Conformal dynamics of 0-branes

M. Cadoni* and P. Carta†
Università di Cagliari, Dipartimento di Fisica, Cittadella Universitaria, 09042 Monserrato, Italy and INFN, Sezione di Cagliari, 09123 Cagliari, Italy

M. Cavaglià‡
Center for Theoretical Physics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139-4307 and INFN, Sede di Presidenza, Roma, Italy

S. Mignemi§
Università di Cagliari, Dipartimento di Matematica, Viale Merello 92, 09123 Cagliari, Italy and INFN, Sezione di Cagliari, 09123 Cagliari, Italy

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We investigate the dynamics of dilatonic D-dimensional 0-branes in the near-horizon regime. The theory is given in a twofold form: two-dimensional dilaton gravity and the nonlinear sigma model. Using asymptotic symmetries, duality relations, and sigma model techniques we find that the theory has three conformal points which correspond to (a) the asymptotic (anti–de Sitter) region of the two-dimensional spacetime, (b) the horizon of the black hole, and (c) the infinite limit of the coupling parameter. We show that the conformal symmetry is perturbatively preserved at one loop, identify the corresponding conformal field theories, and calculate the associated central charges. Finally, we use the conformal field theories to explain the thermodynamical properties of the two-dimensional black holes.

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I. INTRODUCTION

Dilatonic 0-branes are solutions of D-dimensional supergravity coupled to $U(1)$ gauge fields that describe effective low-energy approximations to the D0-brane solutions of string theory. Their investigation is relevant to understanding the anti–de Sitter/conformal field theory (AdS/CFT) correspondence [1] in two spacetime dimensions [2–5]. Though the $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence is quite well known in the dilaton gravity context [3,5], little is known in the more general framework of string theory.

In a recent paper [6] Youm has shown that in the dual-frame near-horizon regime D-dimensional dilatonic 0-branes can be described by an effective two-dimensional dilaton gravity model with nonconstant dilaton and asymptotically AdS_2 black hole solutions. The SL(2,R) isometry group of AdS_2 is thus broken, a feature which has prevented any attempt to using the asymptotic symmetries of AdS_2 to generate an infinite-dimensional conformal symmetry associated with the dynamics of the 0-brane. However, previous investigations [7–10] of the Jackiw-Teitelboim (JT) model [11] (which describes the near-horizon behavior of a specific 0-brane) have shown that the breaking of conformal symmetry due to a nontrivial dilaton can actually be controlled and is essential to understanding features of the CFT such as the existence of a nonvanishing central charge in the Virasoro algebra [7]. Thus, applying similar arguments we can investigate the existence of conformal symmetries for a general 0-brane.

In this paper we show that the asymptotic symmetries of the near-horizon 0-brane solutions are generated by a Virasoro algebra. Using a canonical realization of the asymptotic symmetries we calculate the central charge of the algebra and give an explicit realization of the conformal symmetry in terms of the fields that describe deformations of the boundary of AdS_2. For a particular range of the coupling parameter a we identify the one-dimensional conformal mechanics that lives on the boundary of AdS_2 and realizes the conformal symmetry. In the limit $a\rightarrow\infty$ the dilaton gravity model is shown to be equivalent to a free CFT. Thanks to a previous result by Carlip [12] we also argue that the horizon of the two-dimensional black hole defines a CFT with well-defined central charge. In the sigma model formulation the existence of these three conformal points is recovered, at the classical level, by relating the asymptotic symmetries of the gravitational theory to the conformal symmetries of the sigma model and by implementing the duality symmetries of the theory. The calculation of one-loop beta functions shows that the conformal symmetry is perturbatively preserved. Finally, the CFT results are used to discuss the thermodynamical behavior of the two-dimensional black holes.

The structure of the paper is the following. In Sec. II we review the two-dimensional dilaton gravity model that describes the near-horizon regime of D-dimensional dilatonic 0-branes. We consider the (asymptotically AdS_2) black hole solutions and discuss different limiting cases in the moduli space of the theory. In Sec. III we investigate the asymptotic symmetry group (ASG) of the solutions. We show that the ASG is generated by a Virasoro algebra and we calculate the central charge. We discuss the dynamics induced on the

*Email address: cadoni@ca.infn.it
†Email address: carta@ca.infn.it
‡Email address: cavaglia@mitlns.mit.edu
§Email address: mignemi@ca.infn.it

AdS_2 boundary by the bulk gravity theory. In Secs. IV and V we identify the CFT that lives on the horizon of the black hole and we use the sigma model formulation to describe the relation between the conformal symmetries of the sigma model and the asymptotic symmetry group of dilaton gravity. We also show that the weak-coupling region can be described by a free CFT. Section VI deals with the duality symmetries of the theory. The sigma model approach is used to prove that the horizon of the black hole defines a CFT. In Sec. VII we calculate the one-loop beta functions of the three CFTs and show that the conformal symmetry is preserved at one-loop at the (classical) conformal points. In Sec. VIII we discuss the thermodynamical properties of the two-dimensional black holes. Finally, we state our conclusions in Sec. IX.

II. EFFECTIVE THEORY OF DILATONIC 0-BRANES

In the Einstein frame the bosonic part of the supergravity action that describes dilatonic 0-branes solutions in D dimensions is

$$S=\frac{1}{2\kappa_D^2}\int d^Dx\sqrt{-g}\left[R-\frac{4}{D-2}(\partial\phi)^2-\frac{1}{4}e^{2d\phi}F^2\right], \quad (1)$$

where κ_D is the D-dimensional gravitational coupling constant, ϕ is the dilaton field, d is the dilaton coupling parameter, and F is the field strength of the $U(1)$ gauge field.

In the dual frame [13] the static solution of the model has the near-horizon form $AdS_2\times S^{D-2}$. The 0-brane admits an effective description in terms of a two-dimensional dilaton gravity theory. Following Ref. [6] we write

$$S=\frac{1}{2\kappa_2^2}\int d^2x\sqrt{-g}e^{\delta\phi}[R+\gamma(\partial\phi)^2+\Lambda]. \quad (2)$$

Setting $\kappa_2=1$ and redefining the dilaton $\Phi=e^{\delta\phi}$, Eq. (2) is cast in the form

$$S=\frac{1}{2}\int\sqrt{-g}\,d^2x\,\Phi\left(R+\alpha\frac{(\partial\Phi)^2}{\Phi^2}+\Lambda\right), \quad (3)$$

where

$$\alpha=\frac{\gamma}{\delta^2}=\frac{1}{D-2}\left[D-1-\frac{4}{d^2}\left(\frac{D-3}{D-2}\right)^2\right]. \quad (4)$$

The field equations are

$$R+\Lambda+\frac{\alpha}{\Phi^2}(\nabla\Phi)^2-\frac{2\alpha}{\Phi}\nabla^2\Phi=0, \quad (5)$$

$$\begin{aligned}T_{\mu\nu}=&\frac{\alpha}{\Phi}\left(\nabla_\mu\Phi\nabla_\nu\Phi-\frac{1}{2}g_{\mu\nu}(\nabla\Phi)^2\right)-\nabla_\mu\nabla_\nu\Phi+g_{\mu\nu}\nabla^2\Phi\\&-\frac{1}{2}g_{\mu\nu}\Lambda\Phi\\&=0.\end{aligned} \quad (6)$$

The trace of Eq. (6) gives

$$\nabla^2\Phi-\Lambda\Phi=0, \quad (7)$$

which is independent from α.

