Horizon holography

Ivo Sachs* and Sergey N. Solodukhin†
Theoretische Physik, Ludwig-Maximilians Universität, Theresienstrasse 37, D-80333, München, Germany
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A holographic correspondence between horizon data and space-time physics is investigated. We find similarities with the AdS-CFT correspondence, based on the observation that the optical metric near the horizon describes a Euclidean, asymptotically anti–de Sitter space. This picture emerges for a wide class of static space-times with a nondegenerate horizon, including Schwarzschild black holes as well as de Sitter space-time. We reveal an asymptotic conformal symmetry at the horizon. We compute the conformal weights and 2-point functions for a scalar perturbation and discuss possible connections with a conformal field theory located on the horizon. We then reconstruct the scalar field and the metric from the data given on the horizon. We show that the solution for the metric in the bulk is completely determined in terms of a specified metric on the horizon. From the general relativity point of view our solutions present a new class of space-time metrics with nonspherical horizons. The horizon entropy associated with these solutions is also discussed.

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I. INTRODUCTION

According to the holographic principle the information about physics in space-time would be encoded on a lower dimensional surface, the "holographic screen" [1,2]. Given the data on the screen we can reconstruct the events in the rest of space-time. A concrete realization of the holographic description was found recently for string theory on anti–de Sitter (AdS) space-time. Here the holographic screen is the timelike boundary of AdS and the holographic data are given in the form of a conformal field theory (CFT) living on the screen [3–5]. According to the AdS-CFT correspondence, these data are then enough to reconstruct the physics in the bulk of the anti–de Sitter space. For example, in order to reconstruct the space-time metric in the bulk one has to specify the metric on the screen. This metric in turn couples to the stress energy tensor of the boundary CFT. The quantum expectation value of the dual stress tensor is another piece of the CFT data which is necessary for the reconstruction. The explicit reconstruction of the metric is given in [6].

In more general situations, when the bulk space-time is not necessarily asymptotically anti–de Sitter space, but has a horizon, the holographic information may be encoded on this horizon. In this paper we investigate this situation in some detail. Concretely, we suggest that horizon holography may be related to holography on AdS space by observing that the optical metric near an arbitrary static (nondegenerate) horizon has the form of the direct product of $\mathbb{R}$ (time) and Euclidean anti–de Sitter space. Therefore, some elements of the holographic AdS-CFT dictionary can be transcribed to the present case. Formulating this dictionary we proceed in two steps. First, we specify the data on the holographic screen sufficient to reconstruct the physics in the bulk. Second, one would like to show that the holographic data can be described in terms of some field theory, specifically a conformal field theory. In this paper we mainly concentrate on the first step; that is, we analyze in detail the reconstruction of the matter fields and metric from the data on the horizon. Concerning the second step we compute the correlation functions on the horizon along the same lines as in AdS-CFT. These correlation functions are indeed the same as expected to arise in some conformal field theory. This computation supports the proposed correspondence but further work will be required to firmly establish such a duality and to reveal the details of the underlying CFT. Unfortunately, in the present situation we do not have a string theory formulation of the problem as a guiding principle.

Another motivation for the present work is to understand black hole entropy [7] as arising from some field theory living on the horizon. This was carried out explicitly for the (2+1)-dimensional black hole in [8]. In more general situations one then has to investigate the general structure of the space-time metric near horizons and to reveal the corresponding near-horizon asymptotic symmetries. This idea appeared in recent attempts to understand the black hole entropy in terms of degrees of freedom at the horizon [9,10]. By analyzing a general spherically symmetric metric with horizon [10] or imposing certain fall-off condition [9] for the metric coefficients near horizon one finds the presence of two-dimensional conformal symmetry. This could hint to a construction of a Hilbert space for quantum black holes as a representation of the conformal symmetry group at the horizon. However, in order to incorporate the full dynamics near the horizon it is important to analyze the most general class of metrics admitting the interpretation in terms of horizon and to find the corresponding symmetries. In this paper we give up the condition of spherical symmetry and propose general form for the static metric with Killing horizon. (A similar form was suggested earlier in [11] as a off-shell description of black hole metric near horizon.) Remarkably, this metric still can be a solution to the Einstein equations. We show this by expanding the metric near horizon and observing that all terms of the expansion are uniquely determined by specifying arbitrary metric on the horizon (see [12] for a related discussion for spherically symmetric metrics). Since the metric on the horizon can be chosen arbitrarily the

*Email address: ivo@theorie.physik.uni-muenchen.de
†Email address: soloduk@theorie.physik.uni-muenchen.de

solutions we have found present a new class of space-times (with nonspherical horizons) of general relativity. Furthermore, the asymptotic symmetry near the horizon is conformal but of different type than observed in [9,10].

This paper is organized as follows. In Sec. II we analyze the optical metric, exhibit the Euclidean anti–de Sitter space near the horizon and compute the conformal weights and 2-point functions on the horizon. In Sec. III we consider the general form of the static metric with nondegenerate horizon and reveal the conformal asymptotic symmetry. The reconstruction of the scalar field is considered in Sec. IV and the reconstruction of the metric is carried out in Sec. V. The entropy associated with the nonspherical horizon of our solutions is discussed in Sec. VI. The conclusions are then presented in Sec. VII. The reader mainly interested in application of our results to general relativity may go directly to Secs. III, V, and VI.

II. LOOKING THROUGH THE OPTICAL METRIC

We begin this section by finding adapted coordinates that exhibit the similarities of the geometry near event horizons and Euclidean AdS spaces. We then compute some relevant correlation functions in these coordinates in the second part.

A. Uncovering AdS space near the horizon

The key observation important for the present analysis is the universal appearance of the asymptotically anti–de Sitter space in the optical metric near the horizon. To see how this comes along let us for simplicity first consider a $(d+2)$-dimensional static, spherically symmetric metric and transform it to the optical metric

$$ds^2=-f(r)dt^2+f^{-1}(r)dr^2+r^2d\Omega^2_{(d)}$$
$$=f(r)ds^2_{opt}, \tag{2.1}$$

where $d\Omega^2_{(d)}$ is the standard metric on the unit sphere S^d, in this section we will use θ to denote the various angle coordinates on S^d. In terms of the new variable $z=-\int^r f^{-1}(r)dr$ the optical metric may be written in the form

$$ds^2_{opt}=-dt^2+ds^2_{spt}$$

$$ds^2_{spt}=dz^2+\frac{r^2(z)}{f(z)}d\Omega^2_{(d)}. \tag{2.2}$$

In the case of interest, when the metric (2.1) describes a space-time with a non-degenerate horizon at $r=r_+$ the metric function $f(r)$ has a simple root at $r=r_+$, $f(r)=(2/\beta_H)(r-r_+)+O(r-r_+)^2$. We interpret $2\pi\beta_H$ as the inverse Hawking temperature T_H^{-1} as defined with respect to the Killing vector ∂_t. The horizon surface Σ is the d-dimensional sphere of radius r_+. For r close to r_+ we have

$$(r-r_+)\sim e^{-2z/\beta_H},\quad f(z)\sim e^{-2z/\beta_H},$$

so that the spatial part (3.2) of the optical metric takes the asymptotic form

$$ds^2_{spt}=dz^2+C^2e^{2z/\beta_H}d\Omega^2_{(d)}, \tag{2.3}$$

with C is some irrelevant constant. It is not difficult to see that the metric (2.3) is identical with the asymptotic metric of Euclidean anti–de Sitter space. Note that the radius of the anti–de Sitter space is β_H, the inverse Hawking temperature of the horizon in the original metric (2.1).

