi \partial_{N'} \Phi + V_{+}(\Phi) \right] n_M + \epsilon (\partial_{\phi_{+}} \bar{\mathcal{L}}_0 + \lambda_{\phi}) \partial_M \Phi, \quad (30)$$

$$F_{-M} = -\sqrt{|q|} \left[\frac{1}{2} g^{M'N'} \partial_{M'} \Phi \partial_{N'} \Phi + V_{-}(\Phi) \right] n_M - \epsilon \lambda_{\phi} \partial_M \Phi, \quad (31)$$

where the right hand sides of Eqs. (30) and (31) are evaluated at $x_{\pm}^M = Z_{\pm}^M(y)$, respectively. Hence, by using

$$\lambda_{\phi} = -\partial_{\phi_{+}} \bar{\mathcal{L}}_0 - \epsilon \sqrt{|q|} n_{+}^M \partial_M \Phi \Big|_{x_{+}=Z_{+}}$$

$$= -\epsilon \sqrt{|q|} n_{-}^M \partial_M \Phi \Big|_{x_{-}=Z_{-}}, \quad (32)$$

which is part of the equations of motion, it is confirmed that Eq. (25) is satisfied. Thus, the consistency condition (25) is actually compatible with the equations of motion of the scalar field.

It is easy to extend the above analysis to an arbitrary number of scalar fields.

V. SUMMARY AND DISCUSSION

We have presented an action principle of singular hypersurfaces in general relativity in any dimension without assuming any symmetry. Since an arbitrary number of scalar fields can be consistently included as shown in Sec. IV, the action principle is applicable to a wide class of scalar-tensor type theories of gravity in the Einstein frame. In addition to the scalar fields, any kind of matter Lagrangian density on the hypersurface, which may depend also on the pullback of the scalar fields, can be included. The action principle is manifestly doubly covariant in the sense that coordinate systems on and off a hypersurface are disentangled and can be independently specified. More precisely, there are three independent coordinate systems: that on the hypersurface and those in the two regions separated by the hypersurface. We

have shown that, including variation of the metric, the position of the hypersurface, and the matter fields, the variational principle gives the correct set of equations of motion: the Einstein equation off the hypersurface, Israel's junction condition in a doubly covariant form, and the equations of motion of matter fields including the scalar fields. It is worth mentioning that the position of the hypersurface measured from one side of the hypersurface and that measured from another side can be independently varied as required by the double covariance.

Now let us discuss the application of the doubly covariant action principle to the brane world scenario. In Refs. [24–30] it was shown that the standard cosmology can be realized in the Randall-Sundrum brane world scenario for a spatially homogeneous and isotropic brane. After that, many authors investigated cosmological perturbations in the brane-world scenario [31,22,32–41].

In particular, four independent equations for scalar perturbations on the brane in the plane symmetric ($K=0$) background were derived recently in Ref. [32]. The number of independent equations is the same as in standard cosmology, and it was shown that at low energy these sets of equations differ only by the nonlocal effects due to gravitational waves in the bulk.

In the derivation of the four equations in Ref. [32] the author took advantage of the doubly gauge invariant formalism developed in Refs. [31,22]. It was essential that the formalism include perturbation of the position of a brane as a dynamical variable. In fact, as already discussed in Ref. [31], if we fix the position of the brane by hand as in the Gaussian normal coordinate system, then it is in general inconsistent with convenient gauge choices in the bulk, such as a generalized Regge-Wheeler gauge.² In other words, as was done in Refs. [33,31], we can construct D -gauge invariant variables from the perturbation of the position of the brane, and they are physical degrees of freedom independent of the D -gauge invariant variables in the bulk. The former gauge invariant variables are concise in the sense that they are localized on the brane, and the later variables can be expressed most concisely by the master variables introduced in Ref. [27]. Hence, the inclusion of the brane position as a dynamical variable provides us with the most concise configuration space.

