Bounds in proton-proton elastic scattering at low momentum transfer

A. T. Bates* and N. H. Buttimore†
School of Mathematics, University of Dublin, Trinity College, Dublin 2, Ireland
(Received 21 December 2000; published 12 December 2001)

We present a bound on the imaginary part of the single helicity-flip amplitude for spin 1/2–spin 1/2 scattering at small momentum transfer. The variational method of Lagrange multipliers is employed to optimize the single-flip amplitude using the values of $\sigma_{\rm tot}$, $\sigma_{\rm el}$ and diffraction slope as equality constraints in addition to the inequality constraints resulting from unitarity. Such bounds provide important information related to the determination of the polarization of a proton beam. In the case of elastic proton collisions the analyzing power at small scattering angles offers a method of measuring the polarization of a proton beam, the accuracy of the polarization measurement depending on knowledge of the single helicity-flip amplitude. The bound obtained on the imaginary part of the single helicity-flip amplitude indicates that the analyzing power for proton-proton collisions in the Coulomb nuclear interference region should take positive values at high energies.

DOI: 10.1103/PhysRevD.65.014015 PACS number(s): 13.88.+e, 13.90.+i, 45.10.Db

I. INTRODUCTION

The proton spin puzzle has intrigued experimentalists and theorists since the surprising result from the European Muon Collaboration (EMC) experiment at CERN in 1988, which found a smaller than expected contribution to the spin of the proton from the component quarks. The question, "where does the spin of the proton come from?" remains unanswered [1,2]. Recent data suggests a value of $\sim 20-30\%$ for the fraction of the spin carried by the *up*, *down* and *strange* quarks. The contribution from the gluons and from the orbital angular momentum of the quarks and gluons is not completely known. The Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory plans to probe the proton structure using the deep inelastic scattering of protons at high center-of-mass energies ($\sqrt{s}=50-500$ GeV) and momentum transfers ($p_T \geqslant 10$ GeV/c) [3]. To measure the contribution of the gluons to the spin of the proton, with sufficient accuracy, a polarized proton beam with a beam polarization error of 5% is necessary. One method of measuring the polarization of a proton beam uses the analyzing power in elastic proton collisions at small scattering angles.

In polarized proton-proton elastic collisions the transverse single spin asymmetry $\mathcal{A}$ can be measured by counting the scatters with the beam polarized up ($N^{\uparrow}$) and then down ($N^{\downarrow}$), where

$$\mathcal{A}=\frac{N^{\uparrow}-N^{\downarrow}}{N^{\uparrow}+N^{\downarrow}}. \tag{1.1}$$

The polarization P of a proton beam is related to the measured asymmetry $\mathcal{A}$—through the analyzing power A_N:

$$\mathcal{A}=A_N P, \tag{1.2}$$

the transverse asymmetry $\mathcal{A}$ being equal to the analyzing power when the beam is 100% polarized. Therefore, accurate knowledge of the analyzing power A_N enables the beam polarization to be calculated, given a measured value of the single spin asymmetry. In this paper we present a method of finding a bound on the range of values which the analyzing power can assume for a particular center-of mass (c.m.) energy at low momentum transfer. Lagrangian optimization provides a technique of deriving bounds on scattering amplitudes in proton-proton elastic collisions particularly the single helicity-flip amplitude, which consequently, limits the size of the analyzing power at low momentum transfers inside the Coulomb nuclear interference (CNI) region. The Lagrange multiplier variational technique, extended by Einhorn and Blankenbecler [4] to include equality and inequality constraints in the context of scattering theory, is used to derive a bound on the single helicity-flip amplitude, modified by a kinematical factor. For elastic spin 0–spin 1/2 scattering a number of bounds on the helicity amplitudes have been found using the Lagrangian optimization technique [5–7]. A unitarity representation for helicity amplitudes has been used to derive an asymptotic bound on the single helicity-flip amplitude at low momentum transfers which limits its growth at high energies [8].

A review of *s*-channel helicity amplitudes, appropriate for a discussion of bounds and CNI polarimetry, is presented in Sec. II. We shall indicate how the magnitude of the single helicity-flip amplitude can limit the value of transverse single spin asymmetry at both high energy and small collision angles. Expressions for the scattering observables—the total cross section, the elastic cross section and the hadronic slope parameter—which are used as constraints in the optimization of the single helicity-flip, are exhibited in Sec. III. The inequality unitarity constraints are also introduced. A detailed calculation of the bound is presented in Sec. IV followed by a treatment of errors in Sec. V. A discussion of the bound with its consequences for polarimetry is provided in the concluding Sec. VI.

*Email address: atbates@maths.tcd.ie
†Email address: nhb@maths.tcd.ie

II. AMPLITUDES AND ASYMMETRY

For the elastic scattering of two protons with helicities λ_1 and λ_2, at c.m. energy $\sqrt{s}$, there are sixteen helicity amplitudes ϕ_i in general, each a function of s and t. The number of independent amplitudes reduces to five under the following relations [9–11], λ_1' and λ_2' referring to the outgoing proton helicities taking values $\pm 1/2$:

Parity conservation,

$$\langle \lambda_1'\lambda_2'|\phi|\lambda_1\lambda_2\rangle = (-1)^{\mu-\lambda}\langle -\lambda_1' - \lambda_2'|\phi|-\lambda_1-\lambda_2\rangle. \tag{2.1}$$

Time reversal invariance,

$$\langle \lambda_1'\lambda_2'|\phi|\lambda_1\lambda_2\rangle = (-1)^{\mu-\lambda}\langle \lambda_1\lambda_2|\phi|\lambda_1'\lambda_2'\rangle. \tag{2.2}$$

Identical particle scattering,

$$\langle \lambda_1'\lambda_2'|\phi|\lambda_1\lambda_2\rangle = (-1)^{\lambda-\mu}\langle \lambda_2'\lambda_1'|\phi|\lambda_2\lambda_1\rangle, \tag{2.3}$$

where $\lambda = \lambda_1 - \lambda_2$, $\mu = \lambda_1' - \lambda_2'$. The five helicity amplitudes include two non helicity-flip amplitudes $\phi_1 = \langle ++|\phi|++\rangle$ and $\phi_3 = \langle +-|\phi|+-\rangle$, two double helicity-flip amplitudes $\phi_2 = \langle ++|\phi|--\rangle$ and $\phi_4 = \langle +-|\phi|-+\rangle$, and one single helicity-flip amplitude $\phi_5 = \langle ++|\phi|+-\rangle$, with partial wave expansions [9,10]:

$$\phi_1(s,t) = \frac{\sqrt{s}}{2k}\sum_J (2J+1)[f_0^J(s) + f_{11}^J(s)]d_{00}^J(\theta) \tag{2.4}$$

$$\phi_3(s,t) = \frac{\sqrt{s}}{2k}\sum_J (2J+1)[f_1^J(s) + f_{22}^J(s)]d_{11}^J(\theta) \tag{2.5}$$

$$\phi_2(s,t) = \frac{\sqrt{s}}{2k}\sum_J (2J+1)[f_{11}^J(s) - f_0^J(s)]d_{00}^J(\theta) \tag{2.6}$$

$$\phi_4(s,t) = \frac{\sqrt{s}}{2k}\sum_J (2J+1)[f_{22}^J(s) - f_1^J(s)]d_{1-1}^J(\theta) \tag{2.7}$$

$$\phi_5(s,t) = \frac{\sqrt{s}}{2k}\sum_J (2J+1)f_{21}^J(s)d_{10}^J(\theta) \tag{2.8}$$

where $f_i^J(s)$ $(i=0,1,11,22,21)$ denote s-channel partial wave amplitudes and the c.m. momentum is $k = \sqrt{s-4m^2}/2$. The c.m. scattering angle is given by

$$\cos\theta = 1 + t/2k^2. \tag{2.9}$$

The analyzing power A_N expressed in terms of the s-channel helicity amplitudes is

$$A_N\frac{d\sigma}{dt} = -\frac{4\pi}{s(s-4m^2)}\mathrm{Im}[\phi_5^*(\phi_1+\phi_2+\phi_3-\phi_4)]. \tag{2.10}$$

