Renormalization-group calculation of the color-Coulomb potential

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We report here on the application of the perturbative renormalization-group to the Coulomb gauge in QCD. We use it to determine the high-momentum asymptotic form of the instantaneous color-Coulomb potential $V(\vec{k})$ and of the vacuum polarization $P(\vec{k},k_4)$. These quantities are renormalization-group invariants in the sense that they are independent of the renormalization scheme. A scheme-independent definition of the running coupling constant is provided by $\vec{k}^2V(\vec{k})=x_0g^2(\vec{k}/\Lambda_{\rm Coul})$, and of $\alpha_s\equiv g^2(\vec{k}/\Lambda_{\rm Coul})/4\pi$, where $x_0=12N/(11N-2N_f)$, and $\Lambda_{\rm Coul}$ is a finite QCD mass scale. We also show how to calculate the coefficients in the expansion of the invariant β function $\beta(g)\equiv|\vec{k}|(\partial g/\partial|\vec{k}|)=-(b_0g^3+b_1g^5+b_2g^7+\ldots)$, where *all* coefficients are scheme independent.

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I. INTRODUCTION

In QCD the Wilson loop is the basic gauge-invariant observable. A rectangular Wilson loop of dimension $R\times T$ has, asymptotically at large T, the form $W(R,T)\sim\exp[-TV_W(R)]$. The Wilson potential $V_W(R)$ is the total energy of the state of lowest energy that contains a pair of infinitely massive external quarks at separation R. If dynamical quarks are present in the theory, the pair of external quarks polarizes the vacuum and extracts a pair of dynamical quarks from it, so that, for R not too small, a pair of mesons is formed at separation R, each meson being formed of one external quark and one dynamical quark, plus doses of other constituents. In this case $V_W(R)$ represents the total energy of the pair of mesons at separation R. Clearly the Wilson potential $V_W(R)$ is not a color-confining potential, but rather a residual potential that remains after color has been saturated by vacuum polarization. In this respect $V_W(R)$ should be regarded as a QCD analogue of the van der Waals interatomic potential. Clearly it is not the van der Waals force that holds the atom together, but rather the electrostatic Coulomb potential. In the present article we shall be concerned with the QCD analog of the electrostatic Coulomb potential, which we regard as responsible for confinement of color charge.

This quantity is the color-Coulomb potential $V(R)$. Just as the electrostatic Coulomb potential is the instantaneous part of the 44-component of the photon propagator in the Coulomb gauge, likewise $V(R)$ is the instantaneous part of the 44-component of the gluon propagator in the (minimal) Coulomb gauge, defined in Eq. (3) below. It is not gauge invariant like the Wilson potential $V_W(R)$, but it is a more elementary quantity, in terms of which one may hope to understand the dynamics of the underlying theory. Indeed we expect that its linear rise (or not) at large R may serve as an order parameter for the confinement of color charge, even when $V_W(R)$ is not linearly rising due to vacuum polarization. In the present article we report on a renormalization-group calculation of $V(R)$ in the high-momentum regime, more precisely, a calculation of its Fourier transform $V(|\vec{k}|)$ at large $|\vec{k}|$.

To be sure, confinement manifests itself rather at low momentum. For information in this region we have turned to numerical study, and in an accompanying article on $SU(2)$ lattice gauge theory without quarks [1], we present a numerical determination of $V(|\vec{k}|)$ and also of the equal-time would-be physical 3-dimensionally transverse gluon propagator $D^{\rm tr}_{ij}(\vec{k})$. The reader may find further references in [1], and a confrontation of the confinement scenario in the Coulomb gauge with the numerical data. This confinement scenario is also discussed in [2], and in Sec. III of the present article.

The Coulomb gauge is a ''physical gauge'' in the sense that the constraints are solved exactly, including in particular Gauss's law, $D_iE_i=\rho_{\rm qu}$, which is essential for color confinement. In a confining theory, all physical states are bound states, and the Coulomb gauge is the preferred gauge for calculations of bound states. Binding is provided by the instantaneous color-Coulomb potential $V(R)$ that is treated non-perturbatively, while everything else is regarded as a perturbation [3,4]. However, the Coulomb gauge offers no particular advantage for purely perturbative calculations.

In the present article we shall apply the perturbative renormalization group in the Coulomb gauge. This presents new challenges. However, they are amply rewarded by particularly strong results. These are a consequence of the fact that in the Coulomb gauge the 4-component of the vector potential is invariant under renormalization [2]:

$$g_0A_{0,4}(x)=g_rA_{r,4}(x), \tag{1}$$

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where the subscripts 0 and r refer to unrenormalized and renormalized quantities. This is not true in a Lorentz-covariant gauge.

The Coulomb gauge is traditionally defined by the condition $\partial_i A_i = 0$. This is an incomplete gauge fixing because gauge transformations $g(t)$ that depend on t are not fixed, and consequently in higher-order calculations one encounters singular expressions whose evaluation is ambiguous [5,6]. No doubt this has been an obstacle to the use of the Coulomb gauge in QCD. These difficulties are overcome, however, by defining the Coulomb gauge as the limit of the interpolating gauge characterized by the condition $\partial_i A_i + \lambda_4 A_4 = 0$, where λ is a real positive parameter [7]. Calculations are done at finite λ, and at the end one takes the limit $\lambda \to 0$. The interpolating gauge is a renormalizable gauge that may be treated by standard Becchi-Rouet-Stora-Tyutin (BRST) methods. The gauge-fixing term breaks Lorentz invariance, but it is BRST-exact. An extension of the BRST operator to include infinitesimal Lorentz transformations assures Lorentz invariance for BRST-invariant observables [7]. This definition of the Coulomb gauge is sufficient for the purposes of perturbation theory, and for the perturbative renormalization group that is the subject of the present article. In fact the only calculations that we will do explicitly (in Appendix B) are one loop, and for these one may set $\lambda = 0$ directly. However in our discussion of the renormalization group we rely on the existence of the limit $\lambda \to 0$ in every order of perturbation theory.

[At the non-perturbative level with, say, lattice regularization, the interpolating gauge condition is still subject to the Gribov ambiguity. This is resolved by the further specification of the *minimal* Coulomb gauge, whose lattice definition is given in [1]. Its continuum analogue would be to first minimize $F_{\mathrm{hor},A}(g) \equiv \int d^3x \, {}^g A_i^2$, for each t, with respect to all local gauge transformations $g(x,t)$, where ${}^g A_\mu = g^{-1} A_\mu g + g^{-1} \partial_\mu g$ is the gauge transform of A_μ. At the minimum, the Coulomb gauge condition $\partial_i A_i = 0$ is satisfied for each t, and the Faddeev-Popov operator $M(A) = -D_i(A)\partial_i$ is positive. Next $F_{\mathrm{ver},A}(g) \equiv \int d^4x \, {}^g A_4^2$ is minimized with respect to t-dependent gauge transformations $g(t)$. At the minimum $\int d^3x A_4^a(\vec{x},t) = C$, where C is a constant that satisfies $C \leqslant 2\pi V/T$. Here V is the spatial 3-volume and T is the temporal extent of the Euclidean 4-volume.]