The two-dimensional dilaton gravity model, Eq. (3), has been extensively investigated in the literature [14,15]. For the sake of completeness we briefly summarize the main results. If $\alpha<1$ Eq. (3) admits the asymptotically AdS black hole solutions

$$\begin{aligned}&ds^2=-[b^2r^2-A^2(br)^{2h}]dt^2+[b^2r^2-A^2(br)^{2h}]^{-1}dr^2,\\&\Phi=\Phi_0(br)^{1-2h},\end{aligned} \quad (8)$$

where

$$h=\frac{\alpha}{2(\alpha-1)},\quad b^2=\frac{\Lambda}{2(1-h)(1-2h)},\quad h<\frac{1}{2}. \quad (9)$$

The integration constant A in Eq. (8) is related to the black hole mass m_{bh} by the relation

$$m_{bh}=\frac{1}{2}(1-2h)\Phi_0A^2b. \quad (10)$$

The scalar curvature is

$$R=-2[b^2+h(1-2h)A^2(br)^{2h}r^{-2}]. \quad (11)$$

If $h\neq 0,1/2$ the metric (8) has a curvature singularity at $r=0$. Since the geometry is asymptotically AdS the boundary at $r=\infty$ is timelike.

The thermodynamical behavior of the black hole is characterized by the power-law mass-temperature relation

$$m_{bh}=\frac{1-2h}{2(1-h)}\Phi_0[b(1-h)]^{2h-1}(2\pi T)^{2(1-h)}. \quad (12)$$

Below we shall restrict attention to $-1\leqslant h\leqslant 1/2$. In this case we have $m_{bh}\sim T^s$, $1\leqslant s\leqslant 4$. The entropy of the black hole is

$$S=2\pi(\Phi_0)^{1/2(1-h)}\left[\frac{2m_{bh}}{(1-2h)b}\right]^{(1-2h)/2(1-h)}. \quad (13)$$

The model (3) includes two interesting special cases: $\alpha=0$ ($h=0$) and $\alpha\to-\infty$ ($h=1/2$). The former is the so-called Jackiw-Teitelboim (JT) model [11]. The spacetime has constant curvature, i.e., is locally AdS_2, and the dilaton is linear. Since the JT model has been widely investigated in the literature (see, e.g., [16,3]) we shall not discuss it here. The second case deserves a brief discussion. Taking the limit $h=1/2$ in Eq. (8), we obtain the AdS_2 spacetime with constant dilaton

$$\begin{aligned}&ds^2=-(b^2r^2-A^2br)dt^2+(b^2r^2-A^2br)^{-1}dr^2,\\&\Phi=\Phi_0,\quad b^2=\Lambda/2.\end{aligned} \quad (14)$$

Since the dilaton is constant we can interpret the solution (14) as the near-horizon regime of the extremal Reissner-Nordström black hole (or its string theory generalizations).

Setting $h=1/2$ in Eq. (10) and Eq. (13) we find states of zero energy which are characterized by constant nonzero entropy

$$S=2\pi\Phi_0. \tag{15}$$

Equation (15) describes the typical behavior of extremal Reissner-Nordström black holes in the near-horizon regime [4,17]. The $h=1/2$ model is equivalent to a two-dimensional free CFT. In the limit $\alpha\to-\infty$ the field equations (5) and (6) become

$$R+\Lambda=0, \tag{16}$$

$$T_{\mu\nu}=\nabla_\mu\Phi\nabla_\nu\Phi-\frac{1}{2}g_{\mu\nu}(\nabla\Phi)^2=0. \tag{17}$$

Equation (17) describes the energy-momentum tensor of a free two-dimensional CFT of a single boson Φ.

III. ASYMPTOTIC SYMMETRIES

If $h\neq 0,1/2$ the spacetime (8) is not maximally symmetric; it admits a single Killing vector $\mathcal{T}$ which generates translations in time. In contrast, if $h=0,1/2$ Eq. (8) describes the maximally symmetric AdS_2 spacetime with the $SL(2,R)$ isometry group. For $h=0$ a suitable choice of boundary conditions [3] shows that the ASG, i.e., the isometry group that preserves the asymptotic form of the metric, is generated by a Virasoro algebra. For a generic value of h the discussion of the group of asymptotic symmetries is more involved. The boundary conditions must indeed allow both an ASG which is larger than $\mathcal{T}$ and finite associated charges [18]. We will see below that these requirements are fulfilled only for $0\leqslant h\leqslant 1/2$.

From now on we shall restrict attention to $-1\leqslant h\leqslant 1/2$ and discuss $-1\leqslant h\leqslant 0$ and $0<h\leqslant 1/2$ separately. In the first case suitable boundary conditions for the metric and the dilaton are

$$\begin{aligned}
g_{tt}&=-(br)^2+\gamma_{tt}+\mathcal{O}(r^{2h}),\\
g_{tr}&=\gamma_{tr}(t)(br)^{-3}+\mathcal{O}(r^{2h-3}),\\
g_{rr}&=(br)^{-2}+\gamma_{rr}(br)^{-4}+\mathcal{O}(r^{2h-4}),\\
\Phi&=\Phi_0[\rho(br)^{1-2h}+\gamma_{\Phi\Phi}(br)^{-1-2h}+\mathcal{O}(r^{-1})],
\end{aligned} \tag{18}$$

where the fields $\rho(t)$ and $\gamma(t)$ describe deformations of the dilaton and of the timelike boundary of the spacetime. Both $\mathcal{O}(1)$ deformations in g_{tt} and $\mathcal{O}(r^{-4})$ in g_{rr} dominate the deformations that generate the black hole in Eq. (8). The $\mathcal{O}(1)$ terms in the boundary conditions are essential to extending the isometry group of the metric to an ASG generated by a Virasoro algebra. However, their presence leads to divergent charges associated with the generators of the symmetry.

If $0<h\leqslant\frac{1}{2}$ the boundary conditions are

$$\begin{aligned}
g_{tt}&=-(br)^2+\tilde{\gamma}_{tt}(t)(br)^{2h}+\mathcal{O}(1),\\
g_{tr}&=\tilde{\gamma}_{tr}(t)(br)^{2h-3}+\mathcal{O}(r^{-3}),\\
g_{rr}&=(br)^{-2}+\tilde{\gamma}_{rr}(t)(br)^{2h-4}+\mathcal{O}(r^{-4}),\\
\Phi&=\Phi_0[\rho(br)^{1-2h}+\tilde{\gamma}_{\Phi\Phi}(br)^{-1}+\mathcal{O}(r^{-1-2h})].
\end{aligned} \tag{19}$$

The $\mathcal{O}(1)$ deformations in g_{tt} and the $\mathcal{O}(r^{-4})$ deformations in g_{rr} are subleading with respect to deformations that generate the black hole in Eq. (8). They become of the same order only for $h=0$. In this case we have an ASG which is characterized by a Virasoro algebra and finite charges.

The Killing vectors that generate the ASG are

$$\xi^t=\epsilon(t)+\frac{\ddot{\epsilon}(t)}{2b^4r^2}+\mathcal{O}(r^{-4+\delta}), \tag{20}$$

$$\xi^r=-r\dot{\epsilon}(t)-\frac{\alpha^r(t)}{2}(br)^{-1+\delta}+\mathcal{O}(r^{-3+\delta}), \tag{21}$$

where $\delta=0$ if $-1\leqslant h\leqslant 0$ and $\delta=2h$ if $0<h\leqslant\frac{1}{2}$. The function $\alpha^r(t)$ describes diffeomorphisms of the two-dimensional gravity theory that die off rapidly as r goes to infinity ("pure" gauge diffeomorphisms). The leading terms of the Killing vectors (20) and (21) are identical to the JT case [3]. The generators L_k of the ASG satisfy the Virasoro algebra

$$[L_k,L_l]=(k-l)L_{k+l}+\frac{c}{12}(k^3-k)\delta_{k+l,0}, \tag{22}$$

where we allow for a nonvanishing central charge. We shall see below that the ASG has a natural realization in terms of the conformal group in one dimension (the Diff_1 group) which describes reparametrizations of either the circle or the line, depending on the topology of the $r\to\infty$ boundary.