Now let us illustrate the above arguments about D -gauge-invariant variables by using some equations. For simplicity we consider perturbations around a background with three-dimensional plane symmetry in five-dimensions. This is, following the notation in Ref. [32], we consider the metric

$$ds_5^2 = g_{MN} dx^M dx^N = (g_{MN}^{(0)} + \delta g_{MN}) dx^M dx^N \quad (33)$$

and the embedding relation

$$x^M = Z^M(y) = Z^{(0)M}(y) + \delta Z^M(y), \quad (34)$$

where the background is specified by the plane symmetric background metric

$$g_{MN}^{(0)} dx^M dx^N = \gamma_{ab} dx^a dx^b + r^2 \sum_{i=1}^3 (dx^i)^2 \quad (35)$$

and such background embedding functions $Z^{(0)M}(y)$ that $Z^{(0)a}$ depends only on y^0 and that $Z^{(0)i} = y^i$. Here, the two-dimensional metric γ_{ab} and the function r^2 are assumed to depend only on the two-dimensional coordinates $\{x^a\}$. As for perturbations, since in linear order the perturbations of the position of the hypersurface are decoupled from vector and tensor perturbations, we consider scalar perturbations:

$$\begin{aligned} \delta g_{MN} dx^M dx^N &= \int d^3 \mathbf{k} [h_{ab} Y dx^a dx^b + 2h_{(L)a} V_{(L)i} dx^a dx^i \\ &\quad + (h_{(LL)} T_{(LL)ij} + h_{(Y)} T_{(Y)ij}) dx^i dx^j], \\ \delta Z_M dx^M &= \int d^3 \mathbf{k} [z_a Y dx^a + z_{(L)} V_{(L)i} dx^i], \end{aligned} \quad (36)$$

where $Y = \exp(-i\mathbf{k} \cdot \mathbf{x})$, $V_{(L)i} = \partial_i Y$, $T_{(LL)ij} = 2\partial_i \partial_j Y + (2\mathbf{k}^2/3)\delta_{ij} Y$, and $T_{(Y)ij} = \delta_{ij} Y$, and all coefficients are supposed to depend only on the two-dimensional coordinates $\{x^a\}$ of the orbit space. Here, \mathbf{x} denotes coordinates $\{x^i\}$ of the three-dimensional plane ($i=1,2,3$), and \mathbf{k} represents the momentum $\{k_i\}$ along the plane. Hereafter, we omit \mathbf{k} in most cases. It is easy to see how the coefficients $\{h, z\}$ transform under the five-gauge transformation and to construct five-gauge-invariant variables. Therefore, we obtain the following five-gauge-invariant variables:

$$\phi_a = z_a + X_a, \quad (37)$$

and

$$\begin{aligned} F_{ab} &= h_{ab} - \nabla_a X_b - \nabla_b X_a, \\ F &= h_{(Y)} - X^a \partial_b r^2 + \frac{2k^2}{n} h_{(LL)}, \end{aligned} \quad (38)$$

where $X_a = h_{(L)a} - r^2 \partial_a (r^{-2} h_{(LL)})$ and ∇_a represents the covariant derivative compatible with the two-dimensional metric γ_{ab} . The variables (37) correspond to perturbations of the physical position of the hypersurface Σ , and its normal component $\phi_a n^{(0)a}$ appears in the doubly-gauge-invariant junction condition, where $n^{(0)a}$ is the background unit normal to the hypersurface. The variables (38) correspond to gravitational perturbations in the bulk and can be most concisely expressed in terms of the master variable Φ as

$$\begin{aligned} F_{ab} &= \frac{1}{r} \left(\nabla_a \nabla_b \Phi - \frac{2}{3} \nabla^2 \Phi \gamma_{ab} + \frac{1}{3l^2} \Phi \gamma_{ab} \right), \\ F &= \frac{r}{3} \left(\nabla^2 \Phi - \frac{2}{l^2} \Phi \right). \end{aligned} \quad (39)$$

The perturbed Einstein equation in the bulk is reduced to the following simple equation called the master equation:

$$r^2 \nabla^a [r^{-1} \nabla_a (r^{-1} \Phi)] - \mathbf{k}^2 r^{-2} \Phi = 0. \quad (40)$$

²In the literature this is sometimes called a generalized longitudinal gauge.