An immediate consequence of Eq. (1) is that the 44-component of the gluon propagator is also invariant under renormalization. This gives the relations

$$D_{44}(\vec{k},k_4,\Lambda_{QCD}) = D_{0,44}(\vec{k},k_4,g_0,\Lambda) = D_{r,44}(\vec{k},k_4,g_r,\mu). \tag{2}$$

(We use the same symbol to designate a quantity and its Fourier transform, but distinguish them by their argument x or k.) Here Λ_{QCD} is a finite QCD scale, Λ is the ultraviolet cutoff, and μ is the renormalization mass. We have written the finite quantity $D_{44}(\vec{k},k_4,\Lambda_{QCD})$ without a subscript 0 or r because it is a renormalization-group invariant in the sense that it is independent of Λ and μ and of the regularization and renormalization schemes. The finite scale Λ_{QCD}, for example $\Lambda_{QCD} = \Lambda_{\mathrm{latt}}$ or $\Lambda_{QCD} = \Lambda_{\overline{ms}}$, appears in $D_{0,44}(\vec{k},k_4,g_0,\Lambda)$ only through its dependence on $g_0 = g_0(\Lambda/\Lambda_{QCD})$, and similarly for $D_{r,44}(\vec{k},k_4,g_r,\mu)$. The scale is defined within the particular regularization scheme that is used for calculations, for example lattice or dimensional regularization. To relate two different scales, one calculates the same renormalization-group invariant in the two schemes.

In the Coulomb gauge $D_{44}(\vec{x},t)$ has the decomposition into an instantaneous part, proportional to $\delta(t)$ and a non-instantaneous part $P(\vec{x},t)$ that is less singular at $t=0$,

$$D_{44}(\vec{x},t) = V(\vec{x})\,\delta(t) + P(\vec{x},t), \tag{3}$$

where $V(\vec{x})$ is what we call the instantaneous color-Coulomb potential, and $P(\vec{x},t)$ is a vacuum polarization term. [See Eqs. (27)–(29).] We shall see that $V(\vec{x})$ is anti-screening, whereas $P(\vec{x},t)$ is screening. In momentum space this decomposition reads

$$D_{44}(\vec{k},k_4) = V(\vec{k}) + P(\vec{k},k_4). \tag{4}$$

The instantaneous part is independent of k_4. The k_4-independent part is defined, without reference to a diagrammatic expansion, by

$$V(\vec{k}) \equiv \lim_{k_4 \to \pm\infty} D_{44}(\vec{k},k_4). \tag{5}$$

Here it is assumed that the limit (5) exists and is independent of the sign of k_4. This implies that

$$P(\vec{k},k_4) \equiv D_{44}(\vec{k},k_4) - V(\vec{k}) \tag{6}$$

vanishes at large k_4,

$$\lim_{k_4 \to \pm\infty} P(\vec{k},k_4) = 0. \tag{7}$$

The only tool available to calculate these quantities analytically is perturbation theory. Perturbatively, in $d<4$ dimensions, the separation into instantaneous and vacuum polarization parts may be made diagram by diagram. Each diagram is either independent of k_4 or satisfies Eq. (7). [See the decomposition (27)–(29) below.] However, in $d=4$ dimensions, individual vacuum polarization diagrams contain powers of $\ln(\Lambda k_4)$ associated with divergences, and the limit (5) does not exist in any finite order of perturbation theory. As a result there is an ambiguity in the separation of the instantaneous and vacuum polarization parts in any finite order of perturbation theory. (This may be regarded as a mixing of V and P due to renormalization.) However, because D_{44} is a renormalization-group invariant, it follows from the above definition that $V(\vec{k})$ and $P(\vec{k},k_4)$ are separately renormalization-group invariants. This will allow us to resolve the ambiguity by use of the perturbative renormalization group. We shall then find the asymptotic form of $V(\vec{k})$ at large $|\vec{k}|$ and of $P(\vec{k},k_4)$ at large k_4.

As a result of a renormalization-group calculation, we find that in an $SU(N)$ gauge theory with N_f quarks in the

fundamental representation, the instantaneous color-Coulomb potential is given by

$$\vec{k}^2 V(\vec{k}) = x_0 g^2(|\vec{k}|/\Lambda_{\rm Coul}), \tag{8}$$

where

$$x_0 = \frac{12N}{11N-2N_f}, \tag{9}$$

and $\Lambda_{\rm Coul}$ is a finite QCD mass scale specified below. The running coupling constant $g(|\vec{k}|/\Lambda_{\rm Coul})$ is found as the solution of the renormalization-group equation

$$|\vec{k}|\frac{\partial g}{\partial|\vec{k}|} = \beta(g), \tag{10}$$

where the β function has the weak-coupling expansion

$$\beta(g) = -(b_0 g^3 + b_1 g^5 + b_2 g^7 + \dots), \tag{11}$$

where *all* coefficients b_n are scheme independent, not just b_0 and b_1. The coefficients b_0 and b_1 are also gauge invariant and have their standard values. We shall show how to calculate the remaining coefficients perturbatively in the Coulomb gauge. This allows a calculation of $V(\vec{k})$ to arbitrary accuracy in the high-momentum region. Its leading asymptotic behavior is given by

$$\vec{k}^2 V(|\vec{k}|) \sim \frac{x_0}{2b_0 \ln(|\vec{k}|/\Lambda_{\rm Coul})}. \tag{12}$$

We suppose that asymptotic freedom holds, which requires that b_0, given explicitly below, be positive, $b_0>0$. This is equivalent to $11N-2N_f>0$. We observe that $x_0>0$, and in fact $x_0>1$.

We have also found the leading asymptotic behavior of the vacuum polarization term $P(\vec{k},k_4)$ at large k_4 namely,

$$\vec{k}^2 P^{\rm as}(|\vec{k}|,k_4,g_0,\Lambda) \sim \frac{y_0}{2b_0 \ln(k_4/\Lambda'_{\rm Coul})}, \tag{13}$$

where $\Lambda'_{\rm Coul}$ is another finite QCD mass scale, and $y_0 = 1 - x_0 = -(N+2N_f)/(11N-2N_f) < 0$. The negative sign of y_0 shows that the vacuum polarization term is indeed a screening term.

II. FORMULA FOR D_{44} IN THE COULOMB GAUGE

In the minimal Coulomb gauge, the Euclidean partition function is given by the familiar Faddeev-Popov formula

$$Z(\vec{J},J_4) \equiv \int_G d^4A\ \delta(\partial_i A_i)\det(-D_i\partial_i) \times \exp\int d^4x[-(1/4)F_{\mu\nu}^2 - ig_0 J_\mu A_\mu)], \tag{14}$$

where $F_{\mu\nu}$ is the Yang-Mills field. We have introduced sources J_μ, in terms of which the gluon propagator is given by

$$D_{0,\mu\nu}(x-y)\,\delta^{ab} \equiv g_0^2\langle A_\mu^a(x)A_\nu^b(y)\rangle = -Z^{-1}\frac{\delta}{\delta J_\mu^a(x)}\frac{\delta Z}{\delta J_\nu^b(y)}\bigg|_{J=0}. \tag{15}$$

(We shall frequently suppress color indices.) The subscript G on the integral refers to the fact that the integral over $A_i^{\rm tr}$ is restricted to within the Gribov region. This has very important dynamical consequences at long range that have been proposed as a mechanism for the confinement of color charge [8,9], and which are substantiated by numerical studies of the Coulomb gauge [1]. However, in the present article we are interested in the asymptotically high-momentum region where asymptotic freedom reigns, and we need not specify the Gribov region G.

The partition function may also be expressed in terms of the first order or phase-space functional integral, by introducing a Gaussian integral over an independent color-electric field E_i^a. This is done in Appendix A, with the result that in the minimal Coulomb gauge the partition function may be expressed as an integral over the physical, canonical degrees of freedom $A_i^{\rm tr}$ and $E_i^{\rm tr}$ with the canonical action

$$Z(\vec{J},J_4) = \int_G dA^{\rm tr}dE^{\rm tr}\exp\int d^4x(iE_i^{\rm tr}\dot{A}_i^{\rm tr} - \mathcal{H} - ig_0 J_i A_i^{\rm tr}). \tag{16}$$

Here

$$\mathcal{H} = (1/2)(E_i^2 + B_i^2) \tag{17}$$

is the classical Hamiltonian density, $B_1 \equiv \partial_2 A_3 - \partial_3 A_2 + g_0 A_2 \times A_3$, etc., where $A_i^a = A_i^{{\rm tr},a}$, $(X\times Y)^a \equiv f^{abc}X^bY^c$, and the color-electric field E_i is expressed in terms of the canonical variables by solving Gauss's law,

$$D_i E_i = g_0 J_4, \tag{18}$$

where $D_i \equiv \partial_i + g_0 A_i \times$ is the gauge-covariant derivative.

