Casimir effect in 2D stringy black hole backgrounds

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We consider the two-dimensional ''Schwarzschild'' and ''Reissner-Nordström'' stringy black holes as systems of Casimir type. We explicitly calculate the energy-momentum tensor of a massless scalar field satisfying Dirichlet boundary conditions on two one-dimensional ''walls.'' These results are obtained using the Wald's axioms. Thermodynamical quantities such as pressure, specific heat, isothermal compressibility and entropy of the two-dimensional stringy black holes are calculated. A comparison is made between the obtained results and the laws of thermodynamics. The results obtained for the extremal $(Q=M)$ stringy two-dimensional charged black hole are identical in all three different vacua used, a fact that indicates its quantum stability.

I. INTRODUCTION

A significant tool for investigating the thermodynamical properties of black holes is the Casimir effect. As is well known, in 1948 Casimir [1] evaluated the electromagnetic energy localized between two conducting plates. The disturbance to the electromagnetic vacuum induced by the two parallel plates is actually observable. The so-called Casimir effect [2] is viewed as a tractable model of field theoretical effects associated with the geometry of space [3–5].

In order to examine the analogous effects for non-trivial gravitational backgrounds we need the vacuum expectation values of the energy-momentum tensor. There are many procedures [6–9] for calculating the vacuum expectation value of the energy-momentum tensor such as dimensional regularization [10–12], the Green's function method [13,14], heat kernel method [15,16], zeta function regularization [17], point-splitting method [18–20], and Pauli-Villars regularization [21].

We restrict the form of the renormalized energy-momentum tensor of a massless scalar field [22] (without employing the full theory of regularization) by using the trace of $T_{\mu\nu}$ and enforcing Wald's axioms [23,24] which are

(1) The expectation values of the energy-momentum tensor are covariantly conserved.

(2) Causality holds.

(3) In Minkowski spacetime, standard results should be obtained.

(4) Standard results for the off-diagonal elements should also be obtained.

(5) The energy-momentum tensor is a local functional of the metric; i.e., it depends only on the metric and its derivatives which appear through the Riemann curvature tensor and the metric's covariant derivatives up to second order.

In working this procedure a detailed expression for the renormalized energy-momentum tensor is obtained once the stringy two-dimensional ''Schwarzschild'' (massive) [25,26]

and ''Reissner-Nordström'' (charged) [27,28] black hole backgrounds are treated as systems of Casimir type [6,29,30].

The outline of this paper is as follows. In Secs. II and III the vacuum expectation value of the energy-momentum tensor is explicitly evaluated for the above mentioned stringy black hole backgrounds, respectively, in the Boulware vacuum (labeled by η) [31], the Hartle-Hawking vacuum (labeled by v) [32–34] and the Unruh vacuum (labeled by ξ) [35]. The energy density, pressure, energy and the corresponding force between the two ''Dirichlet walls'' are calculated asymptotically. The thermodynamical quantities specific heat, thermal compressibility and entropy exhibit a fictitious violation of the second thermodynamical law. In Sec. IV the results are discussed and conclusions are given.

II. ''SCHWARZSCHILD'' BLACK HOLE

The line element of the stringy two-dimensional ''Schwarzschild'' black hole [36] which is a low-energy solution of an effective string action [25,26] is given as

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} \tag{1}$$

where the metric function is

$$g(r) = 1 - \frac{M}{\lambda}e^{-2\lambda r} \tag{2}$$

the radial coordinate take values $r_H < r < +\infty$ and the event horizon $\mathcal{H}$ is placed at the point

$$r_H = \frac{1}{2\lambda}\ln\left(\frac{M}{\lambda}\right). \tag{3}$$

The line element (1) is written in ''Schwarzschild'' gauge.

In the conformal gauge which we are going to use in our calculations the line element is given by

$$ds^2 = \Omega(x)(-dt^2 + dx^2) \tag{4}$$

the conformal factor is

$$\Omega(x) = \frac{1}{1 + e^{-2\lambda x}} \tag{5}$$

where the conformal variable

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$$x = \frac{1}{2\lambda}\ln[e^{2\lambda(r-r_H)}-1] \tag{6}$$

takes values $-\infty < x < +\infty$ and the corresponding conformal factor takes values

$$0 < \Omega(x) < 1. \tag{7}$$

The non-zero Christoffel symbols are

$$\Gamma^t_{xt} = \Gamma^x_{xx} = \Gamma^x_{tt} = \frac{1}{2\Omega(x)}\frac{d\Omega(x)}{dx} = \lambda\left[\frac{e^{-2\lambda x}}{1+e^{-2\lambda x}}\right]. \tag{8}$$

The Ricci scalar is given as

$$R(x) = 4\lambda^2\left[\frac{e^{-2\lambda x}}{1+e^{-2\lambda x}}\right]. \tag{9}$$

It is well known that the trace $T^\alpha_\alpha(x)$ of the energy-momentum tensor vanishes classically for a conformally invariant theory. However in the semiclassical approximation which is the case to be discussed here the trace is nonzero in the regularization process and specifically in two dimensions is given by [10,11,22,37]

$$T^\alpha_\alpha(x) = \frac{R(x)}{24\pi}. \tag{10}$$

Thus for the two-dimensional "Schwarzschild" black hole background (1)–(3) the trace of the energy-momentum tensor is

$$T^\alpha_\alpha(x) = \frac{\lambda^2}{6\pi}\left[\frac{e^{-2\lambda x}}{1+e^{-2\lambda x}}\right]. \tag{11}$$

