Two-loop polarization contributions to radiative-recoil corrections to hyperfine splitting in muonium

Michael I. Eides*
Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506 and Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188350, Russia

Howard Grotch†
Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506

Valery A. Shelyuto‡
D. I. Mendeleev Institute of Metrology, St. Petersburg 198005, Russia

(Received 24 July 2001; published 7 December 2001)

We calculate radiative-recoil corrections of order $\alpha^2(Z\alpha)(m/M)E_F$ to hyperfine splitting in muonium generated by the diagrams with electron and muon polarization loops. These corrections are enhanced by the large logarithm of the electron-muon mass ratio. The leading logarithm cubed and logarithm squared contributions were obtained a long time ago. The single-logarithmic and nonlogarithmic contributions calculated here improve the theory of hyperfine splitting, and affect the value of the electron-muon mass ratio extracted from the experimental data on the muonium hyperfine splitting.

DOI: 10.1103/PhysRevD.65.013003 PACS number(s): 12.20.Ds, 31.30.Jv, 32.10.Fn, 36.10.Dr

I. INTRODUCTION: LEADING LOGARITHMIC CONTRIBUTIONS OF ORDER $\alpha^2(Z\alpha)(M/M)\tilde{E}_F$

It is well known that the radiative-recoil corrections of order[1] $\alpha^2(Z\alpha)(m/M)\tilde{E}_F$ to hyperfine splitting in muonium are enhanced by the large logarithm of the electron-muon mass ratio cubed [1]. The leading logarithm cube contribution is generated by the graphs[2] in Fig. 1 with insertions of the electron one-loop polarization operators in the two-photon exchange graphs. It may be obtained almost without any calculations by substituting the effective charge $\alpha(M)$ in the leading recoil correction of order $(Z\alpha)(m/M)\tilde{E}_F$, and expanding the resulting expression in the power series over α [2].

Calculation of the logarithm squared term of order $\alpha^2(Z\alpha)(m/M)\tilde{E}_F$ is more challenging [2]. All graphs in Figs. 1, 2, 3, 4, and 5 generate corrections of this order. The contribution induced by the irreducible two-loop vacuum polarization in Fig. 2 is again given by the effective charge expression. Subleading logarithm squared terms generated by the one-loop polarization insertions in Fig. 1 may easily be calculated with the help of the two leading asymptotic terms in the polarization operator expansion and the skeleton integral. The logarithm squared contribution generated by the diagrams in Fig. 3 is obtained from the leading single-logarithmic contribution of the diagrams without polarization insertions by the effective charge substitution. An interesting effect takes place in calculation of the logarithm squared term generated by the polarization insertions in the radiative photon in Fig. 4. One might expect that the high energy asymptote of the electron factor with the polarization insertion is given by the product of the leading constant term of the electron factor $-5\alpha/(4\pi)$ and the leading polarization operator term. However, this expectation turns out to be wrong. One may check explicitly that instead of the naive factor above one has to multiply the polarization operator by the factor $-3\alpha/(4\pi)$. The reason for this effect may easily be understood. The factor $-3\alpha/(4\pi)$ is the asymptote of the electron factor in massless QED and it gives a contribution to the logarithmic asymptotics only after the polarization operator insertion. This means that in massive QED the part $-2\alpha/(4\pi)$ of the constant electron factor originates from the integration region where the integration momentum is of order of the electron mass. Naturally this integration region does not give any contribution to the logarithmic asymptotics

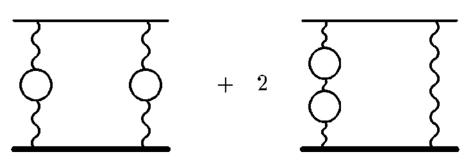

FIG. 1. Graphs with two one-loop polarization insertions.

*Email address: eides@pa.uky.edu, eides@thd.pnpi.spb.ru
†Email address: asdean@pop.uky.edu
‡Email address: shelyuto@vniim.ru

[1]We define the Fermi energy as

$$\tilde{E}_F = \frac{16}{3} Z^4 \alpha^2 \frac{m}{M} \left(\frac{m_r}{m}\right)^3 chR_\infty, \tag{1}$$

where m and M are the electron and muon masses, α is the fine structure constant, c is the velocity of light, h is the Planck constant, R_∞ is the Rydberg constant, and Z is the nucleus charge in terms of the electron charge ($Z=1$ for muonium). The Fermi energy $\tilde{E}_F$ does not include the muon anomalous magnetic moment a_μ which does not factorize in the case of recoil corrections, and should be considered on the same grounds as other corrections to hyperfine splitting.

[2]And by the diagrams with the crossed exchanged photon lines. Such diagrams with the crossed exchanged photon lines are also often omitted in other figures below.

