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Bounds on the cosmological abundance of primordial black holes from diffuse sky brightness: Single mass spectra

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We constrain the mass abundance of unclustered primordial black holes (PBHs), formed with a simple mass distribution and subject to the Hawking evaporation and particle absorption from the environment. Since the radiative flux is proportional to the numerical density, an upper bound is obtained by comparing the calculated and observed diffuse background values (similarly to the Olbers paradox in which point sources are considered) for finite bandwidths. For a significative range of formation redshifts the bounds are better than several values obtained by other arguments $\Omega_{pbh} \leqslant 10^{-10}$; and they apply to PBHs which are evaporating today.

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I. INTRODUCTION

A variety of observations make a compelling case for the existence of dark matter in unknown form(s). Among the discussed possibilities we may mention axions, weakly interacting massive particles (WIMPS), cosmic strings, brown dwarfs and primordial black holes (PBHs), see [1]. It has long been recognized that the abundance of evaporating PBHs [2] may be constrained by a variety of methods. For example, Carr and MacGibbon (see [3] and references therein) used the gamma-ray diffuse background to establish constraints. Similarly, Liddle and Green [4] reviewed a wide variety of methods to study a possible PBH population. Generally speaking, the methods dealing with the evaporation of PBHs apply to the final phase of these objects only, and therefore do not make full use of the radiative thermal flux at larger wavelengths predicted by the Hawking process of quantum evaporation [3,4].

The formation of PBHs from primordial fluctuations was first studied by Carr [5]. In that work he established that large, Gaussian primordial fluctuations from a scale-free primordial spectrum may be responsible for a power-law PBH mass function. A new twist to the problem has been added by the recent work of Niemeyer and Jedamzik [6,7] who used the concepts of critical phenomena to study the initial mass function for Gaussian and ‘‘Mexican hat’’ initial fluctuation spectra. They have argued that in these cases PBHs may form with masses below the horizon mass, and found a mass function which is not a pure power law. Black holes may form at all epochs (see Yokoyama [8], and Kawasaki and Yanagida [9] for recent work on PBH formation from the collapse of large density perturbations in double-inflation and supergravity models) and since there is considerable uncertainty about the actual processes that effectively formed PBHs, we seek here a general method to constrain the population which may be used independently of the specific PBH formation mechanism(s) (see for instance, Hansen *et al.*, and Bousso [10] for a discussion of other possibilities). We shall evaluate a simple (but useful) model consisting in a single-mass scale, which may be associated to the typical mass scale of any given mass function showing such a peak (this situation is very well satisfied by models in which large perturbations collapse to form PBHs).

The simplest reexamination of PBH physics has shown that, as a result of particle absorption from the expanding background, there is a maximum value of the mass (termed ‘‘critical mass’’ in [11]) splitting an evaporating (or subcritical) from a non-evaporating (or supercritical) subset of a general mass distribution. The critical mass separating these regimes for each redshift is therefore a useful boundary to discuss the physics of the PBHs and will be defined in the next section. A detailed quantitative discussion of these issues has been given in Refs. [11,12]. Because of the absorption of particles from the thermal background, PBH evaporation (and therefore some limits to their abundance) have to be reexamined. Actually we shall see that the existence of a regime of $\sim$ constant mass in the life of PBHs is very important for a full evaluation of their survival. Previous studies [13] have not considered the regime of quasi-constant mass and thus need to be updated to account for the evaporation at each evaporation scale labeled by the redshift.

II. FORMALISM

In order to simplify our evaluations, let us consider a homogeneous and isotropic cosmological distribution of PBHs. The radiative thermal flux from an evaporating black hole, as measured by an observer at $z=0$ is given by $F(M)=L(M)/4\pi D_L^2(z)$, where $D_L(z)=(cz/H_0)[1+z(1-q_0)/(\sqrt{1+2q_0z}+1+zq_0)]$ and q_0 is the deceleration factor (see, for instance, [1]).

The luminosity function $L(M)$ is given by $L(M)=L_0(M_{Haw}/M)^2$, with $L_0 \sim 2.6\times 10^{16}\ \mathrm{erg\,s^{-1}}$ and it is simply the Stefan-Boltzmann law applied to an evaporating black hole. We have scaled the mass to a ‘‘natural’’ constant $M_{Haw}=10^{15}$ g (hereafter the Hawking mass), since this is the mass of a PBH which is completing its evaporation today. Thus we write $\mu_* \equiv (M_*/M_{Haw})$ and so on along this work.

The subscript ‘‘*’’ will be used for the initial values of any quantity. Moreover, we shall restrict our considerations to semiclassical PBHs, i.e. those that satisfy $M \gg M_{pl}$, M_{pl} being the Planck mass.

We define $d\mu\,\xi(\mu,z)$ as the numerical abundance of

PBHs dn at redshift z with masses between μ and $\mu+d\mu$ per unit horizon volume; then $\xi(\mu,z)$ denotes the number of PBHs by horizon volume and by mass interval and $\Delta V_{hor}(z)d\mu\xi(\mu,z)$ is the number of PBHs contained within this volume. An upper limit on the mass density of radiating PBHs is obtained from the requirement $\delta F_{total}(PBHs) < \delta F_{back}$, with δF_{back} the value of the measured background flux in a given frequency interval.

To begin the calculation we assume a dilute PBH gas, i.e. a negligible collision term (PBH-PBH) between these objects. In other words we assume that the cosmic expansion rate is much larger than the PBH collision rate. Our aim will be to obtain windows of allowed abundances, for which we limit the PBH mass abundance in function of initial mass versus formation redshift.