The reduced ratio, r_5, of the hadronic helicity single-flip to imaginary hadronic non-flip amplitude is defined as [12]

$$r_5 = \frac{m}{\sqrt{-t}} \times \frac{\phi_5}{\mathrm{Im}\frac{1}{2}(\phi_1+\phi_3)}. \tag{2.11}$$

The analyzing power A_N for the Coulomb nuclear interference (CNI) region can be written as follows, when the transverse total cross section spin difference is neglected [12,13]:

$$A_N = \frac{\sqrt{-t}}{m}\,\frac{(\mu_p - 1 - 2\,\mathrm{Im}\,r_5)(t_c/t) + 2(\rho\,\mathrm{Im}\,r_5 - \mathrm{Re}\,r_5)}{(t_c/t)^2 - 2(\rho+\delta)(t_c/t) + (1+\rho^2)} \tag{2.12}$$

where $t_c = -8\pi\alpha/\sigma_{\mathrm{tot}} \approx -0.0012$ $(\mathrm{GeV}/c)^2$. The electromagnetic and hadronic components of the average helicity non-flip amplitude are of the same magnitude when $t = \sqrt{3}\,t_c$ [14]. The Coulomb phase is δ, $\mu_p = \kappa_p + 1 = 2.7928$ is the magnetic moment of the proton and [12]

$$\rho = \frac{\mathrm{Re}(\phi_1+\phi_3)}{\mathrm{Im}(\phi_1+\phi_3)}. \tag{2.13}$$

Apart from the photon pole term, the t-dependence of helicity nonflip and flip electromagnetic and hadronic amplitudes due to form factor and nuclear slope effects is not expected to play a significant role in the amplitude ratios featuring in the asymmetry. An important contribution to the maximum of A_N, in the CNI region ($|t| < |t_c|$), comes from $\mathrm{Im}\,r_5$ in the form of $\mu_p - 1 - 2\,\mathrm{Im}\,r_5$ as indicated in Eq. (2.12). Therefore, a bound on $\mathrm{Im}\,r_5$ which satisfies $\mu_p - 1 - 2\,\mathrm{Im}\,r_5 > 0$ ensures that the maximum analyzing power in the CNI region is positive.

III. SCATTERING OBSERVABLES

In the CNI region it is convenient to express the five helicity amplitudes in terms of Jacobi polynomials. To relate the $d^J_{\lambda\,\mu}(\theta)$ functions to Jacobi polynomials of the variable $z = \cos\theta$, it is suitable to separate the space of λ and μ into four regions [15]. In the region where $\lambda + \mu \geqslant 0$ and $\lambda - \mu \geqslant 0$, the relation is

$$d^J_{\lambda\,\mu}(\theta) = \sqrt{\frac{(J+\lambda)!(J-\lambda)!}{(J+\mu)!(J-\mu)!}}\left(\frac{1+z}{2}\right)^{(\lambda+\mu)/2} \times \left(\frac{1-z}{2}\right)^{(\lambda-\mu)/2} P_{J-\lambda}^{(\lambda-\mu,\lambda+\mu)}(z), \tag{3.1}$$

and $J - \lambda = 0,1,2,\ldots$. Equivalent forms in the other regions are obtained by use of symmetry relations [10,15];

$$d^J_{\lambda\,\mu}(\theta) = \begin{cases} (-1)^{\lambda-\mu}d^J_{\mu\,\lambda}(\theta), & \lambda+\mu \geqslant 0, \quad \lambda-\mu \leqslant 0, \\ (-1)^{\lambda-\mu}d^J_{-\lambda\,-\mu}(\theta), & \lambda+\mu \leqslant 0, \quad \lambda-\mu \leqslant 0, \\ (-1)^{\lambda-\mu}d^J_{-\lambda\,-\mu}(\theta), & \lambda+\mu \leqslant 0, \quad \lambda-\mu \geqslant 0. \end{cases} \tag{3.2}$$

Expressing the $d^J_{\lambda\,\mu}(\theta)$ functions in terms of Jacobi polynomials, the five independent helicity amplitudes can be written as

$$\phi_1(s,t)=\frac{\sqrt{s}}{2k}\sum_J\,(2J+1)[f_0^J(s)+f_{11}^J(s)]P_J^{(0,0)}(z) \tag{3.3}$$

$$\phi_2(s,t)=\frac{\sqrt{s}}{2k}\sum_J\,(2J+1)[f_{11}^J(s)-f_0^J(s)]P_J^{(0,0)}(z) \tag{3.4}$$

$$\phi_3(s,t)=\frac{\sqrt{s}(1+z)}{4k}\sum_J\,(2J+1)[f_1^J(s)+f_{22}^J(s)]P_{J-1}^{(0,2)}(z) \tag{3.5}$$

$$\phi_4(s,t)=\frac{\sqrt{s}(1-z)}{4k}\sum_J\,(2J+1)[f_{22}^J(s)-f_1^J(s)]P_{J-1}^{(2,0)}(z) \tag{3.6}$$

$$\phi_5(s,t)=\frac{\sqrt{s}\sqrt{1-z^2}}{4k}\sum_J\,(2J+1)\sqrt{\frac{J+1}{J}}f_{21}^J(s)P_{J-1}^{(1,1)}(z) \tag{3.7}$$

where $z=\cos\theta$. The Jacobi polynomials have the properties [16]

$$P_n^{(\alpha,\beta)}(1)=\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)n!}, \tag{3.8}$$

$$\frac{d^m}{dz^m}P_n^{(\alpha,\beta)}(z)=2^{-m}\frac{\Gamma(m+n+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)}P_{n-m}^{(\alpha+m,\beta+m)}(z). \tag{3.9}$$

We will next derive expressions for the observables which are included as equality constraints in the optimization of the single helicity-flip amplitude.

A. Equality constraints

In proton-proton elastic scattering the spin observables can be written in terms of the five helicity amplitudes [11]. The observables are vital to the optimization, since each observable can be included as an equality constraint in the optimized system. The derivation of the bound uses three equality constraints, $\sigma_{\rm tot}$, $\sigma_{\rm el}$ and g, and two inequality constraints related to unitarity.

