Multi-black-hole sectors of AdS_3 gravity

Teresia Månsson* and Bo Sundborg†
Institute of Theoretical Physics, Box 6730, SE-113 85 Stockholm, Sweden
(Received 30 October 2000; revised manuscript received 20 April 2001; published 26 December 2001)

We construct and discuss solutions of $SO(1,2)\times SO(1,2)$ Chern-Simons theory which correspond to multiple BTZ black holes. These solutions typically have additional singularities, the simplest cases being special conical singularities with a 2π surplus angle. There are solutions with singularities inside a common outer horizon, and other solutions with naked conical singularities. Previously such singularities have been ruled out on physical grounds, because they do not obey the geodesic equation. We find however that the Chern-Simons gauge symmetry may be used to locate all such singularities to the horizons, where they necessarily follow geodesics. We are therefore led to conclude that these singular solutions correspond to physically sensible geometries. Boundary charges at infinity are only sensitive to the total mass and spin of the black holes, and not to the distribution among the black holes. We therefore argue that a holographic description in terms of a boundary conformal field theory should represent both single and multiple BTZ solutions with the same asymptotic charges. Then sectors with multiple black holes would contribute to the black hole entropy calculated from a boundary CFT.

DOI: 10.1103/PhysRevD.65.024025 PACS number(s): 04.60.Kz, 04.20.Dw, 04.20.Jb, 04.70.Bw

I. INTRODUCTION

Three-dimensional gravity has been a useful laboratory for exploring quantum gravity in a simplified setting. For a negative cosmological constant there are black hole solutions [1,2], and the Bekenstein-Hawking entropy of these Bañados-Teitelboim-Zanelli (BTZ) black holes have been attributed to boundary degrees of freedom at the horizon [3,4], at infinity [5] or at any intermediate timelike surfaces [6].

Strominger's asymptotic approach makes use of a particular property of asymptotically AdS solutions of three-dimensional gravity discovered by Brown and Henneaux [7]: the asymptotic isometries are represented canonically by a Virasoro algebra. The BTZ mass and spin determine the transformation properties under the conformal transformations (conformal weights). In conformal field theory an argument by Cardy [8] can be used to relate central charges of Virasoro algebras to the densities of states at high weights. Similarly the asymptotic density of states of quantum gravity is fixed by the central charge. It is found to agree with expectations from the BTZ horizon area. This elegant argument is independent of the precise conformal field theory representing quantum gravity, and gives few details about the theory. Carlip has combined it with his horizon approach, to open the way to an understanding of the universal nature of black hole entropy [9], and its relation to horizon area. The price is that the horizon is treated as an input, rather than as a consequence of the global geometry, and again that details of the field theory are lost.

The ability to compute black hole entropy quantum mechanically does not mean that it is fully explained. Even in the simple case of $2+1$ dimensions the gravitational backgrounds that contribute to the entropy are poorly understood. Ideally a correct count of boundary degrees of freedom at the horizon or at infinity should also tell us what bulk geometries are relevant, and how they are excited. They may also be represented differently in different quantum gravity theories. (In $2+1$ dimensions there are several inequivalent quantizations [10].) To start investigating what geometries may represent the entropy we choose to study one description, Chern-Simons theory [11,12]. In Chern-Simons theory the map to the boundary theory is well known and produces a Wess-Zumino-Witten (WZW) theory [13–15].

On the classical level one could ask which constant curvature metrics (solving the equations of motion) look asymptotically like BTZ black holes, and could be expected to be equally important as the standard BTZ solution for the black hole entropy. Bañados [16] (see also [17] and [18]) has given a simple analytic and general characterization of such solutions, but unfortunately the analytic expression of the solution does not give directly the geometric structure of the spacetime. In string theory approaches to black hole entropy, BPS solutions which can be separated into multi-source solutions play a prominent role [19]. This indicates that similar solutions may be of interest also in pure gravity. Indeed, we will find that asymptotically, Chern-Simons multi-source solutions typically are closer than most of the solutions in [16] and [17] to the standard BTZ solutions.

Chern-Simons multi-source solutions have been discussed as candidates for stationary multi-black-hole solutions by Coussaert and Henneaux [20]. Clement [21] found similar solutions in a metrical formulation and generalized them to dynamical solutions with moving sources. His main motivation for doing so was to correct the shortcoming also observed in [20], that the stationary solutions necessarily involve additional conical singularities which typically do not follow geodesics.[1] A non-geodesic behavior signals an un-

*Email address: teresia@physto.se
†Email address: bo@physto.se

[1]The presence of these singularities indicates that such solutions are different from the multi-black-hole solutions with multiple asymptotic regions that have been discussed by Brill [22,23] and the wormhole solutions by Bengtsson *et al.* [24,25].

wanted transport of energy-momentum between the singularity and spacetime.

From our perspective this state of affairs is quite puzzling. The multi-source solutions could be expected to be on equal footing with the BTZ solutions on the basis of their asymptotic behavior, and as Chern-Simons solutions they are no less regular. Hence calculations of black hole entropy based on Chern-Simons theory and its expression in terms of a boundary WZW model, which reproduce Bekenstein-Hawking's result, appear to include unphysical geometries. Is Chern-Simons theory, which has attracted so much attention as a model of quantum gravity, just not sensible as a theory of gravity?

Fortunately, there is a caveat in the above argument. We find that multi-source solutions have similar asymptotic behavior, but also that they represent physical (though singular) geometries. The crucial observation is that one should deal with gauge equivalence classes of solutions to Chern-Simons theory rather than with individual solutions. As we show in Sec. IV B 1, all ''unacceptable'' Coussaert-Henneaux solutions are gauge equivalent to perfectly acceptable solutions.[2] This is possible because the Chern-Simons gauge group is larger than the diffeomorphism group. For a discussion of how the gauge symmetry conspires with the presence of degenerate metrics in the Chern-Simons ''formulation'' see Matschull [26]. In practice, gauge transformations move the static conical singularities to the horizons, where they obey the geodesic equation (by infinite redshift) as already observed by Clement. If all Chern-Simons solutions have similar sensible representative geometries or not is left as an open question. The Coussaert-Henneaux solutions dealt with here certainly constitute an important subclass.

In Sec. II we give our Chern-Simons formulation of the BTZ black hole and in Sec. III we write this Chern-Simons solution in a more geometric way, and are led to a much more general solution, which includes the multi-black-hole solutions, some of which have already been discussed by Coussaert and Henneaux. We also discuss how gauge transformations act on all these these solutions. In Sec. IV we specialize to the case of two black holes (more precisely two excluded regions with closed timelike curves). We study the properties of this solution and the role of degenerate metrics. We solve the problem of the non-geodesic singularities of the Coussaert-Henneaux solutions by choosing a suitable gauge in Sec. IV B 1 and we end with conclusions in Sec. V.

II. THE BTZ BLACK HOLE

In $(2+1)$-dimensional gravity with a negative cosmological constant there exists a black hole solution to Einstein's equations, the BTZ black hole [1]. It can be viewed either as a metric approaching an AdS form asymptotically, or as a quotient of anti–de Sitter space [2]. The BTZ-metric can be written

$$ds^2=-N^2dt^2+N^{-2}dr^2+r^2(N^\phi dt+d\phi)^2 \tag{1}$$

where the lapse function N and the angular shift N^ϕ are

$$N^2=-M+\frac{r^2}{l^2}+\frac{J^2}{4r^2} \qquad M=\frac{r_+^2+r_-^2}{l^2} \tag{2}$$

$$N^\phi=-\frac{J}{2r^2} \qquad J=\frac{2r_+r_-}{l} \tag{3}$$

and $0<r<\infty$, $-\infty<t<\infty$, $0<\phi<2\pi$. M is the mass of the black hole and J is the angular momentum. Both these quantities can be expressed in terms of the values of r (r_+ and r_-) when the lapse function N vanishes. They correspond to the outer (r_+) and the inner (r_-) horizon of the black hole. For the horizons to exist we need $M>0$ and $|J|\leq Ml$. When r_+ coincides with r_-, we get extremal black holes, $|J|=Ml$. We will be concerned mainly with the non-extremal case.