As discussed in Ref. [11], the minimal formalism that describes these features is given by the set of equations

$$H^2=\frac{8\pi G}{3}\varrho_{total}-\frac{K}{R^2}+\frac{\Lambda}{3} \tag{1}$$

$$\dot{\mu}=-\frac{A(\mu)}{\mu^2}+\frac{27\pi G^2}{c^3}\mu^2\varrho_{rad}(t) \tag{2}$$

$$4H\varrho_{rad}+3H\varrho_{pbh}+\dot{\varrho}_{pbh}+\dot{\varrho}_{rad}=\frac{\dot{Q}(t)}{R(t)^3c^2} \tag{3}$$

$$\frac{Df_{pbh}}{Dt}=\frac{\partial f_{pbh}}{\partial t}+\frac{d\mu}{dt}\frac{\partial f_{pbh}}{\partial \mu}-H\vec{\beta}_{pec}\cdot\frac{\partial f_{pbh}}{\partial\vec{\beta}_{pec}}=0 \tag{4}$$

where ϱ_{total} denotes the total density (radiation plus PBHs), $\dot{Q}(t)$ gives the total heat input from the evaporation from the subcritical PBH population $[\dot{Q}(t)=\int dm\,\dot{m}N(m,t)]$, $R(t)$ is the usual scale factor, $f_{pbh}=f_{pbh}(\mu,\beta,t)$ is the distribution function that describes the cosmological evolution of PBHs, their numerical abundances and higher momenta and $\vec{\beta}$ denotes the peculiar velocity of the PBH as given by $\vec{\beta}=(\vec{v}/c)$. In Eq. (2) $A(\mu)\sim10^{26}$ $\mathrm{g}^3\,\mathrm{s}^{-1}h(\mu)$ with $h(\mu>10^2)\sim1$ and $h(\mu\ll1)\sim100$, describing the degrees of freedom of the emitted particles according to the standard model.

The critical mass value (as introduced in [11]) separating the evaporating (subcritical) from the non-evaporating (supercritical) regimes is obtained from the condition $\dot{\mu}=0$ and reads

$$M_c(T)\sim7.3\times10^{25}g\left[\frac{\varrho_{rad}(t_0)}{\varrho_{rad}(t)}\right]^{1/4}\sim\frac{7.3\times10^{25}g}{(T/T_0)} \tag{5}$$

with $T_0\sim2.7$ K the present value of the cosmic microwave background radiation (CMBR) temperature, and the other symbols have standard meaning. In terms of the Hawking mass we write Eq. (5) as $\mu_c(T)\sim10^{11}/(T/T_0)$.

Once the individual fluxes from the evaporating subpopulation are calculated, the total flux can be obtained by integration:

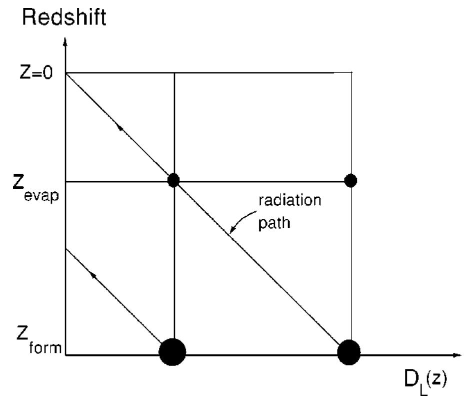

FIG. 1. The fate of primordial black holes in the redshift vs distance-luminosity plane. A PBH formed at z_{form} evaporates and the emitted radiation arrives at the observer when its path (explicitly indicated) crosses the vertical axis. If the PBH mass is initially below the Hawking mass, then the corresponding redshift for complete evaporation z_{evap} is located at $z\geqslant0$.

$$\delta F_{pbh}=\frac{1}{4\pi}\int_0^{z_f}\frac{dz}{D_L^2(z)}\left[\frac{\Delta V_{hor}(z)}{\Delta z}\right]\int_{\theta\mu_{pl}}^{\mu_c(z)}d\mu\,\xi(\mu,z)L(\mu) \tag{6}$$

where z_f denotes the formation redshift and θ is a dimensionless number $O(1)$. primordial black holes above $\mu_c(z)$ do not contribute to the radiation since for them the absorption term is dominant by definition. As it stands from Eq. (6), a suitable evaluation of δF_{pbh} requires integration over the redshift associated to the expansion. This feature is depicted in Fig. 1 which shows the light cone and the received radiation emitted from these objects. All those PBHs with $\mu_*<1$, the Hawking mass, formed at $z_f\gg1$ had evaporated at $z_{evap}(\mu_*)>0$.

Generally speaking the functional form of the amplitude $\xi(\mu,z)$ will depend on the physics of the process that forms the PBHs and can be quite complicated. A reasonable *ansatz* is to factorize the usual volumetric dilution from the mass dependence. Thus, the function $\xi(\mu,z)$ for the original [5] Carr mass function is chosen to have the form

$$\xi(\mu,z)=\frac{\xi_c(\mu,z)}{V_{hor}(z)}=\xi_0\frac{\mu^{-n(z)}}{V_{hor}(z)} \tag{7}$$

and, for instance, the critical Niemeyer-Jedamzik is *not* a power law and we refer to Refs. [6,7] for a full account of their work. A large class of initial mass functions are nonetheless included in the *ansatz* class.