Applying Wald's first axiom, the conservation equation must be fulfilled by the regularized expectation value of the energy-momentum tensor $\langle T^\mu_\nu\rangle_{reg} \equiv T^\mu_\nu$:

$$T^\mu_{\nu;\mu} = 0 \tag{12}$$

which "splits" in two equations:

$$\frac{dT^x_t}{dx} + \Gamma^t_{tx}T^x_t - \Gamma^x_{tt}T^t_x = 0 \tag{13}$$

$$\frac{dT^x_x}{dx} + \Gamma^t_{tx}T^x_x - \Gamma^t_{tx}T^t_t = 0 \tag{14}$$

and since $T^t_x = -T^x_t$ and $T^t_t = T^\alpha_\alpha - T^x_x$, we get

$$\frac{dT^x_t}{dx} + 2\Gamma^t_{tx}T^x_t = 0 \tag{15}$$

$$\frac{dT^x_x}{dx} + 2\Gamma^t_{tx}T^x_x = \Gamma^t_{tx}T^\alpha_\alpha. \tag{16}$$

Substituting the Christoffel symbols (8) into Eqs. (15), (16) and solving them, we get respectively

$$T^x_t(x) = \alpha\Omega^{-1}(x) = \alpha(1+e^{-2\lambda x}) \tag{17}$$

$$T^x_x(x) = \Omega^{-1}(x)[H_2(x)+\beta] \tag{18}$$

where

$$H_2(x) = \frac{1}{2}\int_{x_H}^x \frac{d\Omega(x')}{dx'}T^\alpha_\alpha(x')dx' \tag{19}$$

and the parameters α, β are constants of integration while the point x_H is where the event horizon $\mathcal{H}$ is placed. It can be shown that $H_2(x)$ for the stringy two-dimensional black hole background (1)–(5) becomes

$$H_2(x) = \frac{\lambda^2}{24\pi} - \frac{\lambda^2}{24\pi}(1-\Omega(x))^2. \tag{20}$$

Now the following limiting values of $H_2(x)$ from Eq. (20) are obtained:

$$\text{if } x \to -\infty(r=r_H) \text{ then } H_2(x) = 0[\Omega(x)=0]$$

$$\text{if } x \to +\infty(r \to +\infty) \text{ then } H_2(x) = \frac{\lambda^2}{24\pi}[\Omega(x)=1].$$

Keeping in mind that for any two-dimensional background the most general expression of the regularized energy-momentum tensor is

$$T^\mu_\nu = \begin{bmatrix} T^\alpha_\alpha(x)-\Omega^{-1}(x)H_2(x) & 0 \\ 0 & \Omega^{-1}(x)H_2(x) \end{bmatrix} + \Omega^{-1}(x)\begin{bmatrix} -\beta & -\alpha \\ \alpha & \beta \end{bmatrix} \tag{21}$$

we obtain

$$T^\mu_\nu = \begin{bmatrix} \frac{\lambda^2}{6\pi}[1-\Omega(x)]-\Omega^{-1}(x)\left(\frac{\lambda^2}{24\pi}-\frac{\lambda^2}{24\pi}[1-\Omega(x)]^2\right) & 0 \\ 0 & \Omega^{-1}(x)\left(\frac{\lambda^2}{24\pi}-\frac{\lambda^2}{24\pi}[1-\Omega(x)]^2\right) \end{bmatrix} + \Omega^{-1}(x)\begin{bmatrix} -\beta & -\alpha \\ \alpha & \beta \end{bmatrix} \tag{22}$$

where the stringy background (1)–(3) and relations (11), (20) have been used. In this expression the only unknowns are the parameters α and β; we hope to determine them imposing the third Wald's axiom treating the two-dimensional "Schwarzschild" black hole as a Casimir system [6]. Two one-dimensional "walls" at a proper distance L (between them) are placed at arbitrary points x_1 and x_2. The massless scalar field whose energy-momentum tensor we try to evaluate satisfies the Dirichlet boundary conditions on the "walls," i.e., $\phi(x_1) = \phi(x_2) = 0$.

The standard Casimir energy-momentum tensor in the Minkowski spacetime is already known [6,7]:

$$T_\nu^\mu = \frac{\pi}{24L^2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{23}$$

The two-dimensional "Schwarzschild" black hole is asymptotically flat, i.e. at infinity (towards $\mathcal{J}^+$) is Minkowski spacetime, so the constants of integration α and β are evaluated demanding the regularized energy-momentum tensor given in Eq. (22) to coincide with the standard Casimir energy-momentum tensor (23) at infinity, i.e. $x \to +\infty$, or equivalently setting $\Omega(x) = 1$.

Therefore we get

$$\beta = \frac{\pi}{24L^2} - \frac{\lambda^2}{24\pi}, \quad \alpha = 0 \tag{24}$$

and the regularized energy-momentum tensor has been explicitly calculated

$$T_\nu^{(\eta)\mu} = \begin{bmatrix} \frac{\lambda^2}{6\pi}[1 - \Omega(x)] - \Omega^{-1}(x)\left(\frac{\lambda^2}{24\pi} - \frac{\lambda^2}{24\pi}[1 - \Omega(x)]^2\right) & 0 \\ 0 & \Omega^{-1}(x)\left(\frac{\lambda^2}{24\pi} - \frac{\lambda^2}{24\pi}[1 - \Omega(x)]^2\right) \end{bmatrix}$$

$$+ \Omega^{-1}(x)\left(\frac{\pi}{24L^2} - \frac{\lambda^2}{24\pi}\right)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{25}$$

where η denotes that the regularized energy-momentum tensor has been calculated under the assumption that there are no particles (vacuum state) at infinity (Boulware vacuum). Thus we have obtained the regularized energy-momentum tensor $T_\nu^{(\eta)\mu}$ as a direct sum

$$T_\nu^{(\eta)\mu} = T_{\nu(gravitational)}^\mu + T_{\nu(boundary)}^\mu \tag{26}$$

where the first term denotes the contribution to the vacuum polarization due to the non-trivial topology in which the contribution of the trace anomaly is included and the second term denotes the contribution due to the presence of the two "Dirichlet walls."