The cosmological constant λ is related to the length scale l by the $\lambda=-1/l^2$. We choose units such that $l=1$. To facilitate the Chern-Simons formulation in Sec. II B we rewrite the metric differently the outer region $r>r_+$, the intermediate region $r_+>r>r_-$ and the inner region $r_->r>0$. Thus we make the following Rindler-like coordinate transformation in each region $r>r_+$, $r_+>r>r_-$ and $r_->r>0$:

$$\text{I:}\quad r^2=r_+^2\cosh^2\left(\rho-\alpha-\frac{\pi}{2}\right)-r_-^2\sinh^2\left(\rho-\alpha-\frac{\pi}{2}\right),$$

$$\alpha+\frac{\pi}{2}<\rho<\infty \tag{4}$$

$$\text{II:}\quad r^2=r_-^2\cos^2(\rho-\alpha)+r_+^2\sin^2(\rho-\alpha), \quad \alpha<\rho<\alpha+\frac{\pi}{2} \tag{5}$$

$$\text{III:}\quad r^2=r_-^2\cosh^2(\rho-\alpha)-r_+^2\sinh^2(\rho-\alpha), \quad 0<\rho<\alpha \tag{6}$$

$$\alpha=\operatorname{arctanh}\left(\frac{r_-}{r_+}\right). \tag{7}$$

The constant α is chosen in such a way that $r=0$ corresponds to $\rho=0$. In these coordinates we get a one to one correspondence between r and ρ. This will lead to the following metrics:

$$\text{I:} ds^2=-\sinh^2\left(\rho-\alpha-\frac{\pi}{2}\right)[r_+dt-r_-d\phi]^2+d\rho^2$$
$$+\cosh^2\left(\rho-\alpha-\frac{\pi}{2}\right)[r_-dt-r_+d\phi]^2 \tag{8}$$

[2] It may seem strange that some gauge potentials give meaningful metrics while others in the same class do not, but this property is in fact intrinsic to the Chern-Simons approach. The solutions to the equations of motion are pure gauge, and they may locally be transformed away, giving a completely degenerate metric, unless further conditions on the vector potentials are imposed.

$$\mathrm{II}: ds^2=\sin^2(\rho-\alpha)[r_-dt-r_+d\phi]^2-d\rho^2+\cos^2(\rho-\alpha)\times[r_+dt-r_-d\phi]^2$$

$$\mathrm{III}: ds^2=-\sinh^2(\rho-\alpha)[r_-dt-r_+d\phi]^2+d\rho^2+\cosh^2(\rho-\alpha)[r_+dt-r_-d\phi]^2.$$

If we look at the metric in the inner region III we find that our choice of α causes the coefficient of $d\phi^2$ to vanish precisely when $\rho=0$ and to become negative when $\rho<0$, i.e. we will get closed timelike curves (CTCs). Excluding the negative ρ region in fact removes all CTCs [2]. We also note that t is always a global Killing coordinate, timelike in I and spacelike in II and III. The xy plane is Euclidean in I, Lorentzian in II and Euclidean in III, implying that light cones are drastically tilted inside the black hole. The radial coordinate ρ is spacelike in I, timelike in II and spacelike in III.

Here it makes sense to pause and think about the split into three different coordinate regions. The point we want to make may seem trivial in the metric formulation, but it will reappear in the Chern-Simons formulation. Although the boundaries between the regions happen to coincide with the positions of the inner and outer horizons there is of course nothing special going on locally in these places. So why do we not simply continue our expressions from one side of the boundary to the other instead of changing analytic forms from region to region? The answer is that the analytic expressions of the metric (8) become degenerate at the boundaries of the regions, indicating that the coordinates become singular there. This coordinate singularity just means that the horizon is not covered by these particular coordinates. There exist other coordinates which also cover the horizon and part of the spacetime on the other side. As will become evident, our coordinates are still useful for finding generalized spacetimes with singularities which are physical in a very definite sense detailed in Sec. IV B 1.

A. Chern-Simons formulation of gravity

In the Chern-Simons formulation of three-dimensional gravity [11] isometries of the AdS background are gauged. For AdS the isometry group is $SO(1,2)\times SO(1,2)$ and we call the respective gauge fields of each factor $A=A^k\mathbf{J}_k$ and $\bar{A}=\bar{A}^k\mathbf{J}_k$. The $SO(1,2)$ generators $\mathbf{J}_k$ of a factor of the group are different from those of the other factor, but since they never appear multiplied together we shall not distinguish between them. The commutation rules within each factor are $[\mathbf{J}_k,\mathbf{J}_l]=\epsilon^m_{kl}\mathbf{J}_m$, with the convention $\epsilon^0_{12}=-\epsilon_{012}=-1$, and metric $\eta_{ab}=2\mathrm{Tr}(\mathbf{J}_a\mathbf{J}_b)$ of signature $(-1,1,1)$. The Chern-Simons three-form

$$\mathrm{Tr}\left\{A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right\} \tag{9}$$

and its counterpart for the other factor then serve as Lagrangian densities, which automatically yield a generally covariant action. The equations of motion

$$F=dA+A\wedge A=0 \quad \text{and} \quad F=d\bar{A}+\bar{A}\wedge\bar{A}=0 \tag{10}$$

are then actually equivalent with Einstein's equations, provided the identifications

$$g_{ij}=e^a_i e^b_j \eta_{ab} \tag{11}$$

$$e^a_i=\frac{1}{2}(A^a_i-\bar{A}^a_i) \tag{12}$$

$$\omega^a_i=\frac{1}{2}(A^a_i+\bar{A}^a_i) \tag{13}$$

of the metric, the dreibein and the spin connection are made, and the metric is non-degenerate. Solutions with metrics that are degenerate somewhere need special study. In the present paper we encounter cases where the degeneration corresponds to coordinate singularity or to a conical singularity. In some of the cases the degeneration can be directly associated to horizons, with coordinate singularities in the accompanying "Schwarzschild-like" coordinate systems. Such degenerations may be handled by attaching another coordinate patch with a boundary and gluing them together by the appropriate matching conditions. Then one may find a new coordinate system covering the boundary region, with a metric which is non-degenerate. Thus the degeneration is not a coordinate invariant concept (unless restrictions are imposed on the allowed coordinate transformations at a supposed boundary of spacetime).

B. Chern-Simons representation of the BTZ black hole

Now we want to write down the Chern-Simons fields corresponding to the metric in each region, and then verify that the field strength F vanishes even at the horizons. We need F to vanish everywhere in the interior of our space except at singularities for the solutions to represent a spacetime with constant negative curvature. A non-vanishing field strength at the horizons can only come from a discontinuity in the A field when we glue the different regions together (recall that $F=dA+A\wedge A$, and if A contains a step function the differential gives rise to a delta function). Since derivatives transverse to the boundary only appear in F for the longitudinal components A_t and A_ϕ, it is enough to ensure that these components are continuous.