If $h\neq 1/2$ the black hole solutions (8) are characterized by a nonconstant dilaton. For consistency, the leading term in the asymptotic expansion of the dilaton must be of the form (18) or (19). In Ref. [7] it has been shown that for $h=0$ the ASG of the metric is broken by the nontrivial dilaton and the presence of a nonvanishing central charge in the Virasoro algebra is related to the symmetry breaking. The boundary fields span a representation of the conformal group. This conclusion holds also for negative values of h. For $0<h<1/2$ the central charge vanishes identically. For $h=1/2$ the dilaton is constant, so the boundary conditions imply that its (on-shell) Lie derivative vanishes. Both the SL(2,R) isometry group and the ASG of the metric are preserved.

Using suitable boundary conditions and introducing appropriate boundary fields we could also consider $h<-1$. For instance, for $-2<h<-1$ we could introduce the term $\Gamma_{tt}(br)^{-2}$ in the expansion of g_{tt}, the term $\Gamma_{rr}(br)^{-6}$ in the expansion of g_{rr}, etc. However, larger values of $|h|$ require an increasing number of boundary fields. So in this paper we will consider only $-1\leqslant h\leqslant 1/2$.

A. Transformation laws and equations of motion for the boundary fields

Using the boundary conditions (18) and (19), and the Killing vectors (20) and (21) we find the transformation laws for the boundary fields. They are

$$\delta\rho=\epsilon\dot{\rho}-(1-2h)\dot{\epsilon}\rho, \tag{23}$$

and

$$\delta\gamma_{tt}=\epsilon\dot{\gamma}_{tt}+2\dot{\epsilon}\gamma_{tt}-\frac{\dddot{\epsilon}}{b^2}+b\alpha^r, \tag{24}$$

$$\delta\gamma_{rr}=\epsilon\dot{\gamma}_{rr}+2\dot{\epsilon}\gamma_{rr}+2b\alpha^r, \tag{25}$$

$$\delta\gamma_{\Phi\Phi}=\epsilon\dot{\gamma}_{\Phi\Phi}+(1+2h)\dot{\epsilon}\gamma_{\Phi\Phi}+\frac{\ddot{\epsilon}\dot{\rho}}{2b^2}-\frac{1}{2}(1-2h)b\alpha^r\rho, \tag{26}$$

$$\delta\tilde{\gamma}_{tt}=\epsilon\dot{\tilde{\gamma}}_{tt}+(2-2h)\dot{\epsilon}\tilde{\gamma}_{tt}+b\alpha^r, \tag{27}$$

$$\delta\tilde{\gamma}_{rr}=\epsilon\dot{\tilde{\gamma}}_{rr}+(2-2h)\dot{\epsilon}\tilde{\gamma}_{rr}+(2-2h)b\alpha^r, \tag{28}$$

$$\delta\tilde{\gamma}_{\Phi\Phi}=\epsilon\dot{\tilde{\gamma}}_{\Phi\Phi}+\dot{\epsilon}\tilde{\gamma}_{\Phi\Phi}-\frac{1}{2}(1-2h)b\alpha^r\rho, \tag{29}$$

for $h\leqslant 0$ and $h>0$, respectively.

As was expected, the boundary fields γ and $\tilde{\gamma}$ transform according to a representation of the conformal group which is realized as time reparametrizations $\delta t=\epsilon(t)$ of the boundary. In general the conformal dimensions of the boundary fields depend on the parameter h. Anomalous pieces in the transformation law of the fields imply a nonvanishing central charge in the Virasoro algebra. The boundary fields γ_{tr} and $\tilde{\gamma}_{tr}$ transform as conformal fields as well. However, their deformations are irrelevant (they do not contribute to the charges and do not affect the dynamics of the boundary), so we have omitted their transformation laws for simplicity.

Let us now consider the dynamics of the boundary fields. In Ref. [19] it has been shown that for $h=0$ the two-dimensional gravitational dynamics in the bulk induces a dynamics of the fields on the boundary. This result holds also for generic values of $h<0$. At leading order the boundary fields satisfy the equations

$$\frac{\ddot{\rho}}{b^2}=(1-2h)\rho\gamma_{tt}-(1-2h)^2\rho\gamma_{rr}-2(1-4h)\gamma_{\phi\phi}, \tag{30}$$

$$\frac{\dot{\rho}^2}{b^2\rho}+(1-2h)^2\rho\gamma_{rr}+4(1-2h)\gamma_{\phi\phi}=0, \tag{31}$$

$$(1-2h)\dot{\rho}\gamma_{tt}+\frac{1}{2}(1-2h)^2\rho\dot{\gamma}_{rr}+2(1-2h)\dot{\gamma}_{\phi\phi}+4h\frac{\dot{\rho}}{\rho}\gamma_{\phi\phi}=0, \tag{32}$$

and

$$\rho(\tilde{\gamma}_{rr}-\tilde{\gamma}_{tt})+2\tilde{\gamma}_{\phi\phi}=0, \tag{33}$$

$$(1-h)\dot{\rho}\tilde{\gamma}_{tt}+\frac{1-2h}{2}\rho\dot{\tilde{\gamma}}_{rr}+(2-2h)\dot{\tilde{\gamma}}_{\phi\phi}+\frac{4h(1-h)}{1-2h}\frac{\dot{\rho}}{\rho}\tilde{\gamma}_{\phi\phi}=0, \tag{34}$$

for $h<0$ and $h>0$, respectively. Note that Eq. (30) follows from Eqs. (31) and (32). The equations above determine the dynamics on the boundary, which will be investigated in Sec. III C.

B. Determination of the central charge of the Virasoro algebra

To evaluate the central charge of the Virasoro algebra that generates the ASG we turn to the Hamiltonian formalism [3]. With the parametrization

$$ds^2=-N^2dt^2+\sigma^2(dr+N^r dt)^2, \tag{35}$$

the Hamiltonian of the theory reads

$$H=\int dr(N\mathcal{H}+N^r\mathcal{H}_r). \tag{36}$$

As usual, N and N^r act as Lagrange multipliers and enforce the constraints

$$\begin{aligned}\mathcal{H}&=-\Pi_\Phi\Pi_\sigma+\sigma^{-1}\Phi''-\sigma^{-2}\sigma'\Phi'-\frac{\Lambda}{2}\sigma\Phi\\&\quad+\frac{\alpha}{2}(\sigma\Phi^{-1}\Pi_\sigma^2-\sigma^{-1}\Phi^{-1}\Phi'^2)=0,\\ \mathcal{H}_r&=\Phi'\Pi_\Phi-\sigma\Pi_\sigma'=0,\end{aligned} \tag{37}$$

where

$$\begin{aligned}\Pi_\sigma&=N^{-1}(-\dot{\Phi}+N^r\Phi'),\\ \Pi_\Phi&=N^{-1}(-\dot{\sigma}+(N^r\sigma)')+\alpha\sigma\Phi^{-1}\Pi_\sigma,\end{aligned} \tag{38}$$

are the momenta conjugate to σ and Φ, respectively. Here a prime denotes derivative with respect to r.

For noncompact spacelike surfaces a boundary term J must be added to the Hamiltonian (36) to obtain well-defined variational derivatives [20]. Although the above requirement fixes only the variation δJ, with a suitable choice of asymptotic boundary conditions δJ can be written as a total variation of a functional J of the canonical fields on the boundary. In our case the boundary reduces to a point and the variation of J is

$$\begin{aligned}\delta J=-\lim_{r\to\infty}[&N(\sigma^{-1}\delta\Phi'-\sigma^{-2}\Phi'\delta\sigma-\alpha\sigma^{-1}\Phi^{-1}\Phi'\delta\Phi)\\&-N'(\sigma^{-1}\delta\Phi)+N^r(\Pi_\Phi\delta\Phi-\sigma\delta\Pi_\sigma)].\end{aligned} \tag{39}$$

Let us now consider the symmetries associated with the Killing vectors ξ. These are generated by the phase space functionals [21]

$$H[\xi]=\int dr(\xi^{\perp}\mathcal{H}+\xi^{\parallel}\mathcal{H}_r)+J[\xi], \tag{40}$$

where $\xi^{\perp}=N\xi^t$ and $\xi^{\parallel}=\xi^r+N^r\xi^t$. In general, the Poisson algebra of $H[\xi]$ yields a projective representation of the asymptotic symmetry algebra,

$$\{H[\xi],H[\chi]\}=H[[\xi,\chi]]+K(\xi,\chi), \tag{41}$$

where $K(\xi,\chi)$ is a central charge.