In general, the spatial metric (2.2) approaches the metric of anti–de Sitter space only asymptotically so that in the sub-leading terms the metric will deviate from that of the anti–de Sitter space. Interestingly, the horizon surface Σ, which is the bifurcation point in the metric (2.1), is mapped to infinity of the anti–de Sitter space under the transformation $z(r)$. This supports the idea that a holographic description in terms of a CFT on the boundary of anti–de Sitter space might be applicable to horizons. The dual CFT theory then would be living on the horizon surface Σ.

The horizon in metric (2.1) can be a black hole horizon or a cosmological horizon. The cosmological horizon appearing in de Sitter space is of particular interest. In this case the spatial part of the optical metric describes anti–de Sitter space not just asymptotically but globally. The metric function in this case is $f(r)=1-r^2/l^2$ so that $\beta_H=l$ (l is the radius of de Sitter space). In terms of the new variable $z=-(l/2)\ln[(r+l)/(r-l)]$ we have that

$$r(z)=l\tanh\frac{z}{l},\quad \frac{r^2}{f(z)}=l^2\sinh^2\frac{z}{l}$$

and the spatial part of the optical metric (2.2)

$$ds^2_{spt}=dz^2+l^2\sinh^2\frac{z}{l}d\Omega^2_{(d)}$$

is precisely the metric of anti–de Sitter space for all z.

B. Conformal dimensions and correlation functions on the horizon

Consider now scalar perturbations in the background (2.1) described by the field equation

$$(\Box-m^2)\phi=0, \tag{2.4}$$

where $\Box$ is the wave operator for the metric (2.1). This equation can be re-written as a field equation on the background of the optical metric (2.2). In order to see that we introduce a new field ϕ_{opt} as follows:

$$\phi=f(r)^{-d/4}\phi_{opt}.$$

Then the equation for ϕ_{opt} reads

$$\Box_{opt}\phi_{opt}=m^2(z)\phi_{opt}, \tag{2.5}$$

where

$$\Box_{opt} = -\partial_t^2 + \left(\frac{f}{r^2}\right)^{d/2} \partial_z \left(\left(\frac{r^2}{f}\right)^{d/2} \partial_z \right) + \frac{f}{r^2} \Delta_\theta \tag{2.6}$$

is the wave operator in terms of the optical metric (2.2), and

$$m^2(z) = m^2 f - \left(\frac{d}{4}\right)^2 (f_{,r})^2 + \frac{d}{4} f f_{,rr} + \frac{d^2}{4r} f f_{,r} \tag{2.7}$$

is an effective, z-dependent mass term.

In the near horizon limit, $z \to \infty$, the spatial part of the operator (2.6) is identical with the Laplace operator at asymptotic infinity of Euclidean anti–de Sitter space. To continue we split off the time dependence of the optical field, $\phi_{opt} = e^{\imath \omega t} \phi_\omega(z,\theta)$, so that, as $z \to \infty$, ϕ_ω satisfies equation

$$\Delta_{AdS} \phi_\omega = M^2 \phi_\omega ,$$
$$M^2 = -\omega^2 - \left(\frac{d}{2\beta_H}\right)^2 . \tag{2.8}$$

This is the familiar equation for the scalar perturbations appearing in the context of the AdS-CFT correspondence. The unusual feature of Eq. (2.8) is the negative sign for the effective mass square term. This means that the perturbations are tachyonic.[1] Asymptotically,

$$\phi_\omega \sim \chi_\lambda(\theta) e^{\lambda z/\beta_H}, \quad z \to \infty$$

and from Eq. (2.8) we find two possible roots for $\lambda_{(\omega)}$

$$\lambda^{\pm}_{(\omega)} = -\frac{d}{2} \pm \imath \omega \beta_H . \tag{2.9}$$

In the context of the AdS-CFT correspondence $\lambda^{\pm}_{(\omega)}$ are related to the conformal dimension of a dual operator in the CFT by

$$h^{\pm}_{(\omega)} = d + \lambda^{(\omega)}_{\pm} = \frac{d}{2} \pm \imath \omega \beta_H . \tag{2.10}$$

We see that the conformal weights $h^{\pm}_{(\omega)}$ are complex. So one might worry about unitarity. Note however that, contrary to the usual AdS-CFT correspondence, the conformal field theory in question (if it exists) lives on a Euclidean surface Σ, and does not describe the time evolution of the system.

Near the horizon, the modes with λ^+_ω and λ^-_ω decay in the same way so that we can not discard one of them as decaying faster than another. Therefore, the boundary condition near the horizon (or, equivalently, near infinity of AdS space) contains both modes:

$$\lim_{z \to +\infty} \phi_{opt} = \varphi(t,z,\theta) = e^{-(dz/2\beta_H)} \int_{-\infty}^{+\infty} d\omega e^{-\imath \omega t}$$
$$\times (\varphi^+_\omega(\theta) e^{\imath \omega z} + \varphi^-_\omega(\theta) e^{-\imath \omega z}). \tag{2.11}$$

In terms of the original scalar field $\phi = e^{dz/2\beta_H} \phi_{opt}$ this is the generic behavior and corresponds to the presence of right- and left-moving modes which always appear for any field propagating near a horizon. In the AdS-CFT picture $\varphi^{\pm}_\omega(\theta)$ is dual to an operator $\mathcal{O}^{\pm}_{(-\omega)}$ of conformal dimension $h^{\pm}_{(\omega)}$. Note that the operators $\mathcal{O}^{\pm}_{(\omega)}$ and $\mathcal{O}^{\mp}_{(-\omega)}$ have the same conformal dimension $h^{\pm}_{(\omega)}$.

In order to adjust the prescription of [5] for computing the 2-point functions of dual operators to the present case, let us first consider a hypersurface of constant $z = Z$ as a boundary, afterwards we will take the limit of infinite Z. In this limit the hypersurface approaches the horizon light-cone. Considering Eq. (2.11) as a boundary condition near the horizon we apply Green's formula

$$\phi_{opt}(t,z,\theta) = \int_{-\infty}^{+\infty} dt' \int_{S^d} d\mu(\theta') e^{dz'/\beta_H} (G \partial_{z'} \varphi(z',t',\theta')$$
$$- \partial_{z'} G \varphi(z',t',\theta'))_{z'=Z} , \tag{2.12}$$

where $d\mu(\theta)$ is the integration measure on the sphere S^d. The boundary function $\varphi(z',t',\theta')$ takes the form (2.11) and

$$G(t,t',z,z',\theta,\theta') = \int_{-\infty}^{+\infty} d\omega e^{-\imath \omega (t-t')} G_\omega(z,z',\theta,\theta')$$

is Green's function. Near the horizon, G_ω is a Green function of the operator (2.8) on anti–de Sitter and we can use the AdS expression [13]

$$G_\omega \simeq c_\lambda \frac{e^{-(\lambda_\omega + d) z'/\beta_H}}{\left(\cosh \frac{z}{\beta_H} - \sinh \frac{z}{\beta_H} \cos \gamma\right)^{(\lambda_\omega + d)}} , \tag{2.13}$$

where γ is the geodesic distance between two points θ and θ' on sphere S^d. Again, for λ^+_ω and λ^-_ω the function (2.13) decays in the same way for large z'. Therefore, the complete Green's function is the sum of two contributions. When both points z and z' are close to the boundary of AdS we have

$$G_\omega = e^{-(d/2\beta_H)(z+z')} (G^-_\omega(\theta,\theta') e^{\imath \omega (z+z')}$$
$$+ G^+_\omega(\theta,\theta') e^{-\imath \omega (z+z')}),$$

$$G^-_\omega(\theta,\theta') = \frac{c^+_\omega}{(1-\cos\gamma)^{h^+_\omega}}, \quad G^+_\omega(\theta,\theta') = \frac{c^-_\omega}{(1-\cos\gamma)^{h^-_\omega}}. \tag{2.14}$$