Knowing the metric in the different regions, we may choose corresponding dreibeins and derive the corresponding spin connections from the equation of motions, $de^a+\omega^a_b\wedge e^b=0$ and $d\omega^a+(1/2)\epsilon^a_{bc}\omega^b\wedge\omega^c=-(1/2l^2)\epsilon^a_{bc}e^b\wedge e^c$. The result is unique up to local Lorentz transformations, and a simple choice is

$$\text{I:}\begin{cases} e=-\sinh\left(\rho-\alpha-\frac{\pi}{2}\right)[r_+dt-r_-d\phi]\mathbf{J}_0+\cosh\left(\rho-\alpha-\frac{\pi}{2}\right)[-r_-dt+r_+d\phi]\mathbf{J}_1+d\rho\mathbf{J}_2, \\ \omega=-\sinh\left(\rho-\alpha-\frac{\pi}{2}\right)[-r_-dt+r_+d\phi]\mathbf{J}_0+\cosh\left(\rho-\alpha-\frac{\pi}{2}\right)[r_+dt-r_-d\phi]\mathbf{J}_1, \end{cases}$$

$$\text{II:}\begin{cases} e=d\rho\mathbf{J}_0-\sin(\rho-\alpha)[r_-dt-r_+d\phi]\mathbf{J}_1+\cos(\rho-\alpha)[r_+dt-r_-d\phi]\mathbf{J}_2 \\ \omega=\sin(\rho-\alpha)[r_+dt-r_-d\phi]\mathbf{J}_1-\cos(\rho-\alpha)[r_-dt-r_+d\phi]\mathbf{J}_2 \end{cases}$$

$$\text{III:}\begin{cases} e=\sinh(\rho-\alpha)[-r_-dt+r_+d\phi]\mathbf{J}_0+d\rho\mathbf{J}_1+\cosh(\rho-\alpha)[r_+dt-r_-d\phi]\mathbf{J}_2 \\ \omega=\sinh(\rho-\alpha)[r_+dt-r_-d\phi]\mathbf{J}_0+\cosh(\rho-\alpha)[-r_-dt+r_+d\phi]\mathbf{J}_2. \end{cases} \tag{14}$$

These dreibeins and spin connections can be compared with those of Cangemi *et al.* [27], who use a different radial coordinate [the same as in the metric (1)]. Otherwise the differences are the choices of some of the signs and in the outer and inner regions the Lie algebra components are interchanged. Our choice of α means that closed timelike curves are excluded in the region $\rho>0$, and it corresponds to the boundary condition (at $\rho=0$) that the ϕ -component of the dreibein is lightlike. In effect it relates the tangential components of A and $\bar{A}$ at these boundaries.

From $A=\omega+e$ and $\bar{A}=\omega-e$ we get the Chern-Simons fields

$$\text{I:}A_\phi=A_t=(r_+-r_-)\left[\cosh\left(\rho-\alpha-\frac{\pi}{2}\right)\times\mathbf{J}_1-\sinh\left(\rho-\alpha-\frac{\pi}{2}\right)\mathbf{J}_0\right]$$

$$\text{II:}A_\phi=A_t=(r_+-r_-)[\cos(\rho-\alpha)\mathbf{J}_2+\sin(\rho-\alpha)\mathbf{J}_1]$$

$$\text{III:}A_\phi=A_t=(r_+-r_-)[\cosh(\rho-\alpha)\mathbf{J}_2+\sinh(\rho-\alpha)\mathbf{J}_0], \tag{15}$$

and

$$\text{I:}\bar{A}_\phi=-\bar{A}_t=(r_++r_-)\left[\cosh\left(\rho-\alpha-\frac{\pi}{2}\right)\times\mathbf{J}_1+\sinh\left(\rho-\alpha-\frac{\pi}{2}\right)\mathbf{J}_0\right]$$

$$\text{II:}\bar{A}_\phi=-\bar{A}_t=(r_++r_-)[\cos(\rho-\alpha)\mathbf{J}_2-\sin(\rho-\alpha)\mathbf{J}_1]$$

$$\text{III:}\bar{A}_\phi=-\bar{A}_t=(r_++r_-)[\cosh(\rho-\alpha)\mathbf{J}_2-\sinh(\rho-\alpha)\mathbf{J}_0]. \tag{16}$$

Here we see that our choice of dreibeins makes the longitudinal components of A and $\bar{A}$ continuous when passing between the regions (I→II and so on). The A_ρ and the $\bar{A}_\rho$ just become

$$\text{I:}A_\rho=\mathbf{J}_2,\quad \text{II:}A_\rho=\mathbf{J}_0,\quad \text{III:}A_\rho=\mathbf{J}_1$$

$$\text{I:}\bar{A}_\rho=-\mathbf{J}_2,\quad \text{II:}\bar{A}_\rho=-\mathbf{J}_0,\quad \text{III:}\bar{A}_\rho=-\mathbf{J}_1. \tag{17}$$

Thus the only discontinuous component is A_ρ, which in fact cannot contribute to the field strength since it only depends on ρ and the other components are continuous. $F_{\rho\rho}$ vanishes by antisymmetry and the off-diagonal terms $F_{\phi\rho}$ and $F_{t\rho}$ vanish by relating the discontinuities of $\partial_\rho A_\phi$ and $\partial_\rho A_t$ respectively with that of A_ρ.

There is an important distinction between how the boundaries between the regions are treated in the Chern-Simons formulation and in the metric formulation. A naive analytic continuation of the outer metric (8) to all ρ does not make sense as a solution to Einstein's equations, since it becomes degenerate at the horizon. In contrast, the Chern-Simons formulations seems to leave us with a choice. There is nothing wrong with the expressions for the vector potentials I, II or III, even if they are extended to all ρ. We can take those expressions as they are (giving us a problem in the gravitational interpretation) or we can match solutions and get a Chern-Simons version of the BTZ solution.

From a Chern-Simons perspective the matched discontinuous solutions and the smooth solutions are indistinguishable in the outer region, and they both make equally good sense in the interior. Only imposing boundary conditions in the interior or imposing special gauge conditions may pick out one solution as preferable to the other. Thus a sound gravitational interpretation of the solutions is only possible given special boundary conditions or gauge fixings of the vector potential. In generalizing the BTZ solution we will ensure that the boundaries of different regions are always matched in the same way as in this original BTZ solution.

To prepare for more general solutions let us write the BTZ solution in Cartesian coordinates, $\rho=\sqrt{x^2+y^2}$ and $\phi=\arctan(y/x)$. In the inner region we can write it as

$$A_x = \mathbf{g}\frac{-qy}{x^2+y^2} + \partial_x \rho \mathbf{J}_1 \tag{18}$$

$$A_y = \mathbf{g}\frac{qx}{x^2+y^2} + \partial_y \rho \mathbf{J}_1 \tag{19}$$

where

$$q = r_+ - r_- \tag{20}$$

and

$$\mathbf{g} = [\cosh(\rho-\alpha)\mathbf{J}_2 + \sinh(\rho-\alpha)\mathbf{J}_0]. \tag{21}$$

The second vector potential

$$\bar{A}_x = \bar{\mathbf{g}}\frac{-\bar{q}y}{x^2+y^2} - \partial_x \rho \mathbf{J}_1 \tag{22}$$

$$\bar{A}_y = \bar{\mathbf{g}}\frac{\bar{q}x}{x^2+y^2} - \partial_y \rho \mathbf{J}_1 \tag{23}$$

where

$$\bar{q} = r_+ + r_- \tag{24}$$

and

$$\bar{\mathbf{g}} = [\cosh(\rho-\alpha)\mathbf{J}_2 - \sinh(\rho-\alpha)\mathbf{J}_0]. \tag{25}$$

In Cartesian coordinates it looks as if $\rho=0$ denotes a single point in space. There is no *a priori* justification for this since we chose $\rho=0$ to be special by hand, and all other equations $\rho=\mathrm{const}$ denote topological circles. On the other hand, we excluded $\rho\leqslant 0$ on physical grounds, to get rid of closed timelike curves. Furthermore, calculating F in Cartesian coordinates we get a delta function at the origin which we may formally regard as a source, and in this context we can also regard $\rho=0$ as a single point.

C. Holonomies

In a gauge theory of flat connections, $F=\bar{F}=0$, gauge invariant observables are scarce. The fields are locally pure gauge $A=U^{-1}dU$, for U an element of $SO(1,2)$, and any non-trivial observable has to be associated with the boundaries of spacetime or be topological in nature. The simplest topological observables are holonomies (or Wilson loops) measuring the effect of parallel transport along a closed loop in spacetime. For flat connections the result can only be nonzero if the loop C_x (based at x) is non-contractible. Then the Wilson loop

$$W(C_x) = \mathcal{P}\exp\left(\oint_{C_x} A\right) = U^{-1}(x)U(x+C_x), \tag{26}$$

where $\mathcal{P}$ denotes path ordering of the exponential. As observed by Cangemi *et al.* [27] it is simplest in our case to take the closed curve C_x at constant radial coordinate, i.e. along a level curve of ρ. For two curves C_x and C_y which can be continuously deformed into each other, but are based at two different points x and y, the holonomies are conjugate, $W(C_y)=U(y)^{-1}U(x)W(C_x)U(x)^{-1}U(y)$. The *eigenvalues* of W for two curves which can be continuously deformed into each other are thus equal. These eigenvalues are determined by the parameters q, $\bar{q}$ and the eigenvalues of the $SO(1,2)$ Lie algebra elements $\mathbf{g}$ and $\bar{\mathbf{g}}$. It does not matter in which coordinate patch we follow the level curves, because we have ensured that the connections are flat also at the boundaries between the patches.