The single-mass scale mass function to be studied in this paper is simply

$$\xi(\mu,z)=\xi_0\frac{\delta(\mu-\mu_*)}{V_{hor}(z)} \tag{8}$$

with the horizon volume given by $V_{hor}(z)=4\pi r_0^3/3(1+z)^3$, the numerator denotes the present value for the particle horizon volume and the numerical density in black holes is normalized to the present value when $z=0$. This normalization factor will be canceled by the same factor present in

$\Delta V_{hor}(z)$ in Eq. (6).The subscript "c" denotes comoving values and ξ_0 is the amplitude, interpreted as PBH number by mass interval, i.e. PBHs$\times$g^{-1}. This work will evaluate mass constraints using the mass function of Eq. (8), considered as a basic model. Other examples like the critical collapse spectrum [6,7] discussed recently, or Carr's power-law spectrum can be worked out, although the evaluations are quite involved. However, it is important to stress that all power spectra that produce a substantial number of PBHs present sharp peaks, and therefore look pretty much like delta functions [14].

To proceed with the evaluation of the bounds we must make a connection between the mass function amplitude and the solutions of Eqs. (1)–(4) in the following way: the number density of PBHs is given by

$$n_{PBH}(z) = \int dM \xi(\mu,z). \tag{9}$$

On the other hand, we may be able to evaluate $n_{PBH}(z)$ using the kinetic formalism (see [11]) as given by

$$n_{PBH}(\mu,z) = \frac{g_*}{(2\pi)^3}\int d^3p f_{pbh}(\beta,\mu,z) \tag{10}$$

with $p = mc\beta$ is the non-relativistic momentum.

Some algebraic manipulations enable to cast Eq. (9) in the form

$$n_{PBH}(z) = \frac{g_* c^3 (M_{Haw})^4}{2\pi^2}\int_{\theta\mu_{pl}}^{\mu_c(z)} d\mu \mu^3 \int_0^1 d\beta \beta^2 f_{pbh}(\beta,\mu,z). \tag{11}$$

Here, g_* is the statistical weight and we integrate up to the critical mass $\mu_c(z)$ since above it the evaporation does not take place. Note that for $\mu \geqslant \mu_c(z)$ the third term of Eq. (4) changes sign and the lower limit $\theta\mu_{pl}$ precludes contribution of quantum PBHs near the Planck scale. Now, we may rewrite $n_{PBH}(z)$ using the same normalization factor as

$$n_{PBH}(z) = M_{Haw}\int_{\theta\mu_{pl}}^{\mu_c(z)} d\mu \xi(\mu,z) \tag{12}$$

and by comparing Eqs. (11) and (12) we obtain

$$\xi(\mu,z) = \frac{g_* c^3 (M_{Haw})^3 \mu^3}{2\pi^2}\int_0^1 d\beta \beta^2 f_{pbh}(\beta,\mu,z). \tag{13}$$

Clearly the microphysical framework is a powerful tool, since it can describe all the kinematical effects as explicitly displayed in the limits of the integrals Eqs. (11) and (12) above. A complete analysis of the solutions is beyond the scope of the present work and will be treated elsewhere. Here, we will analyze only the radiation emitted by these objects.

Our general picture is now formally complete, in the sense that solving the formalism for $f_{pbh}(\beta,\mu,z)$ we would be able to set upper limits to the abundance of PBHs using the thermal emission of any mass spectrum. Since the work done in the literature mainly addresses the final explosive phases of these objects, which contributes to the gamma-ray band and to the cosmic ray flux (see for example [3] and references therein), we may view this work as an attempt to generalize those results.

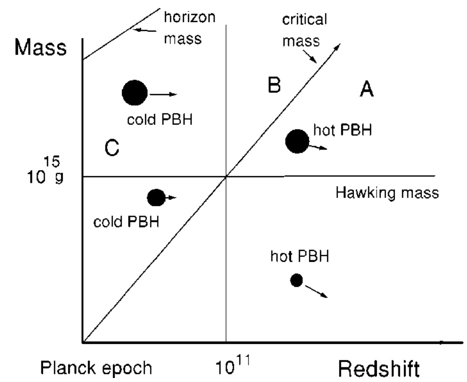

FIG. 2. Mass scales vs redshift of formation plane. The critical mass curve and the Hawking mass constant are explicitly indicated. Note that both cross at $z \sim 10^{11}$. In the region A hot (evaporating) PBHs began to evaporate right after their formation. In the region B the PBHs did not begin to evaporate at formation, but only after crossing the critical mass curve and entering into the region A. In the region C PBHs were formed at very high redshift, greater than $z = 10^{11}$ and followed a similar history. Below $M_{Haw} \sim 10^{15}$ g, PBHs evaporated completely today. Our method of analysis and bounds thus obtained are useful for the PBHs formed at the regions A, B and C (termed "First Regime" in the text) but not below the Hawking mass (see text).

III. APPLICATION TO A SINGLE-SCALE MASS FUNCTION AND COMPARISON WITH OTHER METHODS

To proceed we must now evaluate the last integral found in Eq. (6), for the delta mass-function case where $\xi(\mu,z) \propto \delta(\mu - \mu_*)$. The limits on ξ_0 will thus apply to an unclustered isotropic distribution of PBHs that are subdominant for the total mass balance of the universe. These hypothesis are in fact not very restrictive since for low peculiar velocities and the masses considered, the time scale for clustering is much larger than the Hubble time H_0^{-1}.

As explained above, we will assume that at z_f the mass is concentrated at one discrete value. Figure 2 shows the two important regimes to be considered here: $\mu_* \geqslant 1$ or $\mu_* \leqslant 1$ in terms of the formation redshift and the critical mass $\mu_c(z)$. To visualize the differences we shall discuss them separately.