In the Boulware vacuum the detected negative energy density ρ will asymptotically [$x \to +\infty$, or equivalently setting $\Omega(x) = 1$] be

$$\rho = T_t^{(\eta)t} = -\frac{\pi}{24L^2} \tag{27}$$

the detected pressure p is asymptotically

$$p = -T_x^{(\eta)x} = -\frac{\pi}{24L^2} \tag{28}$$

the detected negative energy E will asymptotically be

$$E(L) = \int_0^L \rho\,dx = -\frac{\pi}{24L}. \tag{29}$$

The corresponding force F (i.e. the force due to the localized energy between the "walls") will be attractive as expected:

$$F(L) = -\frac{\partial E(L)}{\partial L} = -\frac{\pi}{24L^2} < 0. \tag{30}$$

Let us stress that the above formula (29), and the ones that follow regarding the different vacua, consider both the "walls" at the asymptotic region. If one wishes to include more specific effects of the background one has to place them at arbitrary points x_1 and $x_1 + L$. Then the corresponding energy will be

$$E(x_1,L)=\int_{x_1}^{x_1+L}T_t^t(x)dx$$

$$=-\frac{1}{8\lambda}\left(\frac{\pi}{6L^2}-\frac{\lambda^2}{6\pi}\right)e^{-2\lambda x_1}$$

$$-\frac{1}{8\lambda}\left(\frac{\lambda^2}{6\pi}-\frac{\pi}{6L^2}\right)e^{-2\lambda L}e^{-2\lambda x_1}$$

$$-\frac{\pi}{24L}+\frac{\lambda}{16\pi}\log[1+e^{-2\lambda x_1}]-\frac{\lambda}{16\pi}\log[1$$

$$+e^{-2\lambda L}e^{-2\lambda x_1}]\tag{31}$$

yielding Eq. (29) as $x_1\rightarrow+\infty$.

In the Hartle-Hawking vacuum the black hole is in thermal equilibrium with an infinite reservoir of black body radiation at temperature T and the standard Casimir energy-momentum tensor (23) is modified by an additional term [22]:

$$T_\nu^{(v)\mu}=\left[\begin{array}{cc}\frac{\lambda^2}{6\pi}[1-\Omega(x)]-\Omega^{-1}(x)\left(\frac{\lambda^2}{24\pi}-\frac{\lambda^2}{24\pi}[1-\Omega(x)]^2\right) & 0 \\ 0 & \Omega^{-1}(x)\left(\frac{\lambda^2}{24\pi}-\frac{\lambda^2}{24\pi}[1-\Omega(x)]^2\right)\end{array}\right]$$

$$+\Omega^{-1}(x)\left(\frac{\pi}{24L^2}\right)\left[\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right]\tag{35}$$

where v denotes that the regularized energy-momentum tensor has been calculated under the assumption that massless particles (black body radiation) are detected at infinity (towards $\mathcal{J}^+$) (Hartle-Hawking vacuum). Thus the regularized energy-momentum tensor $T_\nu^{(v)\mu}$ is

$$T_\nu^{(v)\mu}=T_{\nu(gravitational)}^\mu+T_{\nu(boundary)}^\mu+T_{\nu(bath)}^\mu\tag{36}$$

where the last term denotes the contribution to the vacuum polarization due to thermal bath at temperature T_H.

In this vacuum the asymptotically $[x\rightarrow+\infty$, or equivalently setting $\Omega(x)=1]$ detected energy density and pressure are

$$\rho=T_t^{(v)t}=-\left(\frac{\pi}{24L^2}+\frac{\lambda^2}{24\pi}\right)\tag{37}$$

$$p=-T_x^{(v)x}=-\left(\frac{\pi}{24L^2}+\frac{\lambda^2}{24\pi}\right).\tag{38}$$

Likewise the detected negative energy is

$$T_\nu^\mu=\frac{\pi T^2}{12}\left[\begin{array}{cc}-2 & 0 \\ 0 & 2\end{array}\right]=\frac{\pi T^2}{6}\left[\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right].\tag{32}$$

Setting T equal to the Hawking temperature of the two-dimensional "Schwarzschild" black hole, i.e. $T_H=\lambda/2\pi$, the Casimir energy-momentum tensor becomes

$$T_\nu^\mu=\frac{\pi}{24L^2}\left[\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right]+\frac{\lambda^2}{24\pi}\left[\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right].\tag{33}$$

Therefore we now obtain

$$\beta=\frac{\pi}{24L^2},\quad\alpha=0\tag{34}$$

and the regularized energy-momentum tensor becomes

$$E(L,T_H)=\int_0^L\rho dx=-\left(\frac{\pi}{24L}+\frac{\lambda^2}{24\pi}L\right)$$

$$=-\left(\frac{\pi}{24L}+\frac{\pi L}{6}T_H^2\right)\tag{39}$$

and the corresponding force F between the "walls" is not always attractive:

$$F(L,T_H)=-\left(\frac{\partial E(L,T_H)}{\partial L}\right)_{T_H}=-\frac{\pi}{24L^2}+\frac{\lambda^2}{24\pi}=-\frac{\pi}{24L^2}$$

$$+\frac{\pi}{6}T_H^2.\tag{40}$$

It is clear that the corresponding force is

(a) attractive

$$L<\frac{1}{2T_H}=\frac{\pi}{\lambda}\tag{41}$$

(b) zero

$$L=\frac{1}{2T_H} \tag{42}$$

(c) repulsive

$$L>\frac{1}{2T_H}. \tag{43}$$

Thus if the last condition is satisfied the outer "wall" moves towards infinity. It can be studied as a "moving mirror" creating particles. The energy rate detected at infinity can be given by the second term in Eq. (39)

$$\frac{dE}{dt}=\frac{\lambda^2}{24\pi}L=\frac{\pi L}{6}T_H^2 \tag{44}$$

and this is (for the massless two-dimensional field) the rate at which the energy is radiated [36,38,39].