The simplest way to calculate the central charge is to evaluate the Poisson brackets (41) using the explicit expression for (40) obtained above. A straightforward calculation gives

$$\begin{aligned}K([\xi,\chi])=&\lim_{r\to\infty}-(\xi^{\perp\prime}\chi^{\perp}-\chi^{\perp\prime}\xi^{\perp})\sigma^{-1}\Pi_\sigma\\&+(\xi^{\perp\prime}\chi^{\parallel}-\chi^{\perp\prime}\xi^{\parallel})\sigma^{-1}\Phi'-(\xi^{\perp}\chi^{\parallel}-\chi^{\perp}\xi^{\parallel})\\&\times\left[\frac{\Lambda}{2}\sigma\Phi+\Pi_\sigma\Pi_\Phi-\frac{\alpha}{2}(\sigma\Phi^{-1}\Pi_\sigma^2\right.\\&\left.+\sigma^{-1}\Phi\Phi'^2)\right].\end{aligned} \tag{42}$$

For the two-dimensional AdS space, however, the boundary at infinity is a point, so the functional derivatives that appear in the Poisson brackets are well defined only for pure gauge transformations, for which $J[\xi]$ vanishes. This problem can be overcome by defining the time-averaged generators $\hat{H}[\xi]$ and charges $\hat{K}[\xi]$ (see Ref. [3])[1]

$$\hat{H}[\xi]=\frac{b}{2\pi}\int_0^{2\pi/b}dt\,H[\xi],\quad \hat{K}[\xi]=\frac{b}{2\pi}\int_0^{2\pi/b}dt\,K[\xi]. \tag{43}$$

Here we assume that the time coordinate t is periodic with period b.

We can obtain the Virasoro algebra (22) by expanding the parameters $\epsilon(t)$ of the Killing vectors (20) and (21) in Fourier modes:

$$\xi(t)=\sum_k \frac{i}{b}e^{ikbt}L_k, \tag{44}$$

where L_k are the generators of the Virasoro algebra. The central charge of the algebra is evaluated by substituting Eq. (44) and the black hole ground state solution ($m_{bh}=0$) in Eq. (42) and then integrating with respect to t,

$$\hat{K}(L_k,L_l)=\lim_{r\to\infty}-2i(1-2h)\Phi_0(br)^{-2h}k^3\delta_{k+l,0}. \tag{45}$$

[1]An analogous problem was encountered in a slightly different context in Ref. [12].

$\hat{K}$ vanishes for $h>0$. In contrast, for $h<0$ we have a divergent contribution. This is due to the infinite energy of the excitations on the boundary. Using Mann's formula for the mass [22], or evaluating directly $H[L_0]$, the energy of the excitations is

$$m=\frac{b\Phi_0}{2}[(1-2h)\tilde{\gamma}_{rr}+4\tilde{\gamma}_{\phi\phi}](br)^{-2h}, \tag{46}$$

which diverges for $r\to\infty$. Note that all generators $H[L_k]$ exhibit the same divergence. Nevertheless, they span a representation of the Virasoro algebra. Indeed, neglecting the divergences and shifting L_0 by a constant, $L_0\to L_0-\Phi_0$, we obtain the Virasoro algebra in the standard form (22) with central charge

$$c=\begin{cases}24(1-2h)\Phi_0, & h\leqslant 0,\\ c=0, & h>0.\end{cases} \tag{47}$$

The central charge can also be evaluated from the algebra of the charges $J[\xi]$ on the constraint surface $H=0$. If $h\leqslant 0$ from Eqs. (18) and (39) we have

$$\begin{aligned}\delta J[\varepsilon]=\Phi_0\Bigg[&b\varepsilon\Bigg((1-h)\tilde{\gamma}_{tt}\delta\rho+\frac{(1-2h)}{2}\rho\,\delta\tilde{\gamma}_{rr}+2\,\delta\tilde{\gamma}_{\phi\phi}\\&+\frac{4h}{1-2h}\frac{\tilde{\gamma}_{\phi\phi}}{\rho}\delta\rho\Bigg)+\frac{\dot{\varepsilon}}{b}\Bigg(\delta\dot{\rho}+\frac{2h}{1-2h}\frac{\dot{\rho}}{\rho}\delta\rho\Bigg)-\frac{\ddot{\varepsilon}}{b}\delta\rho\Bigg]\\&\times(br)^{-2h}.\end{aligned} \tag{48}$$

The equation above is not globally integrable in the full phase space. However, it can be integrated in a neighborhood of the classical solution, i.e., near $\rho=1$ [7]. Using the equations of motion (30)–(32), and expanding Eq. (48) around the classical solutions $\rho=1+\bar{\rho}$ we have, at the leading order in $\bar{\rho}$,

$$J(\varepsilon)=\frac{\Phi_0}{b}(\dot{\varepsilon}\dot{\bar{\rho}}-\ddot{\varepsilon}\bar{\rho})(br)^{-2h}+\varepsilon m, \tag{49}$$

where m is the mass of the excitations (46).

The charge (49) is defined up to an additive constant that has been fixed by setting $J(\varepsilon=1)=m$. Since we are interested in the value of the central charge of the Virasoro algebra (which is independent from m) we consider only $m=0$, i.e., variations near the ground state. Equations (43) imply that J is defined up to a total time derivative. So we write

$$J(\varepsilon)=-2\frac{\Phi_0}{b}(br)^{-2h}\varepsilon\ddot{\bar{\rho}}=\varepsilon\Theta_{tt}. \tag{50}$$

Using the transformation laws (23)–(26) with parameter ω, we obtain

$$\varepsilon\,\delta_\omega\Theta_{tt}=\varepsilon(\omega\dot{\Theta}_{tt}+2\dot{\omega}\Theta_{tt})+K(\varepsilon,\omega), \tag{51}$$

where $K(\varepsilon,\omega)=2(1-2h)b^{-1}\Phi_0(br)^{-2h}(\ddot{\varepsilon}\,\dot{\omega}-\ddot{\omega}\,\dot{\varepsilon})$, in agreement with Eq. (45). Θ_{tt} can be interpreted as the one-dimensional stress-energy tensor associated with the conformal symmetry. $\hat{J}(\varepsilon)$ are the charges which generate the central extension of the Virasoro algebra.

We may repeat the previous calculations for $h>0$. In this case δJ is

$$\delta J[\varepsilon]=\Phi_0\left[b\varepsilon\left((1-h)\gamma_{tt}\delta\rho+\frac{(1-2h)}{2}\rho\delta\gamma_{rr}+2(1-h)\delta\gamma_{\phi\phi}+\frac{4h(1-h)}{1-2h}\frac{\gamma_{\phi\phi}}{\rho}\delta\rho\right)\right]. \quad (52)$$

We integrate again this expression near $\rho=1$. Using the equations of motion (33) and (34), and expanding around the classical solutions $\rho=1+\overline{\rho}$ we obtain, at the leading order in $\overline{\rho}$,

$$J(\varepsilon)=\varepsilon m, \quad (53)$$

where m is the mass of the excitations,

$$m=\frac{b\Phi_0}{2}[(1-2h)\gamma_{rr}+4(1-h)\gamma_{\phi\phi}]. \quad (54)$$

Since J has no anomalous term, the central charge vanishes.

