Seesaw mechanism in three flavors

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(Received 16 March 2001; published 7 December 2001)

We advance a method used to analyze the neutrino properties (masses and mixing) in the seesaw mechanism. Assuming quark-lepton symmetry and hierarchical light neutrino masses, we establish rather simple relations between the light and the heavy neutrino parameters in the favored regions of the solar and the atmospheric neutrino experiments. An empirical condition satisfied by the right-handed mixing angles is obtained.

DOI: 10.1103/PhysRevD.65.013002 PACS number(s): 14.60.Pq, 12.15.Ff

I. INTRODUCTION

Do neutrinos have a nonzero mass? How large would their mixing angles be? Are they like those in the quark sector? These are among the pressing questions in particle physics. The solar [1] and atmospheric [2] neutrino data suggest that neutrinos do have mass and the recent results from Super-Kamiokande (SK) [2] imply a nearly maximal mixing of ν_μ and ν_τ. On the other hand, the fact that neutrinoless double-β decay and other lepton number nonconserving processes are not observed experimentally reflects the smallness of the neutrino masses [3]. The seesaw mechanism [4] has a natural explanation for the small neutrino masses and may enhance lepton mixing up to maximal [5–7].

According to the seesaw mechanism, at $M \gg m_D$, the Majorana mass matrix m^{eff} of the left-handed (LH) neutrino components is given as [5]

$$m^{\mathrm{eff}} = m_D M^{-1} m_D^T. \tag{1}$$

Here M is the Majorana mass matrix of the right-handed (RH) neutrino components and m_D is the neutrino Dirac mass matrix which could be equal to the mass matrix of the up quarks: $m_D = m^{\mathrm{up}}$ according to some kind of quark-lepton symmetry [5,6,8]. In the basis where M^{-1} is diagonal, $M^{-1} = M_i^{-1}\delta_{ij} \equiv R_i^2 \delta_{ij}$ $(i,j=1,2,3)$, m_D can be written as [8]

$$m_D = U_0 m_D^{\mathrm{diag}} V_0. \tag{2}$$

Here U_0 and V_0 are LH and RH rotations, respectively, and $m_D^{\mathrm{diag}} = \mathrm{diag}\{m_1, m_2, m_3\}$.

In this paper, we explore what can be determined about the masses and mixing of the right-handed neutrinos from the low-energy neutrino data. The paper is organized as follows. In Sec. II, a parametrization is introduced and the seesaw mechanism is expressed in two formulas: one of them involves only the neutrino masses and the other involves only some nondimensional parameters, such as mass ratios and mixing angles. Then the RH neutrino masses and mixing angles are derived. In Sec. III, we obtain rather simple relations between the masses and mixing angles entering the seesaw formula in the favored regions of the solar and atmospheric experiments. The numerical results they infer are given thereafter. We summarize and discuss our main results in Sec. IV.

II. GENERAL FRAMEWORK

A. Parametrization

Since the CP-violating effects in neutrino oscillations should be small [9], we shall therefore ignore them and consider U_0 and V_0 to be real orthogonal matrices. For simplicity, we also set $U_0 \sim I$. That is, the left-handed rotations that diagonalize the charged lepton m_1 and neutrino Dirac mass matrices m_D are the same or nearly the same and so the large lepton mixing results from the seesaw transformation [5]. Under these assumptions, it is convenient to write

$$m_D^{\mathrm{diag}} V_0 M^{-1} V_0^T m_D^{\mathrm{diag}} = U_0^T U (N^{\mathrm{diag}})^2 U^T U_0 \approx U (N^{\mathrm{diag}})^2 U^T, \tag{3}$$

or by inverting it,

$$(m_D^{\mathrm{diag}})^{-1} U (N^{\mathrm{diag}})^2 U^T (m_D^{\mathrm{diag}})^{-1} = V_0 M^{-1} V_0^T, \tag{4}$$

where U is LH rotation induced by $M^{-1/2}$ and $N^{\mathrm{diag}} = \mathrm{diag}\{n_1, n_2, n_3\}$ with $n_i^2 = m_i^{\mathrm{eff}}$ $(i=1,2,3)$, the eigenvalues of m^{eff}.

In analogy with the two-flavors case [10], we introduce the following mass parametrization:

$$\xi_3 = \frac{1}{2}\ln\frac{m_2}{m_1}, \quad \xi_8 = \frac{1}{6}\ln\frac{m_3^2}{m_1 m_2}, \tag{5a}$$

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$$\eta_3 = \frac{1}{2}\ln\frac{R_1}{R_2}, \quad \eta_8 = \frac{1}{6}\ln\frac{R_1 R_2}{R_3^2}, \tag{5b}$$

$$\kappa_3 = \frac{1}{2}\ln\frac{n_2}{n_1}, \quad \kappa_8 = \frac{1}{6}\ln\frac{n_3^2}{n_1 n_2}, \tag{5c}$$

and the mixing matrices are parametrized as usual,

$$U = \exp(i\theta_{23}\lambda_7)\exp(i\theta_{13}\lambda_5)\exp(i\theta_{12}\lambda_2), \tag{6a}$$

$$V_0 = \exp(i\beta_{23}\lambda_7)\exp(i\beta_{13}\lambda_5)\exp(i\beta_{12}\lambda_2). \tag{6b}$$