The appearance of the repulsive nature of the corresponding force will be discussed in Sec. IV.

In the Unruh vacuum an outward flux of radiation is detected at infinity (towards $\mathcal{J}^+$). Since the two-dimensional "Schwarzschild" black hole has been proven to radiate and

its spectrum distribution is purely thermal at the Hawking temperature T_H [40,41], the Unruh vacuum state can be identified with the vacuum obtained after the two-dimensional "Schwarzschild" black hole has settled down to an "equilibrium" of temperature T_H. The standard Casimir energy-momentum tensor (23) will be modified by an additional term [22]:

$$T_\nu^\mu=\frac{\pi T_H^2}{12}\begin{bmatrix}-1&1\\1&1\end{bmatrix}=\frac{\lambda^2}{48\pi}\begin{bmatrix}-1&1\\1&1\end{bmatrix}. \tag{45}$$

The Casimir energy-momentum tensor is now given by

$$T_\nu^\mu=\frac{\pi}{24L^2}\begin{bmatrix}-1&0\\0&1\end{bmatrix}+\frac{\lambda^2}{48\pi}\begin{bmatrix}-1&-1\\1&1\end{bmatrix}. \tag{46}$$

Therefore we get

$$\beta=\frac{\pi}{24L^2}-\frac{\lambda^2}{48\pi},\quad \alpha=\frac{\lambda^2}{48\pi} \tag{47}$$

and the regularized energy-momentum tensor becomes

$$T_\nu^{(\xi)\mu}=\begin{bmatrix}\frac{\lambda^2}{6\pi}[1-\Omega(x)]-\Omega^{-1}(x)\left(\frac{\lambda^2}{24\pi}-\frac{\lambda^2}{24\pi}[1-\Omega(x)]^2\right)&0\\0&\Omega^{-1}(x)\left(\frac{\lambda^2}{24\pi}-\frac{\lambda^2}{24\pi}[1-\Omega(x)]^2\right)\end{bmatrix}$$

$$+\Omega^{-1}(x)\begin{bmatrix}-\frac{\pi}{24L^2}+\frac{\lambda^2}{48\pi}&-\frac{\lambda^2}{48\pi}\\\frac{\lambda^2}{48\pi}&\frac{\pi}{24L^2}-\frac{\lambda^2}{48\pi}\end{bmatrix} \tag{48}$$

where ξ denotes that the regularized energy-momentum tensor has been been calculated under the assumption that massless particles are detected at infinity due to the Hawking radiation of the two-dimensional "Schwarzschild" black hole. Thus the regularized energy-momentum tensor $T_\nu^{(\xi)\mu}$ becomes

$$T_\nu^{(\xi)\mu}=T_{\nu(gravitational)}^\mu+T_{\nu(boundary)}^\mu+T_{\nu(radiation)}^\mu \tag{49}$$

where the last term denotes the contribution to the vacuum polarization due to Hawking radiation at temperature T_H.

In this vacuum the detected energy density and pressure are asymptotically [$x\to+\infty$, or equivalently setting $\Omega(x)=1$]

$$\rho=T_t^{(\xi)t}=-\left(\frac{\pi}{24L^2}+\frac{\lambda^2}{48\pi}\right) \tag{50}$$

$$p=-T_x^{(\xi)x}=-\left(\frac{\pi}{24L^2}+\frac{\lambda^2}{48\pi}\right). \tag{51}$$

Likewise for the detected negative energy

$$E(L,T_H)=\int_0^L\rho dx=-\left(\frac{\pi}{24L}+\frac{\lambda^2}{48\pi}L\right)=-\left(\frac{\pi}{24L}+\frac{\pi L}{12}T_H^2\right) \tag{52}$$

and the corresponding force F between the "walls" is not always attractive:

$$F(L,T_H)=-\left(\frac{\partial E(L,TH)}{\partial L}\right)_{T_H}=-\frac{\pi}{24L^2}+\frac{\lambda^2}{48\pi}$$

$$=-\frac{\pi}{24L^2}+\frac{\pi}{12}T_H^2. \tag{53}$$

The corresponding force will now be

(a) attractive

$$L < \frac{1}{\sqrt{2} T_H} = \sqrt{2} \frac{\pi}{\lambda} \qquad (54)$$

(b) zero

$$L = \frac{1}{\sqrt{2} T_H} \qquad (55)$$

(c) repulsive

$$L > \frac{1}{\sqrt{2} T_H}. \qquad (56)$$

Thus if the outer "wall" is placed at a distance L such that the last condition is satisfied then it will move towards infinity. It can be studied as a "moving mirror" creating particles whose energy rate detected at infinity is given by the second term in Eq. (52):

$$\frac{dE}{dt} = \frac{\lambda^2}{48\pi} L = \frac{\pi L}{12} T_H^2. \qquad (57)$$

Thus as before this is for the massless two-dimensional field the rate at which the energy is radiated [36,38,39].

In this vacuum it will be interesting to calculate some thermodynamical quantities and to consider these results with respect to the laws of thermodynamics.