C. Dynamics of the boundary

To get some clue on the origin of the boundary degrees of freedom when $h<0$ we investigate the dynamics that is obeyed by the boundary fields [19]. It is convenient to introduce new fields which are invariant under the pure gauge diffeomorphisms parametrized by α^μ,

$$\beta=\frac{1}{2}(1-2h)\rho\gamma_{rr}+2\gamma_{\phi\phi},$$

$$\gamma=\gamma_{tt}-\frac{1}{2}\gamma_{rr}. \quad (55)$$

In terms of the new fields, the equations of motion (30)–(32) take the form

$$b^{-2}\ddot{\rho}=(1-2h)\rho\gamma-(1-4h)\beta, \quad (56)$$

$$b^{-2}\dot{\rho}^2+2(1-2h)\beta\rho=0, \quad (57)$$

$$(1-2h)(\dot{\rho}\gamma+\dot{\beta})+2h\beta\frac{\dot{\rho}}{\rho}=0. \quad (58)$$

Since the one-form

$$\zeta\equiv\left[(1-2h)\gamma+2h\frac{\beta}{\rho}\right]d\rho+(1-2h)d\beta \quad (59)$$

is not exact, Eqs. (56)–(58) determine a mechanical system with one anholonomic constraint. Equation (57) is a first integral of Eqs. (56) and (58). It implies that the total energy vanishes:

$$E=T+V=\frac{\dot{\rho}^2}{2b^2}+(1-2h)\beta\rho=0. \quad (60)$$

This condition is absent in the JT model [19], where the total energy E is not constrained. E is proportional to the mass of the excitations (46).

The Lagrange equations of the first kind for the fields $\varphi_i=\{\rho,\beta,\gamma\}$ are

$$F_i-m_i\ddot{\varphi}_i+\lambda\zeta_i=0, \quad (61)$$

where F_i is the force that follows from the potential V defined above, $F_i=-\partial_i V$. m_i denote the mass of the fields, λ is a Lagrange multiplier, and ζ_i are the components of the one-form ζ. Setting

$$m_\rho=b^{-2}, \quad m_\beta=m_\gamma=0, \quad (62)$$

the Lagrange equations (61) yield Eq. (56) and fix the Lagrange multiplier as $\lambda=\rho$. The boundary fields φ_i span a representation of the full infinite dimensional group which is generated by the Killing vectors (20) and (21).

The dynamical system (56)–(58) can also be described in terms of a harmonic oscillator coupled to an external source. Introducing the new field $q=\rho^{1/2(1-2h)}$ of conformal dimension $-1/2$ and eliminating β from Eq. (57) by means of Eq. (56), we have

$$\ddot{q}=\frac{b^2}{2}\gamma q. \quad (63)$$

Equation (57) becomes

$$\frac{\dot{q}^2}{2}+\frac{b^2}{4(1-2h)}\beta q^{4h}=0. \quad (64)$$

The equivalence of Eqs. (56)–(58) and Eqs. (63) and (64) is straightforward. Equation (63) is the equation of motion of a harmonic oscillator coupled to the external source γ. It can be derived from the effective action

$$I=\int dt\left[\frac{1}{2}\dot{q}^2+\frac{1}{4}b^2\gamma q^2\right]. \quad (65)$$

The external source γ which couples to the field q is not constant. Rather, it represents an operator of conformal dimension two. (Note that in the calculation of δI, γ, which is an external source, must not be varied.) The action (65) can be shown to be invariant (up to a total derivative) under the conformal transformations (23)–(26). It is interesting to compare this result with the JT case, where the action has an extra potential term [19] of de Alfaro–Fubini–Furlan type [23].

For $h>0$ Eqs. (33) and (34) are nondynamical, in agreement with the vanishing of central charge. (Although boundary degrees of freedom do exist. See, e.g., [24–26].)

IV. CONFORMAL FIELD THEORY AT THE HORIZON

In the previous sections we have shown that the ASG of the metric (8) is generated by a Virasoro algebra. Hence, the asymptotic region of the black hole is described by a CFT. In addition to $r\to\infty$, we also expect the region near the black hole horizon to be described by a CFT. By investigating the algebra of constraints in the presence of a boundary Carlip has shown that when the boundary is a Killing horizon we can impose a natural set of boundary conditions which leads to a Virasoro algebra [12]. Moreover, on a manifold with boundary the algebra of surface deformations acquires a central term that depends uniquely on the boundary values of the dynamical fields. Consequently, the algebra takes the form given in Eq. (22), where the explicit value of the central charge c depends on the model under consideration. In the case of a general dilaton gravity model the central charge c and the eigenvalue of L_0 are [12]

$$\frac{c}{6}=L_0=\Phi_h, \tag{66}$$

where Φ_h is the value of the dilaton at the horizon r_h.

We conclude that the black hole geometry (8) has two conformal regions which are associated to the spacetime boundaries: $r=\infty$ and $r=r_h$. In both cases the algebra of surface deformations has the form of a centrally extended Virasoro algebra with a central charge given in Eqs. (47) and (66), respectively. The existence of the first conformal point ($r=\infty$) follows from the AdS asymptotic behavior of the metric. The existence of the second conformal point ($r=r_h$) does not depend on the details of the solution, being simply a consequence of the existence of a Killing horizon. Using the sigma model formulation of dilaton gravity theories we can interpret the two conformal regions as different coupling regimes of the gravitational theory. In Secs. V–VII we will show that $r=\infty$ and $r=r_h$ correspond to the weak-coupled regime ($\Phi\to\infty$) and to the strong-coupled ($\Phi\to\Phi_h$) regime of the gravitational theory, respectively.

V. SIGMA MODEL FORMULATION AND CONFORMAL SYMMETRIES

The conformal structure described in the previous sections can be traced back to the conformal invariance of the two-dimensional dilaton gravity theory (3). Classically, the generic two-dimensional dilaton gravity theory

$$S=\frac{1}{2}\int d^2x\sqrt{-g}\left[\Phi R[g]-\frac{d\ln|W(\Phi)|}{d\Phi}(\nabla\Phi)^2+V(\Phi)\right], \tag{67}$$

is invariant under the generalized (conformal) Weyl transformation [27]

$$V(\Phi)\to V(\Phi)/\Omega(\Phi),\quad W(\Phi)\to\Omega(\Phi)W(\Phi),$$
$$g_{\mu\nu}\to\Omega(\Phi)g_{\mu\nu}. \tag{68}$$

Here $W(\Phi)$, $V(\Phi)$, and $\Omega(\Phi)$ are arbitrary function of the dilaton field. The classical conformal invariance is generally broken by quantum effects. It is, however, (perturbatively) preserved for models with vanishing beta function.

The conformal invariance of the theory can be made manifest by implementing the canonical transformation $(f,\Phi)\to(M,\Phi)$ [28], where f is the conformal degree of freedom of the two-dimensional metric, $g_{\mu\nu}=f(x)\eta_{\mu\nu}$, and

$$M=N(\Phi)-W(\Phi)(\nabla\Phi)^2,$$
$$N(\Phi)=\int^{\Phi}d\Phi'[W(\Phi')V(\Phi')]. \tag{69}$$

The new field M is invariant under Weyl and gauge transformations and is locally conserved. Apart from a constant normalization factor, M coincides (on-shell) with the ADM mass of the system. Neglecting inessential surface terms, in the new canonical chart the action (67) reads

$$S_\sigma=\frac{1}{2}\int d^2x\sqrt{-g}\,\frac{\nabla_\mu\Phi\nabla^\mu M}{N(\Phi)-M}. \tag{70}$$