Here, $\lambda_2, \lambda_5, \lambda_7$ are the Gell-Mann matrix. One can see that η_3 and η_8 describe the hierarchy of the RH neutrino masses and are always non-negative. Especially, $\eta_3 = 0$ implies $M_1 = M_2$ while $\eta_3 = 3\eta_8$ implies $M_2 = M_3$. Using the diagonal Gell-Mann matrix λ_3 and λ_8, the mass matrices involved now can be rewritten as

$$m_D^{\text{diag}} = (m_1 m_2 m_3)^{1/3} e^{-\xi_3\lambda_3 - \sqrt{3}\xi_8\lambda_8}, \tag{7a}$$

$$M^{-1} = (R_1^2 R_2^2 R_3^2)^{1/3} e^{2\eta_3\lambda_3 + 2\sqrt{3}\eta_8\lambda_8}, \tag{7b}$$

$$(N^{\text{diag}})^2 = (n_1^2 n_2^2 n_3^2)^{1/3} e^{-2\kappa_3\lambda_3 - 2\sqrt{3}\kappa_8\lambda_8}. \tag{7c}$$

This parametrization shows clearly that the relevant variables in the diagonalization of M^{-1} are θ_{12}, θ_{13}, θ_{23}, κ_3, κ_8, ξ_3, and ξ_8. Of these, it is usually assumed that ξ_3 and ξ_8 can be identified with the corresponding quantities of the up sector of quarks as stated before, and $\theta_{12}, \theta_{13}, \theta_{23}, \kappa_3, \kappa_8$ can be obtained, at least approximately, from the low-energy neutrino data. Now let us denote

$$\begin{aligned}\overline{X}(\kappa,\xi,\theta) &= (R_1^2 R_2^2 R_3^2)^{1/3} V_0 e^{2\eta_3\lambda_3 + 2\sqrt{3}\eta_8\lambda_8} V_0^T \\ &= (m_1 m_2 m_3)^{-2/3} (n_1^2 n_2^2 n_3^2)^{1/3} X(\kappa,\xi,\theta).\end{aligned} \tag{8}$$

Here

$$X(\kappa,\xi,\theta) = e^{\xi_3\lambda_3 + \sqrt{3}\xi_8\lambda_8} U e^{-2\kappa_3\lambda_3 - 2\sqrt{3}\kappa_8\lambda_8} U^T e^{\xi_3\lambda_3 + \sqrt{3}\xi_8\lambda_8} \tag{9}$$

and κ, ξ, and θ refer to κ_3, κ_8; ξ_3, ξ_8; and $\theta_{12}, \theta_{13}, \theta_{23}$; respectively. Equation (8) is equivalent with the following two equations:

$$R_1^2 R_2^2 R_3^2 = (m_1 m_2 m_3)^{-2} (n_1^2 n_2^2 n_3^2), \tag{10a}$$

$$X(\kappa,\xi,\theta) = V_0 e^{2\eta_3\lambda_3 + 2\sqrt{3}\eta_8\lambda_8} V_0^T. \tag{10b}$$

The first relation is just the equality of the determinations of both sides of Eq. (8). Taking the total term $(R_1^2 R_2^2 R_3^2)^{1/3} = (m_1 m_2 m_3)^{-2/3} (n_1^2 n_2^2 n_3^2)^{1/3}$ out from Eq. (8) we get the second relation. For later use, we present here the expression of the inverse of $X(\kappa,\xi,\theta)$. It is easy to see from Eq. (9) that

$$\begin{aligned}X^{-1}(\kappa,\xi,\theta) &= e^{-\xi_3\lambda_3 - \sqrt{3}\xi_8\lambda_8} U \\ &\quad \times e^{2\kappa_3\lambda_3 + 2\sqrt{3}\kappa_8\lambda_8} U^T e^{-\xi_3\lambda_3 - \sqrt{3}\xi_8\lambda_8},\end{aligned} \tag{11}$$

so that

$$X^{-1}(\kappa,\xi,\theta) = X(-\kappa,-\xi,\theta) \equiv Y(\kappa,\xi,\theta) \tag{12}$$

and we have

$$Y(\kappa,\xi,\theta) = V_0 e^{-2\eta_3\lambda_3 - 2\sqrt{3}\eta_8\lambda_8} V_0^T. \tag{13}$$

We will start from Eqs. (10b) and (13) to derive the expressions of η_3, η_8, and V_0. Then from Eq. (10a), $M_i (i = 1,2,3)$ can be obtained. In the following discussion, we shall omit the variables κ, ξ, θ in X and Y.

B. Determination of the Majorana masses

In this subsection we deduce two equations about the hierarchy, η_3 and η_8, of the RH neutrino masses. Taking the trace of both sides of Eq. (10b) we obtain

$$\mathrm{Tr}(V_0 e^{2\eta_3\lambda_3 + 2\sqrt{3}\eta_8\lambda_8} V_0^T) = \mathrm{Tr} e^{2\eta_3\lambda_3 + 2\sqrt{3}\eta_8\lambda_8} = \mathrm{Tr}\, X, \tag{14}$$

that is,

$$e^{2\eta_3 + 2\eta_8} + e^{-2\eta_3 + 2\eta_8} + e^{-4\eta_8} = X_{11} + X_{22} + X_{33} \equiv A. \tag{15}$$

Similarly, taking the trace of both sides of Eq. (13) we get

$$e^{-2\eta_3 - 2\eta_8} + e^{2\eta_3 - 2\eta_8} + e^{4\eta_8} = Y_{11} + Y_{22} + Y_{33} \equiv B. \tag{16}$$

It is sufficient for solving η_3 and η_8 from Eqs. (15) and (16) since X_{ii} and Y_{ii} $(i = 1,2,3)$ are known. Once η_3 and η_8 are solved, inserting $M_1 = M_3 e^{-2\eta_3 - 6\eta_8}$, $M_2 = M_3 e^{2\eta_3 - 6\eta_8}$, $n_1^2 = n_3^2 e^{-2\kappa_3 - 6\kappa_8}$, and $n_2^2 = n_3^2 e^{2\kappa_3 - 6\kappa_8}$ in Eq. (10a), we obtain the following expressions of the RH neutrino masses,

$$M_1 = F e^{-2\eta_8 - 2\eta_3}, \quad M_2 = F e^{-2\eta_8 + 2\eta_3}, \quad M_3 = F e^{4\eta_8}. \tag{17}$$

Here $F = (m_t^2 / m_3^{\text{eff}}) e^{4\kappa_8 - 4\xi_8}$ and we have identified $m_i (i = 1,2,3)$ with the masses of up quarks.