Gauss's law is solved by separating the transverse and longitudinal components of E_i according to

$$E_i = E_i^{\rm tr} - \partial_i\phi. \tag{19}$$

Here ϕ is the color-Coulomb potential in terms of which Gauss's law reads $-D_i\partial_i\phi + g_0 A_i^{\rm tr}\times E_i^{\rm tr} = g_0 J_4$, or

$$M(A^{\rm tr})\phi = \rho_{\rm Coul} + g_0 J_4, \tag{20}$$

where $M(A^{\rm tr}) \equiv -D_i(A^{\rm tr})\partial_i$ is the 3-dimensional Faddeev-Popov operator, and $\rho_{\rm Coul} \equiv -g_0 A_i^{\rm tr}\times E_i^{\rm tr}$ is the color-charge density of the dynamical degrees of freedom. If we had included dynamical quarks then, in addition to a Dirac term $\overline{q}(\gamma_i D_i + m)q$ in the Hamiltonian density, there would also be a quark contribution to the color-charge density

$$\rho_{\rm Coul}^a = \rho_{\rm gl}^a + \rho_{\rm qu}^a = -g_0 f^{abc}A_i^{{\rm tr},b}E_i^{{\rm tr},c} + g_0\overline{q}\gamma_0 t^a q. \tag{21}$$

The solution of Gauss's law is given by

$$\phi(\vec{x},x_4)=\int d^3y\, M^{-1}[\vec{x},\vec{y},A^{\rm tr}(x_4)](\rho_{\rm Coul}+g_0J_4)(\vec{y},x_4). \tag{22}$$

Because $M(A^{\rm tr})$ involves only spatial derivatives, the inverse operator $M^{-1}(A^{\rm tr})$ acts instantaneously in time. The Hamiltonian is given by

$$H=\int d^3x\,\mathcal{H}=(1/2)\int d^3x(E_i^2+B_i^2)$$

$$=(1/2)\int d^3x[E_i^{\rm tr2}+(\partial_i\phi)^2+B_i^2] \tag{23}$$

$$=(1/2)\int d^3x(E_i^{\rm tr2}+B_i^2)+(1/2)\int d^3x\,d^3y(\rho_{\rm Coul}+g_0J_4)$$

$$\times(\vec{x})\mathcal{V}(\vec{x},\vec{y};A^{\rm tr})(\rho_{\rm Coul}+g_0J_4)(\vec{y}). \tag{24}$$

Here

$$\mathcal{V}(\vec{x},\vec{y};A^{\rm tr})\equiv[M(A^{\rm tr})^{-1}(-\partial^2)M(A^{\rm tr})^{-1}]_{\vec{x},\vec{y}} \tag{25}$$

is a color-Coulomb potential-energy functional that depends on $A^{\rm tr}$. It acts instantaneously, and couples universally to color charge including, in the present case, the source J_4. Its physical significance will be discussed in the next section.

We now come to the important point. We use Z from Eq. (16) and H from Eq. (24), and obtain

$$Z^{-1}\frac{\delta Z}{\delta J_4^a(x)}=-g_0\int d^3z\langle\mathcal{V}^{ac}(\vec{x},\vec{z};A^{\rm tr})$$

$$\times[\rho^c_{\rm Coul}(\vec{z},x_4)+g_0J_4^c(\vec{z},x_4)]\rangle. \tag{26}$$

To calculate $D_{0,44}(x-y)$ we apply $\delta/\delta J_4^b(y)$ and we obtain two terms,

$$D_{0,44}(x-y)=V_0(x-y)+P_0(x-y). \tag{27}$$

The first,

$$V_0(x-y)\,\delta^{ab}\equiv g_0^2\langle\mathcal{V}^{ab}[\vec{x},\vec{y};A^{\rm tr}(x_4)]\rangle\,\delta(x_4-y_4), \tag{28}$$

comes from the $J_4^c(\vec{z},x_4)$ that appears explicitly in Eq. (26). It is manifestly instantaneous. The second term,

$$P_0(x-y)\,\delta^{ab}\equiv-g_0^2\langle(\mathcal{V}\rho_{\rm Coul})^a(x)(\mathcal{V}\rho_{\rm Coul})^b(y)\rangle, \tag{29}$$

where $(\mathcal{V}\rho)^a(x)=(\mathcal{V}\rho)^a(\vec{x},x_4)\equiv\int d^3z\,\mathcal{V}^{ab}[\vec{x},\vec{z};A^{\rm tr}(x_4)]\rho^b(\vec{z},x_4)$ represents polarization of the vacuum. The expansion of V_0 and P_0 up to one loop is given in Appendix B. The significance of these two terms for confinement is discussed in the next section.

III. PHYSICAL INTERPRETATION

We comment briefly on the physical meaning of the decomposition (27)–(29). We showed in [1] that the Faddeev-Popov operator $M(A^{\rm tr})$ is a positive operator in the minimal Coulomb gauge. It follows that $\mathcal{V}(A^{\rm tr})=M^{-1}(-\partial^2)M^{-1}$ is also a positive operator. The P_0 term represents ordinary vacuum polarization that also occurs in QED. The minus sign that appears in front of P_0 indicates that it corresponds to screening. In QED, the color-Coulomb potential energy functional $\mathcal{V}(\vec{x},\vec{y},A^{\rm tr})$ would be replaced by the electrostatic Coulomb potential $(4\pi|\vec{x}-\vec{y}|)^{-1}$. In [1] we argued that $\mathcal{V}(A^{\rm tr})$ is long-range for a typical configuration $A^{\rm tr}$, and we presented numerical evidence that this is true.

According to the confinement scenario in the minimal Coulomb gauge, $\mathcal{V}(A^{\rm tr})$ is predominantly long range because $M^{-1}(A^{\rm tr})$ is long range. This happens because the Gribov region is bounded by $M(A^{\rm tr})\geqslant 0$, and entropy favors dense population close to the boundary. The boundary occurs where $M(A^{\rm tr})$ has a zero eigenvalue, so it has a small eigenvalue close to the boundary. As a result $M^{-1}(A^{\rm tr})$ is enhanced close to the boundary where the population is dense. The same color-Coulomb potential energy functional $\mathcal{V}(A^{\rm tr})$ appears in both the instantaneous term V_0 and the vacuum polarization term P_0, so both are long range.

The instantaneous term V_0 is responsible for confinement, whereas the vacuum polarization term P_0 causes screening. In any physical process both V_0 and P_0 contribute of course, and one or the other may dominate. Consider the Wilson loop which is a model of a pair of infinitely heavy external quarks. In a theory of pure glue without dynamical quarks, it is believed that at long range there is a linearly rising potential $V_W(R)$ in the Wilson loop, characterized by a string tension. In this case, according to the confinement scenario in the Coulomb gauge, the instantaneous term V_0 dominates. However, if dynamical quarks are present then it is believed that the string ''breaks'' at some distance because it is energetically favorable to produce pairs of dynamical quarks from the vacuum. In this case the vacuum polarization term P_0 dominates. However, in both cases color charge itself is confined. According to the confinement scenario in the Coulomb gauge, that is because the instantaneous term $V_0(R)$ is long range and presumably linearly rising even when $V_W(R)$ is not. Indeed it is precisely the long range of $V_0(R)$ that makes it energetically favorable to produce dynamical quark pairs from the vacuum, thereby causing color confinement. Thus in the Coulomb gauge the instantaneous part provides an order-parameter for the confinement of color charge even when dynamical quarks are present.

