Quantum inequalities for the electromagnetic field

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A quantum inequality for the quantized electromagnetic field is developed for observers in static curved spacetimes. The quantum inequality derived is a generalized expression given by a mode function expansion of the four-vector potential, and the sampling function used to weight the energy integrals is left arbitrary up to the constraints that it be a positive, continuous function of unit area and that it decays at infinity. Examples of the quantum inequality are developed for Minkowski spacetime, Rindler spacetime and the Einstein closed universe.

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I. INTRODUCTION

Nearly four decades ago, it was shown by Epstein, Glaser and Jaffe [1] that a positive definite energy density was incompatible with the usual postulates of a quantized field theory. Worse yet, it appears that the energy density is not even bounded from below. Thus, all standard quantized field theories are capable of violating all the pointwise and averaged energy conditions in general relativity. However, this does not mean that the energy density can remain negative for an arbitrarily long period of time. Over the last decade, new forms of energy conditions involving various temporal and spatial averagings have been developed [2–21]. One such example is the quantum inequality, which is the weighted temporal average of the energy density along the worldline of an observer. Derived directly from quantum field theory, these inequalities limit the magnitude and temporal duration of the existence of negative energy densities. The quantum inequalities say that if an observer tries to make a measurement of the energy density for some characteristic sampling time τ_0, then the maximal negative energy that he might ever measure is bounded below by an inverse power of the characteristic sampling time. Given an observer's four-velocity u^μ and a sampling (weighting) function $f(\tau)$ of characteristic width τ_0, then the quantum inequality is given by

$$\tilde{\rho} \equiv \int_{-\infty}^{\infty} \langle T_{\mu\nu}(\tau)\rangle_{Ren.}\, u^\mu(\tau) u^\nu(\tau) f(\tau)\, d\tau$$

$$\geqslant -\frac{\alpha}{\tau_0^n} S(\tau_0) + \rho_{vacuum}. \qquad (1)$$

Here α is a dimensionless constant of order unity and n is the dimension of the spacetime. For a massless field in Minkowski spacetime, the function $S(\tau_0)$ is equal to 1 and the vacuum energy density vanishes. For massive fields and/or curved spacetimes, $S(\tau_0)$ represents the modification of the quantum inequality away from its massless, flat space functional form. It has the generic behavior that it is approximately 1 for small τ_0, and in most spacetimes it typically decays for longer characteristic sampling times. However there are some known exceptions, such as four-dimensional de Sitter and Rindler spacetimes, where the function S only grows only as fast as τ_0^2.

The quantum inequalities were first derived by Ford [2] to constrain negative energy fluxes for the quantized, massless, minimally coupled scalar field in Minkowski spacetime. These results were then expanded to the energy density of the massive scalar field in Minkowski space [5,7] and in static curved spacetimes [8,12,13]. In all of these cases, a Lorentzian sampling function

$$f(\tau) = \frac{\tau_0}{\pi} \frac{1}{\tau^2 + \tau_0^2} \qquad (2)$$

was used to simplify the calculations. However, Flanagan [10] showed it was possible to derive optimum quantum inequalities for the massless scalar field in two dimensions for an arbitrary, smooth positive choice of the sampling function. This was followed by the work of Fewster and colleagues [14,15,19] who have established the quantum inequality for the minimally coupled scalar field in static curved spacetimes of any dimension with an arbitrary, smooth positive sampling function.

Although much of the previous work has been for the scalar field, work is now progressing for higher spin fields. Vollick has shown that an optimum quantum inequality can be derived for the Dirac field in two spacetime dimensions [20] for an arbitrary sampling function using the conformal properties of the field theory. More recently, Fewster and Verch have established ''quantum weak energy inequalities'' for the Dirac and Majorana fields of nonzero mass in four-dimensional globally hyperbolic spacetimes [21]. Making use of microlocal analysis techniques, Fewster and collaborators [19,21] have vastly extended the applicability of the quantum inequalities to arbitrary globally hyperbolic spacetimes.

The first quantum inequality for the electromagnetic field was derived by Ford and Roman [7] for a Lorentzian sampling function in flat spacetime. This was immediately generalized to curved static spacetimes by the author [13], although both of these calculations relied on the specific choice of the Lorentzian sampling function. In addition, the

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proof in both cases was mathematically long and at some times quite complicated, particularly in some of the lemmas required.

In this paper, we will show that it is possible to derive a generalized quantum inequality for the quantized electromagnetic field in static curved spacetimes with a length element of the form

$$ds^2 = -|g_{00}(\mathbf{x})|dt^2 + g_{ij}(\mathbf{x})dx^i dx^j. \tag{3}$$

The proof presented here is greatly simplified, in large part due to generalization of a more direct positivity lemma originally developed by Fewster and colleagues [14,15]. In addition, the electromagnetic field quantum inequality is proven for an arbitrary choice of sampling function so long as it is a positive, continuous function of unit area that decays at infinity. The end result of our calculations is the quantum inequality written as a mode function expansion:

$$\tilde{\rho} \geqslant -\frac{1}{2\pi}\int_0^\infty d\nu \sum_\lambda \int d^3\mathbf{k} |\hat{f}^{1/2}[\nu+\omega(\mathbf{k})]|^2 \times \left[\frac{1}{|g_{00}|}\overline{E_i(\lambda,\mathbf{k};\mathbf{x})} g^{ij} E_j(\lambda,\mathbf{k};\mathbf{x}) + \left|\frac{g_{00}}{g}\right| \overline{B_i(\lambda,\mathbf{k};\mathbf{x})} \times (g^{ij})^{-1} B_j(\lambda,\mathbf{k};\mathbf{x})\right] + \rho_{\text{vacuum}}, \tag{4}$$

where $E_j(\lambda,\mathbf{k};\mathbf{x})$ and $B_j(\lambda,\mathbf{k};\mathbf{x})$ are the modes for the electric and magnetic components of the field-strength tensor,

$$\hat{f}^{1/2}(\omega) = \int_{-\infty}^{\infty} f^{1/2}(t) e^{-i\omega t} dt \tag{5}$$

is the Fourier transform of the square root of the sampling function and the summation over λ and integration over $d^3\mathbf{k}$ is over all possible polarizations and momentum eigenstates, respectively. As was the case for the scalar field, the electromagnetic field quantum inequality (4) tells us how much negative energy an observer may measure relative to the vacuum energy of the electromagnetic field.

In Sec. II we will discuss the canonical quantization of the electromagnetic field in curved space and elucidate the particle state structure and the form of the stress-tensor. In particular, two different forms of quantization will be discussed: direct quantization in the classical Coulomb gauge and the more elegant Gupta-Bleuler form of quantization. In Sec. III we develop the positivity lemma for generic inner-products of vector fields, which is a generalization of work developed by Fewster and colleagues [14,15] for the scalar field. In Sec. IV we lay out the remainder of the proof of the quantum inequality, finally arriving at the expression above. Lastly, in Sec. V we will look at the resulting quantum inequalities for Minkowski spacetime, Rindler spacetime and the Einstein closed universe.

We will follow the the convention of Wald [22] where the signature of the metric is $(-,+,+,+)$. Greek indices are summed over (0,1,2,3) while Latin indices denote the spatial components (1,2,3). However, the letter λ has been singled out as the polarization state label, and depending on the context, can represent either the two physical polarization states 1 and 2, or the full set of polarization states 0, 1, 2 and 3 in the Gupta-Bleuler formalism which includes the scalar and axial photon polarization states. Also, the complex conjugate of f, will be denoted by $\overline{f}$. Units of $\hbar = c = G = 1$ will be used throughout.

II. ELECTROMAGNETIC FIELD IN STATIC CURVED SPACETIMES

We begin our discussion of the electromagnetic field by defining the classical Maxwell action for a source free field in curved space,

$$S^{Maxwell} = -\frac{1}{4}\int_V F_{\alpha\beta}F^{\alpha\beta}\sqrt{-g}\,d^4x, \tag{6}$$

where $F_{\alpha\beta}$ is the antisymmetric field-strength tensor related to the four-vector potential, A_μ, by

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha. \tag{7}$$

Here ∇ represents covariant differentiation.

Varying the Maxwell action with respect to the vector potential and setting the variation equal to zero leads to the source free inhomogeneous Maxwell equation for the electromagnetic field in curved spacetime,

$$\nabla^\alpha F_{\alpha\beta} = 0. \tag{8}$$

Due to the Bianchi identities, the electromagnetic field also satisfies the subsidiary condition,

$$\nabla_{[\alpha}F_{\beta\gamma]} = 0, \tag{9}$$

which is the homogeneous Maxwell equation. The combined set of equations represents classical electrodynamics written in covariant form. If we insert the four-vector potential into both expressions, it is found that the homogeneous Maxwell equation (9) is trivially satisfied. The inhomogeneous equation (8) yields the second order wave equation

$$\nabla^\alpha\nabla_\alpha A_\beta - \nabla_\beta(\nabla^\alpha A_\alpha) - R_\beta{}^\alpha A_\alpha = 0. \tag{10}$$

Here $R_{\alpha\beta}$ is the Ricci tensor which arises due to the commutation relation for the covariant derivatives acting on a vector field.