1. Total cross section

The first equality constraint involves the total cross section. The optical theorem is used to write the imaginary part of the spin average helicity non-flip amplitude, defined by

$$\phi_+(s,t)=\frac{\phi_1(s,t)+\phi_3(s,t)}{2}, \tag{3.10}$$

in terms of the total cross section $\sigma_{\rm tot}$:

$$\text{Im}\,\phi_+(s,0)=\frac{k\sqrt{s}}{4\pi}\sigma_{\rm tot}(s). \tag{3.11}$$

The total cross section expressed in terms of Im $\phi_+(s,0)$ has the partial wave expansion

$$\sigma_{\rm tot}(s)=\frac{\pi}{k^2}\sum_J\,(2J+1)\{(a_0^J+a_{11}^J)P_J^{(0,0)}(1)+(a_1^J+a_{22}^J)P_{J-1}^{(0,2)}(1)\} \tag{3.12}$$

where the imaginary and real parts of the partial waves are given by

$$a_i=\text{Im}\,f_i^J,\quad b_i=\text{Re}\,f_i^J,\quad i=0,1,11,22,21. \tag{3.13}$$

Using property (3.8) the normalized dimensionless total cross section can be expressed as a partial wave expansion

$$A_0=\sum_J\,(2J+1)\{a_0^J(s)+a_1^J(s)+a_{11}^J(s)+a_{22}^J(s)\} \tag{3.14}$$

where $A_0=(k^2/\pi)\sigma_{\rm tot}$.

2. Slope of the imaginary non-flip amplitude

The slope of the imaginary non-flip amplitude has been used in bounds for other spin dependent elastic collisions [5–7,17]. We find it convenient to use the imaginary part of the spin-averaged amplitude at a particular value of t. The second equality constraint employs the imaginary spin average non-flip amplitude at a particular t inside the Coulomb nuclear interference region, written as a Taylor expansion

$$\text{Im}\,\phi_+(s,t)\approx\text{Im}\,\phi_+(s,0)+t\left(\frac{d}{dt}\text{Im}\,\phi_+(s,t)\right)\bigg|_{t=0}, \tag{3.15}$$

where $|t|$ is sufficiently small so that inclusion of the linear term in the Taylor expansion is an accurate approximation. Use of properties (3.8) and (3.9) leads to the partial wave expansion for the imaginary non-flip amplitude

$$\text{Im}\,\phi_+(s,t)=\frac{\sqrt{s}}{4k}\sum_J\,(2J+1)\{a_0^J+a_1^J+a_{11}^J+a_{22}^J\}\times\left(1-\frac{\zeta}{4}J(J+1)\right) \tag{3.16}$$

where $\zeta=-t/k^2$. The logarithmic derivative of the imaginary spin average non-flip amplitude,

$$g=\frac{d}{dt}\ln\text{Im}\,\phi_+(s,t)|_{t=0}=\frac{1}{\text{Im}\,\phi_+(s,0)}\left(\frac{d}{dt}\text{Im}\,\phi_+(s,t)\right)\bigg|_{t=0}, \tag{3.17}$$

when combined with the Taylor expansion for Im $\phi_+(s,t)$, given by Eq. (3.15), indicates that

$$\text{Im}\,\phi_+(s,t)=\text{Im}\,\phi_+(s,0)\{1+t\,g\}. \tag{3.18}$$

3. Elastic cross section

The third equality constraint relates to the elastic cross section, expressed as a partial wave expansion by integrating the differential cross section over momentum transfer t:

$$\sigma_{\rm el}(s)=\int_{-4k^2}^{0}dt\,\frac{d\sigma(s,t)}{dt}. \tag{3.19}$$

The expression for momentum transfer

$$t=-2k^2(1-z), \tag{3.20}$$

enables one to equivalently integrate over the z variable. Expressing the elastic cross section in terms of partial waves requires integrals of the form

$$\int_{-1}^{+1}dz|\phi_i(s,t)|^2$$

to be calculated, where $i=1,\ldots,5$. The integration formula [18], with Kronecker delta function δ_{mn},

$$\int_{-1}^{+1}(1-z)^{\alpha}(1+z)^{\beta}P_n^{(\alpha,\beta)}(z)P_m^{(\alpha,\beta)}(z)dz$$
$$=\frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}\delta_{mn} \tag{3.21}$$

may be used to evaluate the integral

$$\int_{-1}^{+1}d^J_{\lambda\mu}(\theta)d^{J'}_{\lambda\mu}(\theta)dz=\frac{2}{2J+1}\delta_{JJ'}, \tag{3.22}$$

leading to a partial wave expansion for the normalized dimensionless elastic cross section, defined as $\Sigma_{\rm el}=(k^2/\pi)\sigma_{\rm el}$:

$$\Sigma_{\rm el}(s)=\sum_J\,(2J+1)\{|f_0^J|^2+|f_1^J|^2+|f_{11}^J|^2+|f_{22}^J|^2+2|f_{21}^J|^2\}. \tag{3.23}$$

B. Imaginary single-flip amplitude

The imaginary single helicity-flip amplitude, modified by a kinematical factor, is the objective function in the system. Before optimization we must first express the single-flip amplitude in a suitable form. The imaginary amplitude, from Eq. (3.7) with the Jacobi polynomial $P_{J-1}^{(1,1)}(z)$ expanded as a Taylor series in the CNI region, can be written as

$$\text{Im}\,\tilde{\phi}_5\approx\frac{\sqrt{s}}{4k}\sum_J\,(2J+1)\sqrt{\frac{J+1}{J}}J\left(1-\frac{\zeta}{8}[J(J+1)-2]\right)a_{21}^J \tag{3.24}$$

where $z=1+t/(2k^2)$ and $\zeta=-t/k^2$. At small collision angles, the ratio $\text{Im}\,r_5$ is approximated as

$$\text{Im}\,r_5=\frac{m}{k}\,\frac{\text{Im}\,\tilde{\phi}_5}{\text{Im}\,\phi_+(s,t)}.$$

C. Unitarity

The partial wave amplitudes obey the following unitarity inequalities [19]

$$U_1^J=a_0^J-|f_0^J|^2\geqslant 0,\quad V_1^J=a_{11}^J-|f_{11}^J|^2-|f_{21}^J|^2\geqslant 0$$
$$U_2^J=a_1^J-|f_1^J|^2\geqslant 0,\quad V_2^J=a_{22}^J-|f_{22}^J|^2-|f_{21}^J|^2\geqslant 0. \tag{3.25}$$

It is useful to define $X^J=U_1^J+U_2^J$ and $W^J=V_1^J+V_2^J$, relating to the inequalities

$$X^J=a_0^J+a_1^J-|f_0^J|^2-|f_1^J|^2\geqslant 0 \tag{3.26}$$

$$W^J=a_{11}^J+a_{22}^J-|f_{11}^J|^2-|f_{22}^J|^2-2|f_{21}^J|^2\geqslant 0. \tag{3.27}$$

For the elastic scattering of spin 0 on spin 1/2 particles there are two independent helicity amplitudes, a flip and a non-flip amplitude, with partial wave expansions whose partial wave amplitudes obey unitarity relations similar to U_1^J and U_2^J in Eq. (3.25). The unitarity relations V_1^J and V_2^J of Eq. (3.25) are characteristic of spin 1/2 – spin 1/2 scattering, the f_{21}^J term coming from the single helicity-flip amplitude ϕ_5.

IV. OPTIMIZATION

Equipped with partial wave expansions for the observables and partial wave inequality relations, representing unitarity, we are in a position to optimize the modified helicity single-flip amplitude $\text{Im}\,\tilde{\phi}_5$. We follow the variational technique of Einhorn and Blankenbecler [4] by constructing a Lagrangian consisting of an objective function and a set of equality and inequality constraints. We use the full set of constraints, $\sigma_{\rm tot}$, $\sigma_{\rm el}$, g and unitarity, although a bound on $\text{Im}\,\tilde{\phi}_5$, with fewer constraints, can be derived [20].