As noted, the term V_0 is instantaneous. It is easy to see that in each order of perturbation theory $V_0(x-y)$ is the sum of diagrams in which the points x and y are continuously connected by instantaneous free gluon propagators, as illustrated in Fig. 1(a). We now show that the diagrams contributing to $P_0(x-y)$ have no instantaneous parts. Indeed the free propagators of $A^{\rm tr}$ and $E^{\rm tr}$ are given by

$$\langle A_i^{\rm tr}A_j^{\rm tr}\rangle_0=P_{ij}(\hat{k})/(\vec{k}^2+k_4^2)$$

$$\langle E_i^{\rm tr}E_j^{\rm tr}\rangle_0=(\vec{k}^2\delta_{ij}-k_ik_j)/(\vec{k}^2+k_4^2) \tag{30}$$

$$\langle E_i^{\rm tr}A_j^{\rm tr}\rangle_0=P_{ij}(\hat{k})k_4/(\vec{k}^2+k_4^2),$$

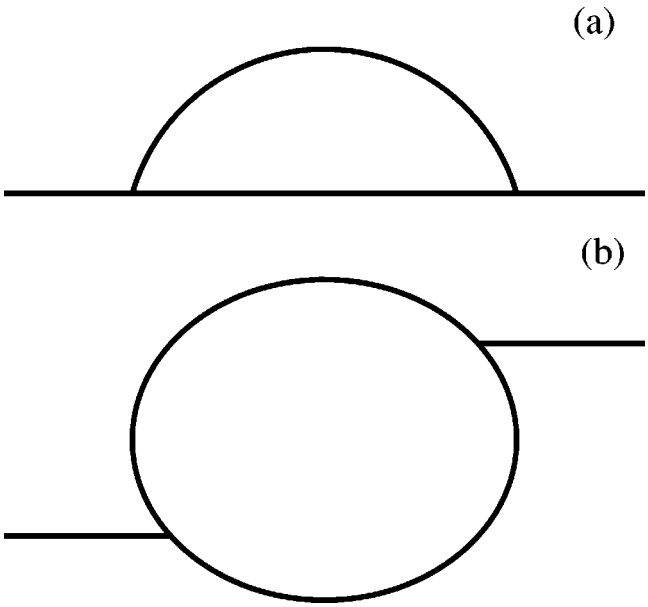

FIG. 1. Horizontal lines correspond to instantaneous propagators $1/\vec{k}^2$. Curved lines correspond to non-instantaneous propagators $1/(\vec{k}^2+k_4^2)$. (a) is a contribution to V_0. (b) is a contribution to P_0.

where $P_{ij}(\hat{k})=(\delta_{ij}-\hat{k}_i\hat{k}_j)$ is the 3-dimensionally transverse projector. These propagators vanish in the limit $k_4\to\infty$, as do the quark free propagators. From the structure of the vacuum polarization term (29), as illustrated in Fig. 1(b), one sees that the diagrams contributing to it must somewhere be connected by the dynamical propagators (30), so P_0 has no instantaneous part.[1] This is apparent in the one-loop calculation presented in Appendix B.

IV. A TECHNICAL DIFFICULTY

In Eq. (5) we have given a definition of the color-Coulomb potential which is independent of perturbation theory. However, since its defining property is false in any finite order or perturbation theory, we must address the problem of how to calculate it.

The perturbative expansion of $D_{0,44}=V_0+P_0$ begins with

$$V_0(|\vec{k}|,g_0,\Lambda)=g_0^2(\vec{k}^2)^{-1}+O(g_0^4) \tag{31}$$

$$P_0(|\vec{k}|,k_4,g_0,\Lambda)=O(g_0^4). \tag{32}$$

The Feynman integrals for these quantities are derived to one-loop order in Appendix B. The integral for $P_0(|\vec{k}|,k_4,g_0,\Lambda)$ is relatively complicated; however, it simplifies considerably for $k_4\gg|\vec{k}|$, as discussed in Appendix B, and one obtains to one-loop order for $SU(N)$ gauge theory,

$$\vec{k}^2V_0(|\vec{k}|,g_0,\Lambda)=g_0^2+g_0^4[v_{11}\ln(\Lambda/|\vec{k}|)+v_{10}] \tag{33}$$

$$\vec{k}^2P_0(|\vec{k}|,k_4,g_0,\Lambda)=g_0^4[p_{11}\ln(\Lambda/k_4)+p_{10}+o(|\vec{k}|/k_4)], \tag{34}$$

where Λ is an ultra-violet cutoff, $\lim_{x\to0}o(x)=0$, and v_{nm} and p_{nm} are constants, with

$$v_{11}=\frac{8N}{(4\pi)^2} \tag{35}$$

$$p_{11}=-\frac{2}{(4\pi)^2}\left(\frac{1}{3}N+\frac{2}{3}N_f\right). \tag{36}$$

Here we have included the vacuum polarization contribution of N_f flavors of quark in the fundamental representation which, to this order, is gauge independent. These coefficients are also found in a Hamiltonian formalism [10].

We obtain asymptotically for $k_4\gg|\vec{k}|$, for $D^{\rm as}_{0,44}=V_0+P^{\rm as}_0$,

$$\vec{k}^2D^{\rm as}_{0,44}(|\vec{k}|,k_4,g_0,\Lambda)=g_0^2+g_0^4[v_{11}\ln(\Lambda/|\vec{k}|)+p_{11}\ln(\Lambda/k_4)+v_{10}+p_{10}]. \tag{37}$$

Observe that p_{10} contributes a $1/\vec{k}^2$ term to $D_{0,44}$. Consequently we cannot assert that the instantaneous part of $D_{0,44}$ in $d=4$ dimensions, namely the color-Coulomb potential V is given by the sum of instantaneous diagrams, as it is in $d<4$ dimensions,

$$V(|\vec{k}|,\Lambda_{QCD})\neq V_0(\vec{k},g_0,\Lambda). \tag{38}$$

We shall see that V_0 and P_0 separately are not renormalization-group invariants, and this will be the key to an unambiguous separation of the instantaneous part V. This implies that the coefficients that appear in V_0 and P_0 may be scheme dependent. In particular, v_{10} and p_{10} are scheme dependent.

Another way to see that the separation cannot be made diagram by diagram is to recall that in higher order, calculations must be done in the interpolating gauge, with gauge condition $\partial_iA_i+\lambda\partial_4A_4=0$. For finite λ no diagram is instantaneous, and a separation criterion is required for the limiting expression at $\lambda=0$.

As a first step we note that in one-loop order $D^{\rm as}_{0,44}(|\vec{k}|,k_4,g_0,\Lambda)$ at large k_4 conveniently separates into a sum of terms that depend, respectively, on $|\vec{k}|$ and on k_4. To get the instantaneous part, namely the part that is independent of k_4, we simply delete the term that depends on k_4, namely $p_{11}\ln(\Lambda/k_4)$. However, the separation of the constant term is ambiguous, as it is in each order. We conclude that to

[1]It is essential to use the phase-space integral with first-order action to separate out the instantaneous diagrams in this way. For one could also integrate out A_4 in the second-order Lagrangian formalism. However if one did that, then $\dot{A}_i^{\rm tr}$ would appear at vertices instead of the independent field variable $E_i^{\rm tr}$. The free propagator $\langle\dot{A}_i^{\rm tr}\dot{A}_j^{\rm tr}\rangle_0=P_{ij}(\hat{k})k_4^2/(\vec{k}^2+k_4^2)$ does not vanish in the limit $k_4\to\infty$, but rather has the limit $\langle\dot{A}_i^{\rm tr}\dot{A}_j^{\rm tr}\rangle_0\to P_{ij}(\hat{k})$ which has an instantaneous non-local part.

one-loop order, the k_4-independent part of $D_{0,44}(|\vec{k}|,k_4,g_0,\Lambda)$ is given by

$$\vec{k}^2 V(|\vec{k}|,g_0,\Lambda)=g_0^2 x_0+g_0^4[v_{11}\ln(\Lambda/|\vec{k}|)+x_1], \quad (39)$$

where x_0 and x_1 are as yet unknown constants.