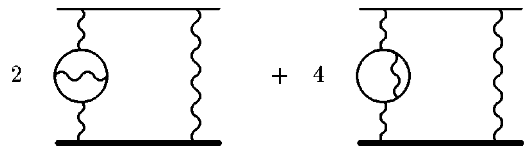

FIG. 2. Graphs with two-loop polarization insertions.

of the radiatively corrected electron factor. The least trivial logarithm squared contribution is generated by the three-loop diagrams in Fig. 5 with the insertions of the light by light scattering block. Their contribution was calculated explicitly in Ref. [2]. Later it was realized that these contributions are intimately connected with the well known anomalous renormalization of the axial current in QED [3]. Because of the projection on the hyperfine splitting (HFS) spin structure in the logarithmic integration region the heavy particle propagator effectively shrinks to an axial vector current vertex, and in this situation calculation of the respective contribution to HFS reduces to substitution of the well known two-loop axial renormalization factor in Fig. 6 [4] in the recoil skeleton diagram. Of course, this calculation reproduces the same contribution as obtained by direct calculation of the diagrams with light by light scattering expressions. From the theoretical point of view it is interesting that one can measure anomalous two-loop renormalization of the axial vector current in the atomic physics experiment.

The sum of all logarithm cubed and logarithm squared contributions of order $\alpha^2(Z\alpha)(m/M)\widetilde{E}_F$ is given by the expression [1,2]

$$\Delta E=\left(-\frac{4}{3}\ln^3\frac{M}{m}+\frac{4}{3}\ln^2\frac{M}{m}\right)\frac{\alpha^2(Z\alpha)}{\pi^3}\,\frac{m}{M}\widetilde{E}_F. \tag{2}$$

It was also shown in Ref. [2] that there are no other contributions with the large logarithm of the mass ratio squared accompanied by the factor α^3, even if the factor Z enters in another manner than in the equation above.

Single-logarithmic and nonlogarithmic terms of order $\alpha^2(Z\alpha)(m/M)\widetilde{E}_F$ are generated by all diagrams in Figs. 1–4, by the graphs with the muon polarization loops, by the graphs with polarization and radiative photon insertions in the muon line, and also by the graphs with two radiative photons in the electron and/or muon lines. Only a partial result for the single-logarithmic and nonlogarithmic corrections generated by the pole part of the graphs with both electron and muon polarization loops is known now [5]. Numerically the respective contribution is about 9 Hz, and may be considered only as an indication of the scale of the respective corrections. Corrections of this scale are phenomenologically relevant for modern experiment and theory [6]. In this paper we calculate all radiative-recoil corrections generated by the diagrams including only the polarization loops, either electronic or muonic, leaving calculation of the other contributions for the future.

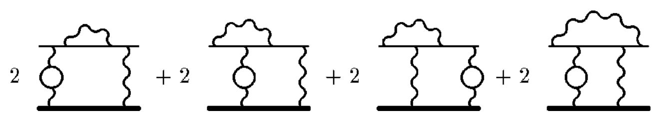

FIG. 3. Graphs with radiative photon insertions.

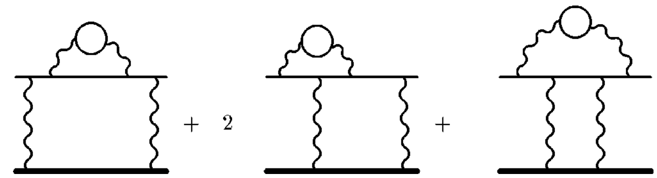

FIG. 4. Graphs with polarization insertions in the radiative photon.

II. TWO-PHOTON EXCHANGE DIAGRAMS: CANCELLATION OF THE ELECTRON AND MUON LOOPS

Calculation of single-logarithmic and nonlogarithmic radiative-recoil corrections of relative order $\alpha^2(Z\alpha)(m/M)$ [and also of orders $(Z^2\alpha)^2(Z\alpha)(m/M)$ and $\alpha(Z^2\alpha)(Z\alpha)(m/M)$] resembles in many respects calculation of the corrections of relative orders $\alpha(Z\alpha)(m/M)$ and $Z^2\alpha(Z\alpha)(m/M)$. It was first discovered in Refs. [7,8] that the contributions of the diagrams with insertions of the electron and muon polarization loops partially cancel, and, hence, it is convenient to treat such diagrams simultaneously.[3] A similar cancellation holds also for the corrections of order $\alpha^2(Z\alpha)(m/M)\widetilde{E}_F$, so we will first remind the reader how it arises when one calculates the polarization contribution of order $\alpha(Z\alpha)(m/M)\widetilde{E}_F$. The nonrecoil contribution in the heavy particle pole of the two-photon exchange diagrams exactly cancels in the sum of the electron and muon polarizations (see for more details Refs. [8,6]). Then the skeleton recoil contribution to the hyperfine splitting generated by the diagrams with two-photon exchanges in Fig. 7 is the result of the subtraction of the heavy pole contribution

$$\Delta E=4\frac{Z\alpha}{\pi}\,\frac{m}{M}\widetilde{E}_F\int_0^\infty\frac{dk}{k}\left[f(\mu k)-f\left(\frac{k}{2}\right)\right], \tag{3}$$

where $\mu=m/(2M)$, and

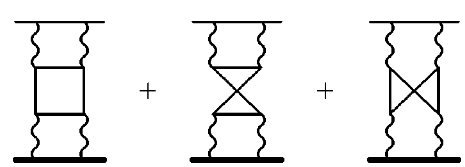

FIG. 5. Graphs with light by light scattering insertions.