1. First regime: $\mu_* \geqslant 1$

In this case, all PBHs formed at z_f are still evaporating today. Here, $M_{Haw} \sim 10^{15}$ g is the Hawking mass, i.e. it is the mass that completely evaporates today. We recall that the time scale for evaporation depends only of μ_* and it is given by $t_{evap} = f(\Omega_0) H_0^{-1}(\mu_*)^3$. We may split this regime in two sub-cases depending on wether z_f the PBH mass is larger or smaller than the critical mass $[\mu(z_f) \geqslant \mu_c(z_f)$ or $\mu(z_f) \leqslant \mu_c(z_f)]$ at z_f. In the first subcase, the PBHs at the onset of evaporation will be delayed until they cross the critical mass curve, as will be described later. Now, we will study the subcase $\mu_*(z_f)\leqslant\mu_c(z_f)$ which consists of initially subcritical PBHs.

A. Subcase $\mu_*(z_f)\leqslant\mu_c(z_f)$

These PBHs will start to evaporate immediately after their formation. The evolution of this spectrum with z will be given by

$$\xi(\mu(z),z\leqslant z_f)=\xi_0\left[\frac{\delta(\mu(z)-\mu_*(z))}{V_{hor}(z)}\right] \tag{14}$$

where $\mu(z)$ is given by the $\mu(z)=\mu_* F(\mu_*,z_{ini},z)$, and the evaporation function $F(\mu_*,z_{ini},z)$ is

$$F(\mu_*,z_{ini},z)\sim\left[1+\left(\frac{1}{\mu_*}\right)^3\int_{z_{ini}}^{z}dz(1+z)^{-5/2}\right]^{1/3} \tag{15}$$

which can be approximated by $(1-2/3\mu_*^3(1+z)^{3/2})^{1/3}\sim 1$, for $z_{ini}\gg 1$.

This form is valid as long as $q_0=1/2$, because we have used that the horizon volume at z is related to the horizon volume at z_f by $V_{hor}(z)=((1+z_f)/(1+z))^3V_{hor}(z_f)$. We do not consider other cases for q_0 in this article, but we checked that the dependence of the results with q_0 is very mild.

Substituting these relations into Eq. (6), we evaluate the radiation received today $(z=0)$ from all these PBHs yielding

$$\delta F_{pbh}=\frac{3L_0M_{Haw}\xi_0}{4\pi}\int_{\epsilon_\odot}^{z_{ini}}dz\frac{[\mu_*F(\mu_*,z)]^{-2}}{D_L^2(z)(1+z)}. \tag{16}$$

Assuming that PBHs have black body spectra, we can apply Wien's law to the mass-temperature relation of black holes: $T(\mu)\sim 2\times10^{11}$ K/(μ) in order to obtain an associated wavelength of the maximum $\lambda(\mu_*)T(\mu_*)=0.29$ cm K. Since we are considering the radiation emitted by PBHs at cosmological distances, we must correct for the expansion of the received wavelength. We know that these relations would be satisfied by $\lambda_{obs}(z)=\lambda_{em}(z')[(1+z')/(1+z)]$. Substituting into the mass-temperature relation

$$\lambda_{obs}(z=0,\mu(z'))=\lambda_0\mu(z')(1+z') \tag{17}$$

and considering that at any time the mass is given by $\mu(z')=\mu_*F(\mu_*,z_{ini},z')$ we obtain

$$\lambda_{obs}(z=0,\mu_*)=\lambda_0\mu_*F(\mu_*,z_{ini},z')(1+z'). \tag{18}$$

The prefactor in front of $(1+z')$ is the emitted wavelength when the initial PBH with mass μ_* had the mass $\mu(z)$ at the correspondent temperature, and $\lambda_0\sim1.5\times10^{-12}$ cm is a reference scale. Then, we recover $\mu_*F(\mu_*,z_{ini},z')=\lambda_{obs}(z=0,\mu_*)/\lambda_0(1+z')$. By its very definition $F(\mu_*,z_{ini},z'=z_{ini})=1$, and thus

$$\frac{\lambda_0}{\lambda_{obs}(z=0,\mu_*)}=\frac{1}{\mu_*(1+z_{ini})}. \tag{19}$$

After all these manipulations the total flux obtained is given by

$$\delta F_{pbh}=\frac{3L_0M_{Haw}\xi_0}{4\pi}[\mu_*(1+z_{ini})]^{-2}\int_{\epsilon_\odot}^{z_{ini}}dz\frac{(1+z)}{D_L^2(z)} \tag{20}$$

where $z_{ini}=z_f$ if $\mu_*\leqslant\mu_c(z_f)$ or $z_{ini}=z_{cross}(\mu_*)$ if $\mu_*\geqslant\mu_c(z_f)$.

An inspection of Eq. (20) shows that there is a potential mathematical divergence at the lower limit of the integral. Therefore we have integrated formally the expression up to a minimum parameter $\epsilon_\odot$, which acts as a cutoff. The integral $I(\epsilon_\odot,z_f)=\int_{\epsilon_\odot}^{z_f}dz[(1+z)/D_L^2(z)]$ admits the expansion

$$I(\epsilon_\odot,z_f)\sim Y(q_0)(H_0/c)^2\left[\frac{1}{\epsilon_\odot}+\ln(z_f/\epsilon_\odot)+\cdots\right] \tag{21}$$

with $Y(q_0)\sim O(1)$ and $\epsilon_\odot\ll 1$. The question arises about how this $\epsilon_\odot$ affects the final results. We shall show below that a careful consideration of the different regimes of the PBHs makes unnecessary the imposition of any $\epsilon_\odot$, and in practice there is no physical divergence of the quantities, contrary to the naive expectation from Eq. (20).