A. Lagrange formalism

The normalized dimensionless total cross section A_0, expressed as an equality constraint, is included in the Lagrange function along with the normalized dimensionless elastic cross section $\Sigma_{\rm el}$, written as an equality constraint, the imaginary spin average non helicity-flip amplitude $\text{Im}\,\phi_+(s,t)$ at a fixed small $|t|$ value, also expressed as an equality constraint, and the partial wave unitarity relations written as inequality constraints. The modified single helicity-flip amplitude $\text{Im}\,\tilde{\phi}_5$ is introduced as the objective function in the Lagrange function:

$$\mathcal{L}=\text{Im}\,\tilde{\phi}_5+\alpha\left[A_0-\sum_J (2J+1)\{a_0^J+a_1^J+a_{11}^J+a_{22}^J\}\right]$$
$$+\beta\left[\Sigma_{\text{el}}-\sum_J (2J+1)(|f_0^J|^2+|f_1^J|^2+|f_{11}^J|^2\right.$$
$$\left.+|f_{22}^J|^2+2|f_{21}^J|^2)\right]+\gamma\left[\text{Im}\,\phi_+-\frac{\sqrt{s}}{4k}\sum_J (2J+1)\right.$$
$$\left.\times\{a_0^J+a_1^J+a_{11}^J+a_{22}^J\}\left(1-\frac{\zeta}{4}J(J+1)\right)\right]$$
$$+\sum_J (2J+1)\mu_J(a_{11}^J+a_{22}^J-|f_{11}^J|^2-|f_{22}^J|^2-2|f_{21}^J|^2)$$
$$+\sum_J (2J+1)\lambda_J(a_0^J+a_1^J-|f_0^J|^2-|f_1^J|^2) \tag{4.1}$$

where α, β and γ are equality multipliers. The inequality multipliers, λ_J and μ_J, are by definition non-negative and $\zeta=-t/(k^2)$. In the high energy or large J limit only the leading order J terms need be included and Eq. (4.1) may be written with $2J$ replacing $2J+1$. The system is optimized by taking first and second derivatives with respect to the real and imaginary partial wave amplitudes, b_i^J and a_i^J. This gives the optimized set of partial waves, at some fixed t in the CNI region;

$$b_i^J=0 \quad \text{for all } i \text{ implying } f_i^J=ia_i^J, \tag{4.2}$$

$$a_0^J=a_1^J=\frac{r_1+r_2(1-\zeta J^2/4)+\tilde{\lambda}_J}{1+2\tilde{\lambda}_J}, \tag{4.3}$$

$$a_{11}^J=a_{22}^J=\frac{r_1+r_2(1-\zeta J^2/4)+\tilde{\mu}_J}{1+2\tilde{\mu}_J} \tag{4.4}$$

and

$$a_{21}^J=\frac{J(1-\zeta J^2/8)}{8\beta(1+2\tilde{\mu}_J)} \tag{4.5}$$

where $\tilde{\lambda}_J=\lambda_J/2\beta$, $\tilde{\mu}_J=\mu_J/2\beta$, $r_1=-\alpha/(2\beta)$, $r_2=-\gamma/(4\beta)$ and $\beta>0$ for a maximum (or $\beta<0$ for a minimum).

B. Unitarity classes

The imaginary partial wave amplitudes, optimized under the four constraints, obey the following unitarity inequalities

$$X^J=a_0^J-a_0^{J\,2}\geqslant 0, \quad W^J=a_{11}^J-a_{11}^{J\,2}-a_{21}^{J\,2}\geqslant 0. \tag{4.6}$$

It is natural to divide the partial waves into two classes, one with contributions from the interior unitarity class I and the other with contributions from the boundary unitarity class B. For the X^J unitarity inequality the interior and boundary unitarity classes are defined as

$$I^X\equiv\{J|X_J>0,\tilde{\lambda}_J=0\}, \quad B^X\equiv\{J|X_J=0,\tilde{\lambda}_J\geqslant 0\}. \tag{4.7}$$

Likewise for the W^J unitarity inequality the interior and boundary unitarity classes are

$$I^W\equiv\{J|W_J>0,\tilde{\mu}_J=0\}, \quad B^W\equiv\{J|W_J=0,\tilde{\mu}_J\geqslant 0\}. \tag{4.8}$$

1. I^X and B^X unitarity classes

The interior unitarity class,

$$I^X\equiv\{J|X^J=a_0^J-a_0^{J\,2}>0,\tilde{\lambda}_J=0\}, \tag{4.9}$$

under the four constraints, is expressed as

$$I^X\equiv\{J|0<a_0^J<1,\tilde{\lambda}_J=0\}. \tag{4.10}$$

Equation (4.3) with $\tilde{\lambda}_J$ set to zero enables us to write the imaginary partial wave amplitude a_0^J, in the interior unitarity class, as

$$a_0^J=r_1+r_2\left(1-\frac{\zeta}{4}J^2\right). \tag{4.11}$$

The constraint $0<a_0^J<1$ restricts the values of the equality multipliers, r_1 and r_2, to $0<r_1+r_2<1$ and $r_2>0$. The number of partial waves J is thus limited to

$$0\leqslant J^2<\frac{4}{\zeta}\left(1+\frac{r_1}{r_2}\right). \tag{4.12}$$

The boundary unitarity class B^X splits into two sub-classes, B^{X_0} and B^{X_1}:

$$B^X\equiv\{J|X_J=a_0^J-a_0^{J\,2}=0,\tilde{\lambda}_J\geqslant 0\}$$
$$\to\begin{cases} B^{X_0}\equiv\{J|a_0^J=0,\tilde{\lambda}_J\geqslant 0\}, \\ B^{X_1}\equiv\{J|a_0^J=1,\tilde{\lambda}_J\geqslant 0\}. \end{cases} \tag{4.13}$$

In the boundary unitarity class B^{X_0} the imaginary partial wave amplitude a_0^J is equal to zero and from Eq. (4.3) the inequality multiplier $\tilde{\lambda}_J$ is given by

$$\tilde{\lambda}_J=-(r_1+r_2)+r_2\frac{\zeta}{4}J^2\geqslant 0. \tag{4.14}$$

The B^{X_0} class begins at $J^2=M_1^2=4/\zeta(1+r_1/r_2)$, and for $J\geqslant M_1+1$, with $0<r_1+r_2<1$ and $r_2>0$, the inequality multiplier $\tilde{\lambda}_J$ is positive. Therefore the boundary unitarity class B^{X_0} is non-empty for $J\geqslant M_1+1$ but with $a_0^J=0$, for all J in this unitarity class, there are no contributions to the observables from this unitarity class. The imaginary partial wave amplitude a_0^J is equal to unity in the boundary unitarity class B^{X_1} and from Eq. (4.3) the inequality multiplier $\tilde{\lambda}_J$ is given by

$$\tilde{\lambda}_J=(r_1+r_2)-1-r_2\frac{\zeta}{4}J^2. \tag{4.15}$$

By definition $\tilde{\lambda}_J \geqslant 0$ and the value of J, in the boundary unitarity class, is limited to

$$J^2 \leqslant \frac{4}{\zeta}\left(\frac{(r_1+r_2)-1}{r_2}\right) \tag{4.16}$$

but with $0<r_1+r_2<1$ and $r_2>0$, J^2 is negative, or J is complex and therefore the boundary unitarity class B^{X_1} is empty.