[3] We always consider the external muon as a particle with charge Ze; this makes the origin of different contributions more transparent. However, somewhat inconsequently we omit the factor Z in the case of the closed muon loops. The reason for this apparent inconsistency is just the cancellation which we discuss now.

FIG. 6. Renormalization of the fifth current.

$$f(k)=\frac{1}{k}(\sqrt{1+k^2}-k-1)-\frac{1}{2}\left(k\sqrt{1+k^2}-k^2-\frac{1}{2}\right),$$

$$f(k)_{k\to 0}\to -\frac{3}{4}+\frac{k^2}{2},\quad f(k)_{k\to\infty}\to -\frac{1}{k}. \tag{4}$$

The electron polarization contribution is obtained from the skeleton integral by multiplying the expression in Eq. (3) by the multiplicity factor 2, and inserting the polarization operator $(\alpha/\pi)k^2 I_1(k)$ in the integrand

$$\frac{\alpha}{\pi}k^2 I_1(k)\equiv\frac{\alpha}{\pi}k^2\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)}. \tag{5}$$

The muon polarization contribution is given by a similar expression; the only difference is that

$$I_1(k)\to I_{1\mu}(k)\equiv\int_0^1 dv\frac{v^2(1-v^2/3)}{\mu^{-2}+k^2(1-v^2)}. \tag{6}$$

Then the total recoil contribution induced by the diagrams with both the one-loop electron and muon polarizations in Fig. 8 has the form

$$\Delta E=8\frac{\alpha(Z\alpha)}{\pi^2}\frac{m}{M}\tilde{E}_F\int_0^\infty\frac{dk}{k}\left[f(\mu k)-f\left(\frac{k}{2}\right)\right][k^2 I_1(k)+k^2 I_{1\mu}(k)]. \tag{7}$$

Next we rescale the integration variable $k\to kM/m$ in the muon term and obtain

$$\begin{aligned}\Delta E&=8\frac{\alpha(Z\alpha)}{\pi^2}\frac{m}{M}\tilde{E}_F\int_0^\infty\frac{dk}{k}\left[f(\mu k)-f\left(\frac{k}{2}\right)+f\left(\frac{k}{2}\right)-f\left(\frac{k}{4\mu}\right)\right]k^2 I_1(k)\\&=8\frac{\alpha(Z\alpha)}{\pi^2}\frac{m}{M}\tilde{E}_F\int_0^\infty\frac{dk}{k}\left[f(\mu k)-f\left(\frac{k}{4\mu}\right)\right]k^2 I_1(k).\end{aligned} \tag{8}$$

We see that the electron and muon polarization contributions have partially canceled. Moreover, it is not difficult to check explicitly that the term with $f(k/(4\mu))$ generates only corrections of higher order in μ, so with linear accuracy in the small mass ratio m/M all recoil contributions generated by

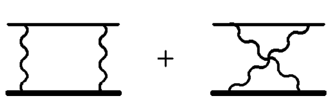

FIG. 7. Diagrams with two-photon exchanges.

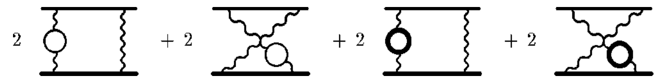

FIG. 8. Diagrams with one-loop polarization insertions.

the diagrams with the one-loop electron and muon polarization insertions in Fig. 8 are given by the integral

$$\Delta E=8\frac{\alpha(Z\alpha)}{\pi^2}\frac{m}{M}\tilde{E}_F\int_0^\infty\frac{dk}{k}f(\mu k)k^2 I_1(k). \tag{9}$$

This integral was calculated in Ref. [8] and we will not discuss its calculation here. Our only goal in this section was to demonstrate the mechanism of the partial cancellation of the electron loop and muon loop contributions.

III. DIAGRAMS WITH EITHER TWO ELECTRON OR TWO MUON LOOPS

The nonrecoil contribution generated by the diagrams with two electron or muon loops in Fig. 1 and Fig. 9 was obtained a long time ago [9]. Although it was not emphasized in that work explicitly, it is easy to check that the result in Ref. [9] includes heavy pole contributions which are due to the diagrams with both the electron and muon polarizations. Repeating the same steps as in the previous section, it is easy to see that the recoil contribution generated by the diagrams in Fig. 1 and Fig. 9 is determined by the integral

$$\Delta E=12\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F\int_0^\infty\frac{dk}{k}f(\mu k)k^4 I_1^2(k), \tag{10}$$

where the numerical factor before the integral is due to the multiplicity of the diagrams, and the whole integral is similar to the integral in Eq. (9). The only significant difference is that now we have the two-loop factor $k^4 I^2(k)$ in the integrand instead of the one-loop factor $k^2 I_1(k)$.