In this limit Green's formula (2.12) indeed gives the function which satisfies the boundary condition (2.11). It also leads to non-local relations between φ^+_ω and φ^-_ω

$$\varphi^+_\omega(\theta) = -2\imath\omega \int_{S^d} d\mu(\theta') G^-_\omega(\theta,\theta') \varphi^-_\omega(\theta')$$

$$\varphi^-_\omega(\theta) = 2\imath\omega \int_{S^d} d\mu(\theta') G^+_\omega(\theta,\theta') \varphi^+_\omega(\theta'). \tag{2.15}$$

[1] Note, that M^2 becomes positive when ω is imaginary. This, in particular, happens for quasi-normal modes.

We note in passing that, as a consequence of Eqs. (2.12) and (2.17) the components of the Green's function (2.14) should satisfy the consistency condition[2]

$$\int_{S^d} d\mu(\theta'') G^-_\omega(\theta,\theta'') G^+_\omega(\theta'',\theta') = \frac{1}{4\omega^2}\delta^{(d)}(\theta,\theta'). \tag{2.16}$$

Now, in order to compute the 2-point functions of dual operators we have to evaluate the boundary action W_B on the solution (2.12). In terms of ϕ_{opt} and the optical metric one has

$$\begin{aligned} W_B[\varphi] &= \frac{1}{2}\int_{-\infty}^{+\infty} dt \int_{S^d} d\mu(\theta) (\phi_{opt} e^{(dz/2\beta_H)} \\ &\quad \times \partial_z (e^{(dz/2\beta_H)}\phi_{opt}))_{z=Z} \\ &= -\frac{1}{2}\int_{S^d} d\mu(\theta)\int_{-\infty}^{+\infty} d\omega\, i\omega (\varphi^+_\omega(\theta)\varphi^+_{-\omega}(\theta) \\ &\quad - \varphi^-_\omega(\theta)\varphi^-_{-\omega}(\theta)) + \cdots, \end{aligned} \tag{2.17}$$

where the periods stand for contact terms which we will discard in the following. The exponent of $W_B[\varphi]$ should be compared with the generating functional

$$\left\langle \exp\left[\int_{S^d} d\mu(\theta)\int_{-\infty}^{+\infty} d\omega (\varphi^+_{-\omega}(\theta)\mathcal{O}^+_\omega(\theta) + \varphi^-_\omega(\theta)\mathcal{O}^-_{-\omega}(\theta))\right]\right\rangle$$

on the CFT side. Using Eq. (2.14) we then read off the correlation functions of the dual operators

$$\langle \mathcal{O}^+_\omega \mathcal{O}^-_{-\omega}\rangle = \omega^2 G^-_\omega \propto \frac{1}{\left(\sin\frac{\gamma}{2}\right)^{2h^+_\omega}}$$

$$\langle \mathcal{O}^-_\omega \mathcal{O}^+_{-\omega}\rangle = \omega^2 G^+_\omega \propto \frac{1}{\left(\sin\frac{\gamma}{2}\right)^{2h^-_\omega}}, \tag{2.18}$$

which are precisely the correlation functions expected arise on sphere S^d for operators with dimension h^+_ω and h^-_ω respectively.

Remarks

(1) Extra boundary. In the above computation of correlation functions we neglected the presence of boundaries different from the horizon. An extra boundary at infinity appears for instance when space-time is asymptotically flat or anti–de Sitter. Then the computation should also produce 2-point functions between different boundary components.

(2) Hawking radiation. We should also notice the appearance of the inverse Hawking temperature β_H in combination with ω in our formulas. Since ω is the energy of the perturbation it is tempting to speculate that the dual operators $\mathcal{O}^\pm_{(\omega)}$ with conformal dimension $h^\pm_{(\omega)}$ could enter in a dual description of the Hawking radiation in terms of a CFT. However, we do not explore this possibility any further in this paper leaving this issue to a future work.

(3) Similarity to 't Hooft's S-matrix. When the horizon is two-dimensional ($d=2$ in the above formulas) we may introduce the function $f_0(\theta,\theta') = -\ln\sin^2(\gamma/2)$ which, for $\theta = \theta'$, is a solution to the equation $\Delta_{S^2} f_0 = 1/2R$ (R is the curvature scalar of S^2). For θ close to θ' it behaves as usual Green's function in two dimensions, $f_0(\theta,\theta') \simeq -\ln|\theta-\theta'|^2$. Then the kernels $G^\pm$ can be written as

$$G^\pm = \exp((1 \pm \imath\omega\beta_H) f_0(\theta,\theta')).$$

For Schwarzschild black hole we have $\omega\beta_H = 4GM\omega$, where M is the mass of the black hole and ω is the energy of a particle falling into black hole. The imaginary part of the exponent in $G^\pm$ resembles the phase appearing in the two-particle amplitude for the gravitational scattering in the eikonal approximation [14,15]:

$$S_{12} = \exp(2\imath G p^-_1 p^+_2 f(y_1,y_2)),$$

$$f(y_1,y_2) = -\ln|y_1-y_2|^2,$$

where $p^\pm_i$ and y_i are the momentum and transversal coordinate of particle *i*. The horizon sphere with angle coordinates θ indeed plays a role of a transversal space for a particle falling into a black hole. Moreover, according to Eq. (2.15) $G^\pm$ relate the in-going and out-going modes at the horizon. Therefore, this similarity may not be accidental and deserves a deeper analysis.

(4) de Sitter space. Our computation of the correlation functions is, in particular, applicable to de Sitter space. The 2-point functions (2.18) are then defined on the cosmological horizon. Recently, there was some interest in quantum gravity and string theory on de Sitter space [16,17]. In particular, some form of dS/CFT duality was proposed in [17]. The

[2]This relation is valid for kernels (2.14) with any h^+ and h^- such that $h^+ + h^- = d$. Most easily it can be proved in flat space (sphere of infinite radius) by doing the Fourier transformation of the kernels. Furthermore, in the particular case where $h^+ = d/2 + \frac{1}{2}$ and $h^- = d/2 - \frac{1}{2}$, the kernels G^- and G^+ are respectively Dirichlet and Neumann correlation function [13]. The proof of Eq. (2.16) in this case was explained to us by A. Barvinsky.

conformal field theory in this proposal lives on a Euclidean space which is a hypersurface of infinite past in de Sitter space. The computation of the 2-point functions in [17] for the CFT defined on infinite past of de Sitter has some similarities with our computation in this section. In particular, the necessity of more general boundary conditions including both modes with h^+ and h^- also arises there. In spite of these similarities the two CFT's are defined in different space-time regions. It would be interesting to understand the relation between these two approaches.