In the generalized Regge-Wheeler gauge where $h_{(L)a} = h_{(LL)} = 0$, the five-gauge-invariant variables are given by $\phi_a = z_a$, $F_{ab} = h_{ab}$, and $F = h_{(Y)}$. On the other hand, in the Gaussian normal gauge where $z_a = z_{(L)} = h_{(L)a} n^{(0)a} = h_{ab} n^{(0)b} = 0$, these are given by $\phi_a = X_a$ and Eq. (38). Note that in the Gaussian normal gauge ϕ_a is expressed in terms of a metric perturbation. Therefore, it is evident that ϕ_a cannot be set zero even in the Gaussian normal gauge since $X_a \neq 0$ in general. Actually, requiring $\phi_a = 0$ in the Gaussian normal gauge is equivalent to requiring $z_a = 0$ in the generalized Regge-Wheeler gauge, which is not possible in general.

Of course, it is always possible to take the Gaussian normal coordinate system. In this coordinate system, as illustrated above, the five-gauge-invariant variable ϕ_a is expressed in terms of metric perturbations. Hence, as was done in Ref. [42] for a static background by a gauge-dependent method, we need to extract the degrees of freedom of $\phi_a n^{(0)a}$ from the metric perturbations. Classically, this procedure should not be difficult since we can use the equations

of motion. However, quantum mechanically, we have to be careful when we use the equations of motion to reduce the action.

The next task in the future is to obtain the second-order action for perturbations by using the doubly covariant action obtained in this paper. After that, we need to obtain the corresponding reduced action by using a formalism to treat constrained systems, e.g., Dirac's method [43] or the Faddeev-Jackiw method [44]. As shown in Ref. [45], the perturbative behavior of the Wheeler-DeWitt wave function can be investigated by the usual quantum field theory in curved spacetime with a reduced action.

ACKNOWLEDGMENTS

The author would like to thank Werner Israel for his continuing encouragement and helpful discussions. This work was done during a stay at the Canadian Institute for Theoretical Astrophysics. The author is grateful to Lev Kofman for his warm hospitality. This work was supported by CITA and a NSERC operating research grant.