All the above results are exact but formal. We need to decouple η_3 and η_8 in Eqs. (15) and (16). From Eq. (15), we have

$$\begin{aligned}A &= e^{2\eta_3 + 2\eta_8} + e^{-2\eta_3 + 2\eta_8} + e^{-4\eta_8} \\ &\geqslant 3(e^{2\eta_3 + 2\eta_8} e^{-2\eta_3 + 2\eta_8} e^{-4\eta_8})^{1/3} = 3.\end{aligned} \tag{18}$$

The equality is satisfied when $\eta_3 = \eta_8 = 0$, that is, when $M_1 = M_2 = M_3$. At $A \gg 3$ (then $B \gg 3$ is also true), Eq. (15) and Eq. (16) can be approximated as follows:

$$e^{2\eta_3 + 2\eta_8} + e^{-2\eta_3 + 2\eta_8} \approx A, \tag{19a}$$

$$e^{2\eta_3 - 2\eta_8} + e^{4\eta_8} \approx B. \tag{19b}$$

Such a case corresponds to at most two degenerate Majorana masses. There are now two possibilities to simplify the above two equations further.

(a) $A>B$. It is easy to know from Eqs. (19a) and (19b) that $A>B$ implies $\eta_3>\eta_8$. So $e^{-2\eta_3+2\eta_8}(<1)$ may be omitted in Eq. (19a),

$$e^{2\eta_3+2\eta_8}=e^{2\eta_3-2\eta_8}e^{4\eta_8}\approx A. \tag{20}$$

Then it is easy to see from Eqs. (19b) and (20) that $e^{2\eta_3-2\eta_8}$ and $e^{4\eta_8}$ are roots of the following quadratic equation:

$$x^2-Bx+A=0 \tag{21}$$

and the three eigenvalues of X are

$$e^{2\eta_3+2\eta_8}\approx A, \tag{22a}$$

$$e^{-2\eta_3+2\eta_8}\approx\frac{2}{B-\sqrt{B^2-4A}}, \tag{22b}$$

$$e^{-4\eta_8}\approx\frac{2}{B+\sqrt{B^2-4A}}. \tag{22c}$$

(b) $A<B$. In this case, we have $\eta_3<\eta_8$. Omitting the term $e^{2\eta_3-2\eta_8}(<1)$ in Eq. (19b), we have

$$e^{4\eta_8}=e^{2\eta_3+2\eta_8}e^{-2\eta_3+2\eta_8}\approx B. \tag{23}$$

Now $e^{2\eta_3+2\eta_8}$ and $e^{-2\eta_3+2\eta_8}$ are roots of the following quadratic equation:

$$x^2-Ax+B=0. \tag{24}$$

Thus one has

$$e^{2\eta_3+2\eta_8}\approx\frac{A+\sqrt{A^2-4B}}{2}, \tag{25a}$$

$$e^{-2\eta_3+2\eta_8}\approx\frac{A-\sqrt{A^2-4B}}{2}, \tag{25b}$$

$$e^{-4\eta_8}\approx\frac{1}{B}. \tag{25c}$$

From Eq. (22) we know that $e^{-2\eta_3+2\eta_8}\sim e^{-4\eta_8}$ (and so $M_2\sim M_3$) when $B^2\sim 4A$ and from Eq. (25) $e^{2\eta_3+2\eta_8}\sim e^{-2\eta_3+2\eta_8}$ (and so $M_1\sim M_2$) when $A^2\sim 4B$. Far beyond these regions, both Eq. (22) and Eq. (25) give the same asymptotic solution: $e^{2\eta_3+2\eta_8}\approx A$, and $e^{-4\eta_8}\approx 1/B$ and $e^{-2\eta_3+2\eta_8}=e^{-2\eta_3-2\eta_8}e^{4\eta_8}\approx B/A$. The solutions are also useful for rough estimation of the Majorana masses even when two of them are degenerate, which can be seen from $e^{2\eta_3+2\eta_8}<e^{2\eta_3+2\eta_8}+e^{-2\eta_3+2\eta_8}<2e^{2\eta_3+2\eta_8}$ and $e^{4\eta_8}<e^{2\eta_3-2\eta_8}+e^{4\eta_8}<2e^{4\eta_8}$. The maximal deviations for $e^{2\eta_3+2\eta_8}$ and $e^{4\eta_8}$ are both 2.

Usually one should have to solve a cubic characteristic equation to obtain the eigenvalues. In the seesaw model, however, one usually encounters a case in which $e^{2\eta_3+2\eta_8}\gg 1$ and $e^{-4\eta_8}\ll 1$ simultaneously. This is a practical difficulty even in numerical calculation. In addition, the solution of a cubic equation is too obscure to see any relation between various physical quantities. By taking the trace of X and its inverse, we decompose the eigenequation in two equations and each contains the main term of $e^{2\eta_3+2\eta_8}$ and $e^{4\eta_8}$, respectively. In concrete calculation, the expressions of A and B can be simplified to such a great extent that their dependence on the parameters can be exhibited explicitly. We will discuss this issue later.