Since the gauge group is a product of two rank one groups it is enough to characterize the eigenvalues by the two traces $\mathrm{Tr}\, W(C)$ and $\mathrm{Tr}\, \bar{W}(C)$. For the BTZ solutions we obtain

$$\mathrm{Tr}\, W(C) = 2\cosh[\pi q], \quad \mathrm{Tr}\, \bar{W}(C) = 2\cosh[\pi \bar{q}], \tag{27}$$

for Wilson loops in the two-dimensional representation of $SO(1,2)$. Via Eq. (3) the holonomies are then related to the mass and spin of the black hole. In the complete classification of conjugacy classes of $SO(2,2)$ Lie algebra elements [2] one finds that holonomies corresponding formally to imaginary q or $\bar{q}$ may occur, and furthermore that the case of coinciding eigenvalues [when $\mathrm{Tr}\, W(C)=2$ or $\mathrm{Tr}\, \bar{W}(C)=2$] allows for non-trivial holonomy matrices (in addition to $W=1$ or $\bar{W}=1$). These cases can be dealt with in the Chern-Simons formulation by modifying the expressions for $\mathbf{g}$ and $\bar{\mathbf{g}}$.

III. MULTI-BLACK-HOLE SOLUTIONS

We will generalize the solution (19) to the case were we have arbitrary many singularites. We will use the same form of the solution as in the inner region, regarding the Lie algebra direction of A.[3] We may then try a solution

$$A = dh\mathbf{J}_1 + (f + dt)\mathbf{g} \tag{28}$$

$$\mathbf{g} = g_0(h)\mathbf{J}_0 + g_2(h)\mathbf{J}_2, \tag{29}$$

where h is a scalar function generalizing the radial coordinate ρ and f is a spatial one-form which is closed except at isolated sources

$$df = 2\pi\sum_{i=1}^{N} q_i \delta^2(\vec{x}-\vec{x}_i)dx\wedge dy. \tag{30}$$

[3] If we exchange $\mathbf{J}_1$ with $\mathbf{J}_2$ we also have to change signs in front of the $\mathbf{J}_0$ component, in order to preserve the commutation relations which govern the equations of motion. If we exchange $\mathbf{J}_1$ with $\mathbf{J}_0$ we have to change the last condition in Eq. (32) below. There will then be a minus sign in front of g_1, in effect exchanging trigonometric and hyperbolic functions. This is precisely the case in Eqs. (15), (16) and (17).

The q_i determine the strength of the sources (the masses and spins of black holes). By integrating Eq. (30) over a large disk D enclosing all sources we obtain

$$\oint_{\partial D} f=\int_D df=2\pi\sum_{i=1}^{N} q_i=2\pi Q. \tag{31}$$

If appropriate boundary conditions on f are assumed, $f\to Qd\phi$ as $\rho\to\infty$. Then we may regard $A_t=Q\mathbf{g}(h)\to A_\phi$ as a natural generalization of the relation $A_t-A_\phi=0$ satisfied by single BTZ black holes. This is consistent with the ansatz (28) after rescaling t.

The equations of motion $dA+A\wedge A=0$ are satisfied by the vector potential (28) outside the sources ($\vec{x}\neq\vec{x}_i$), provided

$$\frac{dg_0}{dh}=g_2,\quad \frac{dg_2}{dh}=g_0. \tag{32}$$

We recognize the equation for the hyperbolic functions entering the BTZ solution, but now their arguments have been generalized from ρ to h. By permuting the Lie algebra elements $\mathbf{J}_i$ in Eqs. (28), (29) one obtains solutions generalizing the BTZ solutions for all three regions, provided the signs in Eqs. (28), (29), (32) are changed accordingly. Matching of the regions works precisely as in the BTZ case. Note that the Lie algebra element $\mathbf{g}$ is spacelike, null or timelike depending on the sign of $\mathrm{Tr}\,\mathbf{g}^2$, and that its sign is necessarily constant all over spacetime for the present solutions. The one-form f/Q generalizes the angular one-form $d\phi$ in the BTZ case.

Although any choices of h and of f satisfying Eq. (30) are consistent with the equations of motion, we will concentrate on boundary conditions and combinations of A and $\bar{A}$ solutions that reduce to ordinary BTZ solutions both for asymptotically large h and close to the sources (regions of closed timelike curves). All the important new features of these generalized solutions are then associated with the fact that they are multi-centered, which in its turn implies that there will be critical points of h and f. Such critical points can give rise to degenerate metrics, a subject we shall return to in Sec. IV B.

The second gauge field $\bar{A}$ has analogous solutions in terms of $\bar{h}$, $\bar{g}_2(\bar{h})$ and $\bar{g}_0(\bar{h})$. In order to get solutions similar to the BTZ solutions we may choose $\bar{h}=-h$, $\bar{g}_2(\bar{h})=g_2(\bar{h})$ and $\bar{g}_0(\bar{h})=g_0(\bar{h})$, guided by Eqs. (15) and (16). In an inner region with g_2 and g_0 even and odd functions respectively, we obtain the vector potentials

$$\begin{aligned}A&=(f+Qdt)g_0(h)\mathbf{J}_0+dh\mathbf{J}_1+(f+Qdt)g_2(h)\mathbf{J}_2\\ \bar{A}&=-(\bar{f}-\bar{Q}dt)g_0(h)\mathbf{J}_0-dh\mathbf{J}_1+(\bar{f}-\bar{Q}dt)g_2(h)\mathbf{J}_2\end{aligned} \tag{33}$$

and the metric

$$\begin{aligned}ds^2=&-g_0^2(h(x,y))\{r_-dt-f_+(x,y)\}^2+g_2^2(h(x,y))\\ &\times\{r_+dt-f_-(x,y)\}^2+dh(x,y)^2,\end{aligned} \tag{34}$$

where

$$f_\pm=\frac{\bar{f}\pm f}{2},\quad r_\pm=\frac{\bar{Q}\pm Q}{2}. \tag{35}$$

The metric is easily compared with the BTZ metric (8) in the inner region (III). The function $h+\alpha$ has replaced the radial coordinate ρ, g_0 and g_2 represent the hyperbolic functions, and $r_\pm d\phi$ is replaced by $f_\pm$. The last change is the most significant one, since two different one-forms are needed to generalize $d\phi$. Only when f_+ and f_- are proportional do we get a direct multi-source generalization of $d\phi$. This happens when the ratio of the two charges at each source is constant. Irrespective of this we can make direct contact with the BTZ-solution very close to a charge, where the effect of the other charges is negligible, or at asymptotically large distances, where the sum of the charges dominate the solution.

In the general case (34) we can still define regions of type I, II and III, between which the solutions have to be matched, and different choices of the function h gives different regions (even their topologies may be different), but they are actually related by gauge transformations, as we proceed to discuss.

A. Gauge transformations

One can check that the gauge transformation

$$\delta A=d\,\delta h\mathbf{J}_1+[A,\delta h\mathbf{J}_1] \tag{36}$$

amounts to a change of h into $h+\delta h$ in the solution for region III, implying that solutions with different functions h are equivalent if only their boundary conditions are the same. Of course, an analogous statement is true for $\bar{A}$. We stress that the solutions are only equivalent in the Chern-Simons formulation of pure gravity. To see this we may study horizons.