The form of the higher-order contributions to P_0 is not known. However, we shall make the simplest assumption, namely that the same procedure may be effected to arbitrary order so that only constant terms in each order in the expansion of $\vec{k}^2 V(|\vec{k}|,g_0,\Lambda)$ are ambiguous,

$$\vec{k}^2 V(|\vec{k}|,g_0,\Lambda) = g_0^2 x_0+\sum_{n=1}^{\infty} g_0^{2n+2}\left[x_n+\sum_{m=1}^{n} v_{nm}\ln^m(\Lambda/|\vec{k}|)\right]. \quad (40)$$

Here the coefficients v_{nm} are calculated from the perturbative expansion of V_0, but the x_n are a set of as yet unknown constants. (In case there are additional ambiguous terms in higher order in the separation of the vacuum polarization part P_0 from $D_{0,44}$, they may also be determined using the renormalization group, because it also restricts the v_{nm}.)

From its definition as the large-k_4 limit of D_{44}, it follows that color-Coulomb potential is a renormalization-group invariant. So when $V(|\vec{k}|)$ is expressed either in terms of unrenormalized or renormalized quantities, it is independent of Λ and μ (and of the regularization and renormalization scheme),

$$V(|\vec{k}|)\equiv V(|\vec{k}|,g_0(\Lambda/\Lambda_{QCD}),\Lambda)=V(|\vec{k}|,g_r(\mu/\Lambda_{QCD}),\mu). \quad (41)$$

Consequently we may set $\Lambda=|\vec{k}|$, and $\mu=|\vec{k}|$, which gives

$$V(|\vec{k}|)=V(|\vec{k}|,g_0(|\vec{k}|/\Lambda_{QCD}),|\vec{k}|) = V(|\vec{k}|,g_r(|\vec{k}|/\Lambda_{QCD}),|\vec{k}|). \quad (42)$$

(Once the functional dependence of $D_{0,44}$ or V_0 on the cutoff Λ is determined—it is a polynomial in $\ln\Lambda$ in each order of perturbation theory—then Λ may be assigned any finite value.) Thus we may set $\Lambda=|\vec{k}|$ in Eq. (40), which gives a simple expansion in terms of the unknown coefficients x_n,

$$\vec{k}^2 V(|\vec{k}|)=x_0 g_0^2(|\vec{k}|/\Lambda_{QCD})+\sum_{n=1}^{\infty} x_n g_0^{2n+2}(|\vec{k}|/\Lambda_{QCD}). \quad (43)$$

V. RENORMALIZATION GROUP IN THE COULOMB GAUGE

The unrenormalized coupling constant $g_0=g_0(\Lambda/\Lambda_{QCD})$ is a function of the cutoff Λ, determined by the flow equation

$$\Lambda\,\partial g_0/\partial\Lambda=\beta_0(g_0). \quad (44)$$

The β function has the expansion

$$\beta_0(g_0)=-(b_0 g_0^3+b_1 g_0^5+b_2 g_0^7+\ldots). \quad (45)$$

In general, the coefficients b_n are both scheme and gauge dependent, except for b_0 and b_1 that are scheme and gauge independent. Similarly the renormalized coupling constant $g_r=g_r(\mu/\Lambda_{QCD})$ is a function of the renormalization mass determined by

$$\mu\,\partial g_r/\partial\mu=\beta_r(g_r). \quad (46)$$

In the Coulomb gauge $D_{0,44}$ has no anomalous dimension coming from multiplicative renormalization. It therefore obeys the simple Callan-Symanzik equation

$$[\Lambda\partial/\partial\Lambda+\beta_0(g_0)\partial/\partial g_0]D_{0,44}(|\vec{k}|,k_4,g_0,\Lambda)=0. \quad (47)$$

As a result, in the Coulomb gauge, the β function may be obtained from the propagator $D_{0,44}$—in fact only $D^{\rm as}_{0,44}$ is needed—whereas in covariant gauges a calculation of the vertex function is necessary. Indeed from the last equation we have

$$\beta_0(g_0)=-\frac{\Lambda\partial D_{0,44}/\partial\Lambda}{\partial D_{0,44}/\partial g_0}, \quad (48)$$

where $D_{0,44}=D_{0,44}(|\vec{k}|,k_4,g_0,\Lambda)$. This holds identically for all values of $|\vec{k}|$, k_4 and Λ, so we may set k_4 to an asymptotically large value and obtain

$$\beta_0(g_0)=-\frac{\Lambda\partial D^{\rm as}_{0,44}/\partial\Lambda}{\partial D^{\rm as}_{0,44}/\partial g_0}. \quad (49)$$

From the one-loop expression (37) for $D^{\rm as}_{0,44}$, we obtain the first coefficient of the β function

$$b_0=\frac{1}{2}(v_{11}+p_{11}). \quad (50)$$

From Eqs. (35) and (36) we obtain the standard expression

$$b_0=\frac{1}{(4\pi)^2}\left(\frac{11}{3}N-\frac{2}{3}N_f\right), \quad (51)$$

without calculating any vertex function. All coefficients b_n may be calculated in this way.

We take the large k_4 limit of the Callen-Symanzik equation (47), and observe that $V(|\vec{k}|,g_0,\Lambda)=\lim_{k_4\to\infty}D_{0,44}(|\vec{k}|,k_4,g_0,\Lambda)$ satisfies the same Callan-Symanzik equation,

$$[\Lambda\partial/\partial\Lambda+\beta_0(g_0)\partial/\partial g_0]V(|\vec{k}|,g_0,\Lambda)=0. \quad (52)$$

Since $g_0=g_0(\Lambda/\Lambda_{QCD})$ is a solution of the flow equation (44), the Callan-Symanzik equation yields

$$\frac{dV(|\vec{k}|,g_0(\Lambda/\Lambda_{QCD}),\Lambda)}{d\Lambda}=0, \quad (53)$$

which is again the statement that $V(|\vec{k}|,g_0(\Lambda/\Lambda_{QCD}),\Lambda)$ is independent of Λ.