The specific heat is given as

$$C_V = -\frac{\partial E}{\partial T} = -\left(\frac{\pi L}{6}\right) T_H = -\left(\frac{L}{12}\right)\lambda \qquad (58)$$

with V the "volume" between the two "walls," i.e. the distance L in this case, and the isothermal compressibility is

$$\kappa_T = -\frac{1}{L}\left(\frac{\partial L}{\partial p}\right)_{T_H} = -\left(\frac{12}{\pi}\right) L^2. \qquad (59)$$

The negative values of the specific heat and the isothermal compressibility are a violation of the second law of thermodynamics which requires $C_V \geq 0$ and $\kappa_T \geq 0$ [42]. This thermodynamical instability at least for C_V is a common feature in black hole physics, using the semiclassical approximation. This may be resolved by a more complete quantum treatment and the inclusion of back reaction effects [43].

The entropy of the stringy two-dimensional "Schwarzschild" black hole seen as a Casimir system is given (applying the first thermodynamical law) by

$$S_{Casimir} = S_{(T=0)} - \left(\frac{\pi L}{6}\right) T_H = S_{(T=0)} - \left(\frac{L}{12}\right)\lambda \qquad (60)$$

and according to the third law of thermodynamics

$$S \to 0 \quad \text{as} \quad T \to 0 \qquad (61)$$

the entropy is

$$S_{Casimir} = -\left(\frac{\pi L}{6}\right) T_H = -\left(\frac{L}{12}\right)\lambda. \qquad (62)$$

The entropy calculated here seems to violate of the second thermodynamical law. This is not true since the entropy calculated here has not been obtained through a statistical counting of microstates. The expression (62) is the part of the thermodynamical entropy S^{TM} due to the vacuum polarization (virtual particles)—it does not need to have a statistical interpretation—and so it is not forbidden to be negative [44].

Being more precise the thermodynamical entropy which gives the contribution of quantum fields (radiation and massive fields) is given as

$$S^{TM} = S^{SM} + S_0 \qquad (63)$$

where S^{SM} is the statistical-mechanical part of the entropy (statistical counting of microstates) which is absent here and in our case $S_0 = S_{Casimir}$ is a quantity which gives the contribution of the vacuum polarization.

Finally the entropy of the two-dimensional Schwarzschild black hole is

$$S_{bh} = S_{classical} + S^{TM} \qquad (64)$$

where $S_{classical}$ is the entropy from the classical gravitational action and the thermodynamical entropy S^{TM} is the one-loop quantum correction [45].

III. "REISSNER-NORDSTRÖM" BLACK HOLE

The line element of the stringy two-dimensional "Reissner-Nordström" (charged) black hole [27,28] in the Schwarzschild gauge is given by

$$ds^2 = -g(r)dt^2 + g^{-1}(r)dr^2 \qquad (65)$$

where

$$g(r) = 1 - \frac{M}{\lambda} e^{-2\lambda r} + \frac{Q^2}{4\lambda^2} e^{-4\lambda r} \qquad (66)$$

with $0 < t < +\infty$, $r_+ < r < +\infty$, r_+ being the future event horizon of the black hole.

Following a parametrization analogous to the four-dimensional case the metric function factorizes as

$$g(r) = (1 - \rho_- e^{-2\lambda r})(1 - \rho_+ e^{-2\lambda r}) \qquad (67)$$

where

$$\rho_\pm = \frac{M}{2\lambda} \pm \frac{1}{2\lambda}\sqrt{M^2 - Q^2} \qquad (68)$$

we can recognize immediately the "outer" event horizon $\mathcal{H}^+$ placed at the point $r_+ = (1/2\lambda)\ln \rho_+$, while the "inner" horizon $\mathcal{H}^-$ is at the point $r_- = (1/2\lambda)\ln \rho_-$.

In the extremal case $(Q=M)$ the two surfaces coincide in a single event horizon at the point

$$r_H = \frac{1}{2\lambda} \ln\left(\frac{M}{2\lambda}\right). \tag{69}$$

The line element (65), (66) in conformal gauge is written

$$ds^2 = \Omega(x)(-dt^2 + dx^2) \tag{70}$$

with the conformal factor

$$\Omega(x) = \frac{X(x)(X(x)+\mu)}{(X(x)+1)^2} \tag{71}$$

$$0 < \Omega(x) < 1 \tag{72}$$

where the conformal variable

$$x = \frac{1}{2\lambda\mu} \ln[X(X+\mu)^{\mu-1}] \tag{73}$$

with $-\infty < x < +\infty$ and the asymmetric variable X is

$$X = e^{2\lambda(r-r_+)} - 1 = \frac{e^{2\lambda r}}{\rho_+} - 1 \tag{74}$$

$$0 < X < +\infty. \tag{75}$$

The new parameter μ is given by

$$\mu = 1 - \frac{\rho_-}{\rho_+}. \tag{76}$$

The non-zero Christoffel symbols are

$$\Gamma^t_{xt} = \Gamma^x_{xx} = \Gamma^x_{tt} = \frac{1}{2\Omega(x)} \frac{d\Omega(x)}{dx} = \lambda \frac{[(X+\mu)+X(1-\mu)]}{(1+X)^2}. \tag{77}$$

The Ricci scalar is given as

$$R(x) = 4\lambda^2 \left[\frac{2(X(x)+\mu-1)+\mu(1-X(x))}{(X(x)+1)^2}\right] \tag{78}$$

and in the semiclassical approximation the trace (or conformal) anomaly for the stringy two-dimensional background (65)–(68) is

$$T^\alpha_\alpha(x) = \frac{\lambda^2}{6\pi}\left[\frac{2(X(x)+\mu-1)+\mu(1-X(x))}{(X(x)+1)^2}\right]. \tag{79}$$