The boundary between region I and II resembles a horizon and it is actually an event horizon for constant charge ratios of all sources, if it consists of a single connected component. This is because we can then use coordinates h and ψ with $d\psi=f$ to obtain the ordinary BTZ metric in the exterior region. The transformation to these coordinates works asymptotically and also in the whole exterior region provided f does not have a zero there. In fact, we will show later that the multi-black hole solutions have singularities at zeroes of f. These singularities may be inside or outside a physical event horizon depending on the choice of the function h. Such a difference could for instance be detected by the propagation of light rays in the background metric. Light rays are of course not included in a Chern-Simons description.

Even if there is little physics in the function h, the multi-black hole solution also depends on the forms f_+ and f_-, which in their turn depend on the positions and charges of the sources. As will be discussed in the next subsection, the charges may be directly measured by holonomies around the sources. The positions of the sources are trickier, and cannot be resolved by the holonomies. Other available observables are the asymptotic charges [28]. The general f_+ and f_- are asymptotic to the corresponding BTZ forms, and the issue is if the approach is fast enough to give finite asymptotic charges, but also slow enough to give non-zero values.

As an example we may compare a single source BTZ solution A_1 with a solution A_2 with sources separated by a small coordinate distance x_0 in the x direction. Then one finds

$$A_2 = A_1 + \delta_{12}A_1 = A_1 + x_0(q_2 - q_1)d\left(\frac{-y}{x^2+y^2}\right)\mathbf{g}. \quad (37)$$

Thus $\delta_{12}A_{1\phi}$ scales as $\mathbf{g}\rho^{-2}$ while $A_{1\phi}$ scales as $\mathbf{g}\rho^{-1}$ with ρ and the change is subleading. If the change $\delta_{12}A_{1\phi}$ can be written as an infinitesimal gauge transform $\delta_{\Lambda_{12}}A_1$ with a decreasing gauge parameter Λ then the separation of the two sources is truly a matter of gauge choice at infinity and it is not detectable by any asymptotic charges. (It will still be detectable by holonomies, corresponding to the fact that the gauge transformations are not defined everywhere, or do not belong to the identity component of the gauge group.) The problem in our case is that the asymptotic behavior of the BTZ solution implies that $\mathbf{g}$ has an exponential dependence on ρ. The same is true for Λ_{12}. Then the boundary values of the fields and the transformation parameters are not well defined. Fortunately, this problem may be circumvented by discussing the vector potentials

$$A' = e^{\rho \mathbf{J}_2} d e^{-\rho \mathbf{J}_2} + e^{\rho \mathbf{J}_2} A e^{-\rho \mathbf{J}_2}, \quad (38)$$

which locally are gauge transforms of A but satisfy different boundary conditions. In fact A'_{BTZ} is a constant and the A' of our generalized multi-source solutions approach constants at infinity. The A' do however give rise to metrics which are everywhere degenerate, and we just regard them as auxiliary solutions which help distinguishing asymptotic gauge transformations and global transformations generated by asymptotic charges. The parameters of global transformations on A' go to constants at infinity while true gauge transformations vanish asymptotically. The effect of both kinds of transformations on the fields A is simply obtained by the mapping inverse to Eq. (38). Conversely, by mapping to A' transformations on A may be classified as gauge transformations or global transformations (or as changing boundary conditions).

Returning to $\delta_{12}A_{1\phi}$, its image $\delta_{12}A'_{1\phi}$ under the map (38) vanishes at infinity, implying that asymptotic charges are left invariant by moving sources apart. In fact, $\delta_{12}A'_{1\phi} = \delta_{\Lambda_{12}}A_1$ for $\Lambda_{12} = -y\mathbf{g}'/(x^2+y^2)$ with a constant $\mathbf{g}'$. Since Λ_{12} diverges at the origin it does not give a globally well defined infinitesimal gauge transformation and there can still be a physical difference between the solutions. In conclusion, solutions with different numbers of sources are inequivalent because of different holonomies, while different positions of the sources may or may not be observable depending on the global properties and boundary conditions of the finite gauge transformations effecting the translations. The asymptotic charges are insensitive to these details, so they may be thought of as generating transformations common to several different sectors labeled by the numbers of sources, and possibly by their positions.

B. Multi-black-hole holonomies

We now wish to calculate holonomies

$$\mathrm{Tr}\left(\mathcal{P}\exp\int_C A\right). \quad (39)$$

We first calculate the ordinary integral over a closed loop,

$$\int_C A = \int\left(\left(f_x\mathbf{g} + \frac{\partial h(x,y)}{\partial x}\mathbf{J}_1\right)dx + \left(f_y\mathbf{g} + \frac{\partial h(x,y)}{\partial y}\mathbf{J}_1\right)dy\right), \quad (40)$$

here written out for an inner-type region. If the function $h(x,y)$ is chosen in such a way that there are closed level curves of $h(x,y)$ the term dh in the integral is zero, and furthermore $\mathbf{g}$ is constant. Then the integral depends on which charges q_i are enclosed by the level curve,

$$\int_C A = \mathbf{g}\int_C f = 2\pi\mathbf{g}\sum_{i\in I_C} q_i = 2\pi\mathbf{g}q_C, \quad (41)$$

where I_C denotes the set of enclosed sources and q_C the enclosed charge. Since the eigenvalues of the traceless real matrix $\mathbf{g}$ are necessarily both real or both imaginary and add up to zero, we may write

$$\mathrm{Tr}\left(\mathcal{P}\exp\int_C A\right) = e^{2\pi q_C\lambda} + e^{-2\pi q_C\lambda} = 2\cosh(2\pi q_C\lambda) \quad (42)$$

where λ is one of the eigenvalues of $\mathbf{g}$, and independent of h. The matrix corresponding to the $\bar{A}$ must also have either both imaginary or both real eigenvalues which we call $\bar{\lambda}$ and $-\bar{\lambda}$. So in general we get three different holonomy types depending on the eigenvalues λ and $\bar{\lambda}$: either one is real and one imaginary, both are real or both are imaginary. When we just have one singularity it is known that these types will correspond to different quotients of anti–de Sitter space. Bañados *et al.* [2] have shown how different spaces are obtained from anti–de Sitter by modding out subgroups of $SO(2,2)$, and that BTZ black holes belong to one of these classes of spaces. They also find three different types of spaces. The correspondence between their eigenvalues λ' and our eigenvalues is $\lambda'_1 = q\lambda - \overline{q\lambda}$ and $\lambda'_2 = q\lambda + \overline{q\lambda}$. In our language the generic BTZ black hole corresponds to the case with two real eigenvalues. When both are imaginary we generally get conical singularities, except in the case $q\lambda = \overline{q\lambda} = i/2$, which curiously corresponds to AdS space.[4] In fact, we may also find ‘‘multi-AdS solutions’’ with several of these AdS charges. They may possibly serve as ground states

[4] $M = -1$ and $J = 0$ are obtained from Eqs. (20), (24) and (3) and the metric (1) then represents AdS.

of multi-black-hole sectors. Note that the holonomy around a single AdS charge is almost trivial, and around two it is entirely trivial.

Notice that we have not mixed holonomy type for the different singularities. It would be interesting to find solutions where the sources give rise to different types of holonomies.

IV. TWO SOURCES

We will study the solutions for the case with two sources in more detail. After verifying that the solutions approach the single-source solution asymptotically and for vanishing separation of the charges, we will continue with a generalization to several sources of the procedure to exclude CTCs, and we will also discuss how the multi-source solutions generically contain additional (mild) singularities.