Let us first discuss the $r=\infty$ conformal region of the black hole geometry (8). Since the (coordinate-dependent) coupling constant of the gravitational model (3) is Φ^{-1} from Eq. (8) it follows that $r=\infty$ corresponds to the weak-coupled regime of the theory. The weak-coupled regime of the action (70) describes a free open string. It is convenient to introduce the Weyl-rescaled frame[2]

$$G_{\mu\nu}=\Phi^{1-a}g_{\mu\nu}, \tag{71}$$

where $a=(1-2h)^{-1}>0$. In this frame the action (3) reads

$$S=\frac{1}{2}\int d^2X\sqrt{-G}[\Phi R[G]+(1+a)\Phi^a\tilde{\lambda}^2], \tag{72}$$

where $\tilde{\lambda}^2=\lambda^2/(1+a)$ and $a=1-\alpha$. Equation (70) takes the form

$$S_\sigma=\frac{1}{2}\int d^2X\sqrt{-G}\frac{\nabla_\mu\Phi\nabla^\mu M}{\tilde{\lambda}^2\Phi^{a+1}-M}. \tag{73}$$

At the tree level in the weak-coupling expansion Eq. (73) describes a bosonic string that propagates in a two-dimensional flat spacetime. When $\Phi\to\infty$ we have

$$S_{string}=\frac{1}{2\pi}\int d^2X\,\partial_\mu\xi^\alpha\partial^\mu\xi_\alpha, \tag{74}$$

where

$$M=\frac{1}{\sqrt{\pi}}(\xi^1+\xi^0),\quad \frac{1}{a\tilde{\lambda}^2}\Phi^{-a}=-\frac{1}{\sqrt{\pi}}(\xi^1-\xi^0). \tag{75}$$

Using the same arguments of Ref. [5] we can work out a nontrivial relationship between the asymptotic symmetries of

[2]From now on we denote with upper-case letters the quantities in the rescaled frame.

the model (3) and the conformal symmetries of the (Dirichlet) open string (74). We find that the Weyl transformation (71) transforms the conformal symmetries of the open string into the asymptotic symmetries of the gravitational theory.

In the Weyl-rescaled frame the solution (8) reads [29]

$$ds^2=-[(\tilde{\lambda}R)^{a+1}-\mu]dT^2+[(\tilde{\lambda}R)^{a+1}-\mu]^{-1}dR^2,$$

$$\Phi=\tilde{\lambda}R, \quad R>0, \tag{76}$$

where

$$\mu=\frac{M}{\tilde{\lambda}^2}. \tag{77}$$

M is related to the black hole mass by $M=2\tilde{\lambda}\Phi_0^a m_{bh}$. Equation (76) can be obtained directly from Eq. (8) by rescaling the metric according to Eq. (71) and setting

$$\tilde{\lambda}R=\Phi_0(br)^{1/a}, \quad T=t\Phi_0^{-a}, \tag{78}$$

where $b=\tilde{\lambda}a$ and $\mu=A^2\Phi_0^{1+a}$. The weak-coupled asymptotic form of the metric (76) is obtained taking the limit $R\to\infty$:

$$ds^2\approx-(\tilde{\lambda}R)^{a+1}dT^2+(\tilde{\lambda}R)^{-a-1}dR^2. \tag{79}$$

The boundary is timelike for $a>0$ and lightlike for $-1<a<0$. In conformal coordinates the asymptotic metric (79) is

$$ds^2\approx(a\tilde{\lambda}X)^{-(a+1)/a}(-dT^2+dX^2), \tag{80}$$

where

$$a\tilde{\lambda}X=(\tilde{\lambda}R)^{-a}, \quad X>0. \tag{81}$$

The usual conformal symmetries of the string correspond to conformal symmetries of the metric (76). Using light-cone coordinates $U=(T+X)/2$ and $V=(X-T)/2$ the conformal Killing vectors of the string are

$$\chi=\chi^U(U)\partial_U+\chi^V(V)\partial_V. \tag{82}$$

Following Ref. [5] and imposing Dirichlet boundary conditions on the string,

$$\partial_T\chi^\mu|_{boundary}=0, \tag{83}$$

we have

$$\chi^{U,V}=\frac{1}{2}\left[\pm\epsilon(T)+\dot{\epsilon}(T)X\pm\frac{\ddot{\epsilon}(T)}{2}X^2+\mathcal{O}(X^3)\right]. \tag{84}$$

In the (T,R) frame the conformal Killing vectors are

$$\chi^T=\epsilon(T)+\frac{1}{2a^2\tilde{\lambda}^2}(\tilde{\lambda}R)^{-2a}\ddot{\epsilon}(T)+\mathcal{O}(R^{-4a}), \tag{85}$$

$$\chi^R=-\frac{1}{a}R\dot{\epsilon}(T)+\mathcal{O}(R^{1-2a}). \tag{86}$$

Performing the change of coordinates (78) and fixing the pure gauge diffeomorphisms the Killing vectors (20) and (21) are recognized to coincide with Eqs. (85) and (86). The Virasoro generators are

$$L_k=-\left[T^{k+1}+\frac{k(k+1)}{2a^2\tilde{\lambda}^2}(\tilde{\lambda}R)^{-2a}T^{k-1}+\mathcal{O}(R^{-4a})\right]\partial_T$$

$$+\left[\frac{1}{a}(k+1)RT^k+\mathcal{O}(R^{1-2a})\right]\partial_R. \tag{87}$$

Using the Weyl transformation (71) in Eqs. (18) and (19) we obtain the boundary conditions for the Weyl-rescaled metric $G_{\mu\nu}$. With a common notation for $h<0$ and $h>0$ we write

$$G_{TT}=G_{TT}^{(0)}(R)+G_{TT}^{(1)}(T,R),$$

$$G_{RR}=G_{RR}^{(0)}(R)+G_{RR}^{(1)}(T,R), \tag{88}$$

$$G_{TR}=G_{TR}^{(0)}(R)+G_{TR}^{(1)}(T,R),$$

where

$$G_{TT}^{(0)}=-(\tilde{\lambda}R)^{a+1}, \quad G_{RR}^{(0)}=(\tilde{\lambda}R)^{-a-1}, \quad G_{TR}^{(0)}=0, \tag{89}$$

and

$$G_{TT}^{(1)}=\gamma_{TT}(T)+\tilde{\gamma}_{TT}(T)(\tilde{\lambda}R)^{-a+1}+\mathcal{O}(R^{-2a})+\mathcal{O}(R^{-3a+1}),$$

$$G_{TR}^{(1)}=\gamma_{TR}(T)(\tilde{\lambda}R)^{-2a-1}+\tilde{\gamma}_{TR}(T)(\tilde{\lambda}R)^{-3a}+\mathcal{O}(R^{-4a-1})$$
$$+\mathcal{O}(R^{-5a}), \tag{90}$$

$$G_{RR}^{(1)}=\gamma_{RR}(T)(\tilde{\lambda}R)^{-2a-2}+\tilde{\gamma}_{RR}(T)(\tilde{\lambda}R)^{-3a-1}$$
$$+\mathcal{O}(R^{-4a-2})+\mathcal{O}(R^{-5a-1}).$$

The asymptotic boundary conditions for the dilaton are

$$\Phi=\Phi_0[\rho(\tilde{\lambda}R)+\gamma_{\Phi\Phi}(\tilde{\lambda}R)^{-a}+\tilde{\gamma}_{\Phi\Phi}(\tilde{\lambda}R)^{-2a+1}+\mathcal{O}(R^{-3a})$$
$$+\mathcal{O}(R^{-4a+1})]. \tag{91}$$

The Killing vectors (85) and (86) generate conformal transformations of the asymptotic fields. At the second order the asymptotic metric (76) is

$$ds^2=-(\tilde{\lambda}R)^{a+1}[1-\mu(\tilde{\lambda}R)^{-a-1}]dT^2$$
$$+(\tilde{\lambda}R)^{-a-1}[1+\mu(\tilde{\lambda}R)^{-a-1}+\mathcal{O}(R^{-2a-2})]dR^2. \tag{92}$$