C. Determination of the RH angles

Once one has solved the three eigenvalues, then the three eigenvectors (and then the three rotation angles) of M^{-1} can be found by the standard procedure of linear algebra. The eigenequation of X is

$$(X-Q_iI)\begin{pmatrix} V_{1i} \\ V_{2i} \\ V_{3i} \end{pmatrix}=0 \quad (i=1,2,3), \tag{26}$$

where $V_{ij}=(V_0)_{ij}$ and we use $Q_i(i=1,2,3)$ satisfying $Q_1>Q_2>Q_3$ to denote the three eigenvalues of X. The eigenvectors, a solution of Eq. (26), can be expressed in

$$V_{21}=\frac{(X_{12}X_{33}-X_{13}X_{23})-Q_1X_{12}}{(X_{23}^2-X_{33}X_{22})+(X_{33}+X_{22})Q_1-Q_1^2}V_{11}, \tag{27a}$$

$$V_{31}=\frac{(X_{13}X_{22}-X_{12}X_{23})-Q_1X_{13}}{(X_{23}^2-X_{33}X_{22})+(X_{22}+X_{33})Q_1-Q_1^2}V_{11}, \tag{27b}$$

etc. We also know that

$$X^{-1}=\frac{1}{\det X}\,\mathrm{adjoint}\,X. \tag{28}$$

Notice $\det X=1$; the inverse of X is just its adjoint matrix. So

$$\begin{aligned} &Y_{11}=X_{22}X_{33}-X_{23}^2, \quad Y_{22}=X_{11}X_{33}-X_{13}^2, \\ &Y_{33}=X_{11}X_{22}-X_{12}^2, \\ &Y_{12}=X_{13}X_{23}-X_{12}X_{33}, \quad Y_{13}=X_{12}X_{23}-X_{13}X_{22}, \\ &Y_{23}=X_{12}X_{13}-X_{11}X_{23}, \end{aligned} \tag{29}$$

and $Y_{ij}=Y_{ji}$. The quadratic terms in Eq. (27) are just the elements of Y. By replacing them with Y_{ij} $(i,j=1,2,3)$, we have

$$V_{21}=\frac{Y_{12}+Q_1X_{12}}{(Y_{11}+Q_1X_{11})-(Q_2^{-1}+Q_3^{-1})}V_{11}, \tag{30a}$$

$$V_{31}=\frac{Y_{13}+Q_1X_{13}}{(Y_{11}+Q_1X_{11})-(Q_2^{-1}+Q_3^{-1})}V_{11}. \tag{30b}$$

TABLE I. $\nu_e \rightarrow \nu_\mu$ solutions to the solar neutrino problem. Here MSW and VO refer to Mikheyev-Smirnov-Wolfenstein matter-enhanced oscillations [12] and vacuum oscillations (the so-called just-so oscillations), respectively. LMA (SMA) stands for a large (small) mixing angle and LOW stands for low probability (or low mass).

Solution	$\Delta m^2_{\rm solar}(\mathrm{eV}^2)$	$\sin^2 2\theta_{12}$
VO	6.5×10^{-11}	0.75
MSW (LMA)	1.8×10^{-5}	0.76
MSW (LOW)	7.9×10^{-8}	0.96
MSW (SMA)	5.4×10^{-6}	6.0×10^{-3}

Here we have used $\mathrm{Tr}\,X=X_{11}+X_{22}+X_{33}=Q_1+Q_2+Q_3$ and $\det X=Q_1Q_2Q_3=1$. Thus all the nondiagonal elements of V_0 can be expressed in a unified form:

$$V_{ij}=\frac{Y_{ij}+Q_jX_{ij}}{(Y_{jj}+Q_jX_{jj})-\hat{Q}_j^{-1}}V_{jj} \quad (i,j=1,2,3 \ \text{ and } \ i\neq j). \tag{31}$$

Here $\hat{Q}_j^{-1}=\mathrm{Tr}\,Y-Q_j^{-1}$. Considering the normalization condition (or unitarity of V_0) $V_0V_0^T=V_0^TV_0=I$, all the elements can be obtained from Eq. (31). Then the deduction of the three RH angles is direct: $\tan\beta_{23}=V_{23}/V_{33}$, $\cos\beta_{13}\sin\beta_{12}=V_{12}$, and $\sin\beta_{13}=V_{13}$.

All the relations obtained, including the masses and the angles, can be easily transformed to express the light neutrino parameters in M^{-1}, m_D, and V_0. The approach is just to make the exchange $\kappa\leftrightarrow-\eta$, $\xi\leftrightarrow-\xi$, and $\theta_{ij}\leftrightarrow\beta_{ij}$ $(1\leqslant i<j\leqslant3)$.