Now, if we impose that $\delta F_{pbh}\leqslant\delta F_{back}$, we obtain a constraint on the amplitude ξ_0

$$\xi_0\leqslant(1+z_{ini})^2\frac{4\pi\mu_*^2}{3L_0M_{Haw}}\frac{\delta F_{back}}{I(\epsilon_\odot,z_f)}. \tag{22}$$

Although the actual limits may be considered for any small interval, we have defined in practice an average background value for the radiation over a large portion of the spectrum as

$$\langle\delta F_{back}\rangle=\left[\frac{\delta F_{back}}{10^{-6}\ \ \mathrm{ergs}^{-1}\ \ \mathrm{cm}^{-2}}\right]. \tag{23}$$

Some care has yet to be taken to take into account the full dependence of the mass density in PBHs $\varrho_{pbh}(z)$ inside the horizon volume. After some algebric manipulation and using the properties of the delta function we derive the expression

$$\varrho_{pbh}(z)=\left(\frac{1+z}{1+z_f}\right)^3\frac{\xi_0(M_{Haw})^2}{V_{hor}(z_f)}F(\mu_*,z)\mu_* \tag{24}$$

which is valid for any $z\geqslant z_{evap}(\mu_*)$. If we divide Eq. (24) by $\varrho_c(z)=\varrho_c(0)(1+z)^3$ (again for $q_0=1/2$), we will obtain $\Omega_{pbh}(z)$ for any z which satisfies the causal relation (that is, $\mu_{pbh}\leqslant\mu_{hor}$),

$$(1+z_f)\leqslant\frac{4.24\times10^{28}}{\sqrt{(\mu_{pbh})}}. \tag{25}$$

This expression precludes the existence of PBHs with masses larger than the one contained within the cosmic horizon, i.e. $\mu_{pbh}(z)\leqslant\mu_{hor}(z)$, but admits the case $\mu_{pbh}\ll\mu_{hor}$ as advocated in Refs. [6,7]. For $\mu_{pbh}\sim10^{11}$, the critical mass value today, we have an upper limit at which PBHs can be formed still satisfying Eq. (25), namely $z_{fmax} \sim 10^{16}$.

To estimate actual bounds we evaluate the expression at $z=0$. Then, we have

$$\varrho_{pbh} = \frac{\xi_0 (M_{Haw})^2 \mu_* F(\mu_*, z_{ini}, z=0)}{(1+z_f)^3 V_{hor}(z_f)}. \tag{26}$$

Now, using Eq. (23) and dividing it by the critical density $\varrho_c(z=0) \sim 2\times 10^{-29}$ g cm^{-3}, we have (imposing $z_{ini}=z_f$)

$$\Omega_{pbh}(0) \leqslant \Omega_{pbh}(z_f,\mu_*) = 8.7\times 10^{-8}(1+z_f)^2 \mu_*^3 F(\mu_*, z_f, z=0)\epsilon_{\odot}\langle \delta F_{back}\rangle \tag{27}$$

where $F(\mu_*, z_f, z=0) \sim O(1)$.

The interpretation of Eq. (27) is the following: the right-hand side sets the maximal abundance in PBHs today ($z=0$) allowed by the sky brightness assumed to be fully produced by the evaporation of PBHs formed at z_f with initial masses given by μ_*. These PBHs were born above the Hawking mass and therefore they are evaporating today. We have also required that their masses were initially below the critical mass value today $\sim 10^{26}$ g, otherwise our constraints do not apply in this form (see below). From now, we shall denote the product $\epsilon_{\odot}\langle \delta F_{back}\rangle$ which appears conspicuously by ϵ_B.

Our bounds based on the sky brightness will be tighter (by construction) inside a finite region in the parameter space defined by μ_* versus z_f. This finite region corresponding to $\Omega_{pbh}(z_f, \mu_*) \leqslant 10^{-10}$ may be further explored to understand which is the physics allowing these bounds. Taking the logarithm on both sides of Eq. (27) we find

$$\log \mu_* + \frac{2}{3}\log(1+z_f) \leqslant -1 - \frac{1}{3}\log \epsilon_B \tag{28}$$

with $\epsilon_B \equiv \epsilon_{\odot}\langle \delta F_{back}\rangle$ as stated above. This inequality displays a relation between the initial mass μ_* and formation redshift z_f bounding the "coolest PBH population," since any PBH population below this line will be contributing so much to the sky brightness that we can certainly rule it out. This curve is a consequence of the Doppler effect, since for z_f above this curve, the cosmic expansion diminished the total flux below the background value and big masses correspond to very cold PBHs, for which the constraint is necessarily very poor. It is clear that the z_f threshold is mass dependent and therefore, the curve must drop at later times.

In terms of the mass we may write Eq. (28) in the form

$$\mu_* \leqslant \frac{0.1}{(1+z_f)^{2/3}\epsilon_B^{1/3}}. \tag{29}$$

Meaning that all those PBHs formed at z_f satisfying both $\mu_* \geqslant 1$ and Eq. (28) above are much more scarce than $\Omega_{pbh}(\mu_*) \leqslant 10^{-10}$, for a given combination value of the z_f and the parameter ϵ_B.