III. GENERAL METRIC AND ASYMPTOTIC SYMMETRIES

What we have found in the previous section is that the optical metric of a spherically symmetric space-time with horizon is a direct product of time and a Euclidean space which near the horizon asymptotically approaches anti–de Sitter space. As we will show, this observation applies in fact to a more general class of metrics with a horizon. Motivated by recent study [18–20,6] on asymptotically anti–de Sitter spaces we consider a general static ansatz of form

$$ds^2=e^{\sigma(x,\rho)}\left(-dt^2+\frac{l^2 d\rho^2}{4\rho^2}+\frac{1}{\rho}g_{ij}(x,\rho)dx^i dx^j\right),$$

$$\sigma(x,\rho)=\ln\rho+\sigma_{(0)}(x)+O(\rho), \tag{3.1}$$

that is the function e^σ has a simple root at $\rho=0$ corresponding to the location of the horizon. For the spatial part of the optical metric in Eq. (3.1) the horizon surface defined by $\rho=0$ is the boundary at infinity. Provided the function $g_{ij}(x,\rho)$ is analytic near $\rho=0$ and approaches $g_{(0)ij}(x)$ when $\rho=0$, we find that the spatial metric in Eq. (3.1) indeed describes asymptotically anti–de Sitter space as ρ goes to zero. The parameter l in Eq. (3.1) determines the radius of this anti–de Sitter. The inverse Hawking temperature for the horizon as defined with respect to $\xi_t=\partial_t$ in the metric (3.1) reads

$$\beta_H^{-1}=l^{-1}\rho\partial_\rho\sigma\big|_{\rho=0}. \tag{3.2}$$

Hence, if $e^\sigma\sim\rho$ for small ρ, then $\beta_H=l$. The metric (3.1) is the general form for a static metric with a Killing horizon. This form was proposed earlier in [11] as off-shell form of the metric near an arbitrary, static black hole horizon. In the next section we will prove that Eq. (3.1) is, in fact, valid also on-shell by determining the functions $\sigma(x,\rho)$ and $g_{ij}(x,\rho)$ which solve the $(d+2)$-dimensional Einstein equations.

First, let us however analyze the $(d+2)$-dimensional diffeomorphisms which preserve the form (3.1). This is again motivated by the analysis done for anti–de Sitter space where the conformal transformations in the boundary originate from a subset of diffeomorphisms in the bulk [21,22] (see also [6]). In general the diffeomorphisms ξ can be functions of x, ρ and time t. However, at present we restrict ourselves to the static case and assume that $\partial_t\xi^\rho=\partial_t\xi^i=0$. This, however, does not mean that ξ^t cannot depend on t. The most general diffeomorphism leaving invariant the form (3.1) is then of the form

$$\xi^t=qt+s,\quad \xi^\rho=\rho\alpha(\rho,x),\quad \xi^i=a^i(x,\rho),$$

$$\alpha(\rho,x)=q\ln\rho+\alpha_0(x) \tag{3.3}$$

where q,s are constant and $a^i(x,\rho)$ is subject to the constraint

$$\partial_\rho a^i(x,\rho)=-\frac{l^2}{4}g^{ij}(x,\rho)\partial_j\alpha_0(x), \tag{3.4}$$

similar to the equation arising in the analysis of diffeomorphisms in anti–de Sitter space [22]. The function $a^i(x,\rho)$ in Eqs. (3.3) and (3.4) is assumed to be regular at the horizon $(\rho=0)$ and moreover we impose the boundary condition that $a^i(x,\rho=0)=0$ (see [22]).

Under these diffeomorphisms the functions $\sigma(x,\rho)$ and $g_{ij}(x,\rho)$ transform as follows:

$$\mathcal{L}_\xi\sigma=a^i(x,\rho)\partial_i\sigma+\alpha\rho\partial_\rho\sigma+2\rho\partial_\rho\alpha, \tag{3.5}$$

$$\mathcal{L}_\xi g_{ij}=\nabla_i(g_{jk}\xi^k)+\nabla_i(g_{ik}\xi^k)-\alpha(g_{ij}-\rho\partial_\rho g_{ij})-2\rho\partial_\rho\alpha g_{ij}. \tag{3.6}$$

The covariant derivative in Eq. (3.6) is with respect to the metric $g_{ij}(x,\rho)$. The metric induced on a surface of constant ρ ($\rho=0$ is the horizon) is $\gamma_{ij}(x,\rho)=e^\sigma(1/\rho)g_{ij}$. Under the diffeomorphisms (3.3) it then transforms as

$$\mathcal{L}_\xi\gamma_{ij}=\nabla_i^{(\gamma)}(\gamma_{jk}a^k)+\nabla_i^{(\gamma)}(\gamma_{ik}a^k)+\alpha\rho\partial_\rho\gamma_{ij}, \tag{3.7}$$

where $\nabla_i^{(\gamma)}$ is the covariant derivative with respect to $\gamma_{ij}(x,\rho)$.

For $q=0$ the diffeomorphisms (3.3) are similar to the those found in [22] for the case of AdS. Since $\alpha(x)$ is not function of ρ the last term in Eqs. (3.5) and (3.6) disappears. On the horizon surface $(\rho=0)$ the transformations of this type act as conformal transformations,

$$\mathcal{L}_\xi\sigma_{(0)}(x)=\alpha_0(x)$$

$$\mathcal{L}_\xi g_{(0)ij}(x)=-\alpha_0(x)g_{(0)ij}(x). \tag{3.8}$$

On the other hand, for the induced metric $\gamma_{ij}(x,\rho)$ we find from Eq. (3.7) that

$$\mathcal{L}_\xi\gamma_{ij}(x)=0, \tag{3.9}$$

i.e., the induced metric on the horizon does not change. Thus, in spite of the fact that the parameter $\alpha_0(x)$ is function on the d-dimensional horizon surface Σ, the transformation (3.3) (with $q=0$) effectively acts as Weyl transform in the two-dimensional (t,ρ) subspace orthogonal to Σ. The induced metric on Σ remains invariant under this.

The remaining diffeomorphisms in Eq. (3.3), parametrized by q lead to the transformations

$$\mathcal{L}_\xi\sigma=q(2+\rho\ln\rho\partial_\rho\sigma)$$

$$\mathcal{L}_\xi g_{ij}=q(-2g_{ij}-\ln\rho g_{ij}+\rho\ln\rho\partial_\rho g_{ij})$$
$$\mathcal{L}_\xi\gamma_{ij}=q\ln\rho\partial_\rho\gamma_{ij}. \tag{3.10}$$

These infinitesimal transformations can be integrated to the finite transformations

$$\sigma(x,\rho)\to\sigma(x,\rho^{e^q})+2q$$
$$g_{ij}(x,\rho)\to e^{-2q}g_{ij}(x,\rho^{e^q})$$
$$\gamma_{ij}(x,\rho)\to\gamma_{ij}(x,\rho^{e^q}) \tag{3.11}$$

and results in simply replacing ρ by ρ^{e^q} in these functions. The interpretation of this transformation is that it changes the definition of the Hawking temperature. Indeed we find from Eqs. (3.2) and (3.11) that

$$\beta_H^{-1}\to e^q\beta_H^{-1}. \tag{3.12}$$

IV. RECONSTRUCTION OF THE SCALAR FIELD

In this section we consider a scalar perturbation on a general background with horizon. The scalar field equation

$$(\Box-m^2)\Phi=0 \tag{4.1}$$

is the same as in Sec. II but now the background is given by Eq. (3.1). To continue we suppose that the functions $\sigma(x,\rho)$ and $g_{ij}(x,\rho)$ in the metric (3.1) are given in terms of an expansion in ρ, i.e.,

$$\sigma(x,\rho)=\ln\rho+\sigma_{(0)}(x)+\sigma_{(1)}(x)\rho+\cdots$$
$$g_{ij}(x,\rho)=g_{(0)ij}(x)+g_{(1)ij}(x)\rho+\cdots. \tag{4.2}$$

Our purpose in this section is to understand what data one should specify on horizon in order to reconstruct the field everywhere in the bulk.

At fixed energy ω the perturbation splits on two sectors

$$\Phi_\omega=e^{\imath\omega t}(\rho^{\imath\omega\beta_H/2}\phi_\omega(x,\rho)+\rho^{-(\imath\omega\beta_H/2)}\psi_\omega(x,\rho)), \tag{4.3}$$

which are the h_ω^+ and h_ω^- sectors discussed in Sec. II. They describe right- and left-moving waves at the horizon. For the functions $\phi_\omega(x,\rho)$ and $\psi_\omega(x,\rho)$ we then have the following expansion:

$$\phi_\omega(x,\rho)=\sum_{k=0}^{\infty}\phi_\omega^{(k)}(x)\rho^k$$
$$\psi_\omega(x,\rho)=\sum_{k=0}^{\infty}\psi_\omega^{(k)}(x)\rho^k. \tag{4.4}$$

The two sectors decouple in the field equation (4.1). Therefore, we will restrict the consideration to the ϕ_ω sector. The analysis for ψ_ω is similar. At fixed ω the equation on $\phi_\omega(x,\rho)$ reads

$$\imath\omega\beta_H\phi_\omega(d(\rho\partial_\rho\sigma-1)+\rho\,\mathrm{Tr}(g^{-1}\partial_\rho g))+\rho\partial_\rho\phi_\omega(4+\imath 4\omega$$
$$+2d(\rho\partial_\rho\sigma-1)+2\rho\,\mathrm{Tr}(g^{-1}\partial_\rho g))+4\rho^2\partial_\rho^2\phi_\omega$$
$$+\rho\nabla_g^2\phi_\omega+\frac{d}{2}\rho g^{ij}\partial_i\sigma\partial_j\phi_\omega-m^2e^\sigma\phi_\omega=0. \tag{4.5}$$

Substituting the expansion (4.4) into Eq. (4.5) and assuming that all coefficients in the expansion (4.2) are known we solve Eq. (4.5) perturbatively at each order in ρ. It proves that all terms of the expansion (4.4) are uniquely determined provided the first term $\phi_\omega^{(0)}(x)$ is given. The same is true also for the ψ_ω sector where one has to fix the function $\psi_\omega^{(0)}(x)$ on the horizon. The first two terms in Eq. (4.4) are determined as follows:

$$\phi_\omega^{(1)}(x)=-\frac{1}{4(1+\imath\omega\beta_H)}\Big[\imath\omega\beta_H(d\sigma_{(1)}+\mathrm{Tr}\,g_{(1)})\phi_\omega^{(0)}$$
$$+\nabla_{(0)}^2\phi_\omega^{(0)}+\frac{d}{2}g_{(0)}^{ij}\partial_i\sigma_{(0)}\partial_j\phi_\omega^{(0)}-m^2e^{\sigma_{(0)}}\phi_\omega^{(0)}\Big], \tag{4.6}$$

$$\phi_\omega^{(2)}(x)=-\frac{1}{8}(d\sigma_{(1)}+\mathrm{Tr}\,g_{(1)})\phi_\omega^{(1)}-\frac{\imath\omega\beta_H}{8(2+\imath\omega\beta_H)}$$
$$\times(2d\sigma_{(2)}+2\,\mathrm{Tr}\,g_{(2)}-\mathrm{Tr}\,g_{(1)}^2)-\frac{1}{8(2+\imath\omega\beta_H)}$$
$$\times\Big[\nabla_{(0)}^2\phi_\omega^{(1)}+\nabla_i(g_{(1)}^{ij}\partial_j\phi_\omega^{(0)})+\frac{1}{2}\partial_i\,\mathrm{Tr}\,g_{(1)}\partial^i\phi_\omega^{(0)}$$
$$+\frac{d}{2}(\partial_i\sigma_{(0)}\partial^i\phi_\omega^{(1)}+\partial_i\sigma_{(1)}\partial^i\phi_\omega^{(0)}$$
$$-g_{(1)}^{ij}\partial_i\sigma_{(0)}\partial_j\phi_\omega^{(0)})-m^2e^{\sigma_{(0)}}(\phi_\omega^{(1)}+\sigma_{(1)}\phi_\omega^{(0)})\Big]. \tag{4.7}$$

At higher orders the expressions become more complicated but the general structure of equation for kth coefficient is always of the form[3]

$$\phi_\omega^{(k)}k(\imath\omega\beta_H+k)+X_{(k)}[\phi_\omega^{(k-1)},\phi_\omega^{(k-2)},\ldots]=0, \tag{4.8}$$

where $X_{(k)}$ is a polynomial in $\phi_\omega^{(p)}$, $p<k$ and their derivatives. Thus, the coefficient $\phi_\omega^{(k)}(x)$ is completely determined by the previous coefficients $\phi_\omega^{(k-1)}$, $\phi_\omega^{(k-2)},\ldots,\phi_\omega^{(0)}$ and ultimately by the function $\phi_\omega^{(0)}(x)$. A separate analysis is needed for zero-energy ($\omega=0$) modes. In this case some terms in Eq. (4.5) disappear. Moreover, instead of the expansion (4.3),(4.4) we have

[3]One can see that for complex ω the has a pole at $\Im\omega=-2\pi T_Hk$, where k is integer. It is tempting to relate it with the quantization of imaginary part of frequency ω for quasi-normal modes. This, however, needs a more careful analysis.

$$\Phi_0=(\phi_{(0)}(x)+\rho\phi_{(1)}(x)+\cdots)+\ln\rho(\psi_{(0)}(x)+\rho\psi_{(1)}(x)+\cdots). \qquad (4.9)$$

This time, ϕ and ψ sectors couple to each other in the field equation. Nevertheless, specifying two functions $\phi_{(0)}(x)$ and $\psi_{(0)}(x)$ on the horizon completely determines all terms in the expansion (4.9). In the first order in ρ we find

$$\psi_{(1)}(x)=-\frac{1}{4}\left(\nabla^2_{(0)}\psi_{(0)}+\frac{d}{2}g^{ij}_{(0)}\partial_i\sigma_{(0)}\partial_j\psi_{(0)}-m^2e^{\sigma_{(0)}}\psi_{(0)}\right),$$

$$\phi_{(1)}(x)=-\frac{1}{3}\psi_{(1)}(x)-\frac{1}{6}(d\sigma_{(1)}+\mathrm{Tr}\,g_{(1)})\psi_{(0)}(x)+\frac{1}{12}\left(\nabla^2_{(0)}\phi_{(0)}+\frac{d}{2}g^{ij}_{(0)}\partial_i\sigma_{(0)}\partial_j\phi_{(0)}-m^2e^{\sigma_{(0)}}\phi_{(0)}\right). \qquad (4.10)$$

Thus, for each energy ω one has to specify a pair of functions $\phi^{(0)}_{\omega}(x)$ and $\psi^{(0)}_{\omega}(x)$ [$\phi_{(0)}(x)$ and $\psi_{(0)}(x)$ if $\omega=0$] on the horizon in order to reconstruct the scalar field everywhere in the bulk. This set of functions forms the "holographic data" on the horizon. In the AdS-CFT correspondence in order to reconstruct a scalar field in the bulk one has to specify two functions on the boundary: a source coupled to the dual CFT operator and the expectation value of the dual operator [6]. The discussion in Sec. II shows that for a fixed $\omega>0$ the functions $\phi^{(0)}_{\omega}(x)$ and $\phi^{(0)}_{-\omega}(x)$ [and $\psi^{(0)}_{\omega}(x)$ and $\psi^{(0)}_{-\omega}(x)$] indeed form such holographic pair.

V. RECONSTRUCTION OF THE METRIC

In this section we show that provided the metric on horizon is specified one can uniquely reconstruct metric everywhere in bulk. For simplicity we consider the static case only. The stationary and time-evolving cases are, however, of great importance and we plan to approach these cases in a separate publication.

We start with the $(d+2)$-dimensional bulk Einstein equations

$$R_{AB}-\frac{1}{2}RG_{AB}=-\frac{d(d+1)}{2L^2}G_{AB}, \quad A,B=(t,\rho,i), \qquad (5.1)$$

where for generality we included a bulk cosmological constant $\Lambda\propto 1/L^2$. Our consideration will cover all cases: $\Lambda>0$, $\Lambda<0$, and $\Lambda=0$. The appropriate case can be recovered by analytic continuation $L^2\to -L^2$ in our formulas.

Our ansatz for the metric G_{AB} is as in Eq. (3.1), i.e.,

$$ds^2=e^{\sigma(x,\rho)}\left(-dt^2+\frac{\beta_H^2d\rho^2}{4\rho^2}+\frac{1}{\rho}g_{ij}(x,\rho)dx^idx^j\right)=e^{\sigma(x,\rho)}\tilde{G}_{AB}dX^AdX^B, \qquad (5.2)$$

where for convenience we separated the optical metric $\tilde{G}_{AB}$. The metric (5.2) should describe a horizon at $\rho=0$, x^i, $i=1,\ldots,d$ being the coordinates on the horizon. This is specified by boundary conditions to be imposed on the function $\sigma(x,\rho)$. This, together with the analyticity condition for the metric $g_{ij}(x,\rho)$ at $\rho=0$, leads us to the following near-horizon expansion:

$$\sigma(x,\rho)=\ln\rho+\sigma_{(0)}(x)+\sigma_{(1)}(x)\rho+\cdots$$

$$g_{ij}(x,\rho)=g_{(0)ij}(x)+g_{(1)ij}(x)\rho+\cdots, \qquad (5.3)$$

which we have already advocated in Sec. IV.

We now show that specifying the function $\sigma_{(0)}(x)$ and the metric $g_{(0)ij}(x)$ on the horizon surface Σ uniquely determines the solution to the Einstein equations (5.1) in bulk for the class of metrics (5.2). Note also that assuming the expansion (5.3) we explicitly break the invariance parametrized by q in Eq. (3.3). As we have seen in Sec. III the function $\sigma_{(0)}(x)$ is pure gauge and can be always removed by using the diffeomorphism (3.3) as is seen from Eq. (3.8). Thus, the only real degrees of freedom living on horizon are those of the metric function $g_{(0)ij}(x)$ or, equivalently, of the induced metric $\gamma_{(0)ij}(x)$ (this is in agreement with conclusion made earlier in [10]). However, in the rest we will keep $\sigma_{(0)}(x)$ arbitrary in order to maintain the invariance under the conformal transformation (3.8). Note also that $\sigma_{(0)}(x)$ and $g_{(0)ij}(x)$ are data on the surface Σ which is the bifurcation point in the horizon light cone. By means of the Killing vector ξ_t these data can be extended over the whole light-cone surface.

In terms of the optical metric $\tilde{G}_{AB}$ (5.2) the Einstein equations take the form[4]

$$\tilde{R}_{AB}=-\frac{d}{4}(-2\tilde{\nabla}_A\tilde{\nabla}_B\sigma+\tilde{\nabla}_A\sigma\tilde{\nabla}_B\sigma)+\tilde{G}_{AB}\left(\frac{d}{4}(\tilde{\nabla}\sigma)^2+\frac{1}{2}\tilde{\nabla}^2\sigma+e^{\sigma}\frac{(d+1)}{L^2}\right). \qquad (5.4)$$

Since we are looking for static solutions, i.e., $\partial_t\sigma=\partial_t g_{ij}=0$, we find that $\tilde{R}_{tt}=0$ and $-2\tilde{\nabla}_t\tilde{\nabla}_t\sigma+\tilde{\nabla}_t\sigma\tilde{\nabla}_t\sigma=0$. Hence, the (tt) component of Eq. (5.4) reduces to the equation

$$\frac{d}{4}(\tilde{\nabla}\sigma)^2+\frac{1}{2}\tilde{\nabla}^2\sigma+e^{\sigma}\frac{(d+1)}{L^2}=0. \qquad (5.5)$$

The remaining equations (5.4) can be represented in the form

$$\tilde{R}_{\mu\nu}=-\frac{d}{4}\pi_{\mu\nu}, \quad \mu,\nu=(\rho,i), \qquad (5.6)$$

where, in terms of $\sigma(x,\rho)$ and $g_{ij}(x,\rho)$ we have

[4]Our curvature conventions are as follows $R^{\mu}_{\nu\alpha\beta}=\partial_\alpha\Gamma^{\mu}_{\nu\beta}+\Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\beta}-\alpha\leftrightarrow\beta$ and $R_{\mu\nu}=R^{\alpha}_{\mu\alpha\nu}$. Note that it differs in sign from the one used in [6,20].

$$\pi_{\rho\rho}=-2\partial_\rho^2\sigma-\frac{2}{\rho}\partial_\rho\sigma+(\partial_\rho\sigma)^2$$

$$\pi_{\rho i}=-2\partial_\rho\partial_i\sigma-\frac{1}{\rho}\partial_i\sigma+\partial_i\sigma\partial_\rho\sigma+(g^{-1}\partial_\rho g)_i^j\partial_i\sigma$$

$$\pi_{ij}=\frac{4}{\beta_H^2}(g_{ij}-\rho\partial_\rho g_{ij})\partial_\rho\sigma+\partial_i\sigma\partial_j\sigma-2\nabla_i^g\nabla_j^g\sigma \tag{5.7}$$

and for the Ricci tensor of optical metric we have

$$\tilde{R}_{\rho\rho}=-\frac{d}{4\rho^2}-\frac{1}{2}\partial_\rho\,\mathrm{Tr}\,(g^{-1}\partial_\rho g)-\frac{1}{4}\mathrm{Tr}(g^{-1}\partial_\rho g)^2$$

$$\tilde{R}_{\rho i}=-\frac{1}{2}(\partial_i\,\mathrm{Tr}(g^{-1}\partial_\rho g)-\nabla_j(g^{-1}\partial_\rho g)_i^j)$$

$$\begin{aligned}\tilde{R}_{ij}=R_{ij}[g]-\beta_H^{-2}\bigg[&\frac{d}{\rho}g_{ij}-(d-2)\partial_\rho g_{ij}\\&-g_{ij}\,\mathrm{Tr}(g^{-1}\partial_\rho g)+\rho(2\partial_\rho^2 g-2\partial_\rho g g^{-1}\partial_\rho g\\&+\partial_\rho g\,\mathrm{Tr}(g^{-1}\partial_\rho g))_{ij}\bigg].\end{aligned} \tag{5.8}$$

Equations (5.6)–(5.8) look similar to the $(d+1)$-dimensional Einstein equations analyzed in [20,6] for AdS space. The essential difference is that $\pi_{\mu\nu}$ in Eq. (5.6) does not represent a contribution of $(d+1)$-dimensional negative cosmological constant. Only asymptotically, when $\rho\to 0$, we have that $-(d/4)\pi_{\mu\nu}=-(d/\beta_H^2)\tilde{G}_{\mu\nu}$, $\mu\nu=(\rho,i)$. That is why the solution to the Eqs. (5.6) indeed describes a space asymptotically approaching AdS space. However, already in the sub-leading terms $\pi_{\mu\nu}$ deviates from the form dictated by the cosmological constant. The complete system (5.5)–(5.8) describes a coupled $(d+1)$-dimensional system of metric $g_{ij}(x,\rho)$ and a scalar field $\sigma(x,\rho)$.

Technically, we solve Eqs. (5.5)–(5.8) in the same way as in the case of AdS space [20,6]. One substitutes the expansion (5.3) into the field equations which can then be solved order by order by matching the coefficients at fixed powers of ρ. This leads to a set of equations sufficient to determine the coefficients $\sigma_{(k)}(x)$ and $g_{(k)ij}(x)$. Compared to the AdS case [6] we have one extra field $\sigma(x,\rho)$ but there is also an extra equation (5.5). So that, the total number of equations is enough to determine all unknown functions from the given holographic data. To the leading order, Eqs. (5.5), (5.6) are satisfied identically. This is a consequence of our choice of the leading term in the expansion of the scalar function $\sigma(x,\rho)$ (5.3).

At the first nontrivial order Eq. (5.5) gives the relation

$$\begin{aligned}\sigma_{(1)}=-\frac{1}{(d+2)}\mathrm{Tr}\,g_{(1)}-\frac{\beta_H^2}{(d+2)}\bigg(&\frac{d}{4}(\nabla_{(0)}\sigma_{(0)})^2+\frac{1}{2}\nabla_{(0)}^2\sigma_{(0)}\\&+e^{\sigma_{(0)}}\frac{(d+1)}{L^2}\bigg)\end{aligned} \tag{5.9}$$

between $\sigma_{(1)}$ and $g_{(1)}$. Another relation comes from the ρ^0 term in the (ij) component of Eq. (5.6)

$$\begin{aligned}&2g_{(1)ij}-(\mathrm{Tr}\,g_{(1)}+d\sigma_{(1)})g_{(0)ij}\\&\quad=\beta_H^2\bigg(\frac{d}{4}(\partial_i\sigma_{(0)}\partial_j\sigma_{(0)}-2\nabla_{(0)i}\nabla_{(0)j}\sigma_{(0)})+R_{(0)ij}\bigg).\end{aligned} \tag{5.10}$$

Equations (5.9) and (5.10) are enough to determine coefficients $\sigma_{(1)}(x)$ and $g_{(1)ij}(x)$. After some algebra we find that

$$\begin{aligned}\sigma_{(1)}(x)=-\frac{1}{4}\beta_H^2R_{(0)}+\beta_H^2\bigg(&\frac{d(d-3)}{16}(\nabla_{(0)}\sigma_{(0)})^2\\&+\frac{(d-1)}{4}\nabla_{(0)}^2\sigma_{(0)}+\frac{(d-2)(d+1)}{4L^2}e^{\sigma_{(0)}}\bigg)\end{aligned} \tag{5.11}$$

$$\begin{aligned}g_{(1)ij}(x)=&\frac{1}{2}\beta_H^2\bigg(R_{(0)ij}+\frac{1}{2}R_{(0)}g_{(0)ij}\bigg)\\&+\frac{d}{4}\beta_H^2\bigg(\frac{1}{2}\partial_i\sigma_{(0)}\partial_j\sigma_{(0)}-\nabla_{(0)i}\nabla_{(0)j}\sigma_{(0)}\bigg)\\&-\frac{d}{4}\beta_H^2g_{(0)ij}\\&\times\bigg(\frac{(d-1)}{4}(\nabla_{(0)}\sigma_{(0)})^2+\nabla_{(0)}^2\sigma_{(0)}\\&+e^{\sigma_{(0)}}\frac{(d+1)}{L^2}\bigg).\end{aligned} \tag{5.12}$$

Not all Einstein equations are independent, some give identities. For example, the $(\rho\rho)$ part of Eqs. (5.6) does not give rise to any new equations on $\sigma_{(1)}$ and $g_{(1)ij}$. An identity appears from (ρi) part of Eq. (5.6):

$$\nabla_{(0)}^j g_{(1)ij}=\partial_i(\mathrm{Tr}\,g_{(1)}+d\sigma_{(1)})-\frac{d}{2}(g_{(1)}+\sigma_{(1)}g_{(0)})_i^j\partial_j\sigma_{(0)}. \tag{5.13}$$

As a check for our formulas one verifies that Eq. (5.13) is indeed valid for $\sigma_{(1)}(x)$ and $g_{(1)ij}(x)$ given by Eqs. (5.11) and (5.12) respectively. In fact, the $(d+2)$-dimensional Bianchi identities imply the (ρi) equation to all orders provided the $(\rho\rho)$, (ij) and (tt) equations are satisfied. For completeness we also give the transformation laws for $\sigma_{(1)}$ and $g_{(1)ij}$ under the remaining diffeomorphisms (3.3). They are

$$\mathcal{L}_\xi \sigma_{(1)}(x) = \alpha_0(x)\sigma_{(1)}(x) - \frac{\beta_H^2}{4} g_{(0)}^{ij}\partial_i\alpha_0\partial_j\sigma_{(0)}$$

$$\mathcal{L}_\xi g_{(1)ij}(x) = -\frac{\beta_H^2}{2}\nabla_{(0)i}\nabla_{(0)j}\alpha_0. \tag{5.14}$$

In the next order in ρ, the $(\rho\rho)$ part of the Einstein equations gives the relation

$$\mathrm{Tr}\, g_{(2)} = \frac{1}{4}\mathrm{Tr}\, g_{(1)}^2 + \frac{d}{4}(\sigma_{(1)}^2 - 4\sigma_{(2)}). \tag{5.15}$$

From this and the equation arising in this order from Eq. (5.5) we can determine

$$\sigma_{(2)} = \frac{1}{16}(-3d\sigma_{(1)}^2 + \mathrm{Tr}\, g_{(1)}^2 - 2\sigma_{(1)}\mathrm{Tr}\, g_{(1)}) - \frac{\beta_H^2}{16}\left(\nabla_{(0)}^2\sigma_{(1)} + \frac{2(d+1)}{L^2}e^{\sigma_{(0)}}\sigma_{(1)}\right) + \cdots, \tag{5.16}$$

where the ellipses represent terms involving derivatives of $\sigma_{(0)}$ which can be set to zero by a suitable diffeomorphism of the form (3.3). Then, from the (ij) part of Eqs. (5.6) we determine

$$\begin{aligned} g_{(2)ij} &= \frac{1}{16}(4g_{(1)}^2 + g_{(0)}(d\sigma_{(1)}^2 - \mathrm{Tr}\, g_{(1)}^2))_{ij} \\ &\quad - \frac{\beta_H^2}{16}(d\nabla_{(0)i}\nabla_{(0)j}\sigma_{(1)} + \nabla_{(0)i}\nabla_{(0)j}\,\mathrm{Tr}\, g_{(1)}) \\ &\quad - \frac{\beta_H^2}{16}(\nabla_{(0)}^2 g_{(1)ij} - \nabla_{(0)}^k\nabla_{(0)i}g_{(1)kj} - \nabla_{(0)}^k\nabla_{(0)j}g_{(1)ki}) \\ &\quad + \cdots. \end{aligned} \tag{5.17}$$

In higher orders the strategy remains the same but the expressions become more complicated. However, the general structure of the equations on coefficients $\sigma_{(k)}$ and $g_{(k)ij}$ can be easily analyzed. From Eq. (5.5) we find that

$$(d+2k)\sigma_{(k)} + \mathrm{Tr}\, g_{(k)} = A_{(k)}[\sigma_{(0)}, g_{(0)}, \ldots, \sigma_{(k-1)}, g_{(k-1)}].$$

The $(\rho\rho)$ equation leads to

$$\mathrm{Tr}\, g_{(k)} + d\sigma_{(k)} = B_{(k)}[\sigma_{(0)}, g_{(0)}, \ldots, \sigma_{(k-1)}, g_{(k-1)}].$$

$A_{(k)}$ and $B_{(k)}$ are some functions of the coefficients $\sigma_{(i)}$, $g_{(i)}$, $i<k$ and their derivatives. These equations determine $\sigma_{(k)}$ and $\mathrm{Tr}\, g_{(k)}$ in terms of $\sigma_{(i)}$, $g_{(i)}$, $i<k$. The (ij) equation takes the form

$$\begin{aligned} &(2kg_{(k)} - g_{(0)}\mathrm{Tr}\, g_{(k)})_{ij} \\ &\quad = C_{(k)ij}[\sigma_{(0)}, g_{(0)}, \ldots, \sigma_{(k-1)}, g_{(k-1)}] \end{aligned}$$

and determines the coefficient $g_{(k)ij}$. We see that no ambiguity arises in this iteration process and all coefficients in the expansion (5.3) are uniquely determined once $\sigma_{(0)}(x)$ and $g_{(0)ij}(x)$ are specified on horizon.

Reconstructing the metric in the AdS-CFT correspondence [6] one has to specify the metric on the boundary and the expectation value of the stress tensor of dual CFT. Therefore, two terms in the expansion in ρ of the bulk metric remain undetermined (for more details see [6]): the metric on the boundary $g_{(0)ij}(x)$ and the term $g_{(2d)ij}(x)$ related to the value of the extrinsic curvature on the boundary. One might have expected the same to happen in our case. However, surprisingly only one tensor $g_{(0)ij}(x)$ needs to be specified, all the other terms in the expansion are then uniquely determined. This is due to the peculiar properties of the horizon (see also [12] for the spherically symmetric case). In particular, the (ij) components of the extrinsic curvature vanish on the horizon.

VI. NON-SPHERICAL HORIZONS, UNIQUENESS AND ENTROPY

The results obtained in the previous section may be interesting for conventional general relativity. What we have found is solution to Einstein equations describing a space-time with a horizon with arbitrary (not necessarily spherically symmetric) metric. It is worth reminding that in general relativity the possible shape of the horizon is constrained by the so-called uniqueness theorem proven by Israel [23] (for a review see [24]). According to this Theorem if the space-time metric is

(1) asymptotically flat;
(2) has an event horizon;
(3) has no singularity on or outside the event horizon;
(4) satisfies the vacuum Einstein equations

then it is spherically symmetric and in empty space coincides with the Schwarzschild metric.

In Sec. V we gave up the condition (1) since the solution we have found is known explicitly only in vicinity of the horizon and apparently cannot be asymptotically flat at infinity. Also, for the ansatz (3.1) we cannot rule out singularities outside the horizon. The only exception from the theorem known before is the so-called topological black holes [25] (for a review see [26]). They are solutions to Einstein equations with cosmological constant. The horizon in this case is a hypersurface of constant curvature which, after some identifications, can be made compact and topologically nontrivial. However, in this case the horizon is still a maximally symmetric space. In the solution found in Sec. V the metric on horizon can be arbitrary and it does not require any bulk cosmological constant for its existence. So that it is a new class of metrics describing space-times with horizon. It may find applications, for instance, in the analysis of fluctuations of the black hole horizon.

Finally, one may wonder whether it is meaningful to discuss thermodynamical properties of the configurations discussed in this paper. For instance, what is the horizon entropy in this case? A standard method to determine the entropy is based on the first law and is not applicable in our case since the solution is known explicitly only locally near horizon. Therefore, its global characteristics like mass are not determined.

The method which does work in this case is the method of conical singularity [11]. It consists in proceeding along the following steps:

(1) go to Euclidean signature, $\tau = \imath t$.

(2) Close Euclidean time τ with arbitrary period $2\pi\beta$, $\beta \neq \beta_H$. Then there appears a conical singularity at horizon surface Σ with angle deficit $\delta = 2\pi(1-\alpha)$, $\alpha = \beta/\beta_H$. The Ricci scalar on such conical manifold has a contribution from the singularity [11]

$$R = 4\pi(1-\alpha)\delta_\Sigma + R_{\alpha=1}, \tag{6.1}$$

where $R_{\alpha=1}$ is the regular part of the curvature.

(3) Compute the Einstein-Hilbert action $W_{EH}[M_\alpha] = -[(1-\alpha)/4G]A_\Sigma - (\alpha/16\pi G)\int_{M_{\alpha=1}} R_{\alpha=1}$ on the conical space and apply the formula

$$S = (\alpha\partial_\alpha - 1)W_{EH}[M_\alpha]|_{\alpha=1} \tag{6.2}$$

to compute the entropy. Since the entropy comes entirely as a contribution of the conical singularity, only the metric near the horizon is essential for the computing the entropy in this method.

The analysis of paper [11] is quite general and requires only the knowing the metric in the form (5.2) near horizon as an expansion (5.3).[5] With this definition, the entropy is found to be universally proportional to the horizon area $A_\Sigma = \int_\Sigma d^d x \sqrt{\gamma_{(0)}}$

$$S_{BH} = \frac{A_\Sigma}{4G}, \tag{6.3}$$

independently on the shape or topology of the horizon surface Σ.

[5]In fact, the results of [11] are even more general since they are still valid when the coefficients in the expansion (5.3) are functions of Euclidean time τ.

An alternative method for computing the entropy is Wald's method of Noether charges [27]. It would be interesting to see if this method applies to the present situation.

VII. CONCLUSIONS

The purpose of this paper was to initiate a systematic study of holographic encoding on a static, nondegenerate, but otherwise arbitrary horizon. Our basic observation is the presence of a universal near horizon structure conformally related to $\mathbb{R}\times$ Euclidean anti–de Sitter space. This is best described in terms of the optical metric. Moreover by analyzing the diffeomorphisms compatible with this near horizon structure we revealed the presence of an asymptotic conformal symmetry. We also computed the 2-point functions for a bulk scalar perturbation and found agreement with the expected form for some conformal field theory located on the horizon. Although, in this paper we have concentrated on static space-times it is conceivable that our results can be extended to more general space-times.

In spite of these similarities with the AdS-CFT correspondence there are also important differences. Notably we find that the conformal weights are complex. In Sec. III we made some speculations about possible interpretations of this phenomenon, but other interpretations are possible and it would be interesting to learn more about the CFT side and to what extent the analogy with the AdS-CFT correspondence is complete. In particular, it would be interesting see if the horizon entropy and Hawking radiation has a CFT representation in this case.

Finally we showed that the metric as well as scalar fields can be unambiguously reconstructed from the horizon data by an iterative process in much the same way as in the anti–de Sitter case.

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