III. NEUTRINO MASSES AND MIXINGS

The deficit of muon neutrinos observed by the super-Kamiokande collaboration and the zenith angle distributions of the data can be explained by oscillation between ν_μ and ν_τ with the best-fit parameters at [2]

$$(\sin^2 2\theta_{23},\Delta m^2_{\rm atm})=(0.95,5.9\times10^{-3}\ \ \mathrm{eV}^2). \tag{32}$$

The $\nu_e-\nu_\mu$ explanation to the solar neutrino problem requires one set of parameters (the best-fit values) in Table I corresponding to the VO and MSW (including LMA, LOW, and SMA), respectively [11]. Here MSW and VO refer to Mikheyev-Smirnov-Wolfenstein matter-enhanced oscillations [12] and vacuum oscillations (so-called just-so oscillation), respectively. LMA (SMA) stands for a large (small) mixing angle and LOW stands for low probability (or low mass). We assume the effective neutrino masses have a hierarchical pattern, that is, $m_1^{\rm eff}\ll m_2^{\rm eff}\ll m_3^{\rm eff}$. So $n_3^2=m_3^{\rm eff}\approx\sqrt{\Delta m^2_{\rm atm}}$ and $n_2^2=m_2^{\rm eff}\approx\sqrt{\Delta m^2_{\rm solar}}$. Little is known about the value of $m_1^{\rm eff}$, which we denote using the parameter $r=m_2^{\rm eff}/m_1^{\rm eff}\gg1$. In the framework of three-flavor neutrino oscillations, the big hierarchy between $\Delta m^2_{\rm atm}$ and $\Delta m^2_{\rm solar}$ together with no observation of the $\bar{\nu}_e\rightarrow\bar{\nu}_e$ oscillation in the CHOOZ experiment implies that the ν_3 component in ν_e is rather small (even negligible) and the upper limit on the value of θ_{13} is [13]

$$\sin^2\theta_{13}\equiv|U_{e3}|^2\leqslant0.015-0.05. \tag{33}$$

We shall therefore set $\theta_{13}=0$. The Dirac masses of neutrinos are taken at the scale $\mu=10^9$ GeV [14]:

$$\begin{aligned} m_D^{\rm diag}(\mu)&=\mathrm{diag}\{m_u(\mu),m_c(\mu),m_t(\mu)\}\\ &=\mathrm{diag}\{1.47\ \ \mathrm{MeV},427\ \ \mathrm{MeV},149\ \ \mathrm{GeV}\}. \end{aligned} \tag{34}$$

These are all the values entering A and B.

IV. ANALYSIS AND RESULT

In this section we start from Eqs. (15) and (16) to get the RH mass hierarchies, η_3 and η_8. Then using Eq. (31), the elements (and then the mixing angles) of the RH mixing matrix would be obtained. The Majorana masses can be obtained from Eq. (17).

Although we have decoupled the Majorana masses and the RH mixing, the expressions of these parameters would be so complicated due to the complicated structure of X that it is hard to see explicitly the relations of various physical parameters. The hierarchical properties of the Dirac and the effective masses of neutrinos make it possible to drop the smaller terms in A and B. In the following, only the leading-order terms of X_{ij} (Y_{ij}) and $A(B)$ will be reserved, respectively.

Instead of calculating the RH Majorana parameters by inserting the values of these parameters, we give a more general analysis in two cases according to whether θ_{12} is large (VO, LMA, and LOW) or small (SMA) and derive the corresponding relations between the masses and mixing of the RH neutrino and the other neutrino parameters.

A. Case I: Large θ_{12}

1. Mass

In this case, all the elements of U have the same order except that $U_{e3}=0$. Reserving the leading-order terms in A and B, we find

$$\begin{aligned} A\approx{}&U_{e2}^2\exp(2\xi_3+2\xi_8+2\kappa_3-2\kappa_8)\\ &+U_{\mu3}^2\exp(4\kappa_8-2\xi_3+2\xi_8), \end{aligned} \tag{35a}$$

$$B\approx U_{\tau1}^2\exp(2\kappa_3+2\kappa_8+4\xi_8). \tag{35b}$$

It is easy to see that both A and B are much larger than 3. Note that when $\Delta m^2_{\rm atm}/\Delta m^2_{\rm solar}\leqslant10^8$, we also have $A<B$. Then from Eq. (25) one has

$$Q_1=e^{2\eta_3+2\eta_8}\approx U_{e2}^2\exp(2\kappa_3-2\kappa_8+2\xi_3+2\xi_8), \tag{36a}$$

$$Q_2=e^{-2\eta_3+2\eta_8}\approx U_{\mu3}^2\exp(4\kappa_8-2\xi_3+2\xi_8), \tag{36b}$$

$$Q_3 = e^{-4\eta_8} \approx \frac{1}{U_{\tau 1}^2} \exp(-2\kappa_3 - 2\kappa_8 - 4\xi_8). \tag{36c}$$

Here we have used the relation $U_{e2}^2 U_{\mu 3}^2 = U_{\tau 1}^2$, which is satisfied when $\theta_{13} = 0$. We should point out that our results would be correct as long as θ_{13} is small enough. Substituting the eigenvalues in Eq. (17), we have

$$M_1 \approx \frac{1}{\sin^2 \theta_{12}} \frac{m_u^2}{m_2^{\mathrm{eff}}}, \quad M_2 \approx \frac{1}{\sin^2 \theta_{23}} \frac{m_c^2}{m_3^{\mathrm{eff}}},$$

$$M_3 \approx \sin^2 \theta_{23} \sin^2 \theta_{12} \frac{m_t^2}{m_1^{\mathrm{eff}}}. \tag{37}$$

The formulas are the same as those given in Ref. [8]. M_1 and M_2 scale as $1/m_2^{\mathrm{eff}}$ and $1/m_3^{\mathrm{eff}}$, respectively, while M_3 scales as $1/m_1^{\mathrm{eff}}$, which gives scales for the two lighter masses, M_1 and M_2, lower and the heaviest one, M_3, higher than one would expect when no mixing occurs.