We calculate the integral in Eq. (10) separating the contributions of small and large momenta with the help of the auxiliary parameter σ such that $1\ll\sigma\ll 1/\mu$:

$$\Delta E=3(B_{11}^{<}+B_{11}^{>})\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F. \tag{11}$$

Then for the small integration momenta region in the leading order in $\mu\sigma$ $(\mu k\leqslant\mu\sigma\ll 1)$ we have

$$B_{11}^{<}=4\int_0^\sigma\frac{dk}{k}f(\mu k)k^4 I_1^2(k)\simeq -3\int_0^\sigma\frac{dk}{k}\,k^4 I_1^2(k). \tag{12}$$

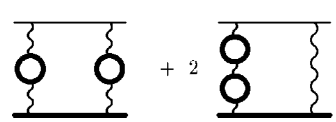

FIG. 9. Graphs with two muon one-loop polarization insertions.

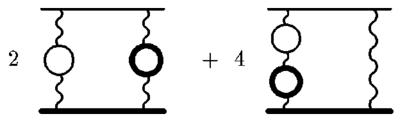

FIG. 10. Graphs with both the electron and muon loops.

We substitute in this integral the closed expression for the polarization $I_1(k)$, and again preserving only the leading contributions in $\mu\sigma \ll 1$ obtain

$$B_{11}^{<} \simeq -\frac{4}{9}\ln^3\sigma + \frac{10}{9}\ln^2\sigma - \frac{25}{27}\ln\sigma - \frac{2}{3}\zeta(3) + \frac{203}{324}. \tag{13}$$

The high momenta contribution is calculated by expanding the polarization operator in $1/k^2 \ll 1/\sigma^2 \ll 1$:

$$\begin{aligned} B_{11}^{>} &= 4\int_\sigma^\infty \frac{dk}{k} f(\mu k) k^4 I_1^2(k) \\ &\simeq 4\int_\sigma^\infty \frac{dk}{k}\left[\frac{1}{\mu k}(\sqrt{1+\mu^2k^2} - \mu k - 1) \right. \\ &\quad \left. - \frac{1}{2}\left(\mu k\sqrt{1+\mu^2k^2} - \mu^2k^2 - \frac{1}{2}\right)\right]\left(\frac{2}{3}\ln k - \frac{5}{9}\right)^2. \end{aligned} \tag{14}$$

For calculation of this integral we use the standard integrals introduced in Ref. [10] as well as some new standard integrals (see Appendix), and obtain

$$\begin{aligned} B_{11}^{>} &= \frac{4}{9}\ln^3(2\mu) - \frac{8}{9}\ln^2(2\mu) + \left(\frac{2\pi^2}{9} + \frac{25}{27}\right)\ln(2\mu) \\ &\quad + \frac{2}{3}\zeta(3) - \frac{4\pi^2}{27} - \frac{41}{18} + \frac{4}{9}\ln^3\sigma - \frac{10}{9}\ln^2\sigma + \frac{25}{27}\ln\sigma. \end{aligned} \tag{15}$$

Now we are ready to write down the total recoil contribution generated by the diagrams in Fig. 1 and Fig. 9:

$$\begin{aligned} \Delta E &= \left[-\frac{4}{3}\ln^3\frac{M}{m} - \frac{8}{3}\ln^2\frac{M}{m} - \left(\frac{2\pi^2}{3} + \frac{25}{9}\right)\ln\frac{M}{m} - \frac{4\pi^2}{9} \right. \\ &\quad \left. - \frac{535}{108}\right]\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F. \end{aligned} \tag{16}$$

The logarithm cubed and logarithm squared terms in this expression are already known [1,2], and the single-logarithmic and nonlogarithmic terms are obtained here.

IV. DIAGRAMS WITH BOTH THE ELECTRON AND MUON LOOPS

Consider now the diagrams with one electron and one muon loop in Fig. 10. We can look at these diagrams as a result of the electron polarization operator insertions in the muon loop diagrams in Fig. 8. The complete analytic expression for the last two diagrams in Fig. 8 has the form

$$\Delta E = 8\frac{\alpha(Z\alpha)}{\pi^2}\frac{m}{M}\tilde{E}_F\int_0^\infty \frac{dk}{k}\left[\tilde{f}(\mu k) - \tilde{f}\left(\frac{k}{2}\right)\right]k^2 I_{1\mu}(k), \tag{17}$$

where

$$\tilde{f}(k) = f(k) + \frac{1}{k}. \tag{18}$$

Unlike Eq. (7) we have restored in Eq. (17) the heavy particle pole contribution, which in the case of the muon polarization loop also carries the recoil factor. To simplify further calculations we rescale the integration momentum $k \to kM/m$:

$$\Delta E = 8\frac{\alpha(Z\alpha)}{\pi^2}\frac{m}{M}\tilde{E}_F\int_0^\infty \frac{dk}{k}\left[\tilde{f}\left(\frac{k}{2}\right) - \tilde{f}\left(\frac{k}{4\mu}\right)\right]k^2 I_1(k), \tag{19}$$

and note that with the linear accuracy in m/M we may omit the second term in the square brackets in the integrand. Then the muon loop diagrams in Fig. 8 are described by the expression

$$\Delta E = 8\frac{\alpha(Z\alpha)}{\pi^2}\frac{m}{M}\tilde{E}_F\int_0^\infty \frac{dk}{k}\tilde{f}\left(\frac{k}{2}\right)k^2 I_1(k). \tag{20}$$

