Black hole entropy from nonperturbative gauge theory

Daniel Kabat*
Department of Physics, Columbia University, New York, New York 10027

Gilad Lifschytz†
Department of Physics, Princeton University, Princeton, New Jersey 08544
and Department of Mathematics and Physics, University of Haifa at Oranim, Tivon 36006, Israel

David A. Lowe‡
Department of Physics, Brown University, Providence, Rhode Island 02912
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We present the details of a mean-field approximation scheme for the quantum mechanics of N D0-branes at finite temperature. The approximation can be applied at strong 't Hooft coupling. We find that the resulting entropy is in good agreement with the Bekenstein-Hawking entropy of a ten-dimensional nonextremal black hole with a 0-brane charge. This result is in accord with the duality conjectured by Itzhaki, Maldacena, Sonnenschein and Yankielowicz. We study the spectrum of single-string excitations within quantum mechanics, and find evidence for a clear separation between light and heavy degrees of freedom. We also present a way of identifying the black hole horizon.

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I. INTRODUCTION

The physics of black holes has played a prominent role in our quest to understand quantum gravity. Semiclassical considerations have shown that the horizon of a black hole has an associated thermodynamic entropy [1], and a key test of any proposed theory of quantum gravity should be to provide a microscopic explanation of this entropy.

Dramatic progress was made a few years ago, when certain extremal black holes were realized as collections of D-branes in string theory. This description led to a precise counting of microstates, which was in exact agreement with semiclassical black hole thermodynamics [2]. Unfortunately this counting relied on supersymmetric nonrenormalization theorems, and therefore could only be applied to certain classes of extremal black holes.

A more general understanding of black hole entropy requires a non-perturbative definition of string theory. This is now available, at least in certain backgrounds, thanks to the M(atrix) and Maldacena conjectures [3,4] (for reviews see [5]). These conjectures relate non-perturbative string theories to dual strongly coupled large-N gauge theories. In this framework, black hole entropy is identified with the entropy of the density matrix which describes the gauge theory at finite temperature.

In principle, one can use these dualities to understand black hole physics in terms of gauge theory dynamics. In practice, however, this requires two things: a precise map between gravity and gauge theory quantities, and a tractable calculational scheme in the gauge theory. Some progress has been made on the first issue [6], although even such basic properties as spacetime locality are still obscure from the gauge theory point of view.

In this paper we will focus on the second issue, of developing practical methods for doing gauge theory calculations. The gauge theory is strongly coupled whenever semiclassical gravity is valid, so we must study the gauge theory non-perturbatively.[1] We do this using techniques from self-consistent mean field theory. This provides us with an approximation to the density matrix which describes the gauge theory at finite temperature. A key test of our approximation is whether it reproduces the semiclassical thermodynamics of the black hole. As we will see, according to this criterion our approximation works quite well, at least over a certain range of temperatures.

For simplicity we will concentrate on the quantum mechanics of N D0-branes, with sixteen supercharges and gauge group $SU(N)$ [9]. At large N and finite temperature, the effective 't Hooft coupling of the quantum mechanics is

$$g_{\rm eff}^2 = g_{YM}^2 N/T^3. \tag{1}$$

Note that the quantum mechanics is strongly coupled at low temperature. This quantum mechanics is dual to a ten-dimensional non-extremal black hole in type IIA supergravity, with N units of 0-brane charge [10]. The metric of the black hole is

*Email addres: kabat@phys.columbia.edu
†Email address: giladl@research.haifa.ac.il
‡Email address: lowe@het.brown.edu

[1]In M(atrix) theory, one can argue that the entropy of certain black holes is not renormalized beyond one loop [7]. The argument assumes Lorentz invariance is recovered in the large N limit, a property which has been checked at leading order in [8].

$$ds^2=\alpha'\left[-h(U)dt^2+h^{-1}(U)dU^2+\frac{c^{1/2}(g_{YM}^2N)^{1/2}}{U^{3/2}}d\Omega_8^2\right] \tag{2}$$

$$h(U)=\frac{U^{7/2}}{c^{1/2}(g_{YM}^2N)^{1/2}}\left(1-\frac{U_0^7}{U^7}\right)$$

where $c=2^7\pi^{9/2}\Gamma(7/2)$ and g_{YM} is the Yang-Mills coupling constant. The horizon of the black hole is at $U=U_0$, which corresponds to a Hawking temperature

$$\begin{aligned}T&=\frac{7}{2\pi\sqrt{30}}(g_{YM}^2N)^{-1/2}\left(\frac{U_0}{2\pi}\right)^{5/2}\\&=0.2034(g_{YM}^2N)^{-1/2}\left(\frac{U_0}{2\pi}\right)^{5/2}.\end{aligned} \tag{3}$$

The dual quantum mechanics is to be taken at the same finite temperature. The black hole has a free energy, which arises from its Bekenstein-Hawking entropy [11]

$$\begin{aligned}\beta F&=-\left(\frac{2^{21}3^25^7\pi^{14}}{7^{19}}\right)^{1/5}N^2\left(\frac{T}{(g_{YM}^2N)^{1/3}}\right)^{9/5}\\&=-4.115N^2\left(\frac{T}{(g_{YM}^2N)^{1/3}}\right)^{1.8}.\end{aligned} \tag{4}$$