2. Angles

Reserving the leading-order terms of the numerators and denominators in Eq. (31), respectively, we obtain

$$V_{21} \approx \frac{U_{\mu 2}}{U_{e2}} e^{-2\xi_3} V_{11}, \quad V_{31} \approx \frac{U_{\tau 2}}{U_{e2}} e^{-\xi_3 - 3\xi_8} V_{11}, \tag{38a}$$

$$V_{12} \approx -\frac{U_{\mu 2}}{U_{e2}} e^{-2\xi_3} V_{22}, \quad V_{32} \approx -\frac{U_{\mu 1}}{U_{\tau 1}} e^{\xi_3 - 3\xi_8} V_{22}, \tag{38b}$$

$$V_{13} \approx \frac{U_{e1}}{U_{\tau 1}} e^{-\xi_3 - 3\xi_8} V_{33}, \quad V_{23} \approx \frac{U_{\mu 1}}{U_{\tau 1}} e^{\xi_3 - 3\xi_8} V_{33}. \tag{38c}$$

Exploiting the unitarity of V_0, it is appropriate to set $V_{ii} \approx 1$. Then the three RH angles are

$$\beta_{12} \approx V_{12} \approx -\frac{m_u}{m_c} \cos\theta_{23} \cot\theta_{12}, \tag{39a}$$

$$\beta_{13} \approx V_{13} \approx \frac{m_u}{m_t} \frac{\cot\theta_{12}}{\sin\theta_{23}}, \tag{39b}$$

$$\beta_{23} \approx V_{23} \approx -\frac{m_c}{m_t} \cot\theta_{23}. \tag{39c}$$

All of the RH angles are small and independent of the effective neutrino masses. Note that, unlike like the LH quark mixing where $\tan\theta \approx \sqrt{m_D/m_s}$ in the two-generation case [15], the RH mixing angles scale linearly with the ratios of the Dirac neutrino masses.

3. Numerical results

a. VO. Inserting the parameters in Eq. (37), we have

$$M_1 \approx 8.0 \times 10^8 \ \mathrm{GeV}, \quad M_2 \approx 4.6 \times 10^9 \ \mathrm{GeV},$$

$$M_3/r \approx 1.5 \times 10^{17} \ \mathrm{GeV}. \tag{40}$$

The mixing angles are easy to obtain from Eq. (39),

$$\beta_{12} \approx -4.6 \times 10^{-3}, \quad \beta_{13} \approx 3.2 \times 10^{-5},$$

$$\beta_{23} \approx -4.3 \times 10^{-3}. \tag{41}$$

b. LMA. In this case we have nearly the same RH angles as in VO and we find

$$M_1 \approx 1.5 \times 10^6 \ \mathrm{GeV}, \quad M_2 \approx 4.6 \times 10^9 \ \mathrm{GeV},$$

$$M_3/r \approx 2.8 \times 10^{14} \ \mathrm{GeV}. \tag{42}$$

c. LOW. We now have

$$M_1 \approx 1.5 \times 10^7 \ \mathrm{GeV}, \quad M_2 \approx 4.6 \times 10^9 \ \mathrm{GeV},$$

$$M_3/r \approx 6.6 \times 10^{15} \ \mathrm{GeV} \tag{43}$$

and

$$\beta_{12} \approx -3.3 \times 10^{-3}, \quad \beta_{13} \approx 2.3 \times 10^{-5},$$

$$\beta_{23} \approx -4.3 \times 10^{-3}. \tag{44}$$

B. Case II: Small $\boldsymbol{\theta_{12}}$ (SMA)

In this case, $U_{e3} = 0$, and U_{e2}, $U_{\mu 1}$, and $U_{\tau 1}$ have the same order 10^{-2} while the other elements of U are of order 1. We have

$$A \approx U_{e1}^2 \exp(-2\kappa_3 - 2\kappa_8 + 2\xi_3 + 2\xi_8) + U_{e2}^2 \exp(2\kappa_3 - 2\kappa_8 + 2\xi_3 + 2\xi_8) \approx X_{11}, \tag{45a}$$

$$B \approx U_{\tau 2}^2 \exp(-2\kappa_3 + 2\kappa_8 + 4\xi_8) + U_{\tau 1}^2 \exp(2\kappa_3 + 2\kappa_8 + 4\xi_8) \approx Y_{33}. \tag{45b}$$

Again, they satisfy $B > A \gg 3$ and $A^2 \gg 4B$, so that

$$Q_1 \approx A \approx f^{-1} U_{e2}^2 \exp(2\kappa_3 - 2\kappa_8 + 2\xi_3 + 2\xi_8), \tag{46a}$$

$$Q_2 \approx \frac{B}{A} \approx U_{\mu 3}^2 \exp(e^{4\kappa_3} + 4\kappa_8 - 2\xi_3 + 2\xi_8), \tag{46b}$$

$$Q_3 \approx \frac{1}{B} \approx f U_{\tau 1}^2 \exp(-2\kappa_3 - 2\kappa_8 - 4\xi_8). \tag{46c}$$

Here $f = [r/(r + \cot^2\theta_{12})]$ and it cannot be omitted since $\cot\theta_{12} \gg 1$. As with case I, we have

TABLE II. Exact numerical and approximate results when $r=10^1$. In each cell we listed the numerical and approximate results above and below, respectively. By solving the eigenequation of X we obtain the eigenvalue(s) that are larger than 1 and the corresponding eigenvector(s). The reciprocal value(s) of the other eigenvalue(s) and the corresponding eigenvector(s) are obtained by solving the eigenequation of Y. Substituting these eigenvalues in Eq. (24) we get the three masses in the Majorana sector (see the text for details).