Taking the Lie derivatives of the metric and of the dilaton with respect to the Killing vectors (85) and (86) we find

$$\delta G_{\mu\nu}=\frac{a-1}{a}\dot{\epsilon}(T)G_{\mu\nu}^{(0)}+\mathcal{O}(G_{\mu\nu}^{(1)}) \tag{93}$$

and

$$\delta\Phi = -\frac{1}{a}\dot{\epsilon}\Phi. \tag{94}$$

(The dilaton has been evaluated on shell.) The conformal transformation reduces to the asymptotic symmetry iff $a=1$ (AdS), i.e., when the Weyl transformation (71) is the identity. The Weyl rescaling (71) transforms the conformal symmetries of the metric $G_{\mu\nu}$ into the asymptotic symmetries of the metric $g_{\mu\nu}$. From Eq. (71) we have

$$\delta g_{\mu\nu} = \Phi^{a-1}\delta G_{\mu\nu} + (a-1)\Phi^{a-2}G_{\mu\nu}\delta\Phi. \tag{95}$$

Finally, substituting Eqs. (93) and (94) in Eq. (95) we obtain

$$\delta_L g_{\mu\nu} = \mathcal{O}(\Phi^{a-1}G^{(1)}_{\mu\nu}). \tag{96}$$

Recalling Eq. (91) it is straightforward to verify that Eq. (96) coincides with Eqs. (18) and (19).

Let us now consider the $r=r_h$ conformal region. The horizon is defined by the equation $\tilde{\lambda}^2\Phi_h^{a+1} - M = 0$. Since on the horizon $\Phi^{-1} \sim (\tilde{\lambda}/m_{bh})^{1/a+1}$ the latter does not belong to the weak-coupled regime (unless we consider macroscopic black holes for which $m_{bh} \gg \tilde{\lambda}$). Moreover, the sigma-model description breaks down at the horizon because the metric of the target space has a curvature singularity at $\Phi = \Phi_h$. So the sigma model is strong-coupled and the action (73) cannot be used to describe the $r=r_h$ conformal region.

VI. DUALITY SYMMETRIES

The black hole horizon can still be described in the sigma model formalism if we make use of duality symmetries. Two-dimensional dilaton gravity possesses a symmetry [30] that acts on the space of (classical) solutions; the field equations (5) and (6) are invariant under duality transformations. The explicit form of the duality transformations depends both on the gauge and on the conformal frame. For instance, in the gauge

$$ds^2 = -Y(\sigma)dt^2 + \Phi^{1-a}(\sigma)d\sigma^2, \tag{97}$$

the duality transformation for the Weyl-rescaled model (72) takes the simple form

$$Y \to \Phi^{a+1}, \quad \Phi \to Y^{1/(a+1)}. \tag{98}$$

In this gauge the black hole solutions (76) read

$$Y = \frac{1}{4}(e^{\tilde{\lambda}\sigma(a+1)/2} - \mu e^{-\tilde{\lambda}\sigma(a+1)/2})^2,$$

$$\Phi = \frac{1}{2}(e^{\tilde{\lambda}\sigma(a+1)/2} + \mu e^{-\tilde{\lambda}\sigma(a+1)/2})^{2/(a+1)}. \tag{99}$$

Applying the duality transformation (98) to Eqs. (99), we find that they only change the mass parameter μ into $-\mu$. So their overall effect is to interchange the zero of Y (the horizon of the black hole) with the zero of Φ (the singularity), namely the value of the coupling at the horizon $\Phi = \Phi_h$ with the coupling at the singularity $\Phi = 0$.

In order to implement the duality transformation in the sigma model action we need to repeat the calculation for the (off-shell) mass function (69). It is convenient to work in the Schwarzschild gauge

$$ds^2 = -Y(r)dt^2 + Y^{-1}(r)dr^2. \tag{100}$$

In this gauge $d\Phi/dr = \tilde{\lambda}$ and the mass function is

$$M = \tilde{\lambda}^2(\Phi^{a+1} - Y). \tag{101}$$

From Eqs. (100) and (97) we have $Y = \Phi^{a-1}(dr/d\sigma)^2$. Substituting this result in the duality relation (98) we obtain

$$Y = \Phi^{a-1}(dr/d\sigma)^2 \to \Phi^{a+1}, \quad \Phi^{a+1} \to \Phi^{a-1}(dr/d\sigma)^2 = Y. \tag{102}$$

So the duality transformation changes the sign of the mass function. Since the latter is a scalar function this statement is gauge independent. Now we can implement the duality transformation in the sigma model action (73). The dual action reads

$$S_{\sigma,dual} = -\frac{1}{2}\int d^2X\sqrt{-G}\frac{\nabla_\mu\Phi\nabla^\mu M}{\tilde{\lambda}^2\Phi^{a+1} + M}. \tag{103}$$

The horizon $\Phi = \Phi_h$ is mapped into $\Phi = 0$. This value belongs to the weak-coupled region of the dual sigma model. Therefore, we can expand the dual action (103) around $\Phi = 0$. At leading order we obtain the free CFT

$$S_{\sigma,dual} = \frac{1}{2}\int d^2X\sqrt{-G}\nabla_\mu\Phi\nabla^\mu\tilde{M}, \tag{104}$$

where $\tilde{M} = -\ln(M/\tilde{\lambda}^2)$. Classically, the black hole horizon is described by a two-dimensional CFT (bosonic string).

VII. PERTURBATIVE CONFORMAL POINTS

We have shown that at the classical level the gravitational dynamics in the limit (a) $\Phi \to \infty$ and (b) $\Phi \to \Phi_h$ is described by free CFTs. Moreover, as we have discussed at the end of Sec. II, the limit (c) $a \to \infty$ $(h \to 1/2)$ also leads to a free CFT. The classical conformal invariance of the sigma model (73) is generally broken at quantum level. Conformal invariance is (perturbatively) preserved only for those values of the coupling with vanishing beta function. Now we show that the classical conformal invariance is preserved at the one-loop level, i.e., that (a), (b), and (c) are three perturbative conformal points of the sigma model.

Let us first consider (a) and (c). Changing coordinates in the target space to the dimensionless fields ξ^α,

$$\Phi = a^2(\xi^1)^{-1/a}, \quad M = a^{2a+1}\tilde{\lambda}^2\xi^0, \tag{105}$$

the sigma-model action (73) is cast in the standard form

$$S_{\sigma}=\frac{1}{2}\int d^2X\sqrt{-G}\,\mathcal{G}_{\alpha\beta}\partial_{\mu}\xi^{\alpha}\partial^{\mu}\xi^{\beta}, \tag{106}$$

where the metric of the target space is

$$\mathcal{G}_{\alpha\beta}=-\frac{a/2}{a-\xi^0(\xi^1)^{(a+1)/a}}\Omega_{\alpha\beta}, \quad \Omega_{\alpha\beta}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{107}$$

For $\xi^1=0$ $(\Phi=\infty)$ and $a=\infty$ the sigma model describes a free two-dimensional CFT. So we can consistently use the perturbation theory around these vacua. The one-loop beta function is

$$\beta_{\alpha\beta}\equiv\frac{1}{2\pi}\mathcal{R}_{\alpha\beta}=-\frac{a+1}{2\pi}\frac{(\xi^1)^{1/a}}{[a-\xi^0(\xi^1)^{(a+1)/a}]^2}\Omega_{\alpha\beta}, \tag{108}$$

where $\mathcal{R}_{\alpha\beta}$ is the Ricci tensor of the target space metric. The beta function is always negative and vanishes for $\xi^1=0$ and $a=\infty$. Hence, $\xi^1=0$ and $a=\infty$ are perturbative (stable) conformal points of the dilaton gravity theory.

Let us now consider the behavior of the sigma model at the horizon. Using the duality symmetries of the model the near-horizon behavior can be described by the dual action (103) in the limit $\Phi\to 0$. Introducing the dimensionless fields

$$M=\tilde{\lambda}^2 e^{-\xi^0}, \quad \Phi=\xi^1, \tag{109}$$

the sigma model action (103) is cast in the standard form (106) with target space metric

$$\mathcal{G}_{\alpha\beta}=\frac{1/2}{1+e^{\xi^0}(\xi^1)^{a+1}}\Omega_{\alpha\beta}. \tag{110}$$

For $\xi^1=0$ the sigma model describes a two-dimensional free conformal field theory and we can consistently use perturbation theory near $\Phi=0$. The one-loop beta function is

$$\beta_{\alpha\beta}=\frac{a+1}{2\pi}\frac{e^{\xi^0}(\xi^1)^a}{[1+e^{\xi^0}(\xi^1)^{a+1}]^2}\Omega_{\alpha\beta}. \tag{111}$$

The beta function is now positive and vanishes for $\xi^1=0$.

The dilaton gravity model has three perturbative conformal points where it behaves as a free two-dimensional conformal field theory. Point (a) corresponds to the weak-coupled regime. Since the beta function is negative this point is stable. The free conformal theory, CFT_∞, describes the gravitational model near the asymptotic region. Point (b) corresponds to the horizon of the black hole. In this case the beta function is positive and the point is perturbatively unstable. The two-dimensional free conformal theory, CFT_h, describes the gravitational model near the horizon of the black hole. Point (c) is of a different nature, since it corresponds to the limit $a\to\infty$; in this case the model becomes *identically* equivalent to a free CFT. Both the existence and structure of the conformal points have important consequences on the derivation of the thermodynamical properties of the black hole. We will discuss this point in the next section.

VIII. MICROSCOPIC ENTROPY OF THE BLACK HOLE

Though we have identified three distinct perturbative conformal points and their corresponding CFTs, a complete description of the 0-brane dynamics is still missing. We are dealing with a strong-coupled system which cannot be solved exactly at the quantum level. Since we do not know how to deal with the regions between the conformal points, the behavior of the system near these points is not sufficient to fully characterize the properties of the theory. Consequently, it is not clear whether the CFTs can be used to explain the (semiclassical) thermodynamical properties of the black hole.

As far as the algebra CFT_∞ is concerned we can reasonably identify the eigenvalue L_0^∞ of L_0 with the energy E of the gravitational configuration, $L_0^\infty=E/b$. The energy E is indeed the eigenvalue of the Killing vector which generates time translations and thus coincides with L_0^∞. The two parameters that characterize the CFT_∞ algebra, L_0^∞, and the central charge c^∞ [see Eq. (47)], are completely determined by the energy and the zero mode of the dilaton, i.e., by the observables which are associated with the gravitational configuration. We can use the Cardy formula [31]

$$S=2\pi\sqrt{\frac{cL_0}{6}} \tag{112}$$

to compute the entropy of the gravitational configuration as a function of the density of states of CFT_∞. Substituting Eq. (47) in Eq. (112) we obtain

$$S=\begin{cases}4\pi\sqrt{\dfrac{(1-2h)E\Phi_0}{b}}, & h\leqslant 0\\ 0, & h>0.\end{cases} \tag{113}$$

Setting $E=m_{bh}$ we should recover the thermodynamical entropy of the black hole. However, for generic nonzero values of h Eq. (113) does not coincide with Eq. (13).[3] This result has different origins depending on the sign of h. We stressed in Sec. III that for $h<0$ the deformations that generate the Virasoro algebra do not generate the black hole. Therefore, we cannot identify E with the black hole mass. Conversely, for $h>0$ the deformations that generate the Virasoro algebra also generate the black hole and we can identify E with m_{bh}. However, the deformations do not correspond to truly dynamical degrees of freedom and the central charge vanishes. This is no surprise, since for $h\neq 0$ there is no obvious reason why the dynamics of a strong-coupled system such as the black hole should be described by a free CFT.[4]

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[3]For $h=0$ (the JT model) Eq. (113) leads to a mismatch of a $\sqrt{2}$ factor between the thermodynamical and the CFT entropy. However, the origin of the discrepancy is known [19].

[4]For $h=0$ one can invoke the AdS/CFT correspondence.

As far as the CFT_h algebra is concerned both L_0^h and c^h are determined by the value of the dilaton at the horizon. Inserting Eq. (66) in Eq. (112) we recover the thermodynamical entropy of the black hole (13) [12]. Although the density of states of CFT_h explains correctly the entropy of the black hole, both L_0^h and c^h depend on the Hawking temperature of the horizon T_h [see Eq. (66)]. Therefore, we cannot identify T_h with the temperature T_{CFT} of CFT_h; the energy-temperature relation for a general CFT is $E \propto c T_{CFT}^2/b$. From this relation and from Eqs. (112) and (66) it follows that T_{CFT} is determined by the curvature of the AdS space, $T_{CFT} \propto b$. Though we expect that the thermodynamical properties of the black hole are described by a CFT at the horizon, the relation between the CFT and the parameters of the black hole is far from being trivial.

The conclusions above have a counterpart in the sigma model formulation. The asymptotic region represents a perturbative stable conformal fixed point. In contrast, the horizon lies deep in the strong-coupled region of the sigma model and its CFT description is obtained through a duality transformation whose physical meaning is not completely clear.

Let us conclude this section by briefly discussing the $a \to \infty$ regime. In this case the dilaton is constant and the solution (14) can be described by a CFT everywhere. The metric (14) possesses a Killing horizon; we can use Eq. (66) to compute L_0 and c. Since $\Phi = \Phi_0$ we obtain $c/6 = L_0 = \Phi_0$. Substituting the latter in the Cardy formula we recover the thermodynamical entropy (15). It should be noticed that if we used the $r \to \infty$ boundary calculation of Sec. III to compute the central charge we would obtain $c=0$, i.e., $S=0$. This is consistent with the fact that there are no dynamical degrees of freedom on the boundary of AdS_2. The degrees of freedom which are responsible for the entropy are not localized on the boundary of AdS_2 but on the Killing horizon of the solution (14). Therefore, we have a realization of holography.

IX. CONCLUSIONS

In this paper we have investigated dilatonic 0-branes in the near-horizon approximation. In the dual frame the solutions have the $\mathrm{AdS}_2 \times \mathrm{S}^{D-2}$ form. Using different approaches such as the canonical realization of asymptotic symmetries as deformation algebra, the sigma model, and dualities, we have found that the dynamics is characterized by three conformal points both at classical and at (one-loop) quantum levels. The ensuing CFTs have been identified by the fields which describe the boundary deformations and/or by the degrees of freedom of the sigma model. We have calculated the central charges of the Virasoro algebra.

The use of the CFTs to describe the thermodynamics of black holes (in particular their entropy) seems problematic. Though the CFT_∞ description is natural from the black hole point of view (the black hole mass is identified with L_0 and the central charge is a function of Φ_0), nevertheless it does not reproduce the black hole entropy for generic couplings. Technically, this may be explained either by the absence of dynamical degrees of freedom on the (AdS) boundary of the spacetime ($h>0$) or by the impossibility of relating them to the degrees of freedom of the black hole ($h<0$). Only in the JT model, where the spacetime is AdS_2 and there is no curvature singularity, CFT_∞ reproduces the black hole thermodynamical relations. In this case we have a genuine realization of the $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence.

In contrast, the CFT at the horizon seems to give a good description of black hole thermodynamical relations. This is sensible because we expect the thermal properties of black holes to be associated with the horizon. However, the horizon lies deep in the strong-coupled region of the sigma model and in order to describe its dynamics by a free CFT we must employ a nonperturbative tool: the duality symmetry of the model. So from the black hole point of view the CFT_h description remains obscure. Both the central charge and the eigenvalue of L_0 depend on the Hawking temperature of the horizon, whereas we would like c to depend on Φ_0 only. Something fundamental is still missing in the picture, hidden perhaps in the full quantum dynamics of the black hole.

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