The stress-tensor for the classical electromagnetic field is found by varying the Maxwell action with respect to the spacetime metric. A straightforward calculation yields

$$T^{Maxwell}_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}. \tag{11}$$

The field-strength tensor, the Maxwell equations and the stress tensor are invariant under the gauge freedom

$$A^{new}_\alpha = A^{old}_\alpha - \nabla_\alpha\Lambda, \tag{12}$$

where $\Lambda = \Lambda(x)$ is an arbitrary scalar function. In classical electromagnetism, the correct choice of gauge can often simplify finding the solution to the field equations. In many cases, it is convenient to choose the Lorentz gauge condition

$$\nabla^\alpha A_\alpha^{new}=0, \tag{13}$$

which immediately removes the middle term in the wave equation (10). This can always be achieved by choosing Λ to satisfy

$$\nabla^\alpha \nabla_\alpha \Lambda = \nabla^\alpha A_\alpha^{old}. \tag{14}$$

It should be noted that there is still a restricted gauge freedom remaining in that we can still add to the vector potential any function that satisfies the homogeneous equation

$$\nabla^\alpha \nabla_\alpha \Lambda^{Hom.}=0. \tag{15}$$

As we shall see below, this restricted gauge freedom will be used to impose the Coulomb gauge. It should be noted that we will drop the identifiers of *new* and *old* in all further calculations.

There is some difficulty in directly quantizing electrodynamics in the form so far described. If one does not specify a gauge, then any four-vector wave equation such as Eq. (10) will in general have four orthonormal solutions (polarization states), $A_\alpha(\lambda;x)$ where $\lambda=0$, 1, 2 or 3. In Minkowski spacetime the $\lambda=0$ solution is typically the scalar photon polarization, $\lambda=1$ and 2 are the two transverse photon polarizations, and $\lambda=3$ is the axial photon polarization. In curved spacetime the "perfect" separation of the modes into these three "distinct" types is not always possible, but we will continue to use the flat space nomenclature. It is found that for one of the polarizations, say $A_\alpha(0;x)$, there does not exist a conjugate momenta when the Hamiltonian is calculated. This is a long known problem in flat spacetime electrodynamics and there are several known approaches which have been developed to quantize the electromagnetic field that can be generalized to curved spacetime. The simplest is to work in a specific gauge [23,24]. A more elegant possibility is to use the Gupta-Bleuler [25–27] formalism of indefinite metrics on the Hilbert space of states. Both of these forms of quantization are discussed below.

A. Direct quantization in the Coulomb gauge

This is probably the simplest and most direct method of quantizing the electromagnetic field. The problem so far stems from the fact that the vector potential has four polarization states, while it is known that the photons of the free field theory only come in two different polarizations. Thus, before the theory is quantized we would like to remove the two superfluous polarizations at the classical level. To do this we require that solutions to the wave equation (10) also satisfy the Lorentz gauge condition

$$\nabla^\alpha A_\alpha=0. \tag{16}$$

This removes one degree of freedom between the components of the vector potential. The next condition that we would like to require is that the time component of the four-vector potential vanish in some frame. To accomplish this we let ξ^α be a timelike vector field. Then we require that A_α satisfy the additional condition

$$\xi^\alpha A_\alpha=0. \tag{17}$$

It is this second condition that can be ensured by the homogeneous part of the gauge freedom. Also, note that is is not true that the Coulomb gauge is noncovariant as is sometimes stated.

In flat spacetime there is no preferred choice of ξ^α, however for the static metric of the form (3), a natural choice is to let ξ^α be the global timelike Killing vector field. This will be the same Killing vector that will be used to define the positive frequency mode functions. Since $\xi^\alpha \propto (1,0,0,0)$, the net effect is to set the A_0-component of the stress tensor equal to zero. This solves two problems simultaneously. First it removes A_0 from the action, thus there is no longer a problem of it not having a conjugate momenta. Secondly, it has reduced the physical degrees of freedom of the solution to the two physically realizable photon states.

Canonical quantization is now straightforward. The metric (3) possesses a timelike Killing vector, which allows us to write the positive frequency mode function solutions of the wave equation (10) as

$$A_\alpha(\lambda,\mathbf{k};\mathbf{x},t)=U_\alpha(\lambda,\mathbf{k};\mathbf{x})e^{-i\omega t}, \tag{18}$$

where $\mathbf{k}$ is the mode label for the propagation vector, λ is the polarization state and $\omega=\omega(\mathbf{k})$. The four-vector functions, $U_\mu(\lambda,\mathbf{k};\mathbf{x})$, are the spatial portion of the solution of the wave equation and carry all the information about the curvature of the spacetime. In addition they satisfy

$$\nabla^\alpha U_\alpha(\lambda,\mathbf{k};\mathbf{x})=0=\nabla^\alpha \overline{U_\alpha(\lambda,\mathbf{k};\mathbf{x})}. \tag{19}$$

The mode functions for the vector potential are normalized such that

$$\begin{aligned}(A(\lambda,\mathbf{k}),A(\lambda',\mathbf{k}'))=-i\int_\Sigma d\Sigma_\mu[&\overline{A_\nu(\lambda,\mathbf{k})}F^{\mu\nu}(\lambda',\mathbf{k}')\\ &-\overline{F^{\mu\nu}(\lambda,\mathbf{k})}A_\nu(\lambda',\mathbf{k}')]\\ =\delta^{\lambda\lambda'}\delta^3(\mathbf{k}-\mathbf{k}'),\end{aligned} \tag{20}$$

where $d\Sigma_\mu=d\sigma\, n_\mu$ is a three-volume element in the Cauchy surface Σ with unit normal n^μ, thus each mode contributes $\frac{1}{2}\omega$ to the vacuum expectation value of the stress tensor before renormalization. The general solution to the vector potential can then be expanded as

$$\begin{aligned}A_\mu(\mathbf{x},t)=\sum_{\lambda=1}^{2}\int d^3\mathbf{k}[&a_\lambda(\mathbf{k})A_\mu(\lambda,\mathbf{k};\mathbf{x},t)\\ &+a_\lambda^\dagger(\mathbf{k})\overline{A_\mu(\lambda,\mathbf{k};\mathbf{x},t)}].\end{aligned} \tag{21}$$

Here $a_\lambda^\dagger(\mathbf{k})$ and $a_\lambda(\mathbf{k})$ are the creation and annihilation operators for the photon which obey the commutation relations

$$[a_\lambda(\mathbf{k}),a_{\lambda'}(\mathbf{k}')]=0=[a_\lambda^\dagger(\mathbf{k}),a_{\lambda'}^\dagger(\mathbf{k}')] \tag{22}$$

and

$$[a_\lambda(\mathbf{k}),a^\dagger_{\lambda'}(\mathbf{k}')]=\delta_{\lambda\lambda'}\delta(\mathbf{k}-\mathbf{k}'). \tag{23}$$

The Fock representation of the number states can now be constructed from the vacuum state denoted by $|0;0\rangle$ where the first slot is for particles of polarization type 1 and the second slot is for polarization type 2. The vacuum state has the property

$$a_\lambda(\mathbf{k})|0;0\rangle=0, \quad \forall\{\lambda,\mathbf{k}\}. \tag{24}$$

One-particle states are obtained by acting on the vacuum with the creation operator,

$$|1_\mathbf{k};0\rangle=a^\dagger_1(\mathbf{k})|0;0\rangle \quad \text{and} \quad |0;1_\mathbf{k}\rangle=a^\dagger_2(\mathbf{k})|0;0\rangle. \tag{25}$$

Multi-particle states can likewise be created by repeated application of the creation operators,

$$|{}^1m_{\mathbf{k}_1},\ldots,{}^jm_{\mathbf{k}_j};{}^1n_{\mathbf{k}_1},\ldots,{}^jn_{\mathbf{k}_j}\rangle = \frac{[a^\dagger_1(\mathbf{k}_1)]^{{}^1m}\ldots[a^\dagger_1(\mathbf{k}_j)]^{{}^jm}[a^\dagger_2(\mathbf{k}_1)]^{{}^1n}\ldots[a^\dagger_2(\mathbf{k}_j)]^{{}^jn}}{({}^1m!\ldots{}^jm!\,{}^1n!\ldots{}^jn!)^{1/2}}|0;0\rangle, \tag{26}$$

where the $\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_j$ are all distinct. The above state contains ${}^1m+{}^2m+\ldots+{}^jm+{}^1n+{}^2n+\ldots+{}^jn$ total particles where ${}^1m$ of them are of momentum $\mathbf{k}_1$ and polarization 1, ${}^1n$ are of momentum $\mathbf{k}_1$ and polarization 2, etc. Effectively, the general number states are a direct product of elements from two different Hilbert spaces, one for each of the polarization states. In order to reduce the index notation to a more manageable form, define the two vectors

$$\mathbf{m}=({}^1m_{\mathbf{k}_1},\ldots,{}^jm_{\mathbf{k}_j}) \quad \text{and} \quad \mathbf{n}=({}^1n_{\mathbf{k}_1},\ldots,{}^jn_{\mathbf{k}_j}), \tag{27}$$

then the states can be written more simply as $|\mathbf{m};\mathbf{n}\rangle$. The most general state that can then be written as a linear superposition of all the possible number states is

$$|\psi\rangle=\sum_{\mathbf{m},\mathbf{n}} c(\mathbf{m},\mathbf{n})|\mathbf{m};\mathbf{n}\rangle, \tag{28}$$

where $c(\mathbf{m},\mathbf{n})$ are complex coefficients and the sum is assumed to range over all the allowed vectors of $\mathbf{m}$ and $\mathbf{n}$. For the state to be properly normalized, the $c(\mathbf{m},\mathbf{n})$'s must satisfy

$$\sum_{\mathbf{m},\mathbf{n}} |c(\mathbf{m},\mathbf{n})|^2=1. \tag{29}$$

B. Gupta-Bleuler formalism

A more elegant form of quantization is to use the Gupta-Bleuler method of imposing an indefinite metric on the Hilbert space of allowable states. We begin by forming the Gupta action,

$$S^{Gupta}=S^{Maxwell}+S^{G.B.}, \tag{30}$$

where $S^{Maxwell}$ is the Maxwell action given by Eq. (6), and the gauge breaking action is given by

$$S^{G.B.}=-\frac{1}{2}\int_V(\nabla^\alpha A_\alpha)^2\sqrt{-g}\,d^4x. \tag{31}$$

Variation of the new action with respect to A^α yields the wave equation,

$$\nabla^\alpha F_{\alpha\beta}+\nabla_\beta(\nabla^\alpha A_\alpha)=0, \tag{32}$$

which can be rewritten in terms of A_α as

$$\nabla^\alpha\nabla_\alpha A_\beta-R_\beta{}^\alpha A_\alpha=0. \tag{33}$$

This would correspond to Maxwell's equations if the field also satisfied the Lorentz gauge condition.