$r=10^1$	M_1(GeV)	M_2(GeV)	M_3(GeV)	β_{12}	β_{13}	β_{23}
VO	6.2×10^8	4.6×10^9	1.9×10^{18}	-3.7×10^{-3}	2.2×10^{-5}	-4.3×10^{-3}
	8.0×10^8	4.6×10^9	1.5×10^{18}	-4.6×10^{-3}	3.2×10^{-5}	-4.3×10^{-3}
LMA	1.2×10^6	4.5×10^9	3.7×10^{15}	-3.2×10^{-3}	2.2×10^{-5}	-4.1×10^{-3}
	1.5×10^6	4.6×10^9	2.8×10^{15}	-4.6×10^{-3}	3.2×10^{-5}	$-4.3\times10-3$
LOW	1.3×10^7	4.6×10^9	7.6×10^{16}	-2.6×10^{-3}	1.8×10^{-5}	-4.3×10^{-3}
	1.5×10^7	4.6×10^9	6.6×10^{16}	-3.3×10^{-3}	2.3×10^{-5}	-4.3×10^{-3}
SMA	7.0×10^6	4.4×10^9	2.1×10^{15}	-9.3×10^{-4}	6.2×10^{-6}	-4.0×10^{-3}
	7.0×10^6	4.6×10^9	2.0×10^{15}	-1.0×10^{-3}	7.2×10^{-6}	-4.3×10^{-3}

$$M_1\approx f\frac{1}{\sin^2\theta_{12}}\frac{m_u^2}{m_2^{\rm eff}},\quad M_2\approx\frac{1}{\sin^2\theta_{23}}\frac{m_c^2}{m_3^{\rm eff}},$$

$$M_3\approx f^{-1}\sin^2\theta_{23}\sin^2\theta_{12}\frac{m_t^2}{m_1^{\rm eff}}. \tag{47}$$

For the mixing angles, we obtain

$$\beta_{12}\approx V_{12}\approx -f\frac{m_u}{m_c}\cos\theta_{23}\cot\theta_{12}, \tag{48a}$$

$$\beta_{13}\approx V_{13}\approx f\frac{m_u}{m_t}\frac{\cot\theta_{12}}{\sin\theta_{23}}, \tag{48b}$$

$$\beta_{23}\approx V_{23}\approx -\frac{m_c}{m_t}\cot\theta_{23}. \tag{48c}$$

Again the factor f appears. Note that the expressions of M_2 and β_{23} are the same as that when θ_{12} is large. Moreover, the SK data suggest strongly that $\theta_{23}\approx\pi/4$. So both M_2 and β_{23} have the same values in all the favored regions considered. It is noteworthy that the factor f makes the value of M_3 remain at a relatively low scale for a wide range of r, which is different from that in Ref. [8]. When $r\gg\cot^2\theta_{12}$ (then $f\approx1$), we have the same expressions of the RH masses and the mixing angles regardless of whether θ_{12} is large or not.

Substituting the values of the parameters from Eq. (47) we have

$$M_1\approx4.7\times10^8 f\ \text{GeV},\quad M_2\approx4.6\times10^9\ \text{GeV},$$

$$M_3\approx3.0\times10^{12}\frac{r}{f}\ \text{GeV}. \tag{49}$$

and from Eq. (48a),

$$\beta_{12}\approx-7.0\times10^{-2}f,\quad \beta_{13}\approx4.9\times10^{-4}f,$$

$$\beta_{23}\approx-4.3\times10^{-3}. \tag{50}$$

Here, with the value of θ_{12} substituted, $f\approx[r/(r+6.6\times10^2)]$.

Comparisons with the exact numerical results are given in Tables II–IV, from which we can see that they fit well. In calculation we take $m_D^{\rm diag}(\mu)$ at $\mu=10^9$ GeV. Note that, although the up-quark masses are running with μ, the Dirac mass hierarchies η_3 and η_8 are almost fixed when μ varies. We find that they satisfy the following approximate relation:

$$\frac{m_u(\mu)m_t(\mu)}{m_c^2(\mu)}\approx1. \tag{51}$$

So the deviation mainly results from $F[=(m_t^2/m_3^{\rm eff})e^{4\kappa_8-4\xi_8}]$ when $m_D^{\rm diag}(\mu)$ is taken at a different scale.

TABLE III. Same as in Table I but for $r=10^2$.

$r=10^2$	M_1(GeV)	M_2(GeV)	M_3(GeV)	β_{12}	β_{13}	β_{23}
VO	7.7×10^8	4.6×10^9	1.5×10^{19}	-5.3×10^{-3}	3.1×10^{-5}	-4.3×10^{-3}
	8.0×10^8	4.6×10^9	1.5×10^{19}	-4.6×10^{-3}	3.2×10^{-5}	-4.3×10^{-3}
LMA	1.5×10^6	4.6×10^9	2.9×10^{16}	-4.4×10^{-3}	3.1×10^{-5}	-4.3×10^{-3}
	1.5×10^6	4.6×10^9	2.8×10^{16}	-4.6×10^{-3}	3.2×10^{-5}	-4.3×10^{-3}
LOW	1.4×10^7	4.6×10^9	6.7×10^{17}	-3.2×10^{-3}	2.3×10^{-5}	-4.3×10^{-3}
	1.5×10^7	4.6×10^9	6.6×10^{17}	-3.3×10^{-3}	2.3×10^{-5}	-4.3×10^{-3}
SMA	6.1×10^7	4.4×10^9	2.4×10^{15}	-9.1×10^{-3}	6.0×10^{-5}	-4.0×10^{-3}
	6.1×10^7	4.6×10^9	2.3×10^{15}	-9.1×10^{-3}	6.4×10^{-5}	-4.3×10^{-3}

TABLE IV. Same as in Table I but for $r=10^3$.