The integral in Eq. (20) turns into the contribution of the diagrams in Fig. 10 after multiplication by the factor 3 and insertion in the integrand of the additional factor

$$\frac{\alpha}{\pi}\left(\frac{k}{2\mu}\right)^2 I_1\left(\frac{k}{2\mu}\right) = \frac{\alpha}{\pi}\left[\frac{2}{3}\ln\frac{k}{2\mu} - \frac{5}{9} + O\left(\frac{\mu^2}{k^2}\right)\right]. \tag{21}$$

This extra factor enters in the asymptotic regime since the characteristic scale of the integration momenta in Eq. (20) is about one, and the parameter μ goes to zero.

Then the contribution to the HFS of the diagrams in Fig. 10 is given by the integral

$$\begin{aligned} \Delta E &= 24\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F\int_0^\infty \frac{dk}{k}\tilde{f}\left(\frac{k}{2}\right)k^2 I_1(k)\left(\frac{2}{3}\ln\frac{k}{2\mu} - \frac{5}{9}\right) \\ &= 24\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F\int_0^\infty dk\left[(\sqrt{4+k^2} - k) \right. \\ &\quad \left. - \frac{k}{2}\left(\frac{k}{4}\sqrt{4+k^2} - \frac{k^2}{4} - \frac{1}{2}\right)\right]\int_0^1 dv\,\frac{v^2(1-v^2/3)}{4+k^2(1-v^2)} \\ &\quad \times\left[\frac{2}{3}\ln\frac{M}{m} + \frac{2}{3}\ln k - \frac{5}{9}\right]. \end{aligned} \tag{22}$$

After calculation we obtain (see the Appendix)

$$\Delta E = \left[\left(\frac{2\pi^2}{3} - \frac{20}{9}\right)\ln\frac{M}{m} + \frac{\pi^2}{3} - \frac{53}{9}\right]\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F. \tag{23}$$

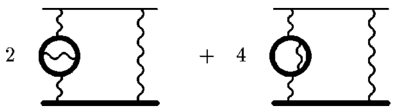

FIG. 11. Graphs with muon two-loop polarization insertions.

V. DIAGRAMS WITH SECOND ORDER POLARIZATION INSERTIONS

The recoil contribution to HFS generated by the diagrams in Fig. 2 and Fig. 11 with two-loop electron and muon polarization insertions is given by the integral [compare Eq. (9)]

$$\Delta E=8\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F\int_0^\infty\frac{dk}{k}f(\mu k)k^2 I_2(k), \tag{24}$$

where $(\alpha^2/\pi^2)k^2 I_2(k)$ is the two-loop polarization operator [11,12]

$$\begin{aligned}I_2(k)=&\frac{2}{3}\int_0^1 dv\,\frac{v}{4+k^2(1-v^2)}\Bigg\{(3-v^2)(1+v^2)\\&\times\left[\mathrm{Li}_2\left(-\frac{1-v}{1+v}\right)+2\,\mathrm{Li}_2\left(\frac{1-v}{1+v}\right)+\frac{3}{2}\ln\frac{1+v}{1-v}\ln\frac{1+v}{2}\right.\\&\left.-\ln\frac{1+v}{1-v}\ln v\right]+\left[\frac{11}{16}(3-v^2)(1+v^2)+\frac{v^4}{4}\right]\ln\frac{1+v}{1-v}\\&+\left[\frac{3}{2}v(3-v^2)\ln\frac{1-v^2}{4}-2v(3-v^2)\ln v\right]\\&+\frac{3}{8}v(5-3v^2)\Bigg\}.\end{aligned} \tag{25}$$

To further simplify calculations we represent the two-loop polarization operator in the form

$$I_2(k)=\frac{3}{4}I_1(k)+\int_0^1 dv\,\frac{R(v)}{4+k^2(1-v^2)}, \tag{26}$$

where

$$\begin{aligned}R(v)=&\frac{2}{3}v\Bigg\{(3-v^2)(1+v^2)\left[\mathrm{Li}_2\left(-\frac{1-v}{1+v}\right)+2\,\mathrm{Li}_2\left(\frac{1-v}{1+v}\right)\right.\\&\left.+\frac{3}{2}\ln\frac{1+v}{1-v}\ln\frac{1+v}{2}-\ln\frac{1+v}{1-v}\ln v\right]\\&+\left[\frac{11}{16}(3-v^2)(1+v^2)+\frac{v^4}{4}\right]\ln\frac{1+v}{1-v}\\&+\left[\frac{3}{2}v(3-v^2)\ln\frac{1-v^2}{4}-2v(3-v^2)\ln v\right]\\&+\frac{3}{4}v(1-v^2)\Bigg\}.\end{aligned} \tag{27}$$

The integral in Eq. (26) decreases as $1/k^2$ at large k. Note that since $R(v)\to 3(1-v)$ at $v\to 1$ this leading term in the asymptotic expansion is not enhanced by the large logarithm $\ln k$. Absence of this logarithm significantly simplifies further calculations.