There are four possible solutions (polarizations) to the above wave equation. First, there are the two physical polarizations which are labeled with $\lambda=1$ or 2. These two polarizations satisfy the wave equation (33) and the Lorentz condition,

$$\nabla^\alpha A_\alpha(\lambda,\mathbf{k};\mathbf{x},t)=0 \quad \text{for} \quad \lambda=1,2. \tag{34}$$

Thus, these two polarizations correspond to the two standard solutions to Maxwell's equations. The remaining two unphysical polarizations, labeled with $\lambda=0$ or 3, also satisfy the wave equation (33), but not necessarily the Lorentz condition. For ultra-static spacetimes, where $|g_{00}|=1$, the most natural choice is to use the scalar photon polarization,

$$A_\alpha(0,\mathbf{k};\mathbf{x},t)=\frac{1}{\omega}(\partial_t,0,0,0)\,\phi(\mathbf{k};\mathbf{x},t), \tag{35}$$

and the longitudinal photon polarization,

$$A_\alpha(3,\mathbf{k};\mathbf{x},t)=\frac{1}{\omega}(0,\partial_{x^1},\partial_{x^2},\partial_{x^3})\,\phi(\mathbf{k};\mathbf{x},t), \tag{36}$$

where $\phi(\mathbf{k};\mathbf{x},t)$ is the solution to the massless, minimally coupled scalar wave equation,

$$\nabla^\alpha\nabla_\alpha\phi(\mathbf{k};\mathbf{x},t)=0. \tag{37}$$

In the more general case of a static spacetime, it is useful to choose the two orthogonal modes which satisfy the condition,

$$A_\alpha(0,\mathbf{k};\mathbf{x},t)+A_\alpha(3,\mathbf{k};\mathbf{x},t)=\frac{1}{\omega}\nabla_\alpha\phi(\mathbf{k};\mathbf{x},t). \tag{38}$$

In both cases, the resulting modes then satisfy

$$\nabla^\alpha A_\alpha(0,\mathbf{k};\mathbf{x},t)=-\nabla^\alpha A_\alpha(3,\mathbf{k};\mathbf{x},t) \tag{39}$$

and

$$F_{\alpha\beta}(0,\mathbf{k};\mathbf{x},t)=-F_{\alpha\beta}(3,\mathbf{k};\mathbf{x},t), \tag{40}$$

for every momenta $\mathbf{k}$.

In addition, if we define the generalized conjugate momenta,

$$\Pi^{\mu\nu} \equiv -(F^{\mu\nu} + g^{\mu\nu}\nabla^{\alpha}A_{\alpha}), \tag{41}$$

then the modes are required to be orthogonal and normalized by

$$\begin{aligned}(A(\lambda,\mathbf{k}),A(\lambda',\mathbf{k}')) &= i\int_{\Sigma} d\Sigma_{\mu}[\overline{A_{\nu}(\lambda,\mathbf{k})}\Pi^{\mu\nu}(\lambda',\mathbf{k}') \\ &\quad - \overline{\Pi^{\mu\nu}(\lambda,\mathbf{k})}A_{\nu}(\lambda',\mathbf{k}')] \\ &= \eta^{\lambda\lambda'}\delta^{3}(\mathbf{k}-\mathbf{k}'),\end{aligned} \tag{42}$$

where $d\Sigma_{\mu} = d\sigma\, n_{\mu}$ is a three-volume element in the Cauchy surface Σ with unit normal n^{μ}, and $\eta^{\lambda\lambda'} = \eta_{\lambda\lambda'} = \mathrm{diag}(-1,1,1,1)$. The general solution to A_{μ} then has the Fourier mode-decomposition

$$\begin{aligned}A_{\mu}(\mathbf{x},t) = \sum_{\lambda=0}^{3}\int d^{3}\mathbf{k}[a_{\lambda}(\mathbf{k})A_{\mu}(\lambda,\mathbf{k};\mathbf{x},t) \\ + a_{\lambda}^{\dagger}(\mathbf{k})\overline{A_{\mu}(\lambda,\mathbf{k};\mathbf{x},t)}].\end{aligned} \tag{43}$$

If we wish to canonically quantize the field A_{μ}, we impose the equal-time commutation relations

$$[A_{\mu}(\mathbf{x},t),A_{\nu}(\mathbf{x}',t)] = 0 = [\Pi^{t\mu}(\mathbf{x},t),\Pi^{t\nu}(\mathbf{x}',t)] \tag{44}$$

and

$$[A_{\mu}(\mathbf{x},t),\Pi^{t\nu}(\mathbf{x}',t)] = \frac{i\,\delta_{\mu}^{\nu}}{\sqrt{-g}}\delta^{3}(\mathbf{x}-\mathbf{x}'). \tag{45}$$

Using the mode decomposition and the normalization condition, we find that the above equal-time commutation relations are equivalent to

$$[a_{\lambda}(\mathbf{k}),a_{\lambda'}^{\dagger}(\mathbf{k}')] = \eta_{\lambda\lambda'}\delta^{3}(\mathbf{k}-\mathbf{k}'), \tag{46}$$

with all other commutators vanishing.

The state structure is similar in form to that found for the Coulomb gauge, except there are now a greater number of allowable states due to the two unphysical polarizations. We now define the vacuum state as $|0;0;0;0\rangle$ where the first slot is for photons of the unphysical polarization $\lambda=0$, the second and third slots are for the two real photon polarizations, and the final slot is for the unphysical polarization with $\lambda=3$. The vacuum state vanishes if any of the four destruction operators act on it, and multi-particle states are again obtained by the repeated application of the creation operators. Unlike the states for the Coulomb gauge quantization, the states of the Gupta-Bleuler formalism have indefinite norm,

$$\begin{aligned}&\langle \mathbf{l},\mathbf{m},\mathbf{n},\mathbf{p}|\mathbf{l}',\mathbf{m}',\mathbf{n}',\mathbf{p}'\rangle \\ &= (-1)^{{}^{1}l+{}^{2}l+\ldots+{}^{j}l}\delta_{\mathbf{l}\,\mathbf{l}'}\delta_{\mathbf{m}\,\mathbf{m}'}\delta_{\mathbf{n}\,\mathbf{n}'}\delta_{\mathbf{p}\,\mathbf{p}'},\end{aligned} \tag{47}$$

where we have added two new vectors, **l** and **p**, for the unphysical photon polarization states. The most general state in the Gupta-Bleuler formulation can be written as a superposition of all the particle number states as

$$|\phi\rangle = \sum_{\mathbf{l},\mathbf{m},\mathbf{n},\mathbf{p}} c(\mathbf{l},\mathbf{m},\mathbf{n},\mathbf{p})|\mathbf{l},\mathbf{m},\mathbf{n},\mathbf{p}\rangle. \tag{48}$$

In order for the Gupta-Bleuler formalism to be equivalent to Maxwell's theory, we need to impose an additional condition on the Hilbert space of states, namely that the expectation value of the Lorentz condition be satisfied for all physically realizable states $|\phi\rangle$,

$$\langle\phi|\nabla^{\alpha}A_{\alpha}(\mathbf{x},t)|\phi\rangle = 0. \tag{49}$$

This condition can be accomplished simply by requiring that the states obey

$$\nabla^{\alpha}A_{\alpha}^{+}(\mathbf{x},t)|\phi\rangle = 0, \tag{50}$$

where A_{α}^{+} is the positive frequency part of A_{α}. The application of this condition to the state $|\phi\rangle$ above means that the $c(\mathbf{l},\mathbf{m},\mathbf{n},\mathbf{p})$'s with the same total number of $\lambda=0$ and 3 photons of the same momenta are related to one another by

$$\begin{aligned}&\sqrt{l_{\mathbf{k}}}\nabla^{\alpha}A_{\alpha}(0,\mathbf{k})c(l_{\mathbf{k}},\mathbf{m},\mathbf{n},(p-1)_{\mathbf{k}}) \\ &+ \sqrt{p_{\mathbf{k}}}\nabla^{\alpha}A_{\alpha}(3,\mathbf{k})c((l-1)_{\mathbf{k}},\mathbf{m},\mathbf{n},p_{\mathbf{k}}) = 0.\end{aligned} \tag{51}$$