Solutions with sources at $x=x_1=x_0$ and $x=x_2=-x_0$ can be written

$$A_t=[(r_{1+}-r_{1-})+(r_{2+}-r_{2-})]\mathbf{g} \tag{43}$$

$$A_x=(f_{1x}+f_{2x})\mathbf{g}+\frac{\partial h(x,y)}{\partial x}\mathbf{J}_1 \tag{44}$$

$$A_y=(f_{1y}+f_{2y})\mathbf{g}+\frac{\partial h(x,y)}{\partial y}\mathbf{J}_1 \tag{45}$$

where

$$f_{1x}=q_1\frac{-y}{(x-x_0)^2+y^2},\quad f_{1y}=q_1\frac{x-x_0}{(x-x_0)^2+y^2} \tag{46}$$

$$f_{2x}=q_2\frac{-y}{(x+x_0)^2+y^2},\quad f_{2y}=q_2\frac{x+x_0}{(x+x_0)^2+y^2}. \tag{47}$$

The conjugate field $\overline{A}$,

$$\overline{A}_x=(\overline{f}_{1x}+\overline{f}_{2x})\overline{\mathbf{g}}+\frac{\partial\overline{h}(x,y)}{\partial x}\mathbf{J}_1 \tag{48}$$

$$\overline{A}_y=(\overline{f}_{1y}+\overline{f}_{2y})\overline{\mathbf{g}}+\frac{\partial\overline{h}(x,y)}{\partial y}\mathbf{J}_1 \tag{49}$$

where

$$\overline{\mathbf{g}}=\overline{g}_2(\overline{h}(x,y))\mathbf{J}_2+\overline{g}_0(\overline{h}(x,y))\mathbf{J}_0. \tag{50}$$

In the BTZ-like inner region with $\overline{h}=-h$, $\overline{g}_2(\overline{h})=g_2(\overline{h})$, $\overline{g}_0(\overline{h})=g_0(\overline{h})$ and g_2 and g_0 even and odd functions respectively, we find the metric

$$\begin{aligned}ds^2=&-g_0^2(h(x,y))\{(r_{1-}+r_{2-})dt-(r_{1+}f_{1x}+r_{2+}f_{2x})\\&\times dx-(r_{1+}f_{1y}+r_{2+}f_{2y})dy\}^2+g_2^2(h(x,y))\\&\times\{(r_{1+}+r_{2+})dt-(r_{1-}f_{1x}+r_{2-}f_{2x})dx\\&-(r_{1-}f_{1y}+r_{2-}f_{2y})dy\}^2\\&+\left\{\frac{\partial h(x,y)}{\partial x}dx+\frac{\partial h(x,y)}{\partial x}dy\right\}^2.\end{aligned} \tag{51}$$

So far the function h has been left unspecified. If, for instance, we choose $h(x,y)=\sqrt{\rho_1\rho_2}-\alpha$ in terms of the radial coordinates ρ_1 and ρ_2 centered on each of the two sources and a function α approaching a constant (7) at infinity and in the limit $x_0\to 0$, we can ensure that the BTZ solution is approached both at infinity and as $x_0\to 0$. To verify this, start by looking at the metric in the outer region

$$\begin{aligned}ds^2=&-\sinh^2(\sqrt{\rho_1\rho_2}-\alpha)((r_{1+}+r_{2+})\\&\times dt-[(r_{1-}f_{1x}+r_{2-}f_{2x})dx+(r_{1-}f_{1y}+r_{2-}f_{2y})dy])^2\\&+\cosh^2(\sqrt{\rho_1\rho_2}-\alpha)((r_{1-}+r_{2-})\\&\times dt-[(r_{1+}f_{1x}+r_{2+}f_{2x})dx+(r_{1+}f_{1y}+r_{2+}f_{2y})dy])^2\\&+\left(\frac{\partial h(x,y)}{\partial x}dx+\frac{\partial h(x,y)}{\partial x}dy\right)^2,\end{aligned} \tag{52}$$

to see how it behaves asymptotically at infinity. In terms of polar coordinates (ρ,ϕ) centered around $(x,y)=(x_1,0)$ (implying $\rho=\rho_1$)

$$\begin{aligned}f_{1x}&=\frac{-y}{\rho^2}=\frac{-\rho\sin\phi}{\rho^2},\quad f_{1y}=\frac{x-x_0}{\rho^2}=\frac{\rho\cos\phi}{\rho^2}\\f_{2x}&=\frac{-y}{\rho_2^2}=\frac{-\rho\sin\phi}{\rho_2^2},\quad f_{2y}=\frac{x+x_0}{\rho_2^2}=\frac{\rho\cos\phi+2x_0}{\rho_2^2}\end{aligned} \tag{53}$$

the metric takes the form

$$\begin{aligned}ds^2_{\text{outer}}=&-\sinh^2(\sqrt{\rho\rho_2}-\alpha)\\&\times\left\{r_+dt-\left(\frac{r_{2-}(\rho^2+2x_0\rho\cos\phi)+r_{1-}\rho_2^2}{\rho_2^2}\right.\right.\\&\left.\left.\times d\phi+\frac{2r_{2-}x_0\sin\phi}{\rho_2^2}d\rho\right)\right\}^2+\cosh^2(\sqrt{\rho\rho_2}-\alpha)\\&\times\left\{r_-dt-\left(\frac{r_{2+}(\rho^2+2x_0\rho\cos\phi)+r_{1+}\rho_2^2}{\rho_2^2}\right.\right.\\&\left.\left.\times d\phi+\frac{2r_{2+}x_0\sin\phi}{\rho_2^2}d\rho\right)\right\}^2\\&+\left\{\frac{\partial h(\phi,\rho)}{\partial\phi}d\phi+\frac{\partial h(\phi,\rho)}{\partial\rho}d\rho\right\}^2.\end{aligned} \tag{54}$$

We see that the metric is asymptotic to the BTZ solution with $r_+=r_{1+}+r_{2+}$ and $r_-=r_{1-}+r_{2-}$ when $\rho\to\infty$ or $x_0\to 0$.

A. Exclusion of closed timelike curves

In the BTZ solution (8) there are closed timelike curves for $\rho<0$, and we expect similar pathologies in the multi-black-hole solutions inside the black holes. It is natural to cut off the range of the coordinates precisely where CTCs are encountered. Here we show how this can be done in the case of two sources. The same procedure can be used for any number of sources. The resulting spacetimes then have singularities in the causal structure if they are continued ''inside'' the sources.

Just as for the BTZ case (8) we need the vector field ∂_ϕ for some periodic coordinate ϕ to become lightlike at each source in order to exclude regions containing closed timelike curves. Coordinates which are periodic around curves enclosing only single sources are readily found. We may use the angle between the line from the source to a point and the positive x direction, or we may use df_+ and df_- to measure angular differences. Close to the sources these measures of angle all agree up to proportionality constants.

To localize the causal singularities to the positions of the sources it is then enough to choose the function α appropriately. In order to encounter closed timelike curves we have to go to the inner region.

First study the metric in the inner region. It is obtained from the outer metric (54) by exchanging r_+ with r_-:

$$\begin{aligned}ds^2_{\rm inner}=&-\sinh^2(\sqrt{\rho\rho_2}-\alpha)\left\{r_-dt-\left(\frac{r_{2+}(\rho^2+2x_0\rho\cos\phi)+r_{1+}\rho_2^2}{\rho_2^2}\times d\phi+\frac{2r_{2+}x_0\sin\phi}{\rho_2^2}d\rho\right)\right\}^2+\cosh^2(\sqrt{\rho\rho_2}-\alpha)\\&\times\left\{r_+dt-\left(\frac{r_{2-}(\rho^2+2x_0\rho\cos\phi)+r_{1-}\rho_2^2}{\rho_2^2}\times d\phi+\frac{2r_{2-}x_0\sin\phi}{\rho_2^2}d\rho\right)\right\}^2\\&+\left\{\frac{\partial h(\phi,\rho)}{\partial\phi}d\phi+\frac{\partial h(\phi,\rho)}{\partial\rho}d\rho\right\}^2.\end{aligned}\tag{55}$$