In terms of the function $R(v)$ the integral for the recoil contribution in Eq. (24) has the form

$$\begin{aligned}\Delta E=&8\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F\int_0^\infty dk k f(\mu k)\left[\frac{3}{4}I_1(k)\right.\\&\left.+\int_0^1 dv\,\frac{R(v)}{4+k^2(1-v^2)}\right]\equiv\Delta E^a+\Delta E^b.\end{aligned} \tag{28}$$

The first contribution on the right hand side is proportional to the well known one-loop contribution in Eq. (9) [13,7,8]:

$$\Delta E^a=\left[-\frac{3}{2}\ln^2(2\mu)+2\ln(2\mu)-\frac{\pi^2}{4}-\frac{7}{3}\right]\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F. \tag{29}$$

Calculation of the second term ΔE^b is a bit more involved. We again introduce the auxiliary parameter σ ($1\ll\sigma\ll 1/\mu$) and consider separately the small and large momenta contributions. For the small integration momenta region in the leading order in $\mu\sigma$ we have

$$\begin{aligned}\Delta E^{b<}=&8\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F\int_0^\sigma dk k f(\mu k)\int_0^1\frac{dv R(v)}{4+k^2(1-v^2)}\\\simeq&-3\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F\int_0^\sigma dk^2\int_0^1\frac{dv R(v)}{4+k^2(1-v^2)}\\\simeq&-3\,\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}E_F\int_0^1 dv\,\frac{R(v)}{1-v^2}\ln\frac{\sigma^2(1-v^2)}{4}\\=&-3\,\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F\Bigg\{\left[\zeta(3)+\frac{5}{24}\right]\ln\frac{\sigma^2}{4}+2\zeta(3)\ln 2\\&+\frac{25}{24}\zeta(3)+\frac{16}{3}\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{2\pi^2}{9}\ln^2 2+\frac{2}{9}\ln^4 2\\&+\frac{5}{12}\ln 2-\frac{5\pi^4}{108}-\frac{223}{144}\Bigg\},\end{aligned} \tag{30}$$

where we used certain integrals for the function $R(v)$ collected in the Appendix.

The high-momentum contribution is given by the integral

$$\Delta E^{b>}=8\,\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F\int_\sigma^\infty dk k f(\mu k)\int_0^1\frac{dv R(v)}{4+k^2(1-v^2)}. \tag{31}$$

First we use the relation $k^2\gg\sigma^2\gg 1$ and the relation $R(v)\to 3(1-v)$ as $v\to 1$ and omit 4 in the denominator in the integrand, and then perform the calculations using the integrals from the Appendix

$$\Delta E^{b>}=4\left[\zeta(3)+\frac{5}{24}\right]\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F\int_{\sigma^2}^{\infty}\frac{dk^2}{k^2}\left[\frac{1}{\mu k}\left(\sqrt{1+\mu^2k^2}-\mu k-1\right)-\frac{1}{2}\left(\mu k\sqrt{1+\mu^2k^2}-\mu^2k^2-\frac{1}{2}\right)\right]$$
$$=2\left[\zeta(3)+\frac{5}{24}\right]\left[3\ln(2\mu)-\frac{9}{2}+3\ln\sigma\right]\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F. \tag{32}$$

The total recoil contribution to the HFS generated by the diagrams with two-loop polarization insertions in Fig. 11 is given by the sum of the contributions in Eqs. (29), (30), and (32):

$$\Delta E=\Delta E^{a}+\Delta E^{b<}+\Delta E^{b>}$$
$$=\left\{-\frac{3}{2}\ln^2\frac{M}{m}-\left[6\zeta(3)+\frac{13}{4}\right]\ln\frac{M}{m}-\frac{97}{8}\zeta(3)-16\,\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{2\pi^2}{3}\ln^2 2-\frac{2}{3}\ln^4 2+\frac{5\pi^4}{36}-\frac{\pi^2}{4}+\frac{7}{16}\right\}\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F. \tag{33}$$

The logarithm squared term in this expression was obtained in Ref. [2], and the single-logarithmic and nonlogarithmic terms are obtained here.