Under this constraint the Hilbert space structure of the state $|\phi\rangle$ takes the form,

$$\begin{aligned}|\phi\rangle &= \cdots + c(0_{\mathbf{k}},\mathbf{m},\mathbf{n},0_{\mathbf{k}})|0_{\mathbf{k}},\mathbf{m},\mathbf{n},0_{\mathbf{k}}\rangle + \cdots + \frac{c(1_{\mathbf{k}},\mathbf{m},\mathbf{n},0_{\mathbf{k}})}{\nabla^{\alpha}A_{\alpha}(3,\mathbf{k})}[\nabla^{\alpha}A_{\alpha}(3,\mathbf{k})|1_{\mathbf{k}},\mathbf{m},\mathbf{n},0_{\mathbf{k}}\rangle - \nabla^{\alpha}A_{\alpha}(0,\mathbf{k})|0_{\mathbf{k}},\mathbf{m},\mathbf{n},1_{\mathbf{k}}\rangle] \\ &\quad + \cdots - \frac{c(1_{\mathbf{k}},\mathbf{m},\mathbf{n},1_{\mathbf{k}})}{\sqrt{2}\nabla^{\alpha}A_{\alpha}(0,\mathbf{k})\nabla^{\alpha}A_{\alpha}(3,\mathbf{k})}\{[\nabla^{\alpha}A_{\alpha}(3,\mathbf{k})]^{2}|2_{\mathbf{k}},\mathbf{m},\mathbf{n},0_{\mathbf{k}}\rangle \\ &\quad - \sqrt{2}\nabla^{\alpha}A_{\alpha}(0,\mathbf{k})\nabla^{\alpha}A_{\alpha}(3,\mathbf{k})|1_{\mathbf{k}},\mathbf{m},\mathbf{n},1_{\mathbf{k}}\rangle - [\nabla^{\alpha}A_{\alpha}(0,\mathbf{k})]^{2}|0_{\mathbf{k}},\mathbf{m},\mathbf{n},2_{\mathbf{k}}\rangle\} + \cdots.\end{aligned} \tag{52}$$

With the definition of a new operator

$$O^{\dagger}(\mathbf{k})=\nabla^{\alpha}A_{\alpha}(3,\mathbf{k})a_0^{\dagger}(\mathbf{k})-\nabla^{\alpha}A_{\alpha}(0,\mathbf{k})a_3^{\dagger}(\mathbf{k}), \tag{53}$$

it is possible to define a new set of states, $|\mathbf{m},\mathbf{n},\mathbf{q}\rangle$, where each ${}^{i}q_{\mathbf{k}_i}$ in $\mathbf{q}$ is the total number of $\lambda=0$ and 3 photons of momenta $\mathbf{k}_i$. The new states are formed by the repeated action of the operator $O^{\dagger}(\mathbf{k})$ acting on the state with zero unphysical photons,

$$|\mathbf{m},\mathbf{n},\{{}^{1}q_{\mathbf{k}_1},{}^{2}q_{\mathbf{k}_2},\ldots,{}^{i}q_{\mathbf{k}_i}\}\rangle = \frac{[O^{\dagger}(\mathbf{k}_1)]^{{}^{1}q}[O^{\dagger}(\mathbf{k}_2)]^{{}^{2}q}\ldots[O^{\dagger}(\mathbf{k}_i)]^{{}^{i}q}}{({}^{1}q!\,{}^{2}q!\ldots{}^{i}q!)^{1/2}}|\mathbf{0},\mathbf{m},\mathbf{n},\mathbf{0}\rangle. \tag{54}$$

The inner products of the new states are

$$\langle\mathbf{m},\mathbf{n},\mathbf{0}|\mathbf{m},\mathbf{n},\mathbf{0}\rangle=1 \tag{55}$$

for the states with no unphysical photons and

$$\langle\mathbf{m},\mathbf{n},q_{\mathbf{k}}|\mathbf{m},\mathbf{n},q_{\mathbf{k}}\rangle=\{|\nabla^{\alpha}A_{\alpha}(0,\mathbf{k})|^2-|\nabla^{\alpha}A_{\alpha}(3,\mathbf{k})|^2\}^{q}=0 \tag{56}$$

for all other states. It is now possible to rewrite the superposition of particle number states (48) with the embodiment of the supplementary condition (50) built in as

$$|\phi\rangle=\sum_{\mathbf{m},\mathbf{n},\mathbf{q}} b(\mathbf{m},\mathbf{n},\mathbf{q})|\mathbf{m},\mathbf{n},\mathbf{q}\rangle. \tag{57}$$

The stress tensor found from the Gupta action is

$$T_{\rho\sigma}^{Gupta}=T_{\rho\sigma}^{Maxwell}+T_{\rho\sigma}^{G.B.}, \tag{58}$$

where $T_{\rho\sigma}^{Maxwell}$ is given by Eq. (11) and the contribution to the stress-tensor from the gauge breaking term is

$$T_{\rho\sigma}^{G.B.}=-A_{\rho}(\nabla_{\sigma}\nabla^{\alpha}A_{\alpha})-A_{\sigma}(\nabla_{\rho}\nabla^{\alpha}A_{\alpha}) +g_{\rho\sigma}\left[A_{\beta}\nabla^{\beta}\nabla^{\alpha}A_{\alpha}+\frac{1}{2}(\nabla^{\alpha}A_{\alpha})^2\right]. \tag{59}$$

Due to the physical photon polarization modes satisfying the Lorentz condition (34) and the Hilbert space of states satisfying the subsidiary condition (50), it is relatively straightforward to show that the expectation value of the normal ordered gauge-breaking portion of the stress-tensor vanishes,

$$\langle\phi|:T_{\rho\sigma}^{G.B.}:|\phi\rangle=0. \tag{60}$$

In addition, due to the relationships between the $c(\mathbf{l},\mathbf{m},\mathbf{n},\mathbf{p})$ coefficients and Eq. (40), it is simple to show for the normal-ordered Maxwell portion of the stress-tensor that the unphysical photon modes do not contribute, thus

$$\langle\phi|:T_{\rho\sigma}^{Maxwell}:|\phi\rangle=\langle\psi|:T_{\rho\sigma}^{Maxwell}:|\psi\rangle, \tag{61}$$

where

$$|\psi\rangle=\sum_{\mathbf{m},\mathbf{n}} c(0,\mathbf{m},\mathbf{n},0)|0,\mathbf{m},\mathbf{n},0\rangle=\sum_{\mathbf{m},\mathbf{n}} c(\mathbf{m},\mathbf{n})|\mathbf{m},\mathbf{n}\rangle. \tag{62}$$

Thus, the only physically observable states are the two physical photon polarization states. In summary we have

$$\langle\phi|:T_{\rho\sigma}^{Gupta}:|\phi\rangle=\langle\phi|:T_{\rho\sigma}^{Maxwell}+T_{\rho\sigma}^{G.B.}:|\phi\rangle =\langle\psi|:T_{\rho\sigma}^{Maxwell}:|\psi\rangle. \tag{63}$$

III. POSITIVITY RESULT

In this section we prove the following inequality: Let M^{ij} be a real, symmetric $n\times n$ matrix with non-negative eigenvalues. Further let $P_i(\lambda,\mathbf{k})$ be a complex n-vector, which is a function of the mode labels $\mathbf{k}$ and λ. Also, let $f(t)$ be a smooth, non-negative function on $\mathbf{R}$ which decays rapidly at infinity, with pointwise square root $f^{1/2}(t)=\sqrt{f(t)}$ and Fourier transform given by

$$\hat{f}(\omega)\equiv\int_{-\infty}^{\infty}dt\, f(t)e^{-i\omega t}. \tag{64}$$

Then in an arbitrary quantum state $|\psi\rangle$, the following inequality holds:

$$\begin{aligned}\mathrm{Re}\sum_{\lambda,\lambda'}\int d^3\mathbf{k}\,d^3\mathbf{k}'\{\hat{f}[\omega(\mathbf{k}')-\omega(\mathbf{k})]\langle a_{\lambda}^{\dagger}(\mathbf{k})a_{\lambda'}(\mathbf{k}')\rangle\\ \times\overline{P_i(\lambda,\mathbf{k})}M^{ij}P_j(\lambda',\mathbf{k}')\pm\hat{f}[\omega(\mathbf{k})+\omega(\mathbf{k}')]\\ \times\langle a_{\lambda}(\mathbf{k})a_{\lambda'}(\mathbf{k}')\rangle P_i(\lambda,\mathbf{k})M^{ij}P_j(\lambda',\mathbf{k}')\}\\ \geqslant-\frac{1}{2\pi}\int_0^{\infty}d\nu\sum_{\lambda}\int d^3\mathbf{k}|\hat{f}^{1/2}[\nu+\omega(\mathbf{k})]|^2\\ \times\overline{P_i(\lambda,\mathbf{k})}M^{ij}P_j(\lambda,\mathbf{k}).\end{aligned} \tag{65}$$

The above inequality is a generalization of the scalar field positivity lemma derived by Fewster and colleagues [14,15]. In order to prove this relation, first define the vector operator

$$[Q_{\nu}^{\pm}]_i=\sum_{\lambda}\int d^3\mathbf{k}\{\overline{g[\nu-\omega(\mathbf{k})]}a_{\lambda}(\mathbf{k})P_i(\lambda,\mathbf{k}) \pm\overline{g[\nu+\omega(\mathbf{k})]}a_{\lambda}^{\dagger}(\mathbf{k})\overline{P_i(\lambda,\mathbf{k})}\}, \tag{66}$$

where

$$g(\omega)=\frac{1}{\sqrt{2\pi}}\hat{f}^{1/2}(\omega). \tag{67}$$

From the definition of the convolution

$$(h_1\star h_2)(\omega)=\int_{-\infty}^{\infty}d\omega'\,h_1(\omega-\omega')h_2(\omega'), \tag{68}$$

it follows that $(g\star g)=\hat{f}$.