Now take a look at the $g_{\phi\phi}$ component,

$$\begin{aligned}g_{\phi\phi}=&-\left(\frac{r_{2+}(\rho^2+2x_0\rho\cos\phi)+r_{1+}\rho_2^2}{\rho_2^2}\right)^2\sinh^2(\sqrt{\rho\rho_2}-\alpha)\\&+\left(\frac{r_{2-}(\rho^2+2x_0\rho\cos\phi)+r_{1-}\rho_2^2}{\rho_2^2}\right)^2\cosh^2(\sqrt{\rho\rho_2}-\alpha)\\&+\frac{x_0^4\rho\sin^2 2\phi}{\rho_2^3}+\left(\frac{\partial\alpha}{\partial\phi}\right)^2+2\frac{x_0^2\sqrt{\rho}\sin 2\phi}{\rho_2^{3/2}}\left(\frac{\partial\alpha}{\partial\phi}\right),\end{aligned}\tag{56}$$

when $\rho=0$. We must choose α in order to make the vector field ∂_ϕ lightlike at $x=x_0$. For α with $\partial_\phi\alpha=0$ when $\rho=0$, the condition that ∂_ϕ becomes lightlike becomes

$$g_{\phi\phi}=-(r_{1+})^2\sinh^2\alpha+(r_{1-})^2\cosh^2\alpha=0$$
$$\Rightarrow\alpha=\text{arctanh}\left(\frac{r_{1-}}{r_{1+}}\right).\tag{57}$$

In the same way we can change to polar coordinates centered around $x=-x_0$ which instead would lead us to the condition

$$g_{\phi\phi}=-(r_{2+})^2\sinh^2\alpha+(r_{2-})^2\cosh^2\alpha=0$$
$$\Rightarrow\alpha=\text{arctanh}\left(\frac{r_{2-}}{r_{2+}}\right).\tag{58}$$

In order to have both these conditions satisfied α can only be a constant in the case $r_{1-}/r_{1+}=r_{2-}/r_{2+}$. Still, there are many ways of choosing an $\alpha(\rho,\phi)$ that does not affect the singularities or the asymptotics of the solutions. We may choose α to be a constant at infinity, for instance

$$\alpha=\text{arctanh}\left(\frac{r_{1-}\rho_2+r_{2-}\rho_1}{r_{1+}\rho_2+r_{2+}\rho_1}\right).\tag{59}$$

We see that in the case $r_{1-}/r_{1+}=r_{2-}/r_{2+}$ this α will reduce to a constant. This will also be the case when $x_0=0$, i.e. when the singularities are in the same point. The requirement $\partial_\phi\alpha=0$ when $\rho=0$ is also easily seen to be fulfilled.

To make the analogy with the BTZ case complete the different regions we had can be generalized to

$$\text{I}: 0<\rho<\alpha\Rightarrow 0<\sqrt{\rho_1\rho_2}<\alpha$$
$$\text{II}: \alpha<\rho<\alpha+\frac{\pi}{2}\Rightarrow\alpha<\sqrt{\rho_1\rho_2}<\alpha+\frac{\pi}{2}$$
$$\text{III}: \alpha+\frac{\pi}{2}<\rho\Rightarrow\alpha+\frac{\pi}{2}<\sqrt{\rho_1\rho_2}.\tag{60}$$

In Fig. 1 we have plotted the ''horizons'' when we have fixed r_+ and r_- but varying distances x_0 between the singularities. Although the equations determining the boundaries of the regions are similar to the single-BTZ case we cannot be certain that we are dealing with true horizons, unless we trace light rays through the new geometries. This explains the quotation marks.

B. Singularities

In this subsection we discuss the nature of the spacetime singularities we necessarily encounter by interpreting gravitationally the multi-source generalizations of the Chern-Simons solutions giving rise to BTZ black hole metrics. Our goal is to find gauge choices in Chern-Simons theory which yield physically sensible geometries. Thus the metrics we consider should be thought of as composite fields in Chern-Simons theory, and the discussion of gauge equivalence is entirely in the Chern-Simons framework. For a clear discus-

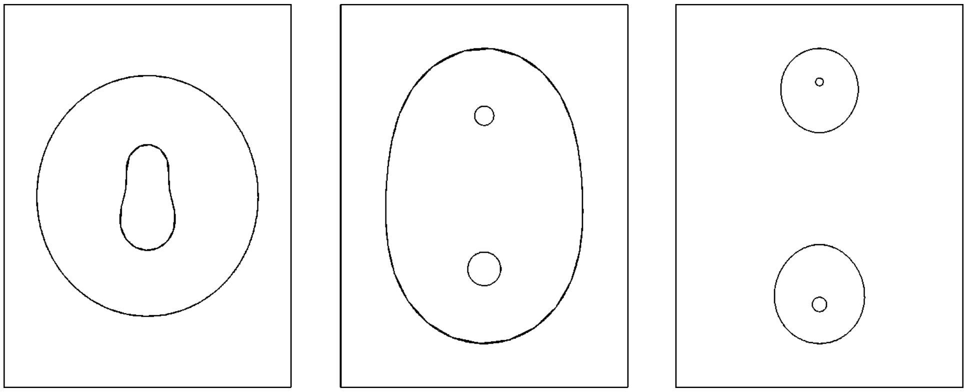

FIG. 1. The inner and outer "horizons" in the xy plane at fixed t for different x_0.

sion of the distinction between gauge equivalence in Chern-Simons theory and in Einstein gravity see Matschull [26].

The metric (34) may locally be written

$$ds^2 = -g_0^2 dT^2 + g_2^2 d\Phi^2 + dh^2, \tag{61}$$

with $dT = r_- dt - f_+$ and $d\Phi = r_+ dt - f_-$, since f_+ and f_- are closed forms. This metric degenerates where g_0 or g_2 vanishes, where one of the functions $T(t,x,y)$, $\Phi(t,x,y)$ or $h(t,x,y)$ has critical point, and where dT, $d\Phi$ and dh are linearly dependent. The coordinate singularities at the BTZ horizons and their multi-black-hole generalizations belong to the first case, but our solutions also display the other types of degeneracies, and we now proceed to investigate their interpretation.

In the case when one of the functions T, Φ or h has a critical point, one may ignore the effects of the functions g_0 or g_2 locally, since they may be absorbed into redefinitions of T, Φ or h only in exceptional cases at the expense of changing the nature of the critical point (but see the following subsection to appreciate the importance of these exceptions). Then the singularity is precisely of the kind discussed by Horowitz [29] for zero cosmological constant. The simplest such singularity occurs between two equal charges separated by some distance.

To see what happens we study the equal charge solutions close to the origin. There $f_+ = f_- = 0$ because the contributions from the two charges cancel by symmetry. The metric (61) then degenerates at the origin at all times, because dT and $d\Phi$ both become parallel to the Killing direction dt. Furthermore, h, which approaches infinity at infinity and assumes local minima at the positions of the charges, has to have a saddle point. Due to gauge invariance (36) the position of the saddle point may be chosen to be at the origin, making the metric on this line (in spacetime) even more degenerate, of rank one. Generically we instead expect degenerations to rank-two metrics on two-dimensional surfaces [29,30]. In fact, we have found that the map $(t,x,y) \to (T,\Phi,h)$ has three singular fold surfaces joined pairwise at three cusp lines if the saddle point of h is displaced slightly. The geometries of such complicated singularities deserve a special study, but for our purposes it is enough to find the simplest singularities in a gauge equivalence class.

Returning to the case of coinciding saddles we proceed to determine the geometry close to the saddles. There we have approximately

$$h = ax^2 - by^2$$

$$r_- f_+ = r_+ f_- = c\,d(xy). \tag{62}$$

By rescaling coordinates and h we find a spatial line element

$$ds^2 = d(xy)^2 + \frac{1}{4} d(x^2 - y^2)^2 = (x^2 + y^2)(dx^2 + dy^2). \tag{63}$$

The area A_O and circumference C_O of circles around the origin are then related by $C_O^2 = 8\pi A_O$ in contrast to the Euclidean relation $C^2 = 4\pi A$. Since the metric is manifestly flat the difference can only be due to a conical singularity at the origin, and we conclude that there is a negative deficit angle of 2π.

We have argued that simple conical singularities with a surplus angle of 2π appear in the geometries with two equal sources provided the gauge is chosen so that saddles of h coincide with zeros of f_+ and f_-. For n sources h typically has $n-1$ saddles since it is chosen to have n local minima at the sources and a maximum (infinity) at infinity. Similarly f_+ and f_- typically have $n-1$ zeros, because of the n sources and the behavior at infinity. If f_+ and f_- are proportional their zeros coincide, and h may be chosen to have saddles at the same points. Fixing the behavior of h appropriately close to its saddles the local calculation is then the same as between two sources, and we conclude that there are $n-1$ conical singularities. Physically the proportionality of f_+ and f_- means that the sources all have the same ratio J/M of spin and mass. Other source distributions generally lead to more complicated singularities in the geometry. Some of these may be removable like the coordinate singularites of the BTZ geometry, but some are likely to be required by global arguments, like the conical singularities we have just discussed.