VI. RENORMALIZATION GROUP TO THE RESCUE

We shall use the renormalization group and our knowledge of the β function to determine the unknown constants x_n. From the Callan-Symanzik equation (52) for V we have

$$\partial V(|\vec{k}|,g_0,\Lambda)/\partial g_0 = \frac{-\Lambda\partial V(|\vec{k}|,g_0,\Lambda)/\partial\Lambda}{\beta_0(g_0)}. \tag{54}$$

This is an identity that holds for all Λ, and we may simplify it by setting $\Lambda=|\vec{k}|$,

$$\partial V(|\vec{k}|,g_0,\Lambda)/\partial g_0|_{\Lambda=|\vec{k}|} = \frac{-\Lambda\partial V(|\vec{k}|,g_0,\Lambda)/\partial\Lambda|_{\Lambda=|\vec{k}|}}{\beta_0(g_0)}. \tag{55}$$

To find all the x_n we substitute the expansion (40) on the left- and right-hand sides. We also expand

$$\beta_0^{-1}(g_0) = -b_0^{-1}g_0^{-3}\left(1+\sum_{p=1}^{\infty} c_p g_0^{2p}\right), \tag{56}$$

where, from Eq. (45), we have $c_1=-b_1/b_0$, etc. The derivative with respect to Λ on the right-hand side of Eq. (55) kills the constants x_n, and we have

$$\sum_{n=0}^{\infty} 2(n+1)g_0^{2n+1}x_n = b_0^{-1}g_0^{-3}\sum_{p=0}^{\infty} c_p g_0^{2p}\sum_{m=1}^{\infty} g_0^{2m+2}v_{m1}, \tag{57}$$

where $c_0\equiv 1$. Equating like powers of g_0 we obtain

$$x_n = [2(n+1)b_0]^{-1}\sum_{m=1}^{n+1} c_{n-m+1}v_{m1}, \tag{58}$$

and in particular

$$x_0 = (2b_0)^{-1}v_{11} \tag{59}$$

$$x_1 = (4b_0)^{-1}(v_{11}c_1+v_{21}). \tag{60}$$

Thus x_0 is found from v_{11} and b_0 which require one-loop calculations, and x_1 requires two-loop calculations. Usually the renormalization group is used to determine higher-order logarithms from lower-order terms. Here instead we have used it to consistently determine an unknown lower-order constant from a known higher-order logarithm. From Eqs. (35) and (51), we obtain

$$x_0 = \frac{12N}{11N-2N_f}. \tag{61}$$

It will be convenient in the following to factorize the coefficient x_0 out of the expansion (43) for V, and we write

$$\vec{k}^2V(|\vec{k}|) = x_0\left[g_0^2(|\vec{k}|/\Lambda_{QCD}) + \sum_{n=1}^{\infty} x_n' g_0^{2n+2}(|\vec{k}|/\Lambda_{QCD})\right], \tag{62}$$

where $x_n'\equiv x_n/x_0$. Remarkably, the leading term is not simply g_0^2, as one would expect from Eq. (33), but rather $x_0g_0^2=[12N/(11N-2N_f)]g_0^2$.

The leading asymptotic form of V may be found from expression (40) to order g_0^4, with $v_{11}=2b_0x_0$, namely,

$$\vec{k}^2V(|\vec{k}|) = x_0g_0^2\{1+g_0^2[2b_0\ln(\Lambda/|\vec{k}|)+x_1']\} \tag{63}$$

which to this order may be written

$$\vec{k}^2V(|\vec{k}|) = x_0\{g_0^{-2}-[2b_0\ln(\Lambda/|\vec{k}|)+x_1']\}^{-1}, \tag{64}$$

where

$$x_1' = \frac{x_1}{x_0} = \frac{1}{2}\left(\frac{v_{21}}{v_{11}}-\frac{b_1}{b_0}\right). \tag{65}$$

From the asymptotic form of $g_0^{-2}\sim 2b_0\ln(\Lambda/\Lambda_{QCD})$, this gives

$$\vec{k}^2V(|\vec{k}|) \sim x_0[2b_0\ln(|\vec{k}|/\Lambda_{QCD})-x_1']^{-1} \tag{66}$$

$$\sim \frac{x_0}{2b_0\ln(|\vec{k}|/\Lambda_{\rm Coul})}, \tag{67}$$

which is valid for large $|\vec{k}|$. Here we have introduced the new mass scale $\Lambda_{\rm Coul}$ characteristic of the Coulomb gauge. It is related to the scale Λ_{QCD} used in the scheme by which the β function was defined according to

$$\Lambda_{\rm Coul} \equiv \exp\left(\frac{x_1'}{2b_0}\right)\Lambda_{QCD}. \tag{68}$$

Normally, a ratio such as $\Lambda_{\rm Coul}/\Lambda_{QCD}$ can be determined from a one-loop calculation, but here a two-loop calculation of v_{21}, the coefficient of a two-loop logarithm that appears in Eq. (65) would be required. For just as the determination of the constant x_0, of 0-loop order, requires the 1-loop calculation which we report here, similarly, to determine the constant x_1' of 1-loop order would require a 2-loop calculation. In both cases the reason is the ambiguity in the separation of the instantaneous part V which, as noted above, cannot be determined from individual perturbative diagrams. Instead it requires the perturbative renormalization group to determine the nth order constant from the $(n+1)$th order logarithm.

Expression (67) for $V(|\vec{k}|)$ exhibits asymptotic freedom and, with x_0 positive, indeed $x_0>1$, the instantaneous part of D_{44} is anti-screening.

VII. ASYMPTOTIC FORM OF THE NON-INSTANTANEOUS PART

We may also determine to this order the asymptotic form of the non-instantaneous part $P=D_{0,44}-V$. It is also a renormalization-group invariant. If we set k_4 to an asymptotically high value we obtain

$$P^{\rm as}(|\vec{k}|,k_4,g_0,\Lambda)\equiv D^{\rm as}_{0,44}(|\vec{k}|,k_4,g_0,\Lambda)-V(|\vec{k}|,g_0,\Lambda). \tag{69}$$

From Eqs. (37) and (39) this gives to one-loop order

$$\vec{k}^2P^{\rm as}(|\vec{k}|,k_4,g_0,\Lambda)=g_0^2y_0+g_0^4[p_{11}\ln(\Lambda/k_4)+y_1], \tag{70}$$

where, by Eqs. (59), (50), (51), and (36),

$$\begin{aligned} y_0&=1-x_0=(2b_0)^{-1}(2b_0-v_{11})=(2b_0)^{-1}p_{11}\\ &=-\frac{N+2N_f}{11N-2N_f}, \end{aligned} \tag{71}$$

and $y_1=p_{10}+v_{10}-x_1$. This gives

$$\vec{k}^2P^{\rm as}(|\vec{k}|,k_4,g_0,\Lambda)=y_0g_0^2\{1+g_0^2[2b_0\ln(\Lambda/k_4)+y_1']\}, \tag{72}$$

where $y_1'=y_1/y_0$. By the reasoning that leads to Eq. (67), we obtain to this order the asymptotic expression

$$\vec{k}^2P^{\rm as}(|\vec{k}|,k_4,g_0,\Lambda)\sim\frac{y_0}{2b_0\ln(k_4/\Lambda'_{QCD})}, \tag{73}$$

where Λ'_{QCD} is another finite QCD mass scale like $\Lambda_{\rm Coul}$. It would take also take a 2-loop calculation to determine $\Lambda'_{QCD}/\Lambda_{QCD}$. Since $y_0<0$ is negative, P is indeed screening.

VIII. INVARIANT COLOR CHARGE

In this section we show how to calculate $V(|\vec{k}|)$ to arbitrary accuracy in the ultra-violet region. We define a new running coupling by constant $g=g(|\vec{k}|/\Lambda_{\rm Coul})$ by

$$\vec{k}^2V(|\vec{k}|)\equiv x_0g^2(|\vec{k}|/\Lambda_{\rm Coul}). \tag{74}$$

Because $V(|\vec{k}|)$ is scheme independent, $g(|\vec{k}|/\Lambda_{\rm Coul})$ is also. From Eq. (62) we have

$$g^2(|\vec{k}|/\Lambda_{\rm Coul})=\left(g_0^2+\sum_{n=1}^{\infty}g_0^{2n+2}x_n'\right)_{g_0=g_0(|\vec{k}|/\Lambda_{QCD})}, \tag{75}$$

so the new coupling constant agrees with g_0 in lowest order. Thus it is a regular redefinition of the coupling constant, and g may be used for perturbative expansions.