In summary, the unitarity classes, I^X and B^{X_0}, are non-empty and the unitarity class B^{X_1} is empty:

$$I^X \equiv \{J | 0<a_0^J<1, 0<J\leqslant M_1\}, \tag{4.17}$$

$$B^{X_0} \equiv \{J | a_0^J=0, M_1+1 \leqslant J \leqslant M_2\} \tag{4.18}$$

where $M_1=F[\sqrt{4/\zeta\,(1+r_1/r_2)}]$, $M_2=F[\sqrt{8/\zeta}]$ and $\zeta=-t/k^2$. The $F[x]$ function gives the greatest integer less than or equal to x.

2. I^W and B^W unitarity classes

The interior unitarity class I^W under the optimization becomes

$$I^W \equiv \{J | W^J = a_{11}^J - a_{11}^{J^2} - a_{21}^{J^2} > 0, \quad \tilde{\mu}_J = 0\}. \tag{4.19}$$

Substituting Eqs. (4.4) and (4.5), with $\tilde{\mu}_J=0$, into the interior constraint $a_{11}^J - a_{11}^{J^2} - a_{21}^{J^2}>0$ leads to the equation;

$$f_2(J) = \tilde{a}_1 + \tilde{a}_2 J^2 + \tilde{a}_3 J^4 + \tilde{a}_4 J^6 > 0 \tag{4.20}$$

where $\tilde{a}_1=(r_1+r_2)\quad[1-(r_1+r_2)]$, $\tilde{a}_2=r_2\zeta[2(r_1+r_2)-1]/4-1/(64\beta^2)$, $\tilde{a}_3=\zeta/(256\beta^2)-r_2^2\zeta^2/16$, $\tilde{a}_4=-\zeta^2/(64\beta)^2$, and only positive J solutions are allowed. The solution can be put in the following form, where it is known that $\eta_2 \sim 1$,

$$0<J^2<\eta_2^2 M_1^2. \tag{4.21}$$

The boundary unitarity class B^W is written as

$$B^W \equiv \{J | W^J = a_{11}^J - a_{11}^{J^2} - a_{21}^{J^2} = 0, \ \tilde{\mu}_J \geqslant 0\}. \tag{4.22}$$

The constraint $a_{11}^J - a_{11}^{J^2} - a_{21}^{J^2}=0$ can be written as a quadratic equation:

$$\tilde{\mu}_J^2 + \tilde{\mu}_J + f_2(J) = 0 \tag{4.23}$$

where

$$f_2(J) = \tilde{a}_1 + \tilde{a}_2 J^2 + \tilde{a}_3 J^4 + \tilde{a}_4 J^6. \tag{4.24}$$

The solutions are

$$\tilde{\mu}_J = \frac{1}{2}\{\pm\sqrt{1-4f_2(J)}-1\}. \tag{4.25}$$

The function $f_2(J)$ is negative for $J>M_1$, where $M_1=F[4/\zeta(1+r_1/r_2)]$ and consequently $\tilde{\mu}_J$ is positive for such J values. By definition $\tilde{\mu}_J \geqslant 0$, therefore the positive solution is chosen;

$$\tilde{\mu}_J = \frac{1}{2}\{\sqrt{1-4f_2(J)}-1\}. \tag{4.26}$$

To summarize, both the classes, I^W and B^W, are non-empty:

$$I^W \equiv \{J | a_{11}^J - a_{11}^{J^2} - a_{21}^{J^2} > 0, \quad 0 \leqslant J \leqslant M_1\}, \tag{4.27}$$

$$B^W \equiv \{J | a_{11}^J - a_{11}^{J^2} - a_{21}^{J^2} = 0, \quad M_1+1 \leqslant J \leqslant M_2\}, \tag{4.28}$$

with $\eta_2=1$, where $M_1=F[\sqrt{4/\zeta(1+r_1/r_2)}]$, $M_2=F[\sqrt{8/\zeta}]$ and $\zeta=-t/k^2$. It is important to notice that with $\eta_2=1$ both interior unitarity classes, I^W and I^X, are non-empty over the same region, $J\in[0,M_1]$. Similarly the boundary unitarity classes, B^{X_0} and B^W, are non-empty over the same region, $M_1+1 \leqslant J \leqslant M_2$. In other words there is no mixing of unitarity classes, all classes being either interior unitarity classes, $I \equiv I^X \cup I^W$, or boundary unitarity classes, $B \equiv B^X \cup B^W$, for a given J.

C. Solution of interior unitarity class

Consider the set of interior unitarity classes, $I \equiv I^X \cup I^W$. The inequality multipliers, $\tilde{\lambda}_J$ and $\tilde{\mu}_J$, in the interior region are equal to zero. The imaginary partial wave amplitudes are therefore written as

$$a_k^J = r_1 + r_2\left(1-\frac{\zeta}{4}J^2\right) \tag{4.29}$$

and

$$a_{21}^J = \frac{J}{8\beta}\left(1-\frac{\zeta}{8}J^2\right), \tag{4.30}$$

$k=0$, 1, 11, 22, with $0 \leqslant J \leqslant M_1$, where $M_1=F[\sqrt{4/\zeta(1+r_1/r_2)}]$ is the maximum J in the interior unitarity class. In this case the contributions to the observables and to the objective function $\mathrm{Im}\tilde{\phi}_5$ solely come from the interior unitarity class I; $A_0^I=A_0$, $\mathrm{Im}\phi_+^I=\mathrm{Im}\phi_+$, $\Sigma_{\mathrm{el}}^I=\Sigma_{\mathrm{el}}$ and $\mathrm{Im}\tilde{\phi}_5^I=\mathrm{Im}\tilde{\phi}_5$. The normalized dimensionless total cross section upon reconstruction is

$$A_0 = 8\sum_{J=0}^{M_1} J\left[r_1 + r_2\left(1-\frac{\zeta}{4}J^2\right)\right]. \tag{4.31}$$

The Euler-Maclaurin expansion [21] for large J is used to write the normalized dimensionless total cross section A_0 as an integration over J, leading to

$$A_0 \approx \frac{M_1^2}{2}\{8(r_1+r_2)-r_2\zeta M_1^2\}. \tag{4.32}$$

In a similar manner the imaginary spin average helicity nonflip amplitude, the dimensionless normalized elastic cross section and the modified imaginary single-flip amplitude are also reconstructed:

$$\mathrm{Im}\,\phi_+ \approx M_1^2\left\{2(r_1+r_2)-(2r_2+r_1)\frac{\zeta}{4}M_1^2+\frac{r_2\zeta^2}{24}M_1^4\right\}, \tag{4.33}$$

$$\Sigma_{\mathrm{el}} \approx \left\{4(r_1+r_2)^2M_1^2-(r_1+r_2)r_2\zeta M_1^4+\frac{r_2^2\zeta^2}{12}M_1^6\right\} + \frac{M_1^4}{64\beta^2}\left\{1-\frac{\zeta}{6}M_1^2+\frac{\zeta^2}{128}M_1^4\right\}, \tag{4.34}$$

$$\mathrm{Im}\,\tilde{\phi}_5 \approx \left\{\Sigma_{\mathrm{el}}-\left(4(r_1+r_2)^2M_1^2 -(r_1+r_2)r_2\zeta M_1^4+\frac{r_2^2\zeta^2}{12}M_1^6\right)\right\}^{1/2} \times\frac{M_1^2}{4}\left\{1-\frac{\zeta}{6}M_1^2+\frac{\zeta^2}{128}M_1^4\right\}^{1/2}. \tag{4.35}$$