The conservation equation (12) again "splits" in two equations

$$\frac{d}{dx}[\Omega(x)T^x_t] = 0 \tag{80}$$

$$\frac{d}{dx}[\Omega(x)T^x_x] = \frac{1}{2}\left(\frac{d\Omega(x)}{dx}\right)T^\alpha_\alpha(x) \tag{81}$$

and by integration we obtain

$$T^x_t(x) = \alpha\Omega^{-1}(x) = \alpha\frac{(X(x)+1)^2}{X(x)(X(x)+\mu)} \tag{82}$$

$$T^x_x(x) = \Omega^{-1}(x)[H_2(x)+\beta]$$

$$= \frac{(X(x)+1)^2}{X(x)(X(x)+\mu)}[H_2(x)+\beta]. \tag{83}$$

Using Eq. (19) $H_2(x)$ is now

$$H_2(x) = \frac{\lambda^2}{12\pi}\int_{X(x_+)}^{X(x)} \frac{[(X+\mu)+X(1-\mu)][2(X+\mu-1)+\mu(1-X)]}{(1+X)^5}dX \tag{84}$$

and the parameters α, β are constants of integration while the point x_+ is where the "outer" event horizon $\mathcal{H}^+$ is placed.

Thus the quantity $H_2(x)$ becomes

$$H_2(x) = \frac{\lambda^2}{24\pi}[\mu^2 + H_1(x)] \tag{85}$$

with

$$H_1(x) = -\frac{4(1-2\mu+\mu^2)}{(1+X)^4} + \frac{4(2-3\mu+\mu^2)}{(1+X)^3} - \frac{(\mu^2-4\mu+4)}{(1+X)^2} \tag{86}$$

with the following limiting values:

$$\text{if } x\to-\infty(X\to 0) \text{ then } H_2(x)=0[H_1(x)=-\mu^2]$$

$$\text{if } x \to +\infty (X \to +\infty) \text{ then } H_2(x) = \frac{\lambda^2}{24\pi}\mu^2[H_1(x)=0].$$

Through the use of relations (79), (85), (86) for the stringy black hole background (65)–(68) Eq. (21) becomes

$$T^\mu_\nu = \begin{bmatrix} \frac{\lambda^2}{6\pi}\left[\frac{2(X(x)+\mu-1)+\mu(1-X(x))}{(X(x)+1)^2}\right] - \Omega^{-1}(x)\left(\frac{\lambda^2}{24\pi}\right)[\mu^2+H_1(x)] & 0 \\ 0 & \Omega^{-1}(x)\left(\frac{\lambda^2}{24\pi}\right)[\mu^2+H_1(x)] \end{bmatrix}$$
$$+ \Omega^{-1}(x)\begin{bmatrix} -\beta & -\alpha \\ \alpha & \beta \end{bmatrix}. \tag{87}$$

In order to find the explicit form of the regularized energy-momentum tensor in the different vacua considered before we follow the analysis of the previous section.

A. Boulware vacuum

In this vacuum there are no particles detected at infinity ($\mathcal{J}^+$) and the regularized energy momentum tensor (87) should coincide at infinity with the standard Casimir energy-momentum tensor (23).

The constants of integration are

$$\beta = \frac{\pi}{24L^2} - \frac{\lambda^2}{24\pi}\mu^2, \quad \alpha = 0 \tag{88}$$

and thus the regularized energy-momentum tensor is

$$T^{(\eta)\mu}_\nu = \begin{bmatrix} \frac{\lambda^2}{6\pi}\left[\frac{2(X(x)+\mu-1)+\mu(1-X(x))}{(X(x)+1)^2}\right] - \Omega^{-1}(x)\left(\frac{\lambda^2}{24\pi}\right)[\mu^2+H_1(x)] & 0 \\ 0 & \Omega^{-1}(x)\left(\frac{\lambda^2}{24\pi}\right)[\mu^2+H_1(x)] \end{bmatrix}$$
$$+ \Omega^{-1}(x)\left(\frac{\pi}{24L^2} - \frac{\lambda^2}{24\pi}\mu^2\right)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{89}$$

The detected energy density, pressure and energy are asymptotically [$x \to +\infty$, or equivalently setting $\Omega(x)=1$] given by

$$\rho = T^{(\eta)t}_t = -\frac{\pi}{24L^2} \tag{90}$$

$$p = -T^{(\eta)x}_x = -\frac{\pi}{24L^2} \tag{91}$$

$$E(L) = \int_0^L \rho \, dx = -\frac{\pi}{24L}. \tag{92}$$

The corresponding force between the "walls" is attractive as expected

$$F(L) = -\frac{\partial E(L)}{\partial L} = -\frac{\pi}{24L^2} < 0. \tag{93}$$

In analogy with Eq. (31) the energy for arbitrary position of the "walls" (x_1, x_1+L) can be evaluated. The corresponding expression is quite complicated and is not presented here but gives Eq. (92) as $x_1 \to +\infty$.

B. Hartle-Hawking vacuum

In this vacuum the stringy black hole (66)–(68) is in thermal equilibrium with an infinite reservoir of black body radiation at temperature T which is equal to the Hawking temperature of the stringy two-dimensional charged black hole [46]:

$$T = T_H = \frac{\lambda}{2\pi}\mu. \tag{94}$$

The regularized energy-momentum tensor (87) should coincide with the modified Casimir energy-momentum tensor (33).