Corresponding to the invariant charge is an invariant β function defined by

$$\beta(g)\equiv|\vec{k}|\frac{\partial g(|\vec{k}|/\Lambda_{\rm Coul})}{\partial|\vec{k}|}. \tag{76}$$

It may be calculated from

$$\beta(g)=\frac{\partial g}{\partial g_0}|\vec{k}|\frac{\partial g_0(|\vec{k}|/\Lambda_{QCD})}{\partial|\vec{k}|} \tag{77}$$

$$=\frac{\partial g}{\partial g_0}\beta_0(g_0)|_{g_0(g)}, \tag{78}$$

where $g_0(g)$ is obtained by inverting Eq. (75). This may be simplified by using

$$\beta(g)=\frac{1}{2g}\frac{\partial g^2}{\partial g_0}\beta_0(g_0) \tag{79}$$

$$=\frac{\vec{k}^2}{2gx_0}\frac{\partial V_0}{\partial g_0}\bigg|_{|\vec{k}|=\Lambda}\beta_0(g_0) \tag{80}$$

$$=-\frac{\vec{k}^2}{2gx_0}\frac{\Lambda\partial V_0}{\partial\Lambda}\bigg|_{|\vec{k}|=\Lambda}, \tag{81}$$

by Eq. (55). From Eq. (40) this gives

$$\beta(g)=-\frac{1}{2g}\left(2b_0g_0^4+\sum_{n=2}^{\infty}v'_{n1}g_0^{2n+2}\right)\bigg|_{g_0(g)}, \tag{82}$$

where $v'_{n1}\equiv v_{n1}/x_0$, and we have used $v_{11}/x_0=2b_0$. To find $V(|\vec{k}|)$ to arbitrary accuracy, one calculates $\beta(g)$ perturbatively and then solves the flow equation (10) for $g(|\vec{k}|/\Lambda_{\rm Coul})$.

Finally we remark that we may choose new unrenormalized and renormalized expansion parameters according to

$$g_0'=g(\Lambda/\Lambda_{\rm Coul}) \tag{83}$$

$$g_r'=g(\mu/\Lambda_{\rm Coul}) \tag{84}$$

so g_0' and g_r' lie on the same invariant trajectory, the only difference between them being the value of the argument.

IX. CONCLUSION

We have successfully applied the perturbative renormalization group to the Coulomb gauge. In this gauge, the 44-component of the gluon propagator $D_{44}(|\vec{k}|,k_4,\Lambda_{QCD})$ is a renormalization-group invariant in the sense that it is independent of the regularization and renormalization schemes, and of the ultra-violet cutoff Λ and renormalization mass μ. With the help of the perturbative renormalization group we have decomposed it into an instantaneous part $V(|\vec{k}|)$, which we call the color-Coulomb potential, and a vacuum polarization part $P(|\vec{k}|,k_4)$ which vanishes at large k_4. Each of these terms is separately a renormalization-group invariant, and

their asymptotic form, at large $|\vec{k}|$ and k_4, respectively, was reported in the Introduction.

The color-Coulomb potential allows us to define an invariant QCD charge $g(|\vec{k}|/\Lambda_{\rm Coul})$ by $\vec{k}^2 V(\vec{k}) = x_0 g^2(|\vec{k}|/\Lambda_{\rm Coul})$, where $x_0 = 12N/(11N-2N_f)$. This invariant charge is the QCD analogue of the invariant charge of Gell-Mann and low in QED. We have shown how to calculate the corresponding invariant β function, $|\vec{k}|(\partial g/\partial|\vec{k}|) = \beta(g)$. Because this charge is scheme independent it may offer some advantage in providing a definition of $\alpha_s(\vec{k}^2) = g^2(|\vec{k}|/\Lambda_{\rm Coul})/4\pi$, whereas the standard definition in current use is scheme-dependent (see, for example, [11]). The color-Coulomb potential $V(|\vec{k}|)$ is also the natural starting point for calculations of bound states such as heavy quarkonium.

The Coulomb gauge provides direct access to quantities of non-perturbative interest. Indeed both $V(|\vec{k}|)$ and $P(|\vec{k}|,k_4)$ have a natural role in a confinement scenario: $V(|\vec{k}|,\Lambda_{QCD})$ is long-range, anti-screening, and responsible for the confinement of color charge, whereas the vacuum polarization term $P(|\vec{k}|,k_4)$ is screening, and responsible for ''breaking of the string'' between external quarks, when dynamical quark pairs are produced from the vacuum. We expect the linear rise (or not) of $V(|\vec{x}|)$ at large $|\vec{x}|$ to provide an order parameter for confinement of color charge, even in the presence of dynamical quarks when the Wilson loop cannot serve this purpose. The accompanying article [1] reports a numerical study of the running coupling constant $g^2(\vec{k}/\Lambda_{\rm Coul})$. The data show a significant enhancement at low $|\vec{k}|$, in agreement with this confinement scenario. However, additional studies at larger values of $\beta \equiv 4/g_0^2$ are necessary before a conclusion can be reached about a linear rise of $V(\vec{x})$ at large $|\vec{x}|$ in the continuum limit, $\beta \to \infty$.

The data also show a strong suppression of the equal-time, 3-dimensionally transverse, would-be physical, gluon propagator $D^{\rm tr}_{ij}(\vec{k})$ at $\vec{k}=0$, and agree with a formula of Gribov that *vanishes* like $|\vec{k}|$ near $|\vec{k}|=0$. The only explanation for this counter-intuitive behavior is the suppression of configurations outside the Gribov horizon in the minimal Coulomb gauge. Since $D^{\rm tr}_{ij}(\vec{k})$ is strongly suppressed, we may understand the main long-range forces between color charge as being due to D_{44} which, as we have seen, is the sum of the attractive instantaneous color-Coulomb potential $V(|\vec{k}|)$ that is anti-screening, and the vacuum polarization term $P(|\vec{k}|,k_4)$ that is screening. According to the confinement scenario discussed in Sec. III and in [2] and [1], both are long range in the minimal Coulomb gauge because entropy favors a high density population close to the Gribov horizon.

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APPENDIX A: RELATION OF FADDEEV-POPOV AND PHASE-SPACE FUNCTIONAL INTEGRALS

We wish to derive the canonical or phase-space functional integral (16) from the Faddeev-Popov formula (14). The argument merely reverses the textbook derivation of the Faddeev-Popov formula from the canonical phase-space functional integral in Coulomb gauge while keeping track of the sources J_μ. We introduce the identity

$$\exp\left[-(1/2)\int d^4x F_{0i}^2\right] = N\int d^3E \exp\int d^4x[iE_i F_{0i} - (1/2)E_i^2], \tag{A1}$$

which is a Gaussian integral over new variables E_i^a that will play the role of independent color-electric field variables. This allows us to rewrite (14) as

$$Z(\vec{J},J_4) = \int_G d^4A\, d^3E\, \delta(\partial_i A_i)\det(-D_i\partial_i) \times \exp\int d^4x[iE_i(\dot{A}_i - D_i A_4) - (1/2) \times (E_i^2 + B_i^2) - ig_0 J_\mu A_\mu)], \tag{A2}$$

where $B_1^a = \partial_2 A_3^a - \partial_3 A_2^a + g_0 f^{abc} A_2^b A_3^c$, etc. Integration on A_4 imposes color-Gauss's law, $D_i E_i = g_0 J_4$, in the form of the constraint $\delta(D_i E_i - g_0 J_4)$,

$$Z(\vec{J},J_4) = \int_G d^3A\, d^3E\, \delta(\partial_i A_i)\det(-D_i\partial_i) \times \delta(D_i E_i - g_0 J_4) \times \exp\int d^4x[iE_i\dot{A}_i^{\rm tr} - (1/2)(E_i^2 + B_i^2) - ig_0 J_i A_i^{\rm tr})]. \tag{A3}$$

The constraint expressed by $\delta(\partial_i A_i)$ has allowed us to replace A_i by its transverse part $A_i^{\rm tr}$ everywhere. We separate the transverse and longitudinal parts of $E_i = E_i^{\rm tr} - \partial_i\phi$, and we have $d^3E = N dE^{\rm tr} d\phi$. The Faddeev-Popov determinant is absorbed by

$$\det(-D_i\partial_i)\,\delta(D_i E_i - g_0 J_4) = \det(M)\,\delta(M\phi - \rho_{\rm Coul} - g_0 J_4) \tag{A4}$$

$$= \delta[\phi - M^{-1}(\rho_{\rm Coul} + g_0 J_4)], \tag{A5}$$

where the symbols are defined as in Eq. (20). We now integrate over ϕ, and in a similar way we integrate out the longitudinal part of A_i, to obtain Eq. (16).