Next, note that

$$M^{ij}=\sum_{\alpha=1}^{n}\kappa^{\alpha}V^{i}_{(\alpha)}V^{j}_{(\alpha)}, \tag{69}$$

where the $V^{i}_{(\alpha)}$ are the eigenvectors of M^{ij}, and the $\kappa^{\alpha}\geqslant 0$ are the corresponding eigenvalues. Now

$$\langle [Q^{\pm}_{\nu}]^{\dagger}_{i}M^{ij}[Q^{\pm}_{\nu}]_{j}\rangle=\sum_{\alpha=1}^{n}\kappa^{\alpha}\langle [Q^{\pm}_{\nu}]^{\dagger}_{i}V^{i}_{(\alpha)}V^{j}_{(\alpha)}[Q^{\pm}_{\nu}]_{j}\rangle$$
$$=\sum_{\alpha=1}^{n}\kappa^{\alpha}|V^{i}_{(\alpha)}[Q^{\pm}_{\nu}]_{i}|\psi\rangle|^{2}\geqslant 0. \tag{70}$$

Furthermore, using the commutation relations and symmetrizing the integrand in $(\lambda,\mathbf{k})$ and $(\lambda',\mathbf{k}')$, we find

$$\int_{0}^{\infty}d\nu\langle [Q^{\pm}_{\nu}]^{\dagger}_{i}M^{ij}[Q^{\pm}_{\nu}]_{j}\rangle$$
$$=\langle S^{\pm}\rangle+\int_{0}^{\infty}d\nu\sum_{\lambda}\int d^{3}\mathbf{k}|g[\nu+\omega(\mathbf{k})]|^{2}$$
$$\times\overline{P_{i}(\lambda,\mathbf{k})}M^{ij}P_{j}(\lambda,\mathbf{k}), \tag{71}$$

where

$$\langle S^{\pm}\rangle=\mathrm{Re}\sum_{\lambda,\lambda'}\int d^{3}\mathbf{k}\,d^{3}\mathbf{k}'[F(\mathbf{k},\mathbf{k}')\langle a^{\dagger}_{\lambda}(\mathbf{k})a_{\lambda'}(\mathbf{k}')\rangle$$
$$\times\overline{P_{i}(\lambda,\mathbf{k})}M^{ij}P_{j}(\lambda',\mathbf{k}')\pm G(\mathbf{k},\mathbf{k}')$$
$$\times\langle a_{\lambda}(\mathbf{k})a_{\lambda'}(\mathbf{k}')\rangle P_{i}(\lambda,\mathbf{k})M^{ij}P_{j}(\lambda',\mathbf{k}')] \tag{72}$$

and the functions F and G are given by

$$F(\mathbf{k},\mathbf{k}')=\int_{0}^{\infty}d\nu\{g[\nu-\omega(\mathbf{k})]\overline{g[\nu-\omega(\mathbf{k}')]}$$
$$+\overline{g[\nu+\omega(\mathbf{k})]}g[\nu+\omega(\mathbf{k}')]\} \tag{73}$$

and

$$G(\mathbf{k},\mathbf{k}')=\int_{0}^{\infty}d\nu\{g[\nu+\omega(\mathbf{k})]\overline{g[\nu-\omega(\mathbf{k}')]}$$
$$+\overline{g[\nu-\omega(\mathbf{k})]}g[\nu+\omega(\mathbf{k}')]\}. \tag{74}$$

The expressions for F and G may be simplified to

$$F(\mathbf{k},\mathbf{k}')=\int_{-\infty}^{\infty}d\nu\,g[\omega(\mathbf{k}')-\nu]g[\nu-\omega(\mathbf{k})]$$
$$=(g\star g)[\omega(\mathbf{k}')-\omega(\mathbf{k})]$$
$$=\hat{f}[\omega(\mathbf{k}')-\omega(\mathbf{k})] \tag{75}$$

and

$$G(\mathbf{k},\mathbf{k}')=\hat{f}[\omega(\mathbf{k})+\omega(\mathbf{k}')]. \tag{76}$$

From Eq. (70) we know that the right hand side of Eq. (71) is manifestly positive, so we conclude that $\langle S^{\pm}\rangle$ obeys the following bound:

$$\langle S^{\pm}\rangle\geqslant-\int_{0}^{\infty}d\nu\sum_{\lambda}\int d^{3}\mathbf{k}|g[\nu+\omega(\mathbf{k})]|^{2}$$
$$\times\overline{P_{i}(\lambda,\mathbf{k})}M^{ij}P_{j}(\lambda,\mathbf{k}),$$
$$=-\frac{1}{2\pi}\int_{0}^{\infty}d\nu\sum_{\lambda}\int d^{3}\mathbf{k}|\hat{f}^{1/2}[\nu+\omega(\mathbf{k})]|^{2}$$
$$\times\overline{P_{i}(\mathbf{k},\lambda)}M^{ij}P_{j}(\lambda,\mathbf{k}), \tag{77}$$

thus proving Eq. (65).

IV. THE QUANTUM INEQUALITY

Consider a stationary observer whose four-velocity is given by

$$u^{\mu}=(|g_{00}|^{-1/2},0,0,0). \tag{78}$$

In both the simple quantization scheme using the Coulomb gauge and in the Gupta-Bleuler quantization scheme, the energy density measured by this observer is given by the Maxwell portion of the stress-tensor,

$$\rho=T^{Maxwell}_{\mu\nu}u^{\mu}u^{\nu}$$
$$=\frac{1}{2}\left[\frac{1}{|g_{00}|}F_{i0}g^{ij}F_{j0}+\frac{1}{2}F_{ij}g^{il}g^{jm}F_{lm}\right]. \tag{79}$$

Now make the identification

$$E_{i}=F_{i0}\quad\text{and}\quad B_{i}=\frac{1}{2}\varrho_{ijk}F_{jk}, \tag{80}$$

where ϱ_{ijk} is the completely antisymmetric Levi-Civita symbol. The energy density can then be written as

$$\rho=\frac{1}{2}|g|^{-1/2}[E_{i}\hat{\epsilon}^{ij}E_{j}+B_{i}(\hat{\epsilon}^{ij})^{-1}B_{j}], \tag{81}$$

where $\hat{\epsilon}$ is an ordinary 3×3 matrix with elements

$$\hat{\epsilon}=\hat{\epsilon}(\mathbf{x})=\frac{\sqrt{-g}}{|g_{00}|}\begin{pmatrix} g^{11} & g^{12} & g^{13}\\ g^{21} & g^{22} & g^{23}\\ g^{31} & g^{32} & g^{33}\end{pmatrix}. \tag{82}$$

The definitions of E_i, B_i and $\hat{\epsilon}$ have been shown to recast the curved space Maxwell field equations into the form of the Maxwell equations inside an anisotropic material medium in Cartesian coordinates. In this interpretation, $\hat{\epsilon}$ plays the role of the dielectric tensor in the constitutive relations. We will not push this interpretation any further and refer the reader to Refs. [28–30] for further discussion.

Upon substitution of the mode function expansion into the stress-tensor, and making use of constitutive relations and the commutation relations we find

$$\rho=|g|^{-1/2}\Bigg\{\mathrm{Re}\sum_{\lambda\lambda'}\int d^3\mathbf{k}\,d^3\mathbf{k}'[a_\lambda^\dagger(\mathbf{k})a_{\lambda'}(\mathbf{k}')\overline{E_i(\lambda,\mathbf{k};\mathbf{x})}\hat{\epsilon}^{ij}E_j(\lambda',\mathbf{k}';\mathbf{x})e^{i(\omega-\omega')t}$$
$$+a_\lambda(\mathbf{k})a_{\lambda'}(\mathbf{k}')E_i(\lambda,\mathbf{k};\mathbf{x})\hat{\epsilon}^{ij}E_j(\lambda',\mathbf{k}';\mathbf{x})e^{-i(\omega+\omega')t}]+\mathrm{Re}\sum_{\lambda\lambda'}\int d^3\mathbf{k}\,d^3\mathbf{k}'[a_\lambda^\dagger(\mathbf{k})a_{\lambda'}(\mathbf{k}')\overline{B_i(\lambda,\mathbf{k};\mathbf{x})}$$
$$\times(\hat{\epsilon}^{ij})^{-1}B_j(\lambda',\mathbf{k}';\mathbf{x})e^{i(\omega-\omega')t}+a_\lambda(\mathbf{k})a_{\lambda'}(\mathbf{k}')B_i(\lambda,\mathbf{k};\mathbf{x})(\hat{\epsilon}^{ij})^{-1}B_j(\lambda',\mathbf{k}';\mathbf{x})e^{-i(\omega+\omega')t}]$$
$$+\frac{1}{2}\sum_{\lambda}\int d^3\mathbf{k}[E_i(\lambda,\mathbf{k};\mathbf{x})\hat{\epsilon}^{ij}\overline{E_j(\lambda,\mathbf{k};\mathbf{x})}+B_i(\lambda,\mathbf{k};\mathbf{x})(\hat{\epsilon}^{ij})^{-1}\overline{B_j(\lambda,\mathbf{k};\mathbf{x})}]\Bigg\}, \tag{83}$$

where

$$E_i(\lambda,\mathbf{k};\mathbf{x})=\partial_i U_0(\lambda,\mathbf{k};\mathbf{x})+i\omega(\mathbf{k})U_i(\lambda,\mathbf{k};\mathbf{x}) \tag{84}$$

and

$$B_i(\lambda,\mathbf{k};\mathbf{x})=\varrho_{ijl}\partial_j U_l(\lambda,\mathbf{k};\mathbf{x}). \tag{85}$$

The last line of Eq. (83) is the vacuum self-energy of the photons. As was the case for the scalar field, we will look at the difference between the energy in an arbitrary state relative to the vacuum energy using the normal order prescription, i.e.,

$$:\rho:=\rho-\rho_{\mathrm{vacuum}}. \tag{86}$$

It is now our intention to show that given a temporal sampling function $f(t)$ then the sampled energy density defined by

$$\Delta\tilde{\rho}=\int_{-\infty}^{\infty}dt\langle:\rho(\mathbf{x},t):\rangle f(t), \tag{87}$$

is bounded from below. Using Eqs. (83) and (86), along with the definitions of $F(\mathbf{k},\mathbf{k}')$ and $G(\mathbf{k},\mathbf{k}')$ given by Eqs. (73) and (74), the sampled energy density is

$$\Delta\tilde{\rho}=\frac{1}{|g_{00}|}\mathrm{Re}\sum_{\lambda\lambda'}\int d^3\mathbf{k}\,d^3\mathbf{k}'\,\omega(\mathbf{k})\omega(\mathbf{k}')[F(\mathbf{k},\mathbf{k}')\langle a_\lambda^\dagger(\mathbf{k})a_{\lambda'}(\mathbf{k}')\rangle\overline{E_i(\lambda,\mathbf{k};\mathbf{x})}g^{ij}E_j(\lambda',\mathbf{k}';\mathbf{x})+G(\mathbf{k},\mathbf{k}')$$
$$\times\langle a_\lambda(\mathbf{k})a_{\lambda'}(\mathbf{k}')\rangle E_i(\lambda,\mathbf{k};\mathbf{x})g^{ij}E_j(\lambda',\mathbf{k}';\mathbf{x})]+\left|\frac{g_{00}}{g}\right|\mathrm{Re}\sum_{\lambda\lambda'}\int d^3\mathbf{k}\,d^3\mathbf{k}'[F(\mathbf{k},\mathbf{k}')\langle a_\lambda^\dagger(\mathbf{k})a_{\lambda'}(\mathbf{k}')\rangle$$
$$\times\overline{B_i(\lambda,\mathbf{k};\mathbf{x})}(g^{ij})^{-1}B_j(\mathbf{k}',\lambda';\mathbf{x})+G(\mathbf{k},\mathbf{k}')\langle a_\lambda(\mathbf{k})a_{\lambda'}(\mathbf{k}')\rangle B_i(\lambda,\mathbf{k};\mathbf{x})(g^{ij})^{-1}B_j(\lambda',\mathbf{k}';\mathbf{x})]. \tag{88}$$

Clearly, both parts of the above expression are of the form $\langle S^{\pm}\rangle$, so we may apply the bound (65) with $M^{ij}=g^{ij}$ and $P_i(\lambda,\mathbf{k})=E_i(\lambda,\mathbf{k};\mathbf{x})$ for the first part of the expression and $M^{ij}=(g^{ij})^{-1}$ and $P_i(\lambda,\mathbf{k})=B_i(\lambda,\mathbf{k};\mathbf{x})$ for the second part of the expression. This yields a difference inequality of

$$\Delta\tilde{\rho}\geqslant-\frac{1}{2\pi}\int_0^\infty d\nu\sum_\lambda\int d^3\mathbf{k}|\hat{f}^{1/2}[\nu+\omega(\mathbf{k})]|^2$$
$$\times\left[\frac{1}{|g_{00}|}\overline{E_i(\lambda,\mathbf{k};\mathbf{x})}g^{ij}E_j(\lambda,\mathbf{k};\mathbf{x})\right.$$
$$\left.+\left|\frac{g_{00}}{g}\right|\overline{B_i(\lambda,\mathbf{k};\mathbf{x})}(g^{ij})^{-1}B_j(\lambda,\mathbf{k};\mathbf{x})\right]. \tag{89}$$

This expression is similar in form to the mode function expansion of the scalar field quantum inequality found by Fewster and colleagues [14,15]. The quantum inequality, Eq. (4), is found by adding the suitably renormalized vacuum energy density to the above expression.

V. EXAMPLES

A. Minkowski spacetime

This quantum inequality is easily evaluated in Minkowski spacetime with no boundaries. Using quantization in the Coulomb gauge, the four-vector mode functions are

$$A_\alpha(\lambda,\mathbf{k};\mathbf{x},t)=(0,\mathbf{A}(\lambda,\mathbf{k};\mathbf{x},t)), \tag{90}$$

where

$$\mathbf{A}(\lambda,\mathbf{k};\mathbf{x},t)=\frac{i}{\sqrt{2\omega(2\pi)^3}}\hat{\varepsilon}_{\mathbf{k}}^{\lambda}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{91}$$

TABLE I. Electromagnetic field quantum inequalities in Minkowski spacetime and difference inequalities in Rindler spacetime calculated for various unit area sampling functions.

Sampling function		Minkowski spacetime	Rindler spacetime
Lorentzian	$\frac{t_0}{\pi(t^2+t_0^2)}$	$\tilde{\rho}\geqslant-\frac{27}{1024\pi^2 t_0^4}$	$\Delta\tilde{\rho}\geqslant-\frac{27}{1024\pi^2\tau_0^4}\left[1+\frac{32}{27}\left(\frac{\tau_0}{\xi}\right)^2\right]$
Lorentzian^2	$\frac{2t_0^3}{\pi(t^2+t_0^2)^2}$	$\tilde{\rho}\geqslant-\frac{3}{16\pi^2 t_0^4}$	$\Delta\tilde{\rho}\geqslant-\frac{3}{16\pi^2\tau_0^4}\left[1+\frac{2}{3}\left(\frac{\tau_0}{\xi}\right)^2\right]$
Gaussian	$\frac{1}{\sqrt{\pi}t_0}e^{-(t/t_0)^2}$	$\tilde{\rho}\geqslant-\frac{3}{32\pi^2 t_0^4}$	$\Delta\tilde{\rho}\geqslant-\frac{3}{32\pi^2\tau_0^4}\left[1+\frac{4}{3}\left(\frac{\tau_0}{\xi}\right)^2\right]$
Cosine^4	$\begin{cases}\frac{4}{3t_0}\cos^4\left(\frac{\pi t}{2t_0}\right) & -t_0<t<t_0\\ 0 & \text{elsewhere}\end{cases}$	$\tilde{\rho}\geqslant-\frac{\pi^2}{96t_0^4}$	$\Delta\tilde{\rho}\geqslant-\frac{\pi^2}{96\tau_0^4}\left[1+\frac{8}{\pi^2}\left(\frac{\tau_0}{\xi}\right)^2\right]$

$\hat{\varepsilon}_{\mathbf{k}}^{\lambda}$ is a unit electric polarization vector and $\omega=\sqrt{\mathbf{k}\cdot\mathbf{k}}$. Due to the Coulomb gauge condition, the propagation vector is orthogonal to the polarization vector, i.e.,

$$\mathbf{k}\cdot\hat{\varepsilon}_{\mathbf{k}}^{\lambda}=\mathbf{0}. \tag{92}$$

A third, orthogonal unit vector along the magnetic field direction is defined by

$$\hat{b}_{\mathbf{k}}^{\lambda}=\hat{\mathbf{k}}\times\hat{\varepsilon}_{\mathbf{k}}^{\lambda}. \tag{93}$$

Inserting the mode functions into Eq. (89), and using $g^{ij}=\delta^{ij}$, we find

$$\begin{aligned}\tilde{\rho}\geqslant&-\frac{1}{(2\pi)^4}\int_0^\infty d\nu\sum_\lambda\int d^3\mathbf{k}\{\hat{f}^{1/2}[\nu+\omega(\mathbf{k})]\}^2\\&\times\omega(\mathbf{k})[\hat{\varepsilon}_{\mathbf{k}}^{\lambda}\cdot\hat{\varepsilon}_{\mathbf{k}}^{\lambda}+\hat{b}_{\mathbf{k}}^{\lambda}\cdot\hat{b}_{\mathbf{k}}^{\lambda}],\\&=-\frac{4}{(2\pi)^3}\int_0^\infty d\nu\int_0^\infty d\omega\,\omega^3\{\hat{f}^{1/2}[\nu+\omega]\}^2,\end{aligned} \tag{94}$$

where we have made a change of variable in the momentum integration to spherical coordinates and have already carried out the angular integration and summation over polarization states. The next step is to make another change of variable

$$u=\nu+\omega,\quad v=\omega, \tag{95}$$

to find

$$\begin{aligned}\tilde{\rho}\geqslant&-\frac{4}{(2\pi)^3}\int_0^\infty du[\hat{f}^{1/2}(u)]^2\int_0^u dv\,v^3,\\&=-\frac{1}{2(2\pi)^3}\int_{-\infty}^\infty du[u^2\hat{f}^{1/2}(u)]^2.\end{aligned} \tag{96}$$

Using Parseval's identity, the quantum inequality is found to be

$$\tilde{\rho}\geqslant-\frac{1}{8\pi^2}\int_{-\infty}^\infty dt\left[\frac{d^2}{dt^2}f^{1/2}(t)\right]^2. \tag{97}$$

This is the most general expression for the quantum inequality in Minkowski spacetime with an arbitrary sampling function. For the choice of a Lorentzian sampling function (2) with characteristic width t_0, it is straightforward to calculate

$$\tilde{\rho}\geqslant-\frac{27}{1024\,\pi^2 t_0^4}. \tag{98}$$

This is a slightly stronger result, by 9/64, than the inequality proven by Ford and Roman [7] using an alternative method. Comparison with the quantum inequality for the scalar field in Minkowski space derived by Fewster and Eveson [14], shows that the electromagnetic field quantum inequality in Minkowski space always differs by a factor of 2. This is a result of the electromagnetic field having two polarization degrees of freedom, unlike the scalar field which has only one, and both the scalar and electromagnetic field modes having the same energy spectrum. Electromagnetic field quantum inequalities for various sampling functions are summarized in Table I.