Duality predicts that the quantum mechanics should have the same free energy. The supergravity description is expected to be valid when the curvature and the dilaton are small near the black hole horizon. This regime corresponds to the 't Hooft large-N limit of the quantum mechanics, when the temperature is such that the dimensionless effective coupling (1) lies in the range [10]

$$1\ll g_{\rm eff}^2\ll N^{10/7}. \tag{5}$$

An outline of this paper is as follows. In Sec. II we develop a mean-field approximation scheme for 0-brane quantum mechanics, building on our earlier work [12]. In Sec. III we present numerical results for the behavior of the gauge theory, focusing on thermodynamic quantities. We compare our results to the black hole predictions, and find good agreement over a certain range of temperatures. Section IV is devoted to a spectral analysis of the propagators, to extract the spectrum of stretched strings that make up the supergravity background. In Sec. V we discuss how local spacetime physics, such as the size of the black hole horizon, may be extracted from the gauge theory. Section VI gives our conclusions and a discussion of possible future directions. A summary of our results has appeared in [13].

II. MEAN-FIELD APPROXIMATION FOR 0-BRANE QUANTUM MECHANICS

The basic idea of our approximation is to treat the $\mathcal{O}(N^2)$ degrees of freedom appearing in 0-brane quantum mechanics as statistically independent, with interactions taken into account via a sort of mean-field approximation. In the rest of this section we present several reasons to believe this simple approximation captures some of the essential physics of the quantum mechanics in the supergravity regime. In the next section we will show that the approximation gives results which are in good agreement with black hole thermodynamics over a certain temperature range.

Let us begin by stating our approach to studying strongly coupled systems in rather general terms. We are presented with a strongly-coupled action S, in our case the action for 0-brane quantum mechanics. We approximate this action with a simpler trial action S_0. All quantities of interest can then be computed as an expansion in powers of $S-S_0$. For instance, the free energy has an expansion [14]

$$\begin{aligned}\beta F&=\beta F_0-\langle e^{-(S-S_0)}-1\rangle_{\mathsf{C},0}\\&=\beta F_0+\langle S-S_0\rangle_0-\frac{1}{2}\langle (S-S_0)^2\rangle_{\mathsf{C},0}\\&\quad+\cdots\end{aligned} \tag{6}$$

where a subscript C,0 denotes a connected expectation value calculated using the trial action S_0. If the trial action comes sufficiently close to capturing the dynamics of the full action, then this expansion should be well-behaved, even if the full action S is strongly coupled.

This sort of approximation relies crucially on an appropriate choice of trial action. In our case, we shall take S_0 to be the most general quadratic action that one can write in terms of the fundamental gauge theory degrees of freedom. This means that our trial action involves an infinite number of adjustable parameters, namely the momentum-dependent two-point functions of all the fundamental fields. One can regard these propagators as providing an infinite set of variational parameters. To fix these parameters we solve a truncated set of Schwinger-Dyson equations. These gap equations provide a non-perturbative approximation to the true two-point functions of the theory, by resumming an infinite set of Feynman diagrams.

As we shall see, this sort of approximation has several attractive features, which initially motivated us to apply these techniques to 0-brane quantum mechanics.

The approximation is non-perturbative in the Yang-Mills coupling constant, and self-consistently cures the infrared divergences which are present in conventional finite-temperature perturbation theory. This makes it possible to apply the approximation at strong coupling, at temperatures where one can make a direct comparison with black hole predictions.

We can formulate the approximation in a way which respects 't Hooft large-N counting, by only keeping planar contributions to the Schwinger-Dyson equations. This means that an overall factor of N^2 in the free energy, as well as the appearance of the gauge coupling only in the combination g_{YM}^2N, is guaranteed. But this is exactly the form (4) of the supergravity result. That is, we are proposing that the overall factor of N^2 in the supergravity free energy can be understood in terms of $\mathcal{O}(N^2)$ elementary quasiparticles, which are in one-to-one correspondence with the degrees of freedom appearing in the fundamental Lagrangian.[2] Incidentally, this means that our approximations are hopeless at couplings [outside the range (5)] which are so strong that 't Hooft scaling breaks down.

A key feature of the approximation is that a quadratic trial action will automatically respect all symmetries that act linearly on the fundamental fields. This is crucial in a problem like 0-brane quantum mechanics, where symmetries play such an important role. By working in a superfield formalism with off-shell supersymmetry, our trial action will have $\mathcal{N}=2$ supersymmetry and $SO(2)\times SO(7)$ rotational symmetry [out of the underlying $\mathcal{N}=16$ supersymmetry and $SO(9)$ rotational symmetry].

Another feature of the approximation is that it avoids certain infrared problems which are present in the full 0-brane quantum mechanics. The difficulty is that the partition function of the full quantum mechanics contains an infrared divergence from the regions in moduli space where the 0-branes are far apart. This leads to a divergent contribution to the entropy with an overall coefficient $\mathcal{O}(N)$. From the supergravity point of view, this corresponds to a thermal gas of gravitons. This divergence may be regulated by putting the system in a finite box. The black hole entropy which is $\mathcal{O}(N^2)$ can then easily be made to dominate over the $\mathcal{O}(N)$ contribution. Our mean-field approximation automatically computes the $\mathcal{O}(N^2)$ piece, while discarding the subleading $\mathcal{O}(N)$ divergence, so no additional infrared regularization is required.

This sort of approximation also has some potential drawