$u_\star(1,1)$ noncommutative gauge theory as the foundation of two-time physics in field theory

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A very simple field theory in noncommutative phase space (X^M,P^M) in $d+2$ dimensions, with a gauge symmetry based on noncommutative $u_\star(1,1)$, furnishes the foundation for the field theoretic formulation of two-time physics. This leads to a remarkable unification of several gauge principles in d dimensions, including Maxwell, Einstein and high spin gauge principles, packaged together into one of the simplest fundamental gauge symmetries in noncommutative quantum phase space in $d+2$ dimensions. A gauge invariant action is constructed and its nonlinear equations of motion are analyzed. In addition to elegantly reproducing the first quantized worldline theory with all background fields, the field theory prescribes unique interactions among the gauge fields. A matrix version of the theory, with a large N limit, is also outlined.

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I. INTRODUCTION

Two-time physics (2T physics) [1–9] is a device that makes manifest many hidden features of one-time physics (1T physics). Until recently, most of the understanding in 2T physics was gained from studying the worldline formalism. This revealed a $d+2$ dimensional holographic origin of certain aspects of 1T physics in d dimensions, including, in particular, higher dimensional hidden symmetries (conformal and others) and new sets of duality-type relations among 1T dynamical systems. While the physical phenomena described by 1T or 2T physics are the same, the space-time point of view is different. The 2T-physics approach in $d+2$ dimensions offers a highly symmetric and unified version of the phenomena described by 1T physics in d dimensions. As such, it raises deep questions about the meaning of space-time.

A noncommutative field theory in phase space introduced recently [9] confirmed the worldline as well as the configuration space field theory [7] results of 2T physics, and suggested more far reaching insights. In this paper the approach of [9] will be taken one step further by showing that it originates from a fundamental gauge symmetry principle based on noncommutative $u_\star(1,1)$. We will see that this phase space symmetry concisely unifies many gauge principles that are traditionally formulated in configuration space separately from each other, including the Maxwell, Einstein and high spin gauge principles.

All new phenomena in 2T physics in the worldline formulation can be traced to the presence of an essential gauge symmetry: sp$(2,R)$ acting on phase space (X^M,P_M) [2]. The 2T feature of space-time (i.e., X^M with two timelike dimensions) is not an input, it is an outcome of the sp$(2,R)$ gauge symmetry. Yet this symmetry is responsible for the effective reduction of the $d+2$ dimensional two-time phase space to (a collection of) d dimensional phase spaces with one-time. Each of the d dimensional phase spaces holographically captures the contents of the $d+2$ dimensional theory, but they do so with holographic pictures that correspond to different 1T dynamics (different 1T Hamiltonians).

In the space of "all worldline theories" for a spinless particle (i.e. all possible background fields), there is a symmetry generated by all canonical transformations [8]. These transformations are above and beyond the local sp$(2,R)$ on the worldline. As observed in [9], the gauging of sp$(2,R)$ in field theory (as opposed to worldline theory) gives rise to a local noncommutative $u_\star(1)$ symmetry in noncommutative phase space that is closely connected to the general canonical transformations. In this paper, we will see that the local sp$(2,R)$ combines with the local $u(1)$ to form a non-Abelian gauge symmetry described by the noncommutative Lie algebra $u_\star(1,1)$ that will form the basis for the gauge theory introduced in this paper (following the notation in [10], we use the star symbol $\star$ in denoting noncommutative symmetry groups).

The $u_\star(1,1)$ gauge principle completes the formalism of [9] into an elegant and concise theory which beautifully describes 2T physics in field theory in $d+2$ dimensions, while resolving some problems that remained open. The resulting theory has deep connections to standard d dimensional gauge theories, gravity and the theory of high spin fields [11]. There is also a finite matrix formulation of the theory in terms of u(N,N) matrices, such that the $N\rightarrow\infty$ limit becomes the $u_\star(1,1)$ gauge theory.

A. Symmetries in the worldline theory

The local symmetries that will play a role in noncommutative field theory make a partial appearance in the worldline formalism. Therefore, for a self-contained set of arguments, we start from basic considerations of the worldline formalism of 2T physics for a spinless scalar particle.

The spinless particle is described in phase space by $X^M(\tau),P_M(\tau)$, interacting with all possible background fields. It is convenient to use the notation $X_1^M\equiv X^M$ and $X_{2M}\equiv P_M$, with $i=1,2$ referring to X_i. We avoid introducing a background metric in D dimensions by defining X_1^M with an upper index and X_{2M} with a lower index, and never raise or lower the M indices in the general setup, in the definitions of gauge symmetries, or the construction of an action. Thus, the formalism is background independent and is not *a priori* committed to any particular signature of space-time. The signature is later determined dynamically by the equations of motion. The worldline action has the form [6,8]

$$I_Q=\int d\tau\left[\dot{X}_1^M X_{2M}-\frac{1}{2}A^{ij}(\tau)Q_{ij}(X_1,X_2)\right], \quad (1)$$

where the symmetric $A_{ij}=A_{ji}$ denotes three Sp(2,R) gauge fields, and the symmetric $Q_{ij}=Q_{ji}$ are three sp(2,R) generators. An expansion of $Q_{ij}(X_1,X_2)$ in powers of X_{2M} in some local domain,

$$Q_{ij}(X_1,X_2)=\Sigma_s(f_{ij}(X_1))^{M_1\cdots M_s}X_{2M_1}\cdots X_{2M_s},$$

defines all the possible background fields $(f_{ij}(X_1))^{M_1\cdots M_s}$ in configuration space. The local sp(2,R) gauge transformations are

$$\delta X_1^M=-\omega^{ij}(\tau)\frac{\partial Q_{ij}}{\partial X_{2M}},\quad \delta X_{2M}=\omega^{ij}(\tau)\frac{\partial Q_{ij}}{\partial X_1^M},$$

$$\delta A^{ij}=\dot{\omega}^{ij}(\tau)+[A,\omega(\tau)]^{ij}. \quad (2)$$

The action I_Q is gauge invariant, with local parameters $\omega^{ij}(\tau)$, provided the $Q_{ij}(X_1,X_2)$ satisfy the sp(2,R) Lie algebra under Poisson brackets. This is equivalent to a set of differential equations that must be satisfied by the background fields $(f_{ij}(X_1))^{M_1\cdots M_s}$ [6,8]. The simplest solution is the free case denoted by $Q_{ij}=q_{ij}$ (no background fields, only the flat metric η_{MN})

$$q_{ij}=X_i^M X_j^N\eta_{MN}:\quad q_{11}=X_1\cdot X_1,\quad q_{12}=X_1\cdot X_2,$$

$$q_{22}=X_2\cdot X_2. \quad (3)$$

Beyond the local sp(2,R) above, if one considers the "space of all worldline theories" of the type I_Q, there is a symmetry that leaves the form of the action invariant [8]. The symmetry can be interpreted as acting in the space of all possible background fields $(f_{ij}(X_1))^{M_1\cdots M_s}$ that obey the sp(2,R) closure conditions. The transformations are given by all canonical transformations that act infinitesimally in the form

$$\delta_0 X_1^M=-\frac{\partial\omega_0(X_1,X_2)}{\partial X_{2M}},\quad \delta_0 X_{2M}=\frac{\partial\omega_0(X_1,X_2)}{\partial X_1^M}, \quad (4)$$

for any $\omega_0(X_1,X_2)$. Then $\delta_0 Q_{ij}$ is derived from Eq. (4) and given by the Poisson brackets $\delta_0 Q_{ij}=\{Q_{ij},\omega_0\}$. Under such transformations the term $\int\dot{X}_1^M X_{2M}$ is invariant, and the action I_Q is mapped to $I_{\tilde{Q}}$ where $\tilde{Q}_{ij}(X_1,X_2)=Q_{ij}(\tilde{X}_1,\tilde{X}_2)$, with $\tilde{X}_i=X_i+\delta_0 X_i$. The new action $I_{\tilde{Q}}$ is in the space of all theories of the form I_Q since, by virtue of canonical transformations, the new $\tilde{Q}_{ij}$ satisfies the sp(2,R) algebra under Poisson brackets if the old Q_{ij} does. By taking advantage of these symmetries all possible $Q_{ij}(X_1,X_2)$, i.e., all possible background fields have been determined up to canonical transformations [8]. The solution will be recapitulated later in this paper in Sec. II B.

B. Field equations from first quantized theory

Instead of using wave functions in configuration space $\psi(X_1^M)$, the quantum theory can be formulated equivalently in phase space, à la Weyl-Wigner-Moyal [12–14], by using distributions in phase space $\varphi(X_1^M,X_{2M})$. The phase space approach is natural in 2T physics, because the sp(2,R) as well as the canonical transformations $\omega_0(X_1,X_2)$ are phase space symmetries that would be cumbersome to discuss (if not impossible) in configuration space. Therefore, we find it beneficial to discuss first quantization in terms of fields in phase space. Sometimes we will use the notation $X^m\equiv(X_1^M,X_{2M})$ with a single index m that takes $2(d+2)$ values. The fields in phase space will be functions of the form $A(X^m)$. Products of fields A,B always involve the associative noncommutative Moyal star product

$$(A\star B)(X)=\exp\left(\frac{i}{2}\theta^{mn}\frac{\partial}{\partial X^m}\frac{\partial}{\partial\tilde{X}^n}\right)A(X)B(\tilde{X})|_{X=\tilde{X}}, \quad (5)$$

where $\theta^{mn}=\hbar\,\delta_N^M\varepsilon_{ij}$, with $i=1,2$, and ε_{ij} is the antisymmetric sp(2,R) invariant metric (note that we have not used any space-time metric in this expression). The star commutator between any two fields is defined by $[A,B]_\star\equiv A\star B-B\star A$. The phase space coordinates satisfy $[X^m,X^n]_\star=i\theta^{mn}$, which is equivalent to the Heisenberg algebra for (X_1^M,X_{2M}).

As shown in [9], first quantization of the worldline theory of Eq. (1) is described by the noncommutative field equations

$$[Q_{ij},Q_{kl}]_\star=i(\varepsilon_{jk}Q_{il}+\varepsilon_{ik}Q_{jl}+\varepsilon_{jl}Q_{ik}+\varepsilon_{il}Q_{jk}), \quad (6)$$

$$Q_{ij}\star\varphi=0. \quad (7)$$

Equation (6) is the quantum version of the sp(2,R) conditions required by the worldline theory. Its general solution was given in [8,9], and will be recapitulated in Eqs. (40)–(43) below. It describes Maxwell, Einstein and high spin *background* gauge fields (i.e., no dynamics). Spinless matter is coupled to these background gauge fields in Eq. (7). The general solution of this equation is a superposition of a basis of fields $\varphi(X_1,X_2)=\Sigma_{nm}c_m^n\varphi_n^m(X_1,X_2)$ where [9]

$$\varphi_n^m(X_1,X_2)=\int d^DY\,\psi_n(X_1)\star e^{-iY^M X_{2M}}\star\chi_m^*(X_1) \quad (8)$$

$$=\int d^DY\,\psi_n\left(X_1-\frac{Y}{2}\right)e^{-iY^M X_{2M}}\chi_m^*\left(X_1+\frac{Y}{2}\right). \quad (9)$$

According to Weyl's correspondence, the $\varphi_n^m(X_1,X_2)$ are related to Hilbert space outer products $\varphi_n{}^m\sim|\psi_n\rangle\langle\chi_m|$. The φ equation (7) is equivalent to the sp(2,R) singlet conditions in the Hilbert space, $Q_{ij}|\psi\rangle=0$, whose solutions form a complete set of physical states $\{|\psi_n\rangle\}$ that are gauge invariant under sp$(2,R)$. The solution space $\{|\psi_n\rangle\}$ is non-empty and is unitary only when space-time has precisely two timelike dimensions, no less and no more [7,9]. In particular, for the free theory (i.e., no backgrounds other than the flat metric η_{MN}, thus $Q_{ij}\to q_{ij}$), the $\{|\psi_n\rangle\}$ form the basis for the unitary singleton or doubleton representation of[1] SO$(d,2)$. Unlike the $\psi_n(X_1)$, the $\chi_m^*(X_1)$ are not restricted by Eq. (7). Therefore, it is reasonable to define φ only up to noncommutative $u_\star^R(1)$ gauge transformations that act from the right $\varphi\to\varphi\star\exp_\star(-i\omega^R)$, or to restrict it by an additional condition on φ from the right side. The φ_n^m automatically satisfy the following closure property under the triple product:

$$\varphi_{n_1}^{m_1}\star(\varphi^\dagger)_{m_2}^{n_2}\star\varphi_{n_3}^{m_3}=\delta_{m_2}^{m_1}\delta_{n_3}^{n_2}\varphi_{n_1}^{m_3}, \tag{10}$$

which follows just from the structure $|\psi_n\rangle\langle\chi_m|$ for orthonormalized states. By fixing a $u_\star^R(1)$ gauge symmetry, φ can be made Hermitian; in this case the set of $\{\chi_m\}$ is the same as the $\{\psi_n\}$. Evidently, there are other choices for the gauge fixing of the right side of φ.

Equations (6) and (7) correctly represent quantum mechanically the 1T physics of a spinless particle in d dimensions interacting with background gauge fields, including the electromagnetic, gravitational and high spin gauge fields [9]. Furthermore, the 2T physics formalism unifies different types of 1T field theories in d dimensions which holographically represent the same $d+2$ dimensional equations, and therefore, in principle it uncovers hidden symmetries and duality type relations among them (this has been explicitly demonstrated in simple cases [1]).

Much of the work in [9] was devoted to developing the noncommutative field theory formalism and the symmetry principles compatible with global and local sp$(2,R)$ symmetry. The goal was to find a field theory, and appropriate gauge principles, from which the free Eqs. (6) and (7) would follow as classical field equations of motion, much in the same way that the Klein-Gordon field theory arises from satisfying τ-reparametrization constraints ($p^2=0$), or string field theory emerges from satisfying Virasoro constraints, etc. This goal was partly accomplished in [9], but as we will explain, by only partially implementing the full gauge principles described by $u_\star(1,1)$.

In the rest of the paper we will complete the goal of [9] by spelling out the gauge principles, and constructing an essentially unique elegant action that results in Eqs. (6) and (7) as *exact background solutions* of nonlinear equations of motion. Expanding the full equations of motion around any background solution provides consistent interactions and propagation for the fluctuating gauge fields. Among other nice features, this theory seems to provide an action principle for high spin gauge fields.

[1]All the SO$(d,2)$ Casimir operators for the singleton or doubleton representation are fixed; in particular, the quadratic Casimir is $C_2(\mathrm{SO}(d,2))=1-d^2/4$. There are many holographic solutions of the $d+2$ dimensional differential equations $q_{ij}\psi_n(X_1)=0$ in the form of d dimensional fields, all of which realize the singleton or doubleton representation. One of the holographic solutions is the Klein-Gordon field in d dimensions which forms a well known representation of the conformal group SO$(d,2)$. Another one is the hydrogen atom in $d-1$ space dimensions, another one is the scalar field in AdS$_d$, and still another one is the scalar field in AdS$_{d-k}\times S^k$ for any $k<d-2$, and more. They all realize the same SO$(d,2)$ representation with the same Casimir eigenvalues, but in different bases [7].

II. $u_\star(1,1)$ GAUGE SYMMETRY

Global sp$(2,R)$ transformations that treat (X_1,X_2) as a doublet are generated by the free $q_{ij}=X_i\cdot X_j$. A complex scalar field in phase space $\varphi(X_1,X_2)$ can transform as a left module, right module, or diagonal module, as explained in [9]. For a left scalar module, the global sp$(2,R)$ transformation is $\delta^{sp}\varphi=-i\omega^{ij}(q_{ij}\star\varphi)$ where ω^{ij} are global parameters. To turn sp$(2,R)$ into a local symmetry, the three ω^{ij} are replaced by arbitrary functions. Then the Hermitian combination $\omega_0=\frac{1}{2}(\omega^{ij}\star q_{ij}+q_{ij}\star\omega^{ij})$ acts from the left like a local noncommutative phase transformation $\delta^0\varphi=-i\omega_0\star\varphi$. Therefore, local sp$(2,R)$ acting on a scalar field from the left is closely related to a noncommutative local $u_\star(1)$. In [9] it is argued that this $u_\star(1)$ acts on $Q_{ij}(X_1,X_2)$ from both sides $\delta^0Q_{ij}=-i[\omega_0,Q_{ij}]_\star$, therefore $\omega_0(X_1,X_2)$ is precisely the quantum version (all powers of $\hbar$) of the canonical transformations, encountered in the worldline formalism, as stated just following Eq. (4).

On a tensor field, global sp$(2,R)$ acts both on its (X_1^M,X_{2M}) dependence, as well as on its indices. For example, for a doublet

$$\delta_{global}^{sp}\varphi_k=\omega_{kl}\varphi^l-i\omega^{ij}(q_{ij}\star\varphi_k). \tag{11}$$

In turning these transformations into local transformations we find that we must have independent local parameters $\omega_{ij}(X_1,X_2)$ and $\omega_0(X_1,X_2)$ because closure cannot be obtained with only the three parameters $\omega_{ij}(X_1,X_2)$. In fact, there is no 3-parameter noncommutative sp$(2,R)$, instead there exists the local four-parameter noncommutative $u_\star(1,1)$ that has sp$(2,R)=$su$(1,1)$ as a global subalgebra.[2] We can collect the 4 parameters in the form of a 2×2 matrix, $\Omega_{ij}=\omega_{ij}+i\omega_0\varepsilon_{ij}$, whose symmetric part $\omega_{ij}(X_1,X_2)$

[2]For local parameters in noncommutative space, the commutator of two transformations with ω_{ij} closes into a transformation that involves both ω_{ij} and ω_0. The minimal noncommutative algebra that includes sp$(2,R)$ in the global limit is the 4-parameter sp$_\star(2,R)$ [10]. This is a subalgebra of the 4-parameter $u_\star(1,1)$ and is obtained from it by introducing a projection in the local space, such as the interchange $X_1\rightleftharpoons X_2$, or mirror reflections $X_2\to-X_2$ as in [10]. Thus, sp$_\star(2,R)$ is embeded in $u_\star(1,1)$ by demanding $\omega_{ij}(X_1,X_2)$ to be symmetric functions and $\omega_0(X_1,X_2)$ to be an antisymmetric function under the projections. Closure is satisfied for the 4 projected functions. Thus, in principle, sp$_\star(2,R)$ would have been the minimal local symmetry to turn global sp$(2,R)$ into a local symmetry in noncommutative space. However, as we will see, the simplest cubic action that we will build for the gauge theory is not symmetric under the parity-like projections. Therefore, the local symmetry appropriate for our purposes is $u_\star(1,1)$ rather than sp$_\star(2,R)$.

becomes sp(2,R) when it is global, while its antisymmetric part generates the local subgroup $u_\star(1)$ with the local parameter $\omega_0(X_1,X_2)$. To act on the doublet one of the indices is raised with the sp(2) metric ε^{ij}, $\delta\varphi_k=\Omega_k^l\star\varphi_l=\omega_k^l\star\varphi_l-i\omega_0\star\varphi_k$; therefore, in matrix form we have

$$\Omega_k^l=\omega_k^l-i\omega_0\delta_k^l=\begin{pmatrix}\omega_{12}-i\omega_0 & \omega_{22}\\ -\omega_{11} & -\omega_{12}-i\omega_0\end{pmatrix}. \quad (12)$$

This matrix satisfies the following Hermiticity conditions:

$$\Omega^\dagger=\varepsilon\Omega\varepsilon. \quad (13)$$

Such matrices close under matrix-star commutators to form $u_\star(1,1)$. It can be easily seen that, for closure under both matrix and star products in commutators, ω_{ij} cannot be separated from the ω_0 and hence they are both integral parts of the local symmetry. The finite $u_\star(1,1)$ group elements are given by exponentiation (using star and matrix products):

$$U=e_\star^\Omega,\quad U^{-1}=(-\varepsilon)U^\dagger\varepsilon=e_\star^{-\Omega}. \quad (14)$$

We can now consider the gauge fields. In [9] it was explained that q_{ij} acting on φ from the left defines a differential operator that is appropriate for building the kinetic terms in the action. To turn these differential operators into covariant differential operators, a gauge potential $A_{ij}(X_1,X_2)$ was introduced and added to the differential operators when acting on φ. Hence the covariant derivatives are $Q_{ij}(X_1,X_2)=q_{ij}+A_{ij}(X_1,X_2)$ acting from the left on φ. These Q_{ij} were shown to play the same role as the sp(2,R) generators encountered in the first quantized worldline theory. This was appropriate for a scalar field, for which only $u_\star(1)$ acts. However, if we consider tensor fields, we must take covariant derivatives with respect to $u_\star(1,1)$. Therefore we need to add only one more gauge field or generator since $u_\star(1,1)$ has 4 parameters. As we will see these will emerge from the following considerations.

We introduce a 2×2 matrix $\mathcal{J}_{ij}=J_{ij}+iJ_0\varepsilon_{ij}$ that parallels the form of the parameters Ω_{ij}. There will be a close relation between the fields J_{ij} and Q_{ij} as we will see soon. When one of the indices is raised, the matrix takes the form

$$\mathcal{J}_i^j=\begin{pmatrix}J_{12}-iJ_0 & J_{22}\\ -J_{11} & -J_{12}-iJ_0\end{pmatrix}. \quad (15)$$

The Hermiticity of the fields J_{ij},J_0 is equivalent to the following $u_\star(1,1)$ condition on this matrix:

$$\mathcal{J}^\dagger=\varepsilon\mathcal{J}\varepsilon. \quad (16)$$

Local gauge transformations are defined by the matrix-star products in the form

$$\delta\mathcal{J}=\mathcal{J}\star\Omega-\Omega\star\mathcal{J}\quad\text{or}\quad\mathcal{J}'=U^{-1}\star\mathcal{J}\star U. \quad (17)$$

Then the matrix form and Hermiticity of $\delta\mathcal{J}$ or $\mathcal{J}'$ are consistent with the matrix form and Hermiticity of $\mathcal{J}$.

Next we consider matter fields. For our purpose we will need to consider the noncommutative group $U_\star^L(1,1)\times U_\star^R(1,1)$. Recall that we already had a hint that the matter field admits independent gauge transformations on its left and right sides. In this notation $\mathcal{J}$ transforms as the adjoint under $U_\star^L(1,1)$ and is a singlet under $U_\star^R(1,1)$, thus it is in the (1,0) representation. For the matter field we take the $(\frac{1}{2},\frac{1}{2})$ representation given by a 2×2 complex matrix $\Phi_i^\alpha(X_1,X_2)$. This field is equivalent to a complex symmetric tensor Z_{ij} and a complex scalar φ, both of which were considered in [9], but without realizing their $U_\star^L(1,1)\times U_\star^R(1,1)$ classification. We define $\bar{\Phi}=\varepsilon\Phi^\dagger\varepsilon$. The $U_\star^L(1,1)\times U_\star^R(1,1)$ transformation rules for this field are

$$\Phi'=U^{-1}\star\Phi\star W,\quad \bar{\Phi}'=W^{-1}\star\bar{\Phi}\star U \quad (18)$$

where $U\in U_\star^L(1,1)$ and $W\in U_\star^R(1,1)$.

We now construct an action that will give the noncommutative field theory equations (6) and (7) in a linearized approximation and prescribe unique interactions in its full version. The action has a resemblance to the Chern-Simons type action introduced in [9], now with an additional field, J_0, while the couplings among the fields obey a higher gauge symmetry

$$S_{\mathcal{J},\Phi}=\int d^{2D}X\,\mathrm{Tr}\Big(-\frac{i}{3}\mathcal{J}\star\mathcal{J}\star\mathcal{J}-\mathcal{J}\star\mathcal{J}+i\bar{\Phi}\star\mathcal{J}\star\Phi - V_\star(\bar{\Phi}\star\Phi)\Big). \quad (19)$$

The invariance under the local $U_\star^L(1,1)\times U_\star^R(1,1)$ transformations is evident.[3] $V(u)$ is a potential function with argument $u=\bar{\Phi}\star\Phi$. Although we will be able to treat the most general potential function $V(u)$ in the discussion below, to illustrate how the model works, it is sufficient to consider the linear function $V(u)=au$, which implies a quadratic form in the field Φ

$$V=a\bar{\Phi}\star\Phi, \quad (20)$$

where a is a constant.

The form of this action is unique as long as the maximum power of $\mathcal{J}$ is three. As we will see, when the maximum power of $\mathcal{J}$ is cubic we will make the connection to the first quantized worldline theory. We have not imposed any conditions on the powers of Φ or interactions between $\mathcal{J},\Phi$, other than obeying the gauge symmetries. A possible linear term in $\mathcal{J}$ can be eliminated by shifting $\mathcal{J}$ by a constant, while the relative coefficients in the action are all absorbed into a renormalization of $\mathcal{J},\Phi$. A term of the form

[3]More generally, the invariance under the local $U_\star^R(1,1)$ could be broken by various terms in the potential $V_\star(\bar{\Phi}\star\Phi)$ or by interactions of Φ with additional fields from its right side.

$\mathrm{Tr}(\mathcal{J}\star\mathcal{J}\star f(\Phi\star\bar{\Phi}))$ that is allowed by the gauge symmetries can be eliminated by shifting $\mathcal{J}\to[\mathcal{J}-\frac{1}{3}f(\Phi\star\bar{\Phi})]$. This changes the term $i\bar{\Phi}\star\mathcal{J}\star\Phi$ by replacing it with interactions of $\mathcal{J}$ with any function of $\bar{\Phi},\Phi$ that preserves the gauge symmetries. However, one can do field redefinitions to define a new Φ so that the interaction with the linear $\mathcal{J}$ is rewritten as given, thus shifting all complications to the function $V_\star(\bar{\Phi}\star\Phi)$. Therefore, with the only assumption being the cubic restriction on $\mathcal{J}$, this unique action will explain the first quantized worldline theory, and will generalize it to an interacting theory based purely on a gauge principle.

It can be checked that the action is Hermitian thanks to the Hermiticity relations for $\mathcal{J},\Phi,\bar{\Phi}$ and cyclicity under the trace and integral signs. Hermiticity is also evident by evaluating the trace explicitly[4]

$$S_{\mathcal{J},\Phi}=\int d^{2D}X\begin{pmatrix} iJ_{11}\star J_{12}\star J_{22}-iJ_{22}\star J_{12}\star J_{11}+\dfrac{2}{3}J_0\star J_0\star J_0+2J_0\star J_0 \\ +(J_{11}\star J_{22}+J_{22}\star J_{11}-2J_{12}\star J_{12})\star(J_0+1)+Tr(i\mathcal{J}\star\Phi\star\bar{\Phi}-V)\end{pmatrix}.$$

The equations of motion are

$$\mathcal{J}\star\Phi=-i\Phi\star V',\quad \mathcal{J}\star\mathcal{J}-2i\mathcal{J}-\Phi\star\bar{\Phi}=0 \tag{21}$$

where $V'(u)=\partial V/\partial u$.

A. Solution and link to worldline theory

One can choose a gauge for the local $U_\star^L(1,1)\times U_\star^R(1,1)$ in which the 2×2 complex matrix Φ is proportional to the identity matrix

$$\Phi_i^{\alpha}=\delta_i^{\alpha}\varphi(X_1,X_2). \tag{22}$$

Then $\bar{\Phi}_\alpha^i=-\delta_\alpha^i\varphi^\dagger$. Thus, 6 gauge parameters are used up in eliminating 6 degrees of freedom from Φ. For a generic φ, the surviving symmetry is a global diagonal $sp^{L+R}(2,R)$ times a local noncommutative $u_\star^L(1)\times u_\star^R(1)$. In this gauge, the equations of motion become

$$\mathcal{J}\star\varphi=-i\varphi\star V'(-\varphi^\dagger\star\varphi),\quad \mathcal{J}\star\mathcal{J}-2i\mathcal{J}+(\varphi\star\varphi^\dagger)1=0. \tag{23}$$

Rewriting the equations of motion in terms of components, we can separate the triplet and singlet parts under the global $sp^{L+R}(2,R)$

$$J_{ij}\star\varphi=0,\quad J_0\star\varphi=\varphi\star V'(-\varphi^\dagger\star\varphi), \tag{24}$$

$$(J_0+1)_\star^2=1+\varphi\star\varphi^\dagger-\frac{1}{2}J_{ij}\star J^{ij} \tag{25}$$

$$\frac{1}{2}J_{(i}^{k}\star J_{j)k}=iJ_{ij}\star(J_0+1)+i(J_0+1)\star J_{ij}. \tag{26}$$

More explicitly, in terms of components (using $J_i^1=J_{i2}$ and $J_i^2=-J_{i1}$) the left hand side of Eq. (26) reduces to commutators $[J_{11},J_{12}]_\star$, $[J_{11},J_{22}]_\star$, and $[J_{11},J_{22}]_\star$. In fact, if in Eq. (26) J_0 on the right hand side were absent, then the commutation relations among the J_{ij} would be precisely those of $sp(2,R)$ as given in Eq. (6). Then we should consider a close relationship between J_{ij} and Q_{ij}. This is also supported by the resemblance of Eq. (24) to Eq. (7). Indeed, as we will see below, the relation is nontrivial and interesting.

The second equation in Eq. (24) can be written in the form $(J_0+1)\star\varphi=\varphi\star[1+V'(-\varphi^\dagger\star\varphi)]$. Applying (J_0+1) on both sides, using $(J_0+1)_\star^2$ given by Eq. (25), and applying $J_{ij}\star\varphi=0$ as in Eq. (24), we obtain an equation purely for φ:

$$\varphi\star\{[1+V'(-\varphi^\dagger\star\varphi)]_\star^2-1-\varphi^\dagger\star\varphi\}=0. \tag{27}$$

It is straightforward to find all the solutions of this equation. Thus consider any $\varphi=\lambda\varphi_n^m$, where λ is a complex constant, and $\varphi_n^m(X_1,X_2)$ is of the form of Eq. (8) which satisfies the triple relation of Eq. (10) by construction. Then

$$\varphi\star\varphi^\dagger\star\varphi=|\lambda|^2\varphi. \tag{28}$$

Inserting such a φ in the equation shows that λ must be a solution of the equation

$$[1+V'(-|\lambda|^2)]^2-1-|\lambda|^2=0. \tag{29}$$

As an illustration, consider the example of the potential in Eq. (20), for which $V'=a$ is a constant. For this case we find $\lambda=\pm(a^2+2a)^{1/2}$. It is evident that, up to $u_\star^R(1)$ gauge transformations, the solutions of Eqs. (24) and (27) are all physi-

[4]In this form we see that $sp_\star(2,R)$ [as opposed to $u_\star(1,1)$] cannot be used as the local symmetry, because all cubic terms change sign under $sp_\star(2,R)$'s parity-like projections, $J_{ij}(X_1,-X_2)=J_{ij}(X_1,X_2)$, and $J_0(X_1,-X_2)=-J_0(X_1,X_2)$, with simultaneous interchange of factors in a star product [10]. It is interesting to note that the action is invariant if the parity properties are exactly the opposite signs than those required by $sp_\star(2,R)$. However, such conditions could not be imposed on J because then they would not be compatible with gauge transformations rules that are required to have a symmetric action.

cal states of the form (8) multiplied by any $|\lambda|$ that solves the equation [the phase of λ can be absorbed away with a $u_\star^R(1)$ transformation].

Next we solve J_0 formally from Eq. (25), $J_0=-1+(1+\varphi\star\varphi^\dagger-\frac{1}{2}J_{ij}\star J^{ij})^{1/2}$, where the square root is understood as a power series with all products replaced by star products. Using $J_{ij}\star\varphi=0=\varphi^\dagger\star J_{ij}$ and $\varphi\star\varphi^\dagger\star\varphi=|\lambda|^2\varphi$, we can simplify each term in the series expansion and obtain the simplified expression

$$J_0=-1+\frac{V'(-|\lambda|^2)}{|\lambda|^2}\varphi\star\varphi^\dagger+\left(1-\frac{1}{2}J_{ij}\star J^{ij}\right)^{1/2} \quad (30)$$

where Eq. (29) has been used. With this form of J_0, all equations involving it, including the last one in Eq. (24), are satisfied. Finally, replacing these results into Eq. (26) and using again $J_{ij}\star\varphi=0=\varphi^\dagger\star J_{ij}$, we find an equation involving only the gauge fields J_{ij}, which we write in components explicitly

$$[J_{11},J_{12}]_\star=i\{J_{11},(1-C_2(J))^{1/2}\}_\star \quad (31)$$

$$[J_{11},J_{22}]_\star=2i\{J_{12},(1-C_2(J))^{1/2}\}_\star \quad (32)$$

$$[J_{12},J_{22}]_\star=i\{J_{22},(1-C_2(J))^{1/2}\}_\star. \quad (33)$$

The right hand side is a star anti-commutator involving the expression

$$C_2(J)=\frac{1}{2}J_{kl}\star J^{kl}=\frac{1}{2}J_{11}\star J_{22}+\frac{1}{2}J_{22}\star J_{11}-J_{12}\star J_{12} \quad (34)$$

which looks like a Casimir operator. However, since Eqs. (31)–(33) are not the sp(2,R) Lie algebra one cannot hastily claim that $C_2(J)$ is a Casimir operator. Indeed, if one attempts to derive the commutation relations between $C_2(J)$ and J_{ij} by repeated use of Eqs. (31)–(33), one finds that $[J_{ij},C_2(J)]_\star$ becomes equal to $[-J_{ij},\{[1-C_2(J)]^{1/2}\}_\star^2]$, and thus obtains an identity. Therefore, these equations do not require that $C_2(J)$ and J_{ij} commute. If they commute, one could renormalize J_{ij} by an appropriate factor to reduce these equations to sp(2,R) commutation relations with the normalization of generators as given by Eq. (6). This is quite interesting, as we will see below. Thus, generally Eqs. (31)–(33) are not the sp(2,R) commutation rules.

Furthermore, if one computes the Jacobi identities by repeated use of Eqs. (31)–(33), one finds

$$[J_{11},[J_{12},J_{22}]_\star]_\star+\text{cyclic}=\frac{1}{2}[J_{ij},[J^{ij},[1-C_2(J)]^{1/2}]_\star]_\star. \quad (35)$$

Under the assumption that the star product is associative, the Jacobi identity is satisfied,[5] and the left side of Eq. (35) vanishes. Therefore associativity of the star product requires the right hand side to vanish, but generally this is a weaker condition than the vanishing of $[J_{ij},C_2(J)]_\star$.

To solve the nonlinear gauge field equations (31)–(33) we will set up a perturbative expansion around a background solution

$$J_{ij}=J_{ij}^{(0)}+gJ_{ij}^{(1)}+g^2J_{ij}^{(2)}+\cdots \quad (36)$$

such that $J_{ij}^{(0)}$ is an exact solution and then analyze the full equation perturbatively in powers of g. For the exact background solution we assume that $\frac{1}{2}J_{kl}^{(0)}\star J^{(0)kl}$ commutes with $J_{ij}^{(0)}$, therefore the background solution satisfies a Lie algebra. Then we can write the exact background solution to Eqs. (31)–(33) in the form

$$J_{ij}^{(0)}=Q_{ij}\star\frac{1}{\sqrt{1+\frac{1}{2}Q_{kl}\star Q^{kl}}} \quad (37)$$

where Q_{ij} satisfies the sp(2,R) algebra of Eq. (6)

$$[Q_{11},Q_{12}]_\star=2iQ_{11},\quad [Q_{11},Q_{22}]_\star=4iQ_{12},$$
$$[Q_{12},Q_{22}]_\star=2iQ_{22}, \quad (38)$$

and $\frac{1}{2}Q_{kl}\star Q^{kl}$ is a Casimir operator that commutes with all Q_{ij} that satisfies the sp(2,R) algebra. The square root is understood as a power series involving the star products and can be multiplied on either side of Q_{ij} since it commutes with the Casimir operator. For such a background, the matter field equations (24) reduce to

$$Q_{ij}\star\varphi=0. \quad (39)$$

This is the matter field equation (7) given by the first quantized theory. Its solution was discussed following Eq. (7).

Summarizing, we have shown that our action $S_{J,\Phi}$ has yielded precisely what we had hoped for. The linearized equations of motion (0th power in g) in Eqs. (38) and (39) are exactly those required by the first quantization of the worldline theory as given by Eqs. (6) and (7). There remains to understand the propagation and self-interactions of the fluctuations of the gauge fields $gJ_{ij}^{(1)}+g^2J_{ij}^{(2)}+\cdots$, which are not included in Eqs. (38) and (39). However, the full field theory, without making the assumption that $C_2(J)$ and J_{ij} commute, includes all the information. In particular the expansion of Eqs. (31)–(33) around the background solution $J_{ij}^{(0)}$ of Eq. (37) should determine uniquely both the propagation and the interactions of the fluctuations involving photons, gravitons, and high spin fields.

B. Explicit background solution

We record the exact solution to the background gauge field and matter field equations (38) and (39), which were obtained in several stages in [7–9]. The solution is given by fixing a gauge with respect to the $u_\star^L(1)$. First, we choose the gauge $Q_{11}=X_1\cdot X_1$. There is remaining $u_\star^L(1)$ symmetry that satisfies $[X_1^2,\omega_0]_\star=0$. Using the conditions imposed on Q_{12}

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[5] It is also interesting to keep in mind the possibility of anomalies, leading to non-associativity (e.g. magnetic fields [15]). If we consider a nonvanishing Jacobian, the mathematical structure of Eqs. (31)–(33) would be a Malchev algebra rather than a Lie algebra.

by the sp(2,R) conditions, one finds that the remaining symmetry is sufficient to fix a gauge for $Q_{12}=X_1\cdot X_2$. Thus, up to $u_\star^L(1)$ gauge transformations $\omega_0(X_1,X_2)$, one can simplify Q_{11},Q_{12} and take the most general Q_{22} as follows:

$$Q_{11}=X_1^M X_1^N \eta_{MN}, \quad Q_{12}=X_1^M X_{2M}, \quad Q_{22}=G(X_1,X_2) \tag{40}$$

where η^{MN} is the flat metric in $d+2$ dimensions, and the general function $G(X_1,X_2)$ is assumed to have a power expansion in X_2 in some domain

$$\begin{aligned}G(X_1,X_2)=&G_0(X_1)+G_2^{MN}(X_1)[X_2+A(X_1)]_M\\ &\times[X_2+A(X_1)]_N+\sum_{s=3}^{\infty}G_s^{M_1\cdots M_s}(X_1)\\ &\times[X_2+A(X_1)]_{M_1}\cdots[X_2+A(X_1)]_{M_s}.\end{aligned} \tag{41}$$

The configuration space fields have the following interpretation: $A_M(X_1)$ is the Maxwell gauge potential, $G_0(X_1)$ is a scalar, $G_2^{MN}(X_1)=\eta^{MN}+h_2^{MN}(X_1)$ is the gravitational metric, and the symmetric tensors $[G_s(X_1)]^{M_1\cdots M_s}$ for $s\geqslant 3$ are high spin gauge fields.[6] The sp(2,R) closure condition in Eq. (38) requires these background fields to be orthogonal to X_1^M and to be homogeneous of degree $(s-2)$:

$$X_1\cdot\partial A_M=-A_M, \quad X_1\cdot\partial G_s=(s-2)G_s, \tag{42}$$

$$X_1^M A_M=X_{1M_1}h_2^{M_1M_2}=X_{1M_1}G_s^{M_1\cdots M_s}=0. \tag{43}$$

The background fields $A,G_0,G_2,G_{s\geqslant 3}$ determine all other background fields $(f_{ij}(X_1))^{M_1\cdots M_s}$ up to $u_\star^L(1)$ gauge transformations $\omega_0(X_1,X_2)$. The full solution of the $d+2$ dimensional equations (42) is given in [8] in terms of d dimensional background fields for Maxwell, dilaton, metric, and higher spin fields. Therefore Eqs. (38) holographically encapsulate all possible *off-shell* arbitrary d dimensional background gauge fields in a $d+2$ dimensional formalism. In the next section we will derive the dynamical equations of motion for the small fluctuations around the backgrounds.

The $u_\star^L(1)$ symmetry of the type $\omega_0(X_1,X_2)$, when expanded in powers of $X_2+A(X_1)$, contains the configuration space gauge transformation parameters for all of the gauge fields [8]

$$\begin{aligned}\omega_0(X_1,X_2)=&\varepsilon_0(X_1)+\varepsilon_1^M(X_1)[X_2+A(X_1)]_M\\ &+\sum_{s=2}^{\infty}\varepsilon_s^{M_1\cdots M_s}(X_1)\\ &\times[X_2+A(X_1)]_{M_1}\cdots[X_2+A(X_1)]_{M_s}.\end{aligned} \tag{44}$$

Since Q_{11},Q_{12} have been gauge fixed, the remaining part of $u_\star^L(1)$ gauge symmetry should not change the form of Q_{11}, Q_{12}, so any surviving gauge parameters ω_0 should satisfy $[X_1^2,\omega_0]_\star=[(X_1\cdot X_2),\omega_0]_\star=0$; this requires the gauge parameters $\varepsilon_{s\geqslant 0}^{M_1\cdots M_s}(X_1)$ to be homogeneous and orthogonal to X_1^M:

$$X_1\cdot\partial\varepsilon_s=s\varepsilon_s, \quad X_{1M_1}\varepsilon_s^{M_1\cdots M_s}=0. \tag{45}$$

The gauge transformation law of the gauge fields $\delta A,\delta G_0,\delta G_2,\delta G_{s\geqslant 3}$ is given by $\delta Q_{22}=i[Q_{22},\omega_0]_\star$. From this it is easy to see that $\varepsilon_0(X_1)$ is the gauge parameter for the Maxwell field, $\varepsilon_1^M(X_1)$ is the infinitesimal general coordinate reparametrizations of all tensor fields, and $\varepsilon_{s\geqslant 2}^{M_1\cdots M_s}(X_1)$ are the gauge parameters for the high spin fields $G_{s+1}^{M_1\cdots M_{s+1}}$. The details of the gauge transformations are given in [8]. This shows that the familiar configuration space gauge principles, Maxwell, Einstein, and high spin, are unified in our approach as being a small part of the $u_\star(1,1)$ gauge symmetry. We will use this remaining $u_\star^L(1)$ gauge symmetry in the analysis of the equations of motion for the small fluctuations of the gauge fields.

III. FLUCTUATIONS AND DYNAMICS OF GAUGE FIELDS

We are interested in analyzing the perturbative expansion of Eqs. (31)–(33) around any background solution. In particular, taking a hint from the form of Eqs. (40) and (41), we will investigate fluctuations $h(X_1,X_2)$ in the direction of J_{22}. More general fluctuations could also be considered,[7] but we will limit the current discussion to fluctuations around the background fields we have identified in the previous section. Those are fluctuations in the direction of J_{22}. Thus, we consider replacing any solution of the background fields $A,G_0,G_2,G_{s\geqslant 3}$ by adding the fluctuations $(A+g\,\delta^{(1)}A)$, $(G_0+g\,\delta^{(1)}G_0)$, $(G_2+g\,\delta^{(1)}G_2)$, $(G_{s\geqslant 3}+g\,\delta^{(1)}G_{s\geqslant 3})$ and then expanding to first order in g. We already know from the form of Eqs. (40) and (41), and the gauge transformations discussed above, that these fluctuations are directly related to

[6] There is no $G_1^M(X_1)$ as the coefficient of the first power of X_2+A, because $A_M(X_1)$ is equivalent to that degree of freedom, as can be seen by re-expanding Q_{22} in powers of X_2 instead of X_2+A. Note also in Q_{12} we really have X_2+A, but A has dropped because we chose to work in the gauge $X_1\cdot A=0$.

[7] The $u_\star^L(1)$ symmetry was sufficient to gauge fix not only Q_{11} but also Q_{12}, because these had to obey the sp(2,R) algebra. However, the J_{ij} obey a more general set of equations and therefore it is not clear whether one could gauge away more general fluctuations around the background. There is certainly the freedom to take vanishing fluctuations in the direction of J_{11}, but it is not clear whether fluctuations in the direction of J_{12} can also be eliminated by gauge choices. This remains to be investigated.

gauge fields for Maxwell, Einstein and high spin gauge symmetries. We wish to analyze the perturbative expansion of Eqs. (31)–(33) in order to determine the equations of motion for the fluctuations.

The J_{ij} including the fluctuations takes the form

$$J_{11}=\frac{1}{\sqrt{1+C_2(Q)}}\star X_1^2,\quad J_{12}=\frac{1}{\sqrt{1+C_2(Q)}}\star(X_1\cdot X_2) \tag{46}$$

$$J_{22}=\frac{1}{\sqrt{1+C_2(Q)}}\star G+gh(X_1,X_2)+\cdots \tag{47}$$

where $h(X_1,X_2)$ is a general function while $G(X_1,X_2)$ describes a specific set of background fields that solve Eqs. (42) and (43). In particular, $G=X_2^2$ corresponds to the free flat background. $C_2(Q)$ is the quadratic Casimir operator for the sp$(2,R)$ algebra of Eq. (38) satisfied by the background, and is given by

$$C_2(Q)=\frac{1}{2}Q_{kl}\star Q^{kl}=\frac{1}{2}(Q_{11}\star Q_{22}+Q_{22}\star Q_{11})-Q_{12}\star Q_{12} \tag{48}$$

$$=\frac{1}{2}\{X_1^2,G\}_\star-(X_1\cdot X_2)\star(X_1\cdot X_2) \tag{49}$$

$$=X_1^2G-\frac{1}{4}\left(\frac{\partial}{\partial X_2}\right)^2G-(X_1\cdot X_2)^2-\frac{1}{4}(d+2) \tag{50}$$

where, in the last line all star products have been evaluated. As long as the background fields satisfy Eqs. (42) and (43) this $C_2(Q)$ commutes with the sp$(2,R)$ generators $Q_{ij}=(X_1^2,(X_1\cdot X_2),G(X_1,X_2))$. The gauge field fluctuations $gh(X_1,X_2)$ up to first order in g will be treated perturbatively in solving the non-linear equations (31)–(33). Unlike the case of $J_{ij}^{(0)}$, in this analysis we will not assume that $C_2(J)$ commutes with J_{ij}, and instead we will derive the conditions that $h(X_1,X_2)$ must satisfy to solve the equations to first order in g. We will develop the equations in the general background $G(X_1,X_2)$ up to a point and eventually, for simplicity, specialize to the free background $G(X_1,X_2)\rightarrow X_2^2$.

First we compute $1-C_2(J)$ for the J_{ij} in Eqs. (46),(47) to first order in g

$$1-C_2(J)=1-C_2(J^{(0)})-\frac{g}{2}(J_{11}^0\star h+h\star J_{11}^0) \tag{51}$$

$$=\frac{1}{1+C_2(Q)}-\frac{g}{2\sqrt{1+C_2(Q)}}\star X_1^2\star h - h\star X_1^2\star\frac{g}{2\sqrt{1+C_2(Q)}}. \tag{52}$$

Next we compute the square root up to first order in g. Because of the orders of factors, this is a complicated expression. In order to get a quick glimpse of the content of the equations, for simplicity, we will proceed under the assumption

$$[C_2(Q),h]_\star=0. \tag{53}$$

Then we get the simple expression

$$\sqrt{1-C_2(J)}=\frac{1}{\sqrt{1+C_2(Q)}}-\frac{g}{4}\{X_1^2,h\}_\star+\cdots. \tag{54}$$

We now insert J_{ij} in Eqs. (31)–(33) and expand both sides up to the first power in g. The zeroth order terms cancel thanks to the properties of J_{ij}^0, and the first power gives the following equations that must be obeyed by $h(X_1,X_2)$:

$$-\frac{i}{4}\{X_1^2,\{X_1^2,h\}_\star\}_\star=0 \tag{55}$$

$$-\frac{i}{2}\{(X_1\cdot X_2),\{X_1^2,h\}_\star\}_\star=[X_1^2,h]_\star \tag{56}$$

$$-\frac{i}{4}\{G,\{X_1^2,h\}_\star\}_\star=[(X_1\cdot X_2),h]_\star-2ih. \tag{57}$$

We can show that the equation $[C_2(Q),h]_\star=0$ that was assumed in arriving at these expressions has no additional information beyond these equations. That is, if we use $C_2(Q)$ as given in Eq. (49) and evaluate the commutator by repeatedly using Eqs. (55)–(57), we find that $[C_2(Q),h]_\star=0$ is identically satisfied. Hence, this is not an additional equation to be taken into account.

To get a quick grasp of the nature of these equations, we will first make a few quick observations by assuming that the right hand side is zero, and in the next paragraph we will analyze them by lifting this assumption. Thus, after inserting the information $[X_1^2,h]_\star=0$, and $[(X_1\cdot X_2),h]_\star=2ih$, we can derive from the left hand side that $X_1^2\star H=0=H\star X_1^2$, and $(X_1\cdot X_2)\star H=0=H\star(X_1\cdot X_2)$, and $\{G,H\}_\star=0$, where we have defined $H=\frac{1}{2}\{X_1^2,h\}_\star=X_1^2h-\partial^2h/(\partial X_2)^2$ after evaluating the star products. The equations satisfied by H are similar to Eqs. (38) and (39) satisfied by a scalar field in a general background, in particular if we consider H similar to $\varphi\star\varphi^\dagger$. As we have already learned, the solution is of the form $H\sim\Sigma|\psi\rangle\langle\psi'|$, where the $\{|\psi\rangle\}$ satisfy the Klein-Gordon equation in configuration space.

These remarks provide a quick indication that the first order equations (55)–(57) for h, or equivalently for H, represent Klein-Gordon type equations for the fluctuations of the gauge fields. In particular, it is worth emphasizing that we have seen a first indication that the original action includes a kinetic term for the fluctuations, although the formalism does not make this immediately apparent. This point will become clearer in the component form discussed in the next paragraph.

Next, we proceed to investigate Eqs. (55)–(57) in component formalism without the assumptions of the previous two paragraphs. However, to make further progress we will specialize to the free background $G=X_2^2$ and take the following expansion in powers of X_2, thus defining the spin components of the fluctuating gauge fields in configuration space:

$$h=h_0(X_1)+h_1^M(X_1)X_{2M}+h_2^{MN}(X_1)\,X_{2M}X_{2N} +\sum_{s=3}^{\infty} h_s^{M_1\cdots M_s}(X_1)(X_{2M_1}\cdots X_{2M_s}). \tag{58}$$

Up to factors of $[1+C_2(Q)]^{1/2}$ we have redefined $h_s^{M_1\cdots M_s}$ as the fluctuations for the gauge fields.[8] In particular, up to some factors, $h_1^M(X_1)$ is the fluctuation of the Maxwell field, and h_2^{MN} is the fluctuation of the gravitational metric. The equations of motion can now be written for the components by evaluating the star products in Eqs. (55)–(57). This is done in the Appendix. Thus, $\{X_1^2,\{X_1^2,h\}_\star\}_\star=0$ gives in component form

$$(X_1^2)^2h_s^{M_1\cdots M_s}-\frac{(s+2)(s+1)}{2}X_1^2\eta_{MN}h_{s+2}^{MNM_1\cdots M_s} +\frac{(s+4)(s+3)}{4}\eta_{KL}\eta_{MN}h_{s+4}^{KLMNM_1\cdots M_s}=0. \tag{59}$$

Similarly $[X_1^2,h]_\star=-(i/2)\{(X_1\cdot X_2),\{X_1^2,h\}_\star\}_\star$ gives

$$\begin{aligned}2(s+1)X_{1N}h_{s+1}^{NM_1\cdots M_s}=&-\frac{2}{s}X_1^{(M_1}\bigg(X_1^2h_{s-1}^{M_2\cdots M_s)}-\frac{s(s+1)}{4}\eta_{MN}h_{s+1}^{MNM_2\cdots M_s)}\bigg)\\&-(s+1)\partial_N\bigg(X_1^2h_{s+1}^{NM_1\cdots M_s}-\frac{(s+3)(s+2)}{4}\eta_{KL}h_{s+3}^{KLNM_1\cdots M_s}\bigg)\end{aligned} \tag{60}$$

and $[(X_1\cdot X_2),h]_\star-2ih=-(i/4)\{X_2^2,\{X_1^2,h\}_\star\}_\star$ gives

$$\begin{aligned}(s-2-X_1\cdot\partial_1)h_s^{M_1\cdots M_s}=&\frac{-2}{s(s-1)}\eta^{(M_1M_2}\bigg(X_1^2h_{s-2}^{M_3\cdots M_s)}-\frac{s(s-1)}{4}\eta_{MN}h_s^{MNM_3\cdots M_s)}\bigg)\\&+\frac{1}{4}\partial_1^2\bigg(X_1^2h_s^{M_1\cdots M_s}-\frac{(s+2)(s+1)}{4}\eta_{MN}h_{s+2}^{MNM_1\cdots M_s}\bigg).\end{aligned} \tag{61}$$

These equations are purely in configuration space X_1^M. The first two equations may be interpreted as subsidiary conditions, while the last one is a second order Klein-Gordon type equation. By construction, they are gauge invariant under the remaining gauge transformations $\omega_0(X_1,X_2)$. Since we have $[C(Q),h]=0$, the remaining gauge symmetry also obeys

$$[C(Q),\omega_0]=0 \tag{62}$$

in addition to Eq. (45), hence they are a subset of the gauge transformations discussed in [8]. These gauge transformations do not change the form of J_{11},J_{12}, while they are applied to the total J_{22} as $\delta J_{22}=i[J_{22},\omega_0]_\star$ from which the transformation properties for the components h_s are obtained.

Note that the double trace of $h_{s\geq 4}$ is restricted by Eq. (59), an important fact for high spin gauge theories [16]. In this connection, we may ask if the double trace would vanish when the $d+2$ dimensional system is holographically viewed in d dimensions. As part of the reduction from $d+2$ dimensions to d dimensions we need to impose the vanishing of X_1^2. Although $X_1^2h_s^{M_1\cdots M_s}$ does not vanish, it appears that $(X_1^2)^2h_s^{M_1\cdots M_s}$ and $\eta_{MN}X_1^2h_s^{MNM_1\cdots M_s}$ may consistently be taken to vanish. Then at the end of the holographic reduction the double trace does indeed vanish in d dimensions.

The main point established in this section is that the full non-linear equations contain information on the propagation of the gauge fields. For simplicity, this was done under the assumption $[C_2(Q),h]_\star=0$. It is desirable to analyze the full form of the perturbative expansion without relying on this assumption. Also, there still remains the completion of this exercise to extract the full form of the kinetic terms and interactions after the reduction to a holographic picture in d dimensions. At that point it will be interesting to compare our equations for the high spin gauge fields to those discussed in other formalisms. In previous investigations equations of motion have been constructed in $d=3,4$ dimensions, including up to cubic interactions that satisfy a truncated (or approximate) form of a high spin gauge symmetry [11]. But the general interaction is not known, and furthermore the construction of an off-shell action has eluded all attempts. By contrast, our approach begins with a complete and unique action (modulo the cubic condition). It is already clear that our theory supplies both the propagation and all interactions

[8]Recall that the power expansion of $G(X_1,X_2)$ did not have G_1^M associated with the first power of X_2, since the Maxwell field A_M was introduced as an independent field instead of G_1^M. The fluctuations of the Maxwell field appear gauge covariantly everywhere in the form $X_2+g\,\delta^{(1)}A$. The various powers of this expression need to be expanded in powers of g to first order. However, there is already one power of g in front of $h_s^{M_1\cdots M_s}$ since it is itself a fluctuation. Therefore, all $g\,\delta^{(1)}A$ drop out to first order in g. However, $g\,\delta^{(1)}A$ also appears in covariantizing the zeroth order quadratic term $Q_{22}\to(X_2+g\,\delta^{(1)}A)^2$. The expansion of this term gives rise to $h_1^M\sim\delta^{(1)}A$ up to factors. Similarly, up to overall factors, $h_s^{M_1\cdots M_s}$ is proportional to the fluctuation $\delta^{(1)}G_s^{M_1\cdots M_s}$.

of the gauge fields. It would be very interesting to investigate the relation between our approach and that of [11]. Related aspects of high spin gauge fields are still under study in our theory, and we hope to report on this topic in a future publication.

IV. MATRIX POINT OF VIEW

In some sense, our current noncommutative field theory is an infinite dimensional matrix theory, and it can be viewed as the large N limit of a finite $2N\times 2N$ matrix theory.

The fields $J_{ij}(X_1,X_2)$ and $J_0(X_1,X_2)$ are constructed from noncommutative $d+2$ dimensional phase space (X_1^M,X_{2M}). Using the Weyl correspondence, it is possible to replace (X_1^M,X_{2M}) by quantum operators acting in a Hilbert space, or equivalently by infinite dimensional matrices. In this sense, our theory is already a ''matrix theory'' for infinite dimensional matrices.

One can introduce a cutoff in the theory by replacing the matrices by finite matrices. The basic Heisenberg commutation rules between (X_1^M,X_{2M}) cannot be obeyed by finite matrices, but by taking special combinations of the basic operators (X_1^M,X_{2M}) one can confine oneself to quantities J_{ij} constructed from them, such that J_{ij} are finite matrices. For example, this is the case on a periodic torus where finite translations in phase space $u_a=\exp(ia\cdot X_1)$ and $v_b=\exp(ib\cdot X_2)$ are indeed represented by finite matrices that obey the algebra $u_a v_b=v_b u_a \omega_{ab}$ when $\omega_{ab}=\exp(-ia\cdot b)$ is a root of unity. Similar considerations apply to the fuzzy sphere in phase space [with $(d,2)$ signature in our case].

Therefore, it is possible to take J_{ij} and J_0 as functions of only u_a,v_b (for a collection of a's and b's), or similar structures, and thus represent them as functions of finite matrices that are closely connected to phase space (X_1^M,X_{2M}). We expect then the non-commutative $\mathrm{u}_\star(1,1)$ to be approximated by the non-compact group $\mathrm{u}(N,N)$ such that the 2×2 noncommutative $\mathcal{J}$ gets replaced by the $2N\times 2N$ matrix representation of $\mathrm{u}(N,N)$. The four $N\times N$ blocks are then identified with the Hermitian J_{ij},J_0 just as in Eq. (15). The form of the action formally remains the same as Eq. (19), except for replacing integration by a trace over matrices. Thus, the equations that $\mathcal{J}$ satisfies are also formally the same, except for replacing star products with matrix products.

We now face again the matrix analog of Eqs. (31)–(33), instead of star products. When $\frac{1}{2}J_{kl}J^{kl}$ commutes with J_{ij} it is possible to construct Q_{ij},Q_0 that satisfy the $\mathrm{u}(N,N)$ algebra, as in Eq. (37). However, the solution for Q_{ij},Q_0 must now be given in terms of u_a,v_b. Indeed it is possible to construct the $\mathrm{u}(N,N)$ algebra in terms of powers of u_a,v_b or similar structures, just like the examples that exist in the literature for $U(2N)$ [17–26]. This would provide the matrix analog of the background solution in Eqs. (40),(41).

Since there are many solutions of the type Eqs. (40),(41) we expect also many solutions for Q_{ij},Q_0 as functions of u_a,v_b or some similar structures. The more general solution of Eqs. (31)–(33) for J's that include propagation of the gauge fields can then be investigated using finite matrix methods.

V. OUTLOOK

We have learned that we can consistently formulate a field theory of 2T physics in $d+2$ dimensions based on a very basic gauge principle in quantum phase space. We have tentatively shown that our equations, compactly written in phase space in the form of Eq. (21), seem to yield a new unified description of various gauge fields in configuration space, including Maxwell, Einstein, and high spin gauge fields interacting with matter and among themselves in d dimensions.

The underlying gauge principle is the noncommutative $\mathrm{u}_\star(1,1)$, and the action that gives rise to the field equations in noncommutative phase space has the rather simple form of Eq. (19). As argued following Eq. (20), the form of the action is unique as long as it is restricted to the maximum cubic power of J. Then, all results are grounded purely in the $\mathrm{u}_\star(1,1)$ gauge principle. With the only assumption being the cubic restriction, the worldline approach is explained by the field theory as *exact background solutions*. This essentially unique action could now be taken as a starting point for a classical as well as quantum analysis of the *interacting* 2T-physics field theory. At this time it is not known what would be the consequences of relaxing the maximum cubic power of J.

Although the analysis of the classical field equations of motion so far has been rudimentary, it was sufficient for showing that the content of the theory is sensible while being very rich and interesting from the point of view of d dimensions. As usual, the 1T-physics content of the theory can be obtained as various holographic images that come from embedding d dimensions in various ways in $d+2$ dimensions. One of the better understood holographic images [27,7] is the field theory in d dimensions in which the Klein-Gordon matter field interacts with various gauge fields, including interactions with the Maxwell field, dilaton, gravitational field, and high spin gauge fields.

The gauge fields propagate and have interactions among themselves. It appears that our approach provides for the first time an action principle that should contribute to the resolution of the long studied but unfinished problem of high spin fields [11,28,8,29]. We have shown that there is a kinetic term for the gauge fields although more study is needed to understand its contents better. The nature and detail of the interactions among the gauge fields can in principle be extracted from our $d+2$ dimensional theory, but this remains as an exercise for the future.[9]

This work can be generalized in several directions. One of these directions is supersymmetry, and one can consider both worldline and space-time supersymmetries.

[9]The nature of interactions for the high spin fields may depend on the background chosen for Q_{ij}. For example, according to previous studies [11] there are no interactions in flat backgrounds, but there are interactions in AdS_d backgrounds, in particular $d=3,4$. In 2T physics, flat backgrounds or AdS_d backgrounds both exist in the same $d+2$ dimensional theory as they emerge from different embeddings of d dimensions in $d+2$ dimensions (see the last reference in [2] and [7]). So it should be interesting to study such issues in detail with regard to high spin field interactions.

In the case of worldline supersymmetry, local sp$(2,R)$ is replaced by local osp$(n|2)$ where n is the number of supersymmetries. This describes 2T physics for spinning particles [3]. Local osp$(n|2)$ on the worldline can be maintained in the presence of background fields, and this has been studied to some extent [7], but more work along the present paper, to build a noncommutative field theory, remains to be done. We may guess that the appropriate gauge group for the supersymmetric noncommutative field theory would be u$_\star(n|1,1)$. Therefore it would be interesting to take the same form of the action in Eq. (19) and repeat the analysis of the current paper for the noncommutative supergroup u$_\star(n|1,1)$. It is likely that the content of this theory is the spinning generalization of what we have discussed in this paper.

In the case of space-time supersymmetry, the worldline action in the absence of background fields has been constructed [4,1]. The local symmetries are richer: in addition to local sp$(2,R)$ they include a $d+2$ dimensional version of kappa supersymmetry and its bosonic generalizations. For the special supersymmetries osp$(N|4)$, su$(2,2|N)$, $F(4)$, and osp$(6,2|N)$ one obtains a $d+2$ dimensional formulation of the superconformal particle in $d=3,4,5,6$ dimensions, respectively. For other supergroups one obtains brane collective coordinates in interaction with superparticle coordinates, giving unitary supersymmetric Bogomol'nyi-Prasad-Sommerfield (BPS) states as the quantum states of the theory. In particular, for osp$(1|64)$ one obtains a toy M model that embodies certain interesting features of M theory [4,1].

The case of background fields in the presence of space-time supersymmetry in the worldline theory remains to be constructed. We expect this to be a rather interesting and rewarding exercise, because kappa supersymmetry is bound to require the background fields to satisfy dynamical equations of motion, as it does in 1T physics [30]. The supersymmetric field equations thus obtained in $d+2$ dimensions should be rather interesting as they would include some long sought field theories in $d+2$ dimensions, among them super-Yang-Mills and supergravity theories. Perhaps one may also attempt directly the space-time supersymmetrization of the present approach, bypassing the background field formulation of the worldline theory.

The matrix approach described above should eventually be considered with space-time supersymmetry. It is conceivable that these methods would lead to a formulation of covariant M(atrix) theory [31]. In this context we expect osp$(1|64)$ to play a crucial role, as some of its attractive features appear to be quite relevant to M theory [32,4,33,34]. In future work we intend to pursue the types of issues that are touched upon in this section.

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APPENDIX

Let $h(X_1,X_2)$ be given by the expansion in Eq. (58). We compute explicitly

$$=X_1^2h-\frac{1}{4}\partial_2^2h=H \tag{A1}$$

$$=\sum_{s=0}^{\infty}\left(X_1^2h_s^{M_1\cdots M_s}-\frac{(s+2)(s+1)}{4}\eta_{MN}h_{s+2}^{MNM_1\cdots M_s}\right)\times(X_{2M_1}\cdots X_{2M_s}). \tag{A2}$$

By applying the formula a second time we compute $\frac{1}{2}\{X_1^2,H\}_\star$. Then the component form of Eq. (55) gives Eq. (59).

Next we compute the commutator

$$[X_1^2,h]_\star=2iX_1\cdot\partial_2h \tag{A3}$$

$$=2i\sum_{s=0}^{\infty}(s+1)X_{1N}h_{s+1}^{NM_1\cdots M_s}(X_{2M_1}\cdots X_{2M_s}) \tag{A4}$$

and anticommutator

$$\frac{1}{2}\{(X_1\cdot X_2),H\}_\star$$

$$=(X_1\cdot X_2)H+\frac{1}{4}(\partial_1\cdot\partial_2)H \tag{A5}$$

$$=\sum_{s=0}^{\infty}\left(\frac{1}{s}X_1^{(M_1}H_{s-1}^{M_2\cdots M_s)}+\frac{(s+1)}{4}\partial_NH_{s+1}^{NM_1\cdots M_s}\right)\times(X_{2M_1}\cdots X_{2M_s}). \tag{A6}$$

We use them in computing the component form of Eq. (56), which gives Eq. (60).

Finally we compute the commutator

$$[(X_1\cdot X_2),h]_\star=i(X_2\cdot\partial_2-X_1\cdot\partial_1)h \tag{A7}$$

$$=i\sum_{s=0}^{\infty}((s-X_1\cdot\partial_1)h_s^{M_1\cdots M_s})(X_{2M_1}\cdots X_{2M_s}) \tag{A8}$$

and anticommutator

$$\frac{1}{2}\{X_2^2,H\}_\star$$

$$=X_2^2H-\frac{1}{4}\partial_1^2H \tag{A9}$$

$$=\sum_{s=0}^{\infty}\left(\frac{2}{s(s-1)}\eta^{(M_1M_2}H_{s-2}^{M_3\cdots M_s)}-\frac{1}{4}\partial_1^2H_s^{M_1\cdots M_s}\right)\times(X_{2M_1}\cdots X_{2M_s}). \tag{A10}$$

By inserting them in Eq. (57) we obtain the component form given in Eq. (61).

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