APPENDIX B: ONE-LOOP EXPANSION

In this appendix we find the one-loop expansion of the quantities V_0 and P_0 defined in Eqs. (28) and (29), and which appear in $D_{0,44}=V_0+P_0$. The Faddeev-Popov operator is written $M(A^{\rm tr})=M_0+M_1(A^{\rm tr})$, where $M_0\equiv-\partial_i^2$ is the negative of the Laplacian, and $(M_1)^{ac}\equiv-g_0f^{abc}A_i^{b,{\rm tr}}\partial_i$. The color-Coulomb potential energy functional $\mathcal{V}(A^{\rm tr})$, defined in Eq. (25), reads

$$\mathcal{V}(\vec{x},\vec{y};A^{\rm tr})=[(M_0+M_1)^{-1}M_0(M_0+M_1)^{-1}]_{\vec{x},\vec{y}}, \quad \text{(B1)}$$

and has the expansion

$$\mathcal{V}(\vec{x},\vec{y};A^{\rm tr})=[M_0^{-1}-2M_0^{-1}M_1M_0^{-1}+3M_0^{-1}M_1M_0^{-1}M_1M_0^{-1}+\ \dots\ ]_{\vec{x},\vec{y}}, \quad \text{(B2)}$$

where $M_0^{-1}|_{\vec{x},\vec{y}}=(2\pi)^{-3}\int d^3k(\vec{k}^2)^{-1}\exp[i\vec{k}\cdot(\vec{x}-\vec{y})]=(4\pi|\vec{x}-\vec{y}|)^{-1}$. From Eq. (28) for V_0 we obtain to one-loop order

$$V_0(x-y)=g_0^2[M_0^{-1}+3M_0^{-1}\langle M_1M_0^{-1}M_1\rangle_0M_0^{-1}]_{\vec{x},\vec{y}}\times\delta(x_4-y_4), \quad \text{(B3)}$$

where we have used $\langle M_1(A^{\rm tr})\rangle=0$, which holds because $M_1(A^{\rm tr})$ is linear in $A^{\rm tr}$. The average designated by $\langle\ \dots\ \rangle_0$ refers to the free-field average, with free-field propagators given in Eq. (30). This gives

$$V_0=V_{0,0}+V_{0,1} \quad \text{(B4)}$$

where the zero-loop piece is given explicitly by

$$V_{0,0}(x-y)\,\delta^{ae}=g_0^2M_0^{-1}(\vec{x}-\vec{y})\,\delta(x_4-y_4)\,\delta^{ae} \quad \text{(B5)}$$

and the one-loop piece by

$$V_{0,1}(x-y)\,\delta^{ae}=3g_0^4\int d^3x'd^3y'f^{abc}f^{cde}\langle A_i^{{\rm tr},b}(\vec{x}',x_4)\times A_j^{{\rm tr},d}(\vec{y}',x_4)\rangle_0M_0^{-1}(\vec{x}-\vec{x}')\times\partial_iM_0^{-1}(\vec{x}'-\vec{y}')\partial_jM_0^{-1}(\vec{y}'-\vec{y})\,\delta(x_4-y_4). \quad \text{(B6)}$$

These terms are illustrated in Fig. 1(a). In momentum space we have $V_{0,0}=g_0^2/\vec{k}^2$, and

$$V_{0,1}(|\vec{k}|)=\frac{3g_0^4N}{(\vec{k}^2)^2}(2\pi)^{-4}\int d^4p\frac{k_i(\delta_{ij}-\hat{p}_i\hat{p}_j)k_j}{(\vec{p}^2+p_4^2)(\vec{p}-\vec{k})^2}. \quad \text{(B7)}$$

The result of this integral is given in Eqs. (33) and (35).

Similarly, for P_0 given in Eq. (29), we have to one-loop order

$$P_0(x-y)\,\delta^{ad}=-g_0^2\langle(M_0^{-1}\rho^a_{\rm Coul})(x)(M_0^{-1}\rho^d_{\rm Coul})(y)\rangle_0, \quad \text{(B8)}$$

where $\rho^a_{\rm Coul}=-g_0f^{abc}A_i^{{\rm tr},b}E_i^{{\rm tr},c}$. This gives

$$P_0(x-y)\,\delta^{ad}=-g_0^2\int d^3x'd^3y'M_0^{-1}(\vec{x}-\vec{x}')\times\langle\rho^a_{\rm Coul}(\vec{x}',x_4)\rho^d_{\rm Coul}(\vec{y}',y_4)\rangle_0M_0^{-1}(\vec{y}'-\vec{y}), \quad \text{(B9)}$$

where

$$\langle\rho^a_{\rm Coul}(x)\rho^d_{\rm Coul}(y)\rangle_0=g_0^2f^{abc}f^{deg}[\langle A_i^{{\rm tr},b}(x)A_j^{{\rm tr},e}(y)\rangle_0\langle E_i^{{\rm tr},c}(x)E_j^{{\rm tr},g}(y)\rangle_0+\langle A_i^{{\rm tr},b}(x)E_j^{{\rm tr},g}(y)\rangle_0\langle E_i^{{\rm tr},c}(x)A_j^{{\rm tr},e}(y)\rangle_0]. \quad \text{(B10)}$$

This term is illustrated in Fig. 1(b). In momentum space it is given by

$$P_{0,1}(k)=\frac{-g_0^4N}{(\vec{k}^2)^2}(2\pi)^{-4}\int d^4p\frac{P_{ij}(\vec{p})}{(\vec{p}^2+p_4^2)}\times\frac{P_{ij}(\vec{p}-\vec{k})}{[(\vec{p}-\vec{k})^2+(p_4-k_4)^2]}[\vec{p}^2-p_4(p_4-k_4)], \quad \text{(B11)}$$

where $P_{ij}(\vec{p})=\delta_{ij}-\hat{p}_i\hat{p}_j$ is the 3-dimensionally transverse projector. The contraction in the numerator gives 2 terms,

$$P_{ij}(\vec{p})P_{ij}(\vec{p}-\vec{k})=J_1+J_2 \quad \text{(B12)}$$

$$J_1=2 \quad \text{(B13)}$$

$$J_2=-\frac{\vec{p}^2\vec{k}^2-(\vec{p}\cdot\vec{k})^2}{\vec{p}^2(\vec{p}-\vec{k})^2}. \quad \text{(B14)}$$

Each term results in a Feynman integral I_1 and I_2. The integral I_2 looks more complicated. However, it is only logarithmically divergent by power counting, and when the integration is performed, the coefficient of the divergent part of I_2 vanishes, so I_2 is finite. As a result $I_2(|\vec{k}|,k_4)$ vanishes in the limit $k_4\to\infty$, and does not contribute to $P_0^{\rm as}(k)$. The result of the I_1 integration is given in Eqs. (34) and (36).

The integrals (B7) and (B11) are evaluated by dimensional regularization, with $p_4\to p_d$, and $\vec{p}=(p_i)$ for $i=1,\dots,(d-1)$.

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