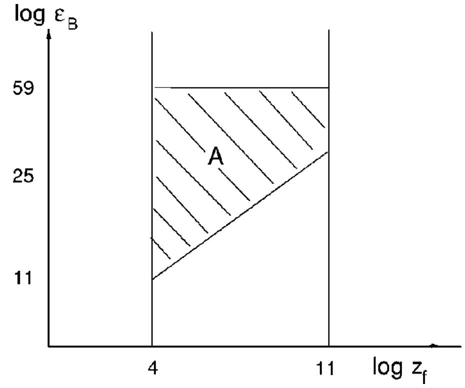

FIG. 3. The region inside which $\Omega_{pbh} < 10^{-10}$ in the ϵ_B vs z_f (hatched) for those PBHs born in region A. The boundaries of this region are defined in the text.

The very fact that $1 \leqslant \mu_* \leqslant \mu_c(z_f)$ leads to some additional constraints avoiding the imposition of an $\epsilon_{\odot}$. Actually, since $\mu_c(z_f) = \mu_c(0)(1+z_f)^{-1}$, we note that these relations can hold only for $(1+z_f) \leqslant 10^{11}$. But the condition $\mu_* > 1$, Eq. (28) requires also that $\epsilon_{\odot}\langle \delta F_{back}\rangle \leqslant 10^{-3}(1+z_f)^{-2}$. Taking the logarithm as before

$$-\frac{1}{3}\log \epsilon_B \geqslant 1 + \frac{2}{3}\log(1+z_f). \tag{30}$$

Using that $(1+z_f) \leqslant 10^{11}$ we obtain an upper bound to the value of the parameter ϵ_B

$$\epsilon_B \leqslant 10^{-25}. \tag{31}$$

Equation (31) is a direct consequence of imposing $\Omega_{pbh}(\mu_*, z_f) \leqslant 10^{-10}$ for $\mu_* < 1$ and $z_f \geqslant 1$. We see that ϵ_B is bounded from above by a very small number, in fact so small that its associated distance $D_L(\epsilon_{\odot}) \sim 1$ cm. This means that we can exclude PBHs in the discussed conditions regardless of the irrelevant value of $\epsilon_{\odot}$, which is a "physical" zero.

Analogously, we must satisfy $\mu_* < \mu_c(z=0)$, which implies $\epsilon_B > 10^{-59}$. Figure 3 displays the region between $\log z_f$ and $\log \epsilon_B$ in which all those PBHs satisfying $1 \leqslant \mu_* \leqslant 0.1/(1+z_f)^{2/3}(\epsilon_B)^{1/3}$ are more scarce than $\Omega_{pbh}(\mu_*) \leqslant 10^{-10}$.

The region in the parameter space $\log z_f$ versus $\log \mu_*$ where $\Omega_{pbh}(\mu_*) < 10^{-10}$ is then bounded by the following inequalities:

$$2\log(1+z_f) + \log \mu_* < 57.2 \tag{32}$$

$$\log \mu_* < \log \mu_c(0) \sim 11 \tag{33}$$

$$\log z_f > \log z_{min} \sim 4 \tag{34}$$

$$\log \mu_* > 0 \tag{35}$$

and Eq. (28). It is clear that Eq. (32) describes the causality constraint, i.e. there can be no PBH with mass greater than the horizon mass at z_f. Eq. (33) states that our method applies to evaporating PBHs only. Equation (34) says that

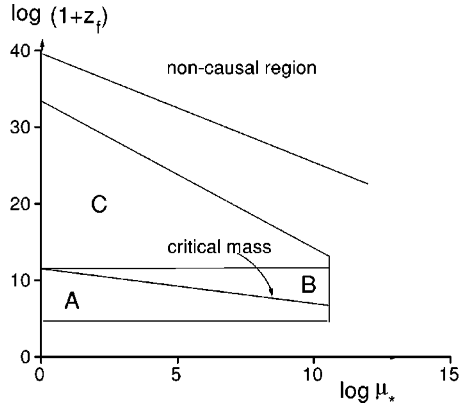

FIG. 4. The compact region in which the $\Omega_{pbh}<10^{-10}$ in the redshift vs mass plane (see text). At $\log \mu_*\sim 11$ the PBHs have a mass of 10^{26} g, the critical mass value today. The inclined straight line bounding region C from above is set by the Doppler effect of the cosmological expansion. We have also drawn the horizon mass line and explicitly indicated the noncausal region above it. The regions A, B and C correspond to the regimes and subcases defined in the text.

PBHs do not form as late as $z<10^4$, since we know that the environment must be very smooth at late times (this arbitrary prescription can be relaxed straightforwardly). Finally, Eq. (35) describes the choice $\mu_*>1$. The compact region where the abundance of PBHs is $\leqslant 10^{-10}$ in this plane is displayed in Fig. 4.