The constants of integration are

$$\beta = \frac{\pi}{24L^2}, \quad \alpha = 0 \tag{95}$$

and thus the regularized energy-momentum tensor is

$$T_\nu^{(v)\mu} = \begin{bmatrix} \frac{\lambda^2}{6\pi}\left[\frac{2(X(x)+\mu-1)+\mu(1-X(x))}{(X(x)+1)^2}\right] - \Omega^{-1}(x)\left(\frac{\lambda^2}{24\pi}\right)[\mu^2 + H_1(x)] & 0 \\ 0 & \Omega^{-1}(x)\left(\frac{\lambda^2}{24\pi}\right)[\mu^2 + H_1(x)] \end{bmatrix}$$
$$+ \Omega^{-1}(x)\left(\frac{\pi}{24L^2}\right)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{96}$$

The detected energy density, pressure and energy are asymptotically [$x \to +\infty$, or equivalently setting $\Omega(x)=1$] given by

$$\rho = T_t^{(v)t} = -\left(\frac{\pi}{24L^2} + \frac{\lambda^2}{24\pi}\mu^2\right) \tag{97}$$

$$p = -T_x^{(v)x} = -\left(\frac{\pi}{24L^2} + \frac{\lambda^2}{24\pi}\mu^2\right) \tag{98}$$

$$E(L,T_H) = \int_0^L \rho\, dx = -\left(\frac{\pi}{24L} + \frac{\lambda^2}{24\pi}\mu^2 L\right)$$
$$= -\left(\frac{\pi}{24L} + \frac{\pi L}{6}T_H^2\right). \tag{99}$$

The corresponding force between the "walls" is not always attractive:

$$F(L,T_H) = -\left(\frac{\partial E(L)}{\partial L}\right)_{T_H} = -\frac{\pi}{24L^2} + \frac{\lambda^2}{24\pi}\mu^2$$
$$= -\frac{\pi}{24L^2} + \frac{\pi}{6}T_H^2. \tag{100}$$

It is obvious again that the corresponding force is

(a) attractive

$$L < \frac{1}{2T_H} = \frac{\pi}{\lambda\mu} \tag{101}$$

(b) zero

$$L = \frac{1}{2T_H} \tag{102}$$

(c) repulsive

$$L > \frac{1}{2T_H}. \tag{103}$$

Thus as in the case of two-dimensional "Schwarzschild" black hole if the last condition is satisfied the outer "wall" moves towards infinity. It can be studied as a "moving mirror" creating particles whose energy rate detected at infinity is given by the second term in Eq. (99):

$$\frac{dE}{dt} = \frac{\lambda^2}{24\pi}\mu^2 L = \frac{\pi L}{6}T_H^2. \tag{104}$$

This is the rate at which energy is radiated for the case of the massless two-dimensional field.

C. Unruh vacuum

In this vacuum an outward flux of radiation is detected at infinity. Thus the stringy two-dimensional charged black hole (66)–(68) radiates and its temperature when the system has settled down to an "equilibrium" state is given as in Eq. (94) [46]. The regularized energy-momentum tensor (87) should now coincide at infinity with the modified Casimir energy-momenum tensor (46).

The constants of integration are

$$\beta = \frac{\pi}{24L^2} - \frac{\lambda^2}{48\pi}\mu^2, \quad \alpha = \frac{\lambda^2}{48\pi}\mu^2 \tag{105}$$

thus the regularized energy-momentum tensor is now given by

$$T_\nu^{(\xi)\mu} = \begin{bmatrix} \frac{\lambda^2}{6\pi}\left[\frac{2(X(x)+\mu-1)+\mu(1-X(x))}{(X(x)+1)^2}\right] - \Omega^{-1}(x)\left(\frac{\lambda^2}{24\pi}\right)[\mu^2+H_1(x)] & 0 \\ 0 & \Omega^{-1}(x)\left(\frac{\lambda^2}{24\pi}\right)[\mu^2+H_1(x)] \end{bmatrix}$$

$$+ \Omega^{-1}(x)\begin{bmatrix} -\frac{\pi}{24L^2}+\frac{\lambda^2}{48\pi}\mu^2 & -\frac{\lambda^2}{48\pi}\mu^2 \\ \frac{\lambda^2}{48\pi}\mu^2 & \frac{\pi}{24L^2}-\frac{\lambda^2}{48\pi}\mu^2 \end{bmatrix}. \tag{106}$$

The detected energy density, pressure and energy are asymptotically $[x\rightarrow+\infty$, or equivalently setting $\Omega(x)=1]$ given by

$$\rho = T_t^{(\xi)t} = -\left(\frac{\pi}{24L^2}+\frac{\lambda^2}{48\pi}\mu^2\right) \tag{107}$$

$$p = -T_x^{(\xi)x} = -\left(\frac{\pi}{24L^2}+\frac{\lambda^2}{48\pi}\mu^2\right) \tag{108}$$

$$E(L,T_H) = \int_0^L \rho\, dx = -\left(\frac{\pi}{24L}+\frac{\lambda^2}{48\pi}\mu^2 L\right)$$

$$= -\left(\frac{\pi}{24L}+\frac{\pi L}{12}T_H^2\right). \tag{109}$$

The corresponding force between the "walls" is not always attractive:

$$F(L,T_H) = -\left(\frac{\partial E(L,T_H)}{\partial L}\right)_{T_H} = -\frac{\pi}{24L^2}+\frac{\lambda^2}{48\pi}\mu^2$$

$$= -\frac{\pi}{24L^2}+\frac{\pi}{12}T_H^2 \tag{110}$$

and is thus

(a) attractive

$$L < \frac{1}{\sqrt{2}T_H} = \sqrt{2}\,\frac{\pi}{\lambda\mu} \tag{111}$$

(b) zero

$$L = \frac{1}{\sqrt{2}T_H} \tag{112}$$

(c) repulsive

$$L > \frac{1}{\sqrt{2}T_H}. \tag{113}$$

Thus if the outer "wall" is placed at a distance L such that the last condition is satisfied then it will move towards infinity. It can be studied as a "moving mirror" creating particles whose energy rate detected at infinity is given by the second term in Eq. (109):

$$\frac{dE}{dt} = \frac{\lambda^2}{48\pi}\mu^2 L = \frac{\pi L}{12}T_H^2 \tag{114}$$

which is for the two-dimensional massless field the rate which the energy is radiated.

It is interesting to evaluate in this vacuum some thermodynamical quantities and to examine these results with respect to the laws of thermodynamics.