-
- [1] J. Polchinski, *String Theory I & II* (Cambridge University Press, Cambridge, England, 1998).
- [2] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429**, 263 (1998); Phys. Rev. D **59**, 086004 (1999).
- [3] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **436**, 257 (1998).
- [4] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).
- [5] K. S. Thorne, R. H. Price and D. A. Macdonald, *Black Holes: The Membrane Paradigm* (Yale University Press, New Haven, CT, 1986).
- [6] E. Poisson and W. Israel, Phys. Rev. D **41**, 1796 (1990).
- [7] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996).
- [8] E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Reading, MA, 1990).
- [9] A. Linde, *Particle Physics and Inflationary Cosmology* (Harwood Academic, Chur, Switzerland, 1990).
- [10] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
- [11] W. Israel, Nuovo Cimento Soc. Ital. Fis., B **44**, 1 (1966); **48**, 463 (1967).
- [12] V.A. Berezin, N.G. Kozimirov, V.A. Kuzumin, and I.I. Tkachev, Phys. Lett. B **212**, 415 (1988).
- [13] V.A. Berezin, Phys. Lett. B **241**, 194 (1990).
- [14] P. Hájíček, Commun. Math. Phys. **150**, 545 (1992).
- [15] P. Hájíček, B.S. Kay, and K.V. Kuchař, Phys. Rev. D **46**, 5439 (1992).
- [16] K. Nakamura, Y. Oshiro, and A. Tomimatsu, Phys. Rev. D **53**, 4356 (1996).
- [17] A.D. Dolgov and I.B. Khriplovich, Phys. Lett. B **400**, 12 (1997).
- [18] P. Hájíček and J. Bičák, Phys. Rev. D **56**, 4706 (1997).
- [19] G. Hayward and J. Louko, Phys. Rev. D **42**, 4032 (1990).
- [20] J.L. Friedman, J. Louko, and S.N. Winters-Hilt, Phys. Rev. D **56**, 7674 (1997).
- [21] V.D. Gladush, gr-qc/0001073.
- [22] S. Mukohyama, Class. Quantum Grav. **17**, 4777 (2000).
- [23] V.P. Frolov, M.A. Markov, and V.F. Mukhanov, Phys. Lett. B **216**, 272 (1989); Phys. Rev. D **41**, 383 (1990).
- [24] J.M. Cline, C. Grojean, and G. Servant, Phys. Rev. Lett. **83**, 4245 (1999).
- [25] E.E. Flanagan, S.H.H. Tye, and I. Wasserman, Phys. Rev. D **62**, 044039 (2000).
- [26] P. Binétruy, C. Deffayet, U. Ellwanger, and D. Langlois, Phys. Lett. B **477**, 285 (2000).
- [27] S. Mukohyama, Phys. Lett. B **473**, 241 (2000).
- [28] P. Kraus, J. High Energy Phys. **12**, 011 (1999).
- [29] D. Ida, J. High Energy Phys. **09**, 014 (2000).
- [30] S. Mukohyama, T. Shiromizu, and K. Maeda, Phys. Rev. D **62**, 024028 (2000); **63**, 029901(E) (2001).
- [31] S. Mukohyama, Phys. Rev. D **62**, 084015 (2000).
- [32] S. Mukohyama, Phys. Rev. D **64**, 064006 (2001).
- [33] H. Kodama, A. Ishibashi, and O. Seto, Phys. Rev. D **62**, 064022 (2000).
- [34] H. Kodama, hep-th/0012132.
- [35] R. Maartens, Phys. Rev. D **62**, 084023 (2000); C. Gordon and R. Maartens, *ibid.* **63**, 044022 (2001); R. Maartens, V. Sahni, and T.D. Saini, *ibid.* **63**, 063509 (2001).
- [36] D. Langlois, Phys. Rev. D **62**, 126012 (2000); Phys. Rev. Lett. **86**, 2212 (2001).
- [37] C. van de Bruck, M. Dorca, R. Brandenberger, and A. Lukas, Phys. Rev. D **62**, 123515 (2000); C. van de Bruck and M. Dorca, hep-th/0012073; M. Dorca and C. van de Bruck, Nucl. Phys. **B605**, 215 (2001).
- [38] K. Koyama and J. Soda, Phys. Rev. D **62**, 123502 (2000); and (to be published), hep-th/0108003.
- [39] D. Langlois, R. Maartens, and D. Wands, Phys. Lett. B **489**, 259 (2000).
- [40] D. Langlois, R. Maartens, M. Sasaki, and David Wands, Phys. Rev. D **63**, 084009 (2001).

- [41] H.A. Bridgman, K.A. Malik, and D. Wands, Phys. Rev. D (to be published), astro-ph/0107245.
- [42] J. Garriga and T. Tanaka, Phys. Rev. Lett. **84**, 2778 (2000).
- [43] P. Dirac, *Lectures on Quantum Mechanics* (Yeshiva Press, New York, 1964).
- [44] L. Faddeev and R. Jackiw, Phys. Rev. Lett. **60**, 1692 (1988); R. Jackiw, hep-th/9306075.
- [45] S. Mukohyama, Phys. Rev. D **56**, 7638 (1997).