B. Subcase $\mu_c(z_f)\leqslant\mu_*\leqslant\mu_{hor}(z_f)$

The range of masses implies $(1+z_f)\leqslant 4.24\times 10^{28}$; therefore we can rewrite the previous expressions to obtain, for ξ_0,

$$\xi_0\leqslant 6.8\times 10^{52}(1+z_{cross}(\mu_*))^2\mu_*^2 h_0^{-2}\epsilon_B. \tag{36}$$

And considering that $(1+z_{cross}\mu_*)^2\sim 10^{22}\mu_*^{-2}$ we insert these relations into Eqs. (24) and (31) to yield

$$\Omega_{pbh}(z=0,\mu_*\geqslant\mu_c(z_f))\leqslant 8.7\times 10^{14}\mu_*\epsilon_B F(\mu_*,z=0,z_{ini}). \tag{37}$$

Requiring that $\Omega_{pbh}(\mu_*)<10^{-10}$ as before, the corresponding masses must satisfy

$$\mu_c(z_f)\leqslant\mu_*\leqslant\frac{10^{-25}}{\epsilon_B}. \tag{38}$$

Analogously to the former case $\mu_*>1$ then the cutoff is limited to

$$\epsilon_B\leqslant 10^{-25}, \tag{39}$$

and the same considerations as before apply. Since $\mu_* < \mu_c(z=0)\sim 10^{11}$ must also hold, we obtain the limit

$$\epsilon_B\geqslant 10^{-34}, \tag{40}$$

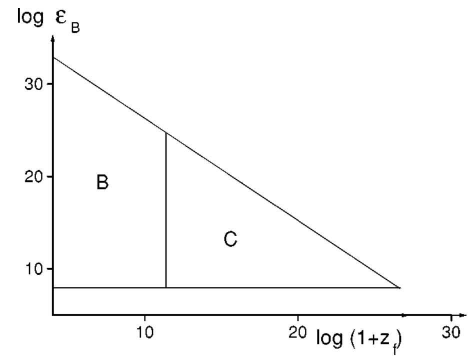

FIG. 5. The same as Fig. 3 for the regimes corresponding to the regions B and C. The region B spans redshifts from 0 up to 10^{11}, when the critical mass equals the Hawking mass. The region C corresponds to any redshift value bigger than 10^{11} up to 10^{28}, the moment when the horizon mass equals the Hawking mass (of academic interest only). The upper diagonal line reflects the combination of the Doppler effect and the value of ϵ_B (see text).

again corresponding to microscopic lengthscales, and therefore not relevant for the bounds so obtained. Finally, requiring that $10^{-25}/\epsilon_B\geqslant\mu_c(z_f)$, we obtain the inequality

$$\log\frac{\epsilon_B}{(1+z_f)}\leqslant -36. \tag{41}$$

Note that Eqs. (38) and (39) are completely independent of z_{ini} because the latter dependence cancels out. Then, it is Eq. (39) the one that sets the maximum value for the lower cutoff in redshift, already shown to be irrelevant. Although we may have considered the subcase $\mu_*>\mu_c(z_f)$ in two parts [the first $1>\mu_c(z_f)$ and the second $1<\mu_c(z_f)$], they actually differ only in the allowed value of z_f: for the first we have $(1+z_f)>10^{11}$ and for the second case $(1+z_f)<10^{11}$.

Figure 5 displays a region in the plane $\log(1+z_f)$ versus ϵ_B in which we find PBHs satisfying $\Omega_{pbh}(\mu_*)\leqslant 10^{-10}$ and for which $\mu_*\geqslant\mu_c(z_f)$. All these PBHs satisfy the conditions $\mu_*\geqslant 1$, $\mu_*\leqslant\mu_c(0)\sim 10^{11}$ and $\log z_f\leqslant 28.6$ as well (for $\log z_f\geqslant 28.6$ we would have $\mu_{hor}\leqslant 1$).

1. Second regime: $\mu_*<1$

When we deal with PBHs born with masses smaller than the Hawking mass initially, we may substitute the lower limit of the integral Eq. (21) by $z_{evap}(\mu_*)$, because no PBH will survive its own timescale for evaporation. Therefore the integral $I(z_{evap}(\mu_*),z_{ini})=\int_{z_{evap}(\mu_*)}^{z_{ini}}dz[(1+z)/D_L^2(z)]$ is explicitly evaluated as

$$I(z_{evap},z_{ini})=(H_0/c)^2\left[G(\mu_*)+\ln\left(\frac{z_{ini}}{z_{evap}(\mu_*)}\right)\right] \tag{42}$$

where

$$G(\mu_*)=\frac{z_{ini}-z_{evap}(\mu_*)}{z_{evap}(\mu_*)z_{ini}}$$

and

$$z_{evap}(\mu_*)=\frac{1}{\mu^2}-1.$$

.

Following the same reasoning as in the previous case we obtain for ξ_0

$$\xi_0 \leq 7.2\times 10^{22} h_0^{-2} \frac{\langle \delta F_{back}\rangle}{G(\mu_*)+\ln(z_{ini}\mu_*^2)} g^{-1} \tag{43}$$

where z_{ini} falls in one of these two subcases: when $\mu_* > \mu_c(z_f)$ we have $z_{ini}=10^{11}/\mu_*$ and when $\mu_* < \mu_c(z_f)$ we have $z_{ini}=z_f$.

Using the expressions above we finally find, after substituting Eq. (43) into Eq. (31) and dividing by $\varrho_c(z)$

$$\begin{aligned}\Omega_{pbh}(z>z_{evap}(\mu_*))&\\ &<9\times 10^{14} F(\mu_*,z_{ini},z)\left[\frac{\mu_*}{G(\mu_*)+\ln(z_{ini}\mu_*^2)}\right]\\ &\quad\times\langle \delta F_{back}\rangle h_0^{-2}.\end{aligned} \tag{44}$$

Since that $F[\mu_*,z_{ini},z=z_{evap}(\mu_*)]=0$, the result Eq. (44) describes correctly the evolution of the abundance from z_f until $z_{evap}(\mu_*)$, taking into account the Hawking evaporation. The most interesting case obtained from Eq. (44) applies to $z=z_f$, from which we derive the maximum abundance allowed at formation still consistent with the sky brightness today $\langle \delta F_{back}\rangle$

$$\Omega_{pbh}(z_f)<9\times 10^{14}\left[\frac{\mu_*}{G(\mu_*)+\ln(z_f\mu_*^2)}\right]\langle \delta F_{back}\rangle h_0^{-2} \tag{45}$$

for $\mu_* \leq \mu_c(z_f)$ and

$$\Omega_{pbh}(z_f)\leq 9\times 10^{14}\left[\frac{\mu_*}{G(\mu_*)+\ln(10^{11}\mu_*)}\right]\langle \delta F_{back}\rangle h_0^{-2} \tag{46}$$

for $\mu_* \geq \mu_c(z_f)$.