The specific heat is given as

$$C_V = -\frac{\partial E}{\partial T} = -\left(\frac{\pi L}{6}\right)T_H = -\left(\frac{L}{12}\right)\lambda\mu \tag{115}$$

and the isothermal compressibility is

$$\kappa_T = -\frac{1}{L}\left(\frac{\partial L}{\partial p}\right)_T = -\left(\frac{12}{\pi}\right)L^2. \tag{116}$$

The comments made in the corresponding "Schwarzschild" case also hold here.

The entropy of the stringy two-dimensional charged black hole seen as a Casimir system using the first thermodynamical law is given by

$$S_{Casimir} = S_{(extremal)} - \left(\frac{\pi L}{6}\right) T_H = S_{(extremal)} - \left(\frac{L}{12}\right) \lambda \mu \tag{117}$$

where $S_{(extremal)}$ is the entropy of the two-dimensional extremal ($Q=M$ or equivalently $\mu=0$) black hole due to vacuum polarization.

The two-dimensional extremal black hole is shown [46] to be obtained as a regular limit of the stringy two-dimensional charged (nonextremal) black hole. The results obtained in the case of the stringy two-dimensional ''Schwarzschild'' black hole are those of the stringy two-dimensional charged black hole when the parameter μ approaches 1, i.e. the electric charge is zero ($Q=0$) [46]. Therefore we expect to get Eq. (62) by setting $\mu=1$ to Eq. (117). In order to achieve this we must set the entropy of the extremal black hole $S_{(extremal)}$ due to vacuum polarization equal to zero. The thermodynamical entropy of the two-dimensional charged black hole is given now by

$$S_{Casimir} = - \left(\frac{\pi L}{6}\right) T_H = - \left(\frac{L}{12}\right) \lambda \mu. \tag{118}$$

This is the part of the thermodynamical entropy due to the vacuum polarization as mentioned before. The statistical-mechanical part S^{SM} of the thermodynamical entropy is absent and since the entropy calculated here does not have a statistical interpretation is not prohibited to be negative.

IV. DISCUSSION

In this paper we have explicitly calculated in the stringy two-dimensional ''Schwarzschild'' and ''Reissner-Nordström'' black hole backgrounds the regularized energy-momentum tensor of a massless scalar field satisfying the Dirichlet boundary conditions. The regularized energy-momentum tensor is separately treated in the Boulware, Hartle-Hawking and Unruh vacua. In the Boulware vacuum the asymptotically detected energy, energy density, pressure acting on the ''walls'' and the corresponding force between the ''walls'' where proved to be the same for both stringy black hole backgrounds (''Schwarzschild'' and ''Reissner-Nordström''). In the other two vacua the expressions obtained for the ''Reissner-Nordström'' black hole are seen to approach the corresponding results for the ''Schwarzschild'' case when $\mu \to 1$, i.e. the electric charge is zero ($Q=0$). We have shown that the corresponding force between the ''Dirichlet walls'' is not always attractive: it can be attractive, zero or repulsive depending on the distance between the ''walls'' being smaller, equal or larger of the inverse Hawking temperature of the black hole. This can be understood if we recall the semi-infinite Witten's cigar which is an interpretation of an Euclidean black hole and which is asymptotic to a cylindrical two-dimensional spacetime. The inverse temperature of the Euclidean black hole can be viewed as the circumference of the Witten's cigar. Therefore if the distance between the ''Dirichlet walls'' is smaller than the circumference of the Witten's cigar then the corresponding force will be dominated by the attractive term due to the boundary effects and not by the repulsive term of the radiation pressure. If the ''Dirichlet walls'' are placed at a distance approximately equal to the circumference of the Witten's cigar the attractive term will be compensated by the repulsive term and the corresponding force will be zero. Finally if the ''Dirichlet walls'' are placed at a distance larger than the circumference then the dominant term will be the repulsive term due to the radiation pressure and the corresponding force acting on the ''walls'' will be repulsive.

We have concluded that the thermodynamical quantities specific heat and thermal compressibility violate the second thermodynamical law since they obtain negative values. The thermodynamical second law instability induced by the specific heat is not suprising since it is a byproduct of the semi-classical approximation (one-loop gravity). This is the reason we reach the same result in the four-dimensional black hole physics. The thermodynamical stability is regained by ''freezing'' the black hole which in the case of the two-dimensional charged black hole means to reach extremality ($Q=M$). The thermodynamical second law instability induced by the negativity of the isothermal compressibility reflects the possibility of extracting energy from the vacuum. The thermodynamical entropy was evaluated in the Unruh vacuum and was shown to be negative. This is not a violation of the second thermodynamical law since the thermodynamical (Casimir) entropy calculated here—the statistical-mechanical part of entropy is absent—is due to the vacuum polarization. Thus we have obtained the one-loop correction (due to vacuum polarization) to the classical entropy obtained from the corresponding classical gravitational action. Same result can be easily obtained for the case of Hartle-Hawking vacumm in contradistinction to the Casimir entropy of the Boulware vacuum which is zero since energy is temperature independent.

In the extremal case ($Q=M$) of the stringy two-dimensional charged black hole the results obtained in the Hartle-Hawking and Unruh vacua coincide with those of stable Boulware vacuum. This is another argument which strengthens our belief that extremal black holes are a stable quantum mechanical ending point for the black holes in the process of their evaporation.

Finally we would also like to note that, for asymptotic position of the ''walls,'' it is natural for the results to depend only on the temperatute of the black hole; however for arbitrary position of the ''walls'' the dependence of the energy on the position is different for the two stringy two-dimensional black holes studied in this work. Possible determination (using this difference) of the nature of a black hole through the Casimir effect is very interesting and we have in mind to come back with a future work.

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