VI. DISCUSSION OF RESULTS

Collecting all contributions in Eq. (16), Eq. (23), and Eq. (33) we obtain

$$\Delta E_t=\left\{-\frac{4}{3}\ln^3\frac{M}{m}-\frac{25}{6}\ln^2\frac{M}{m}-\left[6\zeta(3)+\frac{33}{4}\right]\ln\frac{M}{m}-\frac{97}{8}\zeta(3)-16\,\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{2\pi^2}{3}\ln^2 2-\frac{2}{3}\ln^4 2+\frac{5\pi^4}{36}-\frac{13\pi^2}{36}-\frac{4495}{432}\right\}\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F. \tag{34}$$

The contribution which contains only single logarithms and constants is

$$\Delta E=\left\{-\left[6\zeta(3)+\frac{33}{4}\right]\ln\frac{M}{m}-\frac{97}{8}\zeta(3)-16\,\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{2\pi^2}{3}\ln^2 2-\frac{2}{3}\ln^4 2+\frac{5\pi^4}{36}-\frac{13\pi^2}{36}-\frac{4495}{432}\right\}\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F. \tag{35}$$

The factor in the square brackets is about (-103), and numerically the respective contribution to the muonium HFS is

$$\Delta E_{new}=-0.027\ 7\ \ \mathrm{kHz}. \tag{36}$$

The magnitude of this correction is just in the range we should expect based on the partial result in Ref. [5]. The contribution in Eq. (36) is of the same scale as the logarithmic in $Z\alpha$ corrections of order $(Z\alpha)^3(m/M)E_F$ and $\alpha(Z\alpha)^2(m/M)E_F$, calculated recently in Refs. [14,15].

Collecting the recent results from Refs. [10,14,15] and Eq. (35), and using the experimental value of the muonium hyperfine splitting [16] we may derive a value of the electron-muon mass ratio

$$\frac{M}{m}=206.768\,279\,8\ \ (23)\ \ (16)\ \ (32), \tag{37}$$

where the first error comes from the experimental error of the hyperfine splitting measurement, the second comes from the error in the value of the fine structure constant α, and the third is an estimate of the yet unknown theoretical contributions.

Combining all errors we obtain the mass ratio

$$\frac{M}{m}=206.768\,279\,8\ \ (43),\qquad \delta=2\times10^{-8}, \tag{38}$$

which is almost six times more accurate than the best direct experimental value in Ref. [16].

Estimating the errors in Eq. (37) we assumed that the theoretical error of calculation of the muonium hyperfine splitting is about 70 Hz. This theoretical error is determined by the estimate of the still uncalculated terms which include single-logarithmic and nonlogarithmic radiative-recoil corrections of order $\alpha^2(Z\alpha)(m/M)\tilde{E}_F$ generated by the graphs containing besides the polarization loops also radiative photons, as well as the nonlogarithmic contributions of order $(Z\alpha)^3(m/M)E_F$, $\alpha(Z\alpha)^2(m/M)E_F$, and some other corrections (see a more detailed analysis in Refs. [6,17]). Calculation of all these contributions and reduction of the theoretical uncertainty of the hyperfine splitting in muonium below 10 Hz is the current task of the theory. As the next step towards this goal we hope to present soon the results of calculation of the single-logarithmic and nonlogarithmic radiative-recoil corrections of order $\alpha^2(Z\alpha)(m/M)\tilde{E}_F$ generated by the graphs containing besides the polarization loops also radiative photons.

Note added in proof

We would like to mention one more radiative-recoil correction, namely, the leading logarithmic correction of order $\alpha^3(Z\alpha)$:

$$\Delta E=-\frac{8}{9}\ln^4\frac{M}{m}\frac{\alpha^3(Z\alpha)}{\pi^4}\frac{m}{M}\tilde{E}_F. \tag{39}$$

This correction is generated by the diagrams with four polarization operator insertions in the exchanged photons similar to the diagrams with the three polarization insertions in Fig.

1. It contains the large logarithm to the fourth power, and like the leading logarithm cubed contribution in Eq. (39) may be easily obtained either by direct calculation or by substitution of the effective charge $\alpha(M)$ in the leading recoil correction of order $(Z\alpha)(m/M)\tilde{E}_F$. We mention this correction together with the radiative-recoil corrections of order $\alpha^2(Z\alpha)$ because due to the presence of the large logarithm it is numerically of the same order as these corrections:

$$\Delta E = -1.668\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\tilde{E}_F. \tag{40}$$

ACKNOWLEDGMENTS

This work was supported by the NSF grant PHY-0049059. Work of V. A. S. was also supported in part by the RFBR grant 00-02-16718.

APPENDIX: AUXILIARY INTEGRALS

All contributions to hyperfine splitting in the main body of this paper are written with the help of the function

$$f(\mu k) \equiv \Phi_0^\mu(k) + \frac{1}{2}\Phi_1^\mu(k), \tag{A1}$$

where the standard auxiliary functions $\Phi_n(k)$ were introduced in Ref. [10]:

$$\Phi_0^\mu(k) = W(\xi_\mu) - \frac{1}{\sqrt{\xi_\mu}}, \tag{A2}$$

$$\Phi_1^\mu(k) = -\xi_\mu W(\xi_\mu) + \frac{1}{2}, \tag{A3}$$

and

$$W(\xi_\mu) = \sqrt{1+\frac{1}{\xi_\mu}} - 1, \quad \xi_\mu = \mu^2 k^2. \tag{A4}$$