$r=10^3$	M_1(GeV)	M_2(GeV)	M_3(GeV)	β_{12}	β_{13}	β_{23}
VO	7.9×10^8	4.6×10^9	1.5×10^{20}	-5.5×10^{-3}	3.2×10^{-5}	-4.3×10^{-3}
	8.0×10^8	4.6×10^9	1.5×10^{20}	-4.6×10^{-3}	3.2×10^{-5}	-4.3×10^{-3}
LMA	1.5×10^6	4.5×10^9	2.8×10^{17}	-4.6×10^{-3}	3.2×10^{-5}	-4.3×10^{-3}
	1.5×10^6	4.6×10^9	2.8×10^{17}	-4.6×10^{-3}	3.2×10^{-5}	-4.3×10^{-3}
LOW	1.5×10^7	4.6×10^9	6.6×10^{18}	-3.3×10^{-3}	2.3×10^{-5}	-4.3×10^{-3}
	1.5×10^7	4.6×10^9	6.6×10^{18}	-3.3×10^{-3}	2.3×10^{-5}	-4.3×10^{-3}
SMA	2.8×10^8	4.5×10^9	5.1×10^{15}	-4.4×10^{-2}	2.9×10^{-4}	-4.1×10^{-3}
	2.8×10^8	4.6×10^9	5.0×10^{15}	-4.2×10^{-2}	2.9×10^{-4}	-4.3×10^{-3}

V. SUMMARY AND DISCUSSION

In this paper, we introduce a parametrization which transforms all the involving masses in the seesaw formula to the mass ratios. Then by taking the traces of X and its inverse, we derive the equations of the Majorana mass ratios, η_3 and η_8. The solutions to these equations are obtained under some conditions and the elements of V_0 are expressed in a unified form. Assuming quark-lepton symmetry and hierarchical effective neutrino masses, rather simple relations among the various neutrino parameters entering the seesaw formula are deduced. Finally, setting the Dirac neutrino masses to be equal to the up-quark masses, we present the numerical results in the favored regions of the solar and atmospheric neutrino experiments.

Now let us give a combined analysis of the results obtained and list our main points as follows: M_2 ($\approx 4.6\times10^9$ GeV) and so the product of M_1 and M_3 is nearly independent of θ_{12}; the three RH neutrino masses are hierarchical and $M_3/M_2(\propto m_3^{\rm eff}/m_1^{\rm eff})\gg M_2/M_1(\propto m_1^{\rm eff}/m_2^{\rm eff})$; β_{23} ($\approx -4.3\times10^{-3}$) and $\beta_{12}/\beta_{13}\approx -\frac{1}{2}(m_t/m_c)\sin 2\theta_{23}\approx -\frac{1}{2}(m_t/m_c)$ are also independent of θ_{12}. Moreover, the RH mixing angles satisfy the following condition:

$$\frac{\beta_{12}\beta_{23}}{\beta_{13}}\approx \cos^2\theta_{23}\approx \frac{1}{2}, \tag{52}$$

which is independent of not only θ_{12} and the effective neutrino masses but also the Dirac masses of neutrinos. It is interesting to notice that the (13) elements (U_{e3}, V_{13}, and U_{us}) determined by the third mixing angles of the three corresponding mixing matrices are all small. It is also noteworthy that the third mixing angles in both the Cabibbo-Kobayashi-Maskawa (CKM) matrix of quarks and the RH mixing matrix are of orders of the products of the other two angles, respectively. In the former, we have $|U_{us}U_{ub}/U_{cb}|\approx(\rho^2+\eta^2)^{-1/2}$. Here, ρ and η are smaller than 1 [16].

Numerically, the lightest right-handed neutrino mass can lie between 10^6 GeV and 10^8 GeV while the heaviest right-handed neutrino mass ranges from about 10^{12} GeV to far larger than 10^{17} GeV.

Numerically, all the three RH angles are small although they may contain the contribution from the diagonalization of M^{-1}. The absolute values of β_{12} and β_{13} are about $10^{-3}\sim10^{-2}$ and $10^{-6}\sim10^{-4}$, respectively.

The SMA solution seems especially attractive in the sense that $M_3\sim10^{15}$ GeV for a wide range of r due to the factor f while M_3's for the other three regions (VO, LMA, and LOW) increase rapidly with r and become too large to be viable. Especially, for the VO solution to the solar neutrino problem, both the two mass squared difference splittings (of the order 10^{-3} eV^2 and 10^{-11} eV^2, respectively) and the scale of the heaviest RH neutrino mass M_3 ($\gg10^{17}$ GeV) make it look very unnatural [17].

In this work, we have set $\theta_{13}=0$. Although the small θ_{13} has little effect on the oscillation solution to the solar and the atmospheric neutrino deficits, it may become important in the seesaw mechanism, especially in the SMA region where θ_{13} is comparable with θ_{12}. It may lead to large RH mixing angles owing to the contribution from the diagonalization of M^{-1} as well as degenerate masses. This can also be seen from the fact that the coefficients of U_{e3} in A are much larger than that of the other elements of U. We point out that the method is even valid in such cases in which more skills may be needed. We will discuss this in more detail in a later paper.

ACKNOWLEDGMENTS

The authors would like to express our sincere thanks to Professor T. K. Kuo for pointing out this problem to G. Cheng during his visit at Purdue University and provoking our interesting in it. We are also indebted to him for his warm help in the progress of this research and for his kindness by giving us his papers before publication. We are grateful to Dr. Du Taijiao and Dr. Tu Tao for useful discussions. The authors are also grateful for the help provided by the Chinese High Performance Computing Center at Hefei. The authors are supported in part by the National Science Foundation in China under Grant No. 19875047.

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