1. Geodesic singularities

The stationary conical singularities discussed above have been found before by Clement [21] and by Coussaert and

Henneaux [20]. These authors have also remarked that such singularities do not follow geodesics. This is quite disturbing for the commonly used hypothesis that Chern-Simons theory should be relevant to the counting of black hole states. Already at the classical level would Chern-Simons theory give rise to geometries which seem to leak energy and momentum.

Fortunately, Chern-Simons theory itself contains the answer to the problem. By asking under what precise conditions the singularities are non-geodesic we may find an exception: when the singularity is located at a horizon. This case was already mentioned by Clement, but not in the Chern-Simons context where it becomes truly important. While the set of geometries with singularities fixed to horizons may seem like an exceptional set of measure zero, in Chern-Simons theory they are not exceptional. In fact, large class of solutions (and all those considered by Coussaert and Henneaux) may be written in a gauge such that the singularities are located at the horizon and thus follow geodesics. We now proceed to give some details of this argument.

We need to evaluate the Christoffel symbols Γ^x_{tt} and Γ^y_{tt} which vanish precisely where there are static geodesics. If they can be made to vanish at the conical singularities the puzzle of the unphysical geometries is solved. We study the Coussaert-Henneaux solutions, which are essentially ordinary BTZ solutions, but with the mass and angular momentum distributed in the same proportions on several sources. In our language this means that

$$r_- f_+ = r_+ f_- = r_- r_+ f, \tag{64}$$

where the single form f encodes the source distribution. As has been pointed out several times above this assumption simplifies the interpretation of the solutions considerably. Now

$$\Gamma^x_{tt} = \pm (r_+^2 - r_-^2) \frac{f_x g_i(h) g_i{}'(h)}{f_y \partial_x h - f_x \partial_y h}, \tag{65}$$

where the sign (and the label i) depends on the region. In general this expression and the one for Γ^y_{tt} diverge at a common zero of f and critical point of h, but if g or g' vanishes at the same point the whole expression instead goes to zero. This is what happens if the conical singularity is located at a horizon.

It only remains to argue that the singularities can be moved to a horizon. Indeed, in region III a infinitesimal shift of h is equivalent to an infinitesimal gauge transformation (36), and similar relations exist in the other regions. Assuming that these transformations can be integrated, we conclude that changes of function h are gauge transformations. By adjusting h we can then make g or g' vanish at a conical singularity, i.e. a gauge transformation may take the singularity to a horizon, where it follows a (null) geodesic simply by being stationary.

V. CONCLUSIONS

We have constructed and investigated solutions to three-dimensional AdS gravity which generalize the BTZ solution. While the ordinary BTZ black hole can be viewed as a single source solution in the Chern-Simons formulation, we have constructed multi-source solutions. These solutions give rise to a kind of multi-black-hole solutions, which however also display other singularities. In the simplest cases the additional singularities are fixed conical singularities, but more complicated cases also occur. Einstein's equations break down at these singularities, so they represent geometries which are not allowed in pure Einsteinian gravity. On the other hand, they occur very naturally in the Chern-Simons framework, which is natural for quantization, so we believe that these multi-black-hole solutions should be included in a full Chern-Simons treatment of BTZ black hole entropy.

We have also shown that a large class of these multi-black-hole solutions allow a gauge choice which ensures that the singularities in the corresponding geometries follow geodesics. Geometrically the solutions then precisely encode the BTZ solution outside a number of horizons. These horizons are however all connected with each other, since the conical singularities which join them cannot appear outside the horizons without violating the geodesic equation. The union of all these horizons appears to the outside observer as a single horizon. Only at the horizon (and inside) is the difference to the single black hole solution noticable. In this picture of a single horizon, special light-like geodesics on the horizon are identified pairwise, since they in fact represent the same conical singularity, only approached from two different directions (two different ridges on the saddle point of the function h).

Although we have not attempted in this paper to find the quantum states corresponding to the multi-black-hole solutions, we have provided evidence that such states should be included in the black hole spectrum. Namely, the asymptotics at infinity of the classical solutions approach the single-BTZ solutions so rapidly that the difference cannot be detected by any asymptotic charges. Only non-asymptotic observables like the holonomies distinguish between the solutions. It then seems quite unnatural to exclude the sectors with multiple sources, in particular since the sources may be hidden inside the horizon. Presumably, the additional sectors of the boundary conformal field theory that are required to represent multi-black-hole solutions can also be understood by purely two-dimensional considerations, for instance by the requirement of modular invariance.

ACKNOWLEDGMENTS

We would like to thank Sören Holst and Max Karlovini for useful discussions. It is a pleasure to also thank Marc Henneaux for a conversation about Ref. [20] and Ingemar Bengtsson for one on papers [24,25]. The work of B.S. was financed by the Swedish Science Research Council.

[1] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992).
[2] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993).
[3] S. Carlip, Phys. Rev. D **51**, 632 (1995).
[4] M. Banados and A. Gomberoff, Phys. Rev. D **55**, 6162 (1997).
[5] A. Strominger, J. High Energy Phys. **02**, 009 (1998).
[6] M. Banados, T. Brotz, and M.E. Ortiz, Nucl. Phys. **B545**, 340 (1999).
[7] J.D. Brown and M. Henneaux, Commun. Math. Phys. **104**, 207 (1986).
[8] J.L. Cardy, Nucl. Phys. **B270**, 186 (1986).
[9] S. Carlip, Phys. Rev. Lett. **82**, 2828 (1999).
[10] S. Carlip, gr-qc/9305020.
[11] A. Achucarro and P.K. Townsend, Phys. Lett. B **180**, 89 (1986).
[12] E. Witten, Nucl. Phys. **B311**, 46 (1988).
[13] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[14] G. Moore and N. Seiberg, Phys. Lett. B **220**, 422 (1989).
[15] S. Elitzur, G. Moore, A. Schwimmer, and N. Seiberg, Nucl. Phys. **B326**, 108 (1989).
[16] M. Banados, ‘‘Three-dimensional quantum geometry and black holes,’’ hep-th/9901148.
[17] K. Ezawa, Int. J. Mod. Phys. A **10**, 4139 (1995).
[18] J. Navarro-Salas and P. Navarro, Phys. Lett. B **439**, 262 (1998).
[19] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996).
[20] O. Coussaert and M. Henneaux, in *Geometry of Constrained Dynamical Systems*, edited by John M. Charap (Cambridge University Press, Cambridge, England, 1995), pp. 150–157.
[21] G. Clement, Phys. Rev. D **50**, 7119 (1994).
[22] D. Brill, in *Mathematical and Quantum Aspects of Relativity and Cosmology: Proceedings of the Second Samos Meeting on Cosmology, Geometry and Relativity*, edited by S. Cotsakis and G. W. Gibbons (Springer, Berlin, 2000), gr-qc/9904083.
[23] D. Brill, Ann. Phys. (Leipzig) **9**, 217 (2000).
[24] S. Åminneborg, I. Bengtsson, D. Brill, S. Holst, and P. Peldan, Class. Quantum Grav. **15**, 627 (1998).
[25] S. Åminneborg, I. Bengtsson, and S. Holst, Class. Quantum Grav. **16**, 363 (1999).
[26] H. Matschull, Class. Quantum Grav. **16**, 2599 (1999).
[27] D. Cangemi, M. Leblanc, and R.B. Mann, Phys. Rev. D **48**, 3606 (1993).
[28] M. Banados, Phys. Rev. D **52**, 5816 (1996).
[29] G.T. Horowitz, Class. Quantum Grav. **8**, 587 (1991).
[30] C. Gibson, *Singular Points of Smooth Mappings* (Pitman, London, 1979), p. 57.