The equality multipliers r_1, r_2 and β are found to be

$$r_1=\frac{A_0^3\zeta(1-3\,\mathrm{Im}\phi_+/A_0)}{36(1-2\,\mathrm{Im}\phi_+/A_0)^2}, \tag{4.36}$$

$$r_2=\frac{A_0^2\zeta}{72(1-2\,\mathrm{Im}\phi_+/A_0)^2} \tag{4.37}$$

and

$$\beta=\frac{M_1^2\left\{1-\frac{\zeta}{6}M_1^2+\frac{\zeta^2}{128}M_1^4\right\}^{1/2}}{8\left\{\Sigma_{\mathrm{el}}-\left(4(r_1+r_2)^2M_1^2-(r_1+r_2)r_2\zeta M_1^4+\frac{r_2^2\zeta^2}{12}M_1^6\right)\right\}^{1/2}} \tag{4.38}$$

where $\zeta=-t/k^2$. The equality multiplier β, with solutions for r_1 and r_2, is expressed as

$$\beta=\frac{9(A_0-2\,\mathrm{Im}\,\phi_+)\sqrt{1-2\,\mathrm{Im}\,\phi_+/A_0+36\,\mathrm{Im}\,\phi_+^2/A_0^2}}{2A_0\zeta\sqrt{72\Sigma_{\mathrm{el}}-2A_0^2\zeta/(1-2\,\mathrm{Im}\,\phi_+/A_0)}}. \tag{4.39}$$

The optimized modified imaginary single-flip amplitude, expressed as a function of r_1, r_2 and β, becomes

$$\mathrm{Im}\,\tilde{\phi}_5=\frac{(A_0-2\,\mathrm{Im}\,\phi_+)}{4A_0\zeta}\,\frac{\sqrt{1/2-2(1-\mathrm{Im}\,\phi_+/A_0)\,\mathrm{Im}\,\phi_+}}{\sqrt{36\Sigma_{\mathrm{el}}-A_0^2\zeta/(1-2\,\mathrm{Im}\,\phi_+/A_0)}} \tag{4.40}$$

with

$$J_{\mathrm{max}}=\frac{12}{\zeta}\left(1-2\frac{\mathrm{Im}\,\phi_+}{A_0}\right). \tag{4.41}$$

For low momentum transfers the imaginary spin average non-flip amplitude $\mathrm{Im}\,\phi_+$, expanded to order t, is written as $\mathrm{Im}\,\phi_+\approx(A_0/2)(1+gt)$. Under this approximation the maximum J inside the interior unitarity class is independent of t, and in the limit $t\rightarrow 0$, the number of partial waves is finite where

$$J_{\mathrm{max}}=\sqrt{12g}k. \tag{4.42}$$

The equality multipliers in the low t limit become

$$r_1=\frac{A_0}{72g^2k^2}\left(\frac{1+3gt}{t}\right),\quad r_2=-\frac{A_0}{72g^2k^2t} \tag{4.43}$$

and

$$\beta=\frac{\sqrt{2}9gk^2}{\sqrt{72\Sigma_{\mathrm{el}}-2A_0^2/(gk^2)}}[1+gt(2+9gt/8)]^{1/2}. \tag{4.44}$$

The upper bound on $|\mathrm{Im}\,r_5|$, where $|\mathrm{Im}\,r_5| =m|\mathrm{Im}\,\tilde{\phi}_5|/(k\;\mathrm{Im}\,\phi_+)$, can be expressed analytically:

$$|\mathrm{Im}\,r_5|\leqslant\frac{\sqrt{2}m\,kg}{A_0}\sqrt{18\Sigma_{\mathrm{el}}-\frac{A_0^2}{2gk^2}}\times h(t) \tag{4.45}$$

where

$$h(t)=\frac{[1+gt(2+9gt/8)]^{1/2}}{(1+gt)}. \tag{4.46}$$

The variable $h(t)$ has the value one at $t=0$ and does not vary much over the CNI region. Writing $A_0=k^2\sigma_{\mathrm{tot}}/\pi$ and $\Sigma_{\mathrm{el}}=k^2\sigma_{\mathrm{el}}/\pi$, enables the bound on $|\mathrm{Im}\,r_5|$ to be expressed as

$$|\mathrm{Im}\, r_5| \leqslant m\sqrt{g}\left(\frac{36\pi g\,\sigma_{\mathrm{el}}}{\sigma_{\mathrm{tot}}^2}-1\right)^{1/2}\times h(t). \tag{4.47}$$

D. Results

The value of the bound on $|\mathrm{Im}\, r_5|$ is given in Tables I and II with the values of the equality multipliers. The most noticeable feature of the bound is its size at low momentum transfers, having a value of 0.89 at $\sqrt{s}=52.8$ GeV, $t=-0.001$ $(\mathrm{GeV}/c)^2$. The partial wave series terminates at $J=231$ which is the upper J limit, M_1, for the interior unitarity class I. When considering both the interior and boundary unitarity classes, values of $J>M_1$ are permitted.

The bound on $|\mathrm{Im}\, r_5|$, under the approximation

$$g \approx \frac{\sigma_{\mathrm{tot}}^2}{32\pi\sigma_{\mathrm{el}}}, \tag{4.48}$$

with momentum transfers in the CNI region is expressed as

$$|\mathrm{Im}\, r_5| \leqslant m\sqrt{\frac{g}{8}}\times h(t) \tag{4.49}$$

and in the zero momentum transfer limit, $t\rightarrow 0$, the bound on $|\mathrm{Im} r_5|$ is finite and can be expressed analytically as

$$|\mathrm{Im}\, r_5| \leqslant m\sqrt{\frac{g}{8}}. \tag{4.50}$$

This approximation generates a "stricter" bound on $|\mathrm{Im}\, r_5|$. The results are given in Table III.

E. Solution of interior and boundary classes

Consider the union of the classes $I\cup B \equiv I^W\cup I^X\cup B^W\cup B^X$. The boundary unitarity classes are

$$B^X \equiv \{J|a_{11}^J - a_{11}^{J^2} - a_{21}^{J^2}=0, \quad M_1+1\leqslant J\leqslant M_2\} \tag{4.51}$$

and

$$B^{W_0} \equiv \{J|a_0^J=0, \quad M_1+1\leqslant J\leqslant M_2\} \tag{4.52}$$

where $M_1=F[\sqrt{4/\zeta(1+r_1/r_2)}]$, $M_2=F[\sqrt{8/\zeta}]$ and $\zeta=-t/k^2$. The contribution to $|\mathrm{Im}\, r_5|$ from the boundary unitarity class B can range from 0% to 100% and the contribution to $|\mathrm{Im}\, r_5|$, from the boundary unitarity class, can be selected without violating any of the constraints and this contribution can be made arbitrarily small.