B. Rindler spacetime

Next, we would like to find the quantum inequality in Rindler spacetime. We begin with the Minkowski space length element,

$$ds^2=-dt^2+dx^2+dy^2+dz^2. \tag{99}$$

Next we apply the coordinate transformation

$$\begin{aligned}t&=\xi\sinh\eta,\\x&=\xi\cosh\eta,\end{aligned} \tag{100}$$

to arrive at the Rindler length element

$$ds^2=-\xi^2 d\eta^2+d\xi^2+dy^2+dz^2. \tag{101}$$

In this form, the metric is static, but the g_{00} component is not a constant, so we cannot quantize the theory in the Coulomb gauge but must use the Gupta-Bleuler formalism. Thus we are looking for mode solutions to the vector wave equation (33). These have been calculated by Candelas and Deutsch [31] for the two physical polarizations. The unphysical solutions have also been calculated [32,33]. The modes can be conveniently expressed in terms of the mode solutions to the massless scalar field wave equation in Rindler space,

$$\left(-\frac{1}{\xi^2}\partial_\eta^2+\frac{1}{\xi}\partial_\xi\xi\partial_\xi+\partial_y^2+\partial_z^2\right)\phi(x)=0. \tag{102}$$

The positive frequency scalar mode solutions, normalized for the Klein-Gordon inner product of scalar fields, are [34]

$$\phi(\omega,k_y,k_z;x)=\frac{2}{(2\pi)^2}(\sinh\omega\pi)^{1/2}K_{i\omega}(\beta\xi)e^{i(k_y y+k_z z-\omega\eta)}, \tag{103}$$

where $\beta=(k_y^2+k_z^2)^{1/2}$ and $K_{i\nu}(x)$ are the modified Bessel functions of the second kind (Macdonald functions) of *imaginary order*. The two physical modes that are important to our calculations are the transverse electric modes (TE),

$$A_\alpha(1,\omega,k_y,k_z;x)=\frac{1}{\beta}(0,0,\partial_z,-\partial_y)\phi(\omega,k_y,k_z;x), \tag{104}$$

and the transverse magnetic modes (TM),

$$A_\alpha(2,\omega,k_y,k_z;x)=\frac{1}{\beta}\left(\xi\partial_\xi,\frac{1}{\xi}\partial_\eta,0,0\right)\phi(\omega,k_y,k_z;x). \tag{105}$$

These two modes are properly normalized with respect to Eq. (42) and are also orthogonal. If they are inserted into Eq. (89) for the difference inequality, and after a little algebra, we find

$$\begin{aligned}\Delta\tilde{\rho}\geqslant&-\frac{1}{\pi}\int_0^\infty d\nu\int_0^\infty d\omega|\hat{f}^{1/2}[\nu+\omega]|^2\int_{R^2}dk_y dk_z\bigg[\beta^2\bar{\phi}\phi\\&+\frac{\omega^2}{\xi^2}\bar{\phi}\phi+(\partial_\xi\bar{\phi})(\partial_\xi\phi)\bigg],\\=&-\frac{1}{\pi^4}\int_0^\infty d\nu\int_0^\infty d\omega|\hat{f}^{1/2}[\nu+\omega]|^2\left(\frac{\omega^2}{\xi^2}+\frac{1}{4\xi}\partial_\xi\xi\partial_\xi\right)\\&\times\sinh(\pi\omega)\int_0^\infty d\beta\,\beta K_{i\omega}^2(\beta\xi),\end{aligned} \tag{106}$$

where we have switched to polar coordinates to carry out the angular portion of the $dk_y dk_z$ integrals. With the aid of Eq. 6.521.3 of [35], it is easily demonstrated that

$$\int_0^\infty d\beta\,\beta K_{i\omega}^2(\beta\xi)=\frac{\pi\omega}{2\xi^2\sinh(\pi\omega)}. \tag{107}$$

Thus

$$\begin{aligned}\Delta\tilde{\rho}\geqslant&-\frac{1}{2\pi^3\xi^4}\int_0^\infty d\nu\int_0^\infty d\omega\,\omega(\omega^2+1)|\hat{f}^{1/2}[\nu+\omega]|^2,\\=&-\frac{1}{16\pi^3\xi^4}\bigg\{\int_{-\infty}^\infty|u^2\hat{f}^{1/2}(u)|^2du\\&+2\int_{-\infty}^\infty|u\hat{f}^{1/2}(u)|^2du\bigg\},\\=&-\frac{1}{8\pi^2\xi^4}\bigg\{\int_{-\infty}^\infty\bigg[\frac{d^2}{d\eta^2}f^{1/2}(\eta)\bigg]^2d\eta\\&+2\int_{-\infty}^\infty\bigg[\frac{d}{d\eta}f^{1/2}(\eta)\bigg]^2d\eta\bigg\},\end{aligned} \tag{108}$$

where we have again changed the variables of integration in accordance with Eq. (95) in the second line and used Parseval's identity to arrive at the third line. The quantum inequality is found by adding the Rindler space vacuum energy density [31],

$$\rho_{\text{vacuum}}=-\frac{1}{\pi^2\xi^4}\int_0^\infty d\omega\frac{\omega^3+\omega}{e^{2\pi\omega}-1}=-\frac{11}{240\pi^2\xi^4}, \tag{109}$$

to the above expression. For the Lorentzian sampling function, Eq. (2), and the definition of the proper time of the static observer, $\tau=\xi\eta$, we find

$$\Delta\tilde{\rho}\geqslant-\frac{27}{1024\pi^2\tau_0^4}\left[1+\frac{32}{27}\left(\frac{\tau_0}{\xi}\right)^2\right]. \tag{110}$$

Once again we find that the Rindler space difference inequality for the electromagnetic field is twice that of the scalar field result found by Fewster and Eveson [14] for the same reason as discussed in the previous example. The electromagnetic field quantum inequalities for other sampling functions are also summarized in Table I.

We need to be careful about the interpretation of this quantum inequality in Rindler spacetime, as it appears that both the vacuum energy density and the expression for the difference inequality, Eq. (108), diverge in the limit as $\xi\to 0$. This does not mean the quantum inequality fails on the particle horizon in Rindler spacetime. This divergence is really a pathology of the coordinates and spacetime trajectory used. Recall that the quantum inequality found above is for a static observer in the Rindler coordinates. This trajectory is not that of a geodesic observer but one undergoing constant acceleration. A "static" observer at $\xi=0$ would require a constant infinite acceleration, an impossible scenario. The divergence in the quantum inequality expresses this. We can then ask what is the quantum inequality along the worldline of a geodesic observer in Rindler space? Well, a geodesic observer in Rindler spacetime is the same as a constant velocity geodesic in Minkowski spacetime, with the resulting quantum inequality in the geodesic observer's rest frame already found in the preceding Minkowski space example. It is obvious that there is nothing "unique" happening as the geodesic observer crosses the point in space which is associated with the particle horizon in Rindler coordinates. Thus, in Rindler space, the quantum inequality along a geodesic worldline does not fail.

C. Static Einstein spacetime

Finally, we study the quantum inequality in the static closed universe where the length element is given by

$$ds^2=-dt^2+a^2[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta d\varphi^2)], \tag{111}$$

and a is the radius of the universe. The modes of the electromagnetic field in this spacetime have been studied by various authors [36,37,30]. In terms of the vector potential, the mode solutions are the vector spherical harmonics on S^3 with harmonic time dependence. In a fashion similar to the previous examples, the four-vector mode functions can be found from a scalar function that satisfies the partial differential equation

$$\left(\nabla^\alpha\nabla_\alpha-\frac{2\cos\chi}{a^2\sin\chi}\partial_\chi\right)\psi_{nlm}(t,\chi,\theta,\varphi)=0, \tag{112}$$

which is not the scalar wave equation in the Einstein universe. The scalar mode solutions are

$$\psi_{nlm}(t,\chi,\theta,\varphi)=V_{nl}(\chi)Y_{lm}(\theta,\varphi)e^{-i\omega_n t}, \tag{113}$$

where $\omega_n=n/a$ and $Y_{lm}(\theta,\varphi)$ are the scalar spherical harmonics on S^2. The functions $V_{nl}(\chi)$ are defined as

$$V_{nl}(\chi)=\frac{2^l l!\sqrt{(n-l-1)!}}{\sqrt{l(l+1)\pi(n+l)!}}\sin^{l+1}\chi\, C^{l+1}_{n-l-1}(\cos\chi), \tag{114}$$

where $C^\lambda_\eta(x)$ are the Gegenbauer polynomials as defined in [35]. The primary quantum number n ranges over the integers greater than 1, i.e. $n=2,3,4,\ldots$. For a given n there are n^2-1 harmonic states with the same energy labeled by the quantum numbers, $l=1,\ldots,n-1$ and $n=-l,-l+1,\ldots,0,\ldots l-1,l$.