In terms of these functions all high-momentum contributions to hyperfine splitting may be represented as linear combinations of the standard integrals

$$V_{lmn} = \int_\sigma^\infty \frac{dk^2}{(k^2)^l}(\ln k)^m \Phi_n^\mu(k), \tag{A5}$$

where $l=1$, $m=0,1,2$ and $n=0,1$. Calculation of these integrals was described in Ref. [10], and we present here only the results for the two integrals which were not calculated in Ref. [10]:

$$V_{120} = \frac{2}{3}\ln^3(2\mu) - 2\ln^2(2\mu) + \left(\frac{\pi^2}{3}+4\right)\ln(2\mu) + \zeta(3) - \frac{\pi^2}{3} - 4 + \frac{2}{3}\ln^3\sigma, \tag{A6}$$

$$V_{121} = -\frac{1}{3}\ln^3(2\mu) - \frac{1}{2}\ln^2(2\mu) - \left(\frac{\pi^2}{6}+\frac{1}{2}\right)\ln(2\mu) - \frac{1}{2}\zeta(3) - \frac{\pi^2}{12} - \frac{1}{4} - \frac{1}{3}\ln^3\sigma. \tag{A7}$$

In Sec. IV we encountered the integral

$$\int_0^\infty dk \ln k\left[(\sqrt{4+k^2}-k) - \frac{k}{8}(k\sqrt{4+k^2}-k^2-2)\right] \times \int_0^1 dv \frac{v^2(1-v^2/3)}{4+k^2(1-v^2)} = \frac{\pi^2}{18} - \frac{209}{432}, \tag{A8}$$

which may be calculated by changing the integration variable

$$z = \frac{2}{k+\sqrt{4+k^2}}. \tag{A9}$$

A number of integrals with the function $R(v)$ [see Eq. (27)] used in Sec. V are collected below:

$$\int_0^1 dv\left[\frac{3}{4}v^2\left(1-\frac{v^2}{3}\right) + R(v)\right] = \frac{82}{81}, \tag{A10}$$

$$\int_0^1 dv R(v) = \frac{329}{405}, \tag{A11}$$

$$\int_0^1 dv \frac{R(v)}{1-v^2} = \zeta(3) + \frac{5}{24}, \tag{A12}$$

$$\int_0^1 dv \frac{R(v)}{1-v^2}\ln(1-v^2) = 2\zeta(3)\ln 2 + \frac{25}{24}\zeta(3) + \frac{16}{3}\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{2\pi^2}{9}\ln^2 2 + \frac{2}{9}\ln^4 2 + \frac{5}{12}\ln 2 - \frac{5\pi^4}{108} - \frac{223}{144}. \tag{A13}$$

[1] M. I. Eides and V. A. Shelyuto, Phys. Lett. **146B**, 241 (1984).
[2] M. I. Eides, S. G. Karshenboim, and V. A. Shelyuto, Phys. Lett. B **216**, 405 (1989); Yad. Fiz. **49**, 493 (1989) [Sov. J. Nucl. Phys. **49**, 309 (1989)].
[3] S. G. Karshenboim, M. I. Eides, and V. A. Shelyuto, Yad. Fiz. **52**, 1066 (1990) [Sov. J. Nucl. Phys. **52**, 679 (1990)].
[4] S. L. Adler, Phys. Rev. **177**, 2426 (1969).
[5] G. Li, M. A. Samuel, and M. I. Eides, Phys. Rev. A **47**, 876 (1993).
[6] M. I. Eides, H. Grotch, and V. A. Shelyuto, Phys. Rep. **342**, 63 (2001).
[7] E. A. Terray and D. R. Yennie, Phys. Rev. Lett. **48**, 1803 (1982).
[8] J. R. Sapirstein, E. A. Terray, and D. R. Yennie, Phys. Rev. D **29**, 2290 (1984).
[9] M. I. Eides, S. G. Karshenboim, and V. A. Shelyuto, Phys. Lett. B **229**, 285 (1989); Pis'ma Zh. Éksp. Teor. Fiz. **50**, 3 (1989) [JETP Lett. **50**, 1 (1989)]; Yad. Fiz. **50**, 1636 (1989) [Sov. J. Nucl. Phys. **50**, 1015 (1989)].
[10] M. I. Eides, H. Grotch, and V. A. Shelyuto, Phys. Rev. D **58**, 013008 (1998).
[11] G. Kallen and A. Sabry, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **29**, 17 (1955).
[12] J. Schwinger, *Particles, Sources and Fields* (Addison-Wesley, Reading, MA, 1973), Vol. 2.
[13] W. E. Caswell and G. P. Lepage, Phys. Rev. Lett. **41**, 1092 (1978).
[14] K. Melnikov and A. Yelkhovsky, Phys. Rev. Lett. **86**, 1498 (2001).
[15] R. J. Hill, Phys. Rev. Lett. **86**, 3280 (2001).
[16] W. Liu *et al.*, Phys. Rev. Lett. **82**, 711 (1999).
[17] P. J. Mohr and B. N. Taylor, Rev. Mod. Phys. **72**, 351 (2000).