Consider the case with $\Sigma_{\mathrm{el}}^B=0.1\Sigma_{\mathrm{el}}$, $\Sigma_{\mathrm{el}}^I=0.9\Sigma_{\mathrm{el}}$, at $\sqrt{s}=52.8$ GeV and $t=-0.001$ $(\mathrm{GeV}/c)^2$. The maximum contribution to $|\mathrm{Im}\, r_5|$ is 34.7 where $|\mathrm{Im}\, r_5^I|\leqslant 0.5$ and $|\mathrm{Im}\, r_5^B|\leqslant 34.2$. The case with $\Sigma_{\mathrm{el}}^B=0.01\Sigma_{\mathrm{el}}$, $\Sigma_{\mathrm{el}}^I=0.99\Sigma_{\mathrm{el}}$, leads to $|\mathrm{Im}\, r_5|\leqslant 11.6$ where $|\mathrm{Im}\, r_5^I|\leqslant 0.8$ and $|\mathrm{Im}\, r_5^B|\leqslant 10.6$. The bound on $|\mathrm{Im}\, r_5^B|$ falls when the fraction of Σ_{el} in the boundary unitarity class is reduced. The partial wave amplitudes in this region also become smaller in amplitude and contribute less to the bound on $|\mathrm{Im}\, r_5^B|$. The fraction of Σ_{el} in the boundary unitarity class can be further reduced until the contribution from this class to $|\mathrm{Im}\, r_5|$ is negligible in comparison with the contribution from the interior unitarity class. In this limit the bound is, as before,

$$|\mathrm{Im}\, r_5| \leqslant m\sqrt{g}\left(\frac{36\pi g\;\sigma_{\mathrm{el}}}{\sigma_{\mathrm{tot}}^2}-1\right)^{1/2}\times h(t) \tag{4.53}$$

or, under the approximation $g\approx\sigma_{\mathrm{tot}}^2/(32\pi\sigma_{\mathrm{el}})$,

$$|\mathrm{Im}\, r_5| \leqslant m\sqrt{\frac{g}{8}}\times h(t) \tag{4.54}$$

where

$$h(t)=\frac{[1+gt(2+9/8\,gt)]^{1/2}}{(1+gt)}. \tag{4.55}$$

TABLE I. $|\mathrm{Im}\, r_5|^{\max}$ inside the interior region at $t=-0.001$ $(\mathrm{GeV}/c)^2$.

$\sqrt{s}$ (GeV)	r_1	r_2	β	$J_{\max}$	$\lvert\mathrm{Im}\, r_5\rvert$
19.4	−12.54	12.77	90	81	0.97
23.5	−12.56	12.79	117	98	0.92
30.7	−12.02	12.24	158	131	0.92
44.7	−11.22	11.44	217	195	1.05
52.8	−11.53	11.76	293	231	0.89
62.5	−11.56	11.79	358	276	0.86

TABLE II. $|\mathrm{Im}\, r_5|^{\max}$ inside the interior region at $t=-0.01$ $(\mathrm{GeV}/c)^2$.

$\sqrt{s}$ (GeV)	r_1	r_2	β	$J_{\max}$	$\lvert\mathrm{Im}\, r_5\rvert$
19.4	−1.25	1.49	85	81	0.97
23.5	−1.05	1.27	111	98	0.91
30.7	−1.00	1.22	150	131	0.92
44.7	−0.92	1.14	204	195	1.05
52.8	−0.95	1.17	276	231	0.89
62.5	−0.94	1.18	337	276	0.86

TABLE III. $|\mathrm{Im}\, r_5|^{\max}$, with an approximation for g, over the CNI region.

$\sqrt{s}$ (GeV)	$t=0$ $(\mathrm{GeV}/c)^2$	$t=-0.001$ $(\mathrm{GeV}/c)^2$	$t=-0.01$ $(\mathrm{GeV}/c)^2$
19.4	0.803	0.805	0.825
23.5	0.805	0.808	0.827
30.7	0.819	0.821	0.842
44.7	0.839	0.841	0.864
52.8	0.841	0.843	0.866
62.5	0.846	0.848	0.871

The bound is identical to the bound when only the interior unitarity class is considered. A finite number of partial waves at low momentum transfer ensures a finite value for the sum

$$\mathrm{Im}\,\tilde{\phi}_5=\sum_J J^2 a_{21}^J\left(1-\frac{\zeta}{8}J^2\right) \tag{4.56}$$

and consequently a finite upper bound on $|\mathrm{Im}\,r_5|$ of less than unity.

V. ERROR ON BOUND

The upper bound on the imaginary single helicity-flip amplitude, modified by a kinematical factor at zero momentum transfer, is given by

$$|\mathrm{Im}\,r_5|\leqslant m\sqrt{\frac{g}{8}}\left(\frac{36\pi g\sigma_{\mathrm{el}}}{\sigma_{\mathrm{tot}}^2}-1\right)^{1/2}. \tag{5.1}$$

There are errors on all the experimental quantities in Eq. (5.1) and consequently the upper bound on $|\mathrm{Im}\,r_5|$ has an uncertainty. The experimental quantities g, σ_{tot} and σ_{el} have a nominal value plus an uncertainty: $g\pm\Delta g$, $\sigma_{\mathrm{tot}}\pm\Delta\sigma_{\mathrm{tot}}$ and $\sigma_{\mathrm{el}}\pm\Delta\sigma_{\mathrm{el}}$. What are the values of Δg, $\Delta\sigma_{\mathrm{tot}}$ and $\Delta\sigma_{\mathrm{el}}$? Consider the value of $\mathrm{Im}\,r_5$ at $\sqrt{s}=52.8$ GeV; $|\mathrm{Im}\,r_5|\leqslant 0.891$ with $g=6.435\pm 0.14$ GeV^{-2} [22], $\sigma_{\mathrm{tot}}=42.906$ mb [23] and $\sigma_{\mathrm{el}}=7.407$ mb [23]. The value of Δg is known but we must calculate $\Delta\sigma_{\mathrm{tot}}$ and $\Delta\sigma_{\mathrm{el}}$.

A parametrization for the total and elastic cross section in elastic pp collisions [23] allows a value for the cross sections to be found and a value of their uncertainties to be calculated. Each cross section is parametrized as

$$\sigma(p)=A+Bp^n+C\log^2(p)+D\log(p) \tag{5.2}$$

where σ is in mb and p is the laboratory momentum in GeV/c. The uncertainty in σ is given by

$$\Delta\sigma=\left\{\left(\frac{\partial\sigma}{\partial A}\right)^2(\Delta A)^2+\left(\frac{\partial\sigma}{\partial B}\right)^2(\Delta B)^2+\left(\frac{\partial\sigma}{\partial n}\right)^2(\Delta n)^2+\left(\frac{\partial\sigma}{\partial C}\right)^2(\Delta C)^2+\left(\frac{\partial\sigma}{\partial D}\right)^2(\Delta D)^2\right\}^{1/2}. \tag{5.3}$$

The fitted parameters A, B, n, C and D are given in Table IV.