The two physical four-vector potential modes are the electric J-pole modes,

$$A_\alpha(1,n,l,m;x)=\frac{1}{n}\left(0,\frac{l(l+1)}{\sin^2\chi},\partial_\chi\partial_\theta,\partial_\chi\partial_\varphi\right)\psi_{nlm}(t,\chi,\theta,\varphi), \tag{115}$$

and the magnetic J-pole modes,

$$A_\alpha(2,\omega,k_y,k_z;x)=\left(0,0,\frac{1}{\sin\chi}\partial_\varphi,\sin\chi\partial_\theta\right)\psi_{nlm}(t,\chi,\theta,\varphi), \tag{116}$$

both of which satisfy the vector wave equation (10), the Lorentz gauge and Coulomb gauge conditions, and are orthonormal.

Inserting these modes into Eq. (89) yields

$$\begin{aligned}\Delta\tilde{\rho}\geq&-\frac{1}{\pi}\int_0^\infty d\nu\sum_{n=2}^\infty|\hat{f}^{1/2}[\nu+\omega_n]|^2\\&\times\sum_{l=1}^\infty\frac{1}{a^4\sin^2\chi}\Bigg\{l(l+1)\left[\frac{1}{2}\partial_\chi^2+2n^2\right]V_{nl}^2\\&\times\sum_{m=-l}^{l}\overline{Y_{lm}}Y_{lm}+\frac{1}{2}[(\partial_\chi V_{nl})^2+n^2V_{nl}^2]\\&\times\left(\frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta+\frac{1}{\sin^2\theta}\partial_\varphi^2\right)\sum_{m=-l}^{l}\overline{Y_{lm}}Y_{lm}\Bigg\}.\end{aligned} \tag{117}$$

However, the spherical harmonics satisfy an addition theorem,

$$\sum_{m=-l}^{l}\overline{Y_{lm}}Y_{lm}=\frac{2l+1}{4\pi}, \tag{118}$$

which is independent of the θ and φ coordinates, thus the terms in the expression for the difference inequality involving derivatives with respect to θ and φ will vanish. The remaining terms can then be written as

$$\begin{aligned}\Delta\tilde{\rho}\geq&-\frac{1}{2\pi^2a^4}\int_0^\infty d\nu\sum_{n=2}^\infty|\hat{f}^{1/2}[\nu+\omega_n]|^2\\&\times\left(n^2+\frac{1}{4\sin^2\chi}\partial_\chi^2\sin^2\chi\right)\\&\times\sum_{l=1}^{n-1}\frac{(2l+1)l(l+1)}{\sin^2\chi}V_{nl}^2.\end{aligned} \tag{119}$$

The Gegenbauer polynomials also satisfy an addition theorem, Eq. 8.934.3 of [35], which for our case can be written as

$$\sum_{l=0}^{n-1}\frac{(2l+1)2^{2l}(l!)^2(n-l-1)!}{(n+l)!}[\sin^l\chi\, C^{l+1}_{n-l-1}(\cos\chi)]^2=n. \tag{120}$$

Using this in the difference inequality leads to

$$\begin{aligned}\Delta\tilde{\rho}\geq&-\frac{1}{2\pi^3a^4}\int_0^\infty d\nu\sum_{n=2}^\infty|\hat{f}^{1/2}[\nu+\omega_n]|^2\\&\times\left(n^2+\frac{1}{4\sin^2\chi}\partial_\chi^2\sin^2\chi\right)\left[n-\frac{1}{n}\left(\frac{\sin n\chi}{\sin\chi}\right)^2\right],\\=&-\frac{1}{2\pi^3a^3}\int_0^\infty d\nu\sum_{n=2}^\infty\omega_n(n^2-1)|\hat{f}^{1/2}[\nu+\omega_n]|^2.\end{aligned} \tag{121}$$

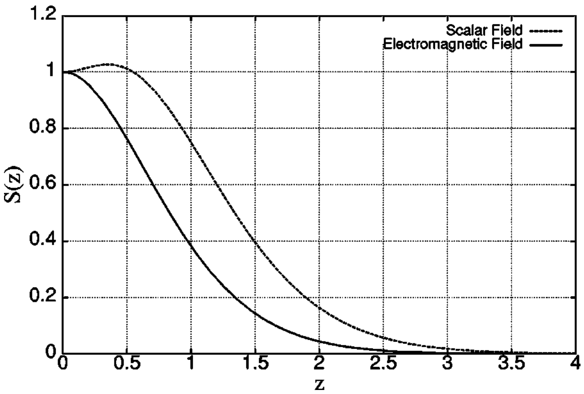

FIG. 1. Plot of the scale functions for a Lorentzian sampling function in the four-dimensional static Einstein universe. The solid line is the electromagnetic field result, while the dotted line is the scalar field result. Note, for small $z = t_0/a$ both scale functions approach 1, while for large z they decay to zero.

The resulting expression is spatially invariant, as expected in a homogeneous and isotropic universe. In addition, it has the general form of a sum over all the energies times the multiplicity for each energy times the Fourier transform of the square root of the sampling function, a form similar to that found by Fewster and Teo [15] for the scalar field in both the three-dimensional closed universe and in the four-dimensional static Robertson-Walker spacetimes. In order to find the quantum inequality, we need to add to the above expression the renormalized vacuum energy density for the electromagnetic field which is found to be [38]

$$\rho_{\text{vacuum}} = \frac{11}{240\pi^2 a^4}. \tag{122}$$

When the difference inequality is evaluated for the Lorentzian sampling function we find

$$\Delta\tilde{\rho} \geqslant -\frac{27}{1024\pi^2 t_0^4} S_{EM}(t_0/a), \tag{123}$$

where $S_{EM}(z)$ is the scale function for the closed universe given by

$$S_{EM}(z) = \frac{2048}{27\pi^2} z^4 \sum_{n=2}^{\infty} n(n^2-1) \int_{nz}^{\infty} K_0^2(u)\,du, \tag{124}$$

and $K_0(u)$ is the zero-order modified Bessel function of the second kind. It is straightforward to evaluate this function numerically and is plotted in Fig. 1. For sampling times very small compared to the radius of the universe, the scale function is approximately 1, for which we effectively recover the flat space quantum inequality. This makes sense because over such sampling times the region of the universe over which the observer moves is indistinguishable from Minkowski space. However, for sampling times on the order of, or larger than the radius of the universe, the observer (and thus the quantum inequality) has time to "see" the large scale structure of the universe. Thus the scale function changes appreciably away from 1.

It should also be pointed out that unlike the Minkowski and Rindler spacetime examples, the quantum inequality for the electromagnetic field is not simply twice that of the scalar field quantum inequality. In both of the previous cases, the spacetimes are flat with the Riemann curvature term in the wave equation vanishing. Therefore, the electromagnetic wave equation (10) in the Lorentz gauge can be reduced to the scalar field wave equation. Thus, the energy spectra are identical for the scalar and electromagnetic fields in each spacetime with the factor of 2 coming from the degeneracy of the electromagnetic field having two orthogonal polarization states. However, for the Einstein universe, and in curved spacetimes in general, the energy spectrum for the scalar and electromagnetic field modes is not the same, thus the scalar and electromagnetic quantum inequalities have different forms.

Using the work of Fewster and Teo [15], the scalar field difference inequality in the Einstein closed universe with a Lorentzian sampling function is

$$\Delta\tilde{\rho} \geqslant -\frac{27}{2048\pi^2 t_0^4} S_{scalar}(t_0/a), \tag{125}$$

where

$$S_{scalar}(z) = \frac{2048}{27\pi^2} z^4 \sum_{n=0}^{\infty} \sqrt{n(n+2)}(n+1)^2 \times \int_{\sqrt{n(n+2)}z}^{\infty} K_0^2(u)\,du. \tag{126}$$

This scale function is also plotted in Fig. 1 where we again see the generic behavior of the scale function being 1 for small values of t_0/a and decaying for large values. However, unlike the electromagnetic case which is a monotonically decreasing function, the scalar case has a bump which peaks at $t_0/a \sim 0.75$ and then smoothly decays. The bump is due to the $n=1$ term in the summation, a term which has no electromagnetic counterpart. If this term is removed from the summation, the remaining portion of the scale function does result in a monotonically decreasing behavior more akin to, but not exactly like the electromagnetic case. At present, it is not known if the bump in the scalar case has any physical meaning, as no state has yet been demonstrated which actually achieves this bound, although it may be a good guess that such a state would include $n=1$ modes. There has also been an alternative conjecture that the bump may be an artifact of the inequalities not being optimal. In either case, further research on the scalar field quantum inequality should eventually clarify this issue.

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