However, when we try to force Eqs. (45) and (46) to yield values $\Omega_{pbh} \leq 10^{-10}$ we did not find physical windows for reasonable values of PBH masses. The reason is that to obtain $\Omega_{pbh} \leq 10^{-10}$, μ_* have to be very small and thus violate our semi-classical assumptions. This limitation is to be expected, since those PBHs evaporated a long time ago when the cosmic radiation was $\sim(1+z)^4$ times more intense than the present background. This means that we cannot use the cosmic isotropic radiation in order to estimate constraints for the sub-Hawking PBH population, and other methods (as described, for example, by Green *et al.* [15]) must be used to obtain useful limits to their abundance.

IV. CONCLUSIONS

We have shown that under a set of circumstances a limit on Ω_{pbh} better than 10^{-10} can be obtained by using the background brightness of the sky. This is quite stringent if we compare it with the values given by other methods [4]. We attempted to evaluate the evolution of maximal abundance $\Omega_{pbh}(z)$ for any z from observations performed today ($z=0$), a process that requires integration over z (and eventually over the actual mass spectrum from whatever the formation process, not attempted here).

The good news here is that PBHs can be excluded in a fairly large window (as seen in Fig. 4) if they were born above the Hawking mass (but below 10^{26} g). We see four physical reasons for this fact:

(1) Many of these PBHs *still* evaporate today, and therefore contribute to the background in various bands.

(2) The cosmic radiation today is much less intense than at high redshift values, and therefore constitutes a useful natural background.

(3) The cosmic expansion today is quite weak; then the Doppler damping of the emitted radiation does not have any killing effect.

(4) PBHs initially above the critical mass at some z_f delay the start of their evaporation; therefore they injected energy later, when the expansion is less effective to damp the radiation.

The constraints so obtained are weakly dependent of q_0 or the cosmological constant, and the mass range probed by this method is larger than previous works. Even though we take into account the existence of the critical mass and the redshift of the emitted radiation through the function $I(\epsilon_\odot,z_f)$, we found that our analysis renders useful limits for PBH masses initially above $\sim 10^{15}$ g, but not for those below it. In addition, PBHs above 10^{26} g must be limited by other methods because they do not evaporate at all. The consideration of the different cases leaded us to conclude that $\epsilon_\odot$ need not to be imposed by hand. Moreover, it happens to be an extremely small number in all cases, with microscopic associated distances $D_L(\epsilon_\odot)$. Physically, this means that the spatial distribution of PBHs does not need to satisfy any specific requirement near the earth, and thus the bounds so obtained are quite general.

The upper limits to the number density of PBHs were obtained from the simple requirement $\delta F_{pbh} < \delta F_{back}$, since other astrophysical mechanisms may be also contributing to the cosmic background brightness as measured by our detectors. The limits have been derived for a Dirac's delta mass function, whose main features are much simpler than any other choice. We can think this case as a first approach to the general problem, likely accurate for peaky distributions of a more general type. Detailed calculations of the latter remain an interesting problem for future investigations.

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[1] J. A. Peacock, *Cosmological Physics* (Cambridge University Press, Cambridge, England, 1998).
[2] S.W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[3] B. Carr and J.H. Mac Gibbon, Phys. Rep. **307**, 141 (1998).
[4] A.R. Liddle and A.M. Green, Phys. Rep. **307**, 125 (1998); A.M. Green and A.R. Liddle Phys. Rev. D **60**, 063509 (1999).
[5] B. Carr, Astrophys. J. **201**, 1 (1975).
[6] J.C. Niemeyer and K. Jedamzik, Phys. Rev. Lett. **80**, 5481 (1998).
[7] J.C. Niemeyer and K. Jedamzik, Phys. Rev. D **59**, 124013 (1999).
[8] J. Yokoyama, Phys. Rep. **307**, 133 (1998); Prog. Theor. Phys. Suppl. **136**, 338 (1999).
[9] M. Kawasaki and T. Yanagida, Phys. Rev. D **59**, 043512 (1999).
[10] R.N. Hansen, M. Christensen, and A.L. Larsen, Phys. Rev. D **61**, 108701 (2000) and references therein; R. Bousso, Phys. Rep. **307**, 117 (1998) and references therein.
[11] P.S. Custódio and J.E. Horvath, Phys. Rev. D **58**, 023504 (1998).
[12] P.S. Custódio and J.E. Horvath, Phys. Rev. D **60**, 083002 (1999).
[13] J.D. Barrow, E.J. Copeland, and A.R. Liddle, Phys. Rev. D **46**, 645 (1992); see also Mon. Not. R. Astron. Soc. **253**, 675 (1991).
[14] This point has been emphasized by A.M. Green, astro-ph/0105253.
[15] A.M. Green, A.R. Liddle, and A. Riotto, Phys. Rev. D **56**, 7559 (1997).