Using Eq. (5.3), and the values of the fitted parameters, the uncertainties $\Delta\sigma_{\mathrm{tot}}$ and $\Delta\sigma_{\mathrm{el}}$ can be written as

$$\Delta\sigma_{\mathrm{tot}}=\sqrt{0.01+2.5\times 10^{-5}\log^4(p)+2.5\times 10^{-3}\log^2(p)} \tag{5.4}$$

and

$$\Delta\sigma_{\mathrm{el}}=\sqrt{0.64+2.89p^{-2.42}+12.819p^{-4.42}+4.41\times 10^{-4}\log^4(p)+0.0676\log^2(p)}. \tag{5.5}$$

A laboratory beam momentum of $p=k\sqrt{s}/m=1485$ GeV/c at $\sqrt{s}=52.8$ GeV gives $\Delta\sigma_{\mathrm{tot}}=0.463$ mb, or 1.08% of σ_{tot} and $\Delta\sigma_{\mathrm{el}}=2.345$ mb, or 31.66% of σ_{el}. The uncertainty in $\mathrm{Im}r_5$ is

$$\Delta\,\mathrm{Im}\,r_5=\sqrt{\left(\frac{\partial \mathrm{Im}\,r_5}{\partial g}\right)^2(\Delta g)^2+\left(\frac{\partial \mathrm{Im}\,r_5}{\partial\sigma_{\mathrm{tot}}}\right)^2(\Delta\sigma_{\mathrm{tot}})^2+\left(\frac{\partial \mathrm{Im}\,r_5}{\partial\sigma_{\mathrm{el}}}\right)^2(\Delta\sigma_{\mathrm{el}})^2}. \tag{5.6}$$

At $\sqrt{s}=52.8$ GeV, $g=6.435\pm 0.14$ GeV^{-2}, $\sigma_{\mathrm{tot}}=42.906\pm 0.463$ mb and $\sigma_{\mathrm{el}}=7.407\pm 2.345$ mb. The uncertainty $\Delta\,\mathrm{Im}\,r_5=0.049$ and the upper bound on $|\mathrm{Im}\,r_5|$ is 0.891 ± 0.049. The approximation

$$g\approx\frac{\sigma_{\mathrm{tot}}^2}{32\pi\sigma_{\mathrm{el}}} \tag{5.7}$$

can be used to write the bound on $|\mathrm{Im}\,r_5|$ as

$$|\mathrm{Im}\,r_5|\leqslant m\sqrt{\frac{g}{8}} \tag{5.8}$$

at zero momentum transfer. The uncertainty $\Delta\,\mathrm{Im}\,r_5$ is simply

$$\Delta\,\mathrm{Im}\,r_5=\sqrt{\left(\frac{\partial \mathrm{Im}\,r_5}{\partial g}\right)^2(\Delta g)^2}. \tag{5.9}$$

At $\sqrt{s}=52.8$ GeV, $g=6.435\pm 0.14$ GeV^{-2}, $|\mathrm{Im}\,r_5|\leqslant 0.846$ and $\Delta\,\mathrm{Im}\,r_5=0.009$ or $|\mathrm{Im}\,r_5|\leqslant 0.846\pm 0.009$. The error on σ_{el} being large has relatively little effect on the uncertainty of the bound.

TABLE IV. Fitted parameters for pp scattering.

Reaction	A	B	n	C	D
σ_{tot}	48.0 ± 0.1			0.522 ± 0.005	-4.51 ± 0.05
σ_{el}	11.9 ± 0.8	26.9 ± 1.7	-1.21 ± 0.11	0.169 ± 0.021	-1.85 ± 0.26

VI. CONCLUSION

A bound on the imaginary part of the single helicity-flip amplitude for proton proton elastic scattering in the CNI region has been obtained using a Lagrangian variational method. As equality constraints, the optimization used the total cross section σ_{tot}, the elastic cross section σ_{el}, and the diffraction slope while partial wave unitarity involved a number of inequality constraints. The bound varies smoothly as the momentum transfer variable $t \to 0$ and, in fact, limits the augmented helicity flip to nonflip amplitude ratio $|\text{Im}\, r_5|$ to values less than unity near $t=0$. With additional information, such as that provided by a double-spin asymmetry, an improved bound could possibly be obtained although it would be necessary to have accurate knowledge of any new observables employed as constraints in the Lagrange function. Use of a double-spin asymmetry, however, requires a double summation over angular momentum that does not easily yield to analytical or computational approaches. Constraints used in the bound developed here involve the much more tractable single summations. As the bound of about 0.84 on the helicity flip amplitude ratio is less than $(\mu_p - 1)/2 = 0.896$ at the high energies considered, the coefficient of (t_c/t) in the expression for the asymmetry is constrained to be positive. The analyzing power is therefore expected to be greater than zero for at least a part of the interference region. Though the bound would have to be improved to $(\mu_p - 1)/40 = 0.0448$ to limit the polarization error to the recommended 5% required for accurate measurement of the contribution of gluons to the spin of the proton, it does, however, encourage the use of proton proton elastic collisions in the CNI region as a relative polarimeter. Calibration of such a polarimeter would be immediate when a method of determining $\text{Im}\, r_5$ from the absolute spin polarization of a high energy proton becomes available.

ACKNOWLEDGMENTS

The authors are grateful to Enterprise Ireland for partial support under Scientific Research Project SC/96/778 and International Programmes IC/1999/075, IC/2000/027.

[1] M. Anselmino, A. Efremov, and E. Leader, Phys. Rep. **261**, 1 (1995).
[2] H. Cheng, Chin. J. Phys. (Taipei) **38**, 753 (2000).
[3] G. Bunce, N. Saito, J. Soffer, and W. Vogelsang, Annu. Rev. Nucl. Part. Sci. **50**, 525 (2000).
[4] M.B. Einhorn and R. Blankenbecler, Ann. Phys. (N.Y.) **67**, 480 (1971).
[5] K.H. Mütter, Nucl. Phys. **B31**, 589 (1971).
[6] D.P. Hodgkinson, Phys. Lett. **39B**, 640 (1972).
[7] G. Mennessier, S.M. Roy, and V. Singh, Nuovo Cimento A **50**, 443 (1979).
[8] S.M. Troshin and N.E. Tyurin, Phys. Lett. B **459**, 641 (1999).
[9] M. Jacob and G.C. Wick, Ann. Phys. (N.Y.) **7**, 404 (1959).
[10] M.L. Goldberger, M.T. Grisaru, S.W. MacDowell, and D.Y. Wong, Phys. Rev. **120**, 2250 (1960).
[11] C. Bourrely, J. Soffer, and E. Leader, Phys. Rep. **59**, 95 (1980).
[12] N.H. Buttimore, B.Z. Kopeliovich, E. Leader, J. Soffer, and T.L. Trueman, Phys. Rev. D **59**, 114010 (1999).
[13] N. Akchurin, N.H. Buttimore, and A. Penzo, Phys. Rev. D **51**, 3944 (1995).
[14] B.Z. Kopeliovich and L.I. Lapidus, Sov. J. Nucl. Phys. **19**, 114 (1974).
[15] M. Andrews and J. Gunson, J. Math. Phys. **5**, 1391 (1964).
[16] W.W. Bell, *Special Functions for Scientists and Engineers* (Van Nostrand, London, 1968).
[17] S.W. MacDowell and A. Martin, Phys. Rev. **135**, B960 (1964).
[18] Bateman Manuscripts, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953).
[19] K.H. Mütter, Nucl. Phys. **B27**, 73 (1971).
[20] A.T. Bates and N.H. Buttimore, Nucl. Phys. B (Proc. Suppl.) **86**, 175 (2000); A.T. Bates, Ph.D. thesis, University of Dublin, 2000.
[21] R.P. Boas and C. Stutz, Am. J. Phys. **39**, 745 (1971).
[22] F. Pereira and E. Ferreira, Phys. Rev. D **61**, 077507 (2000).
[23] Particle Data Group, D.E. Groom *et al.*, Eur. Phys. J. C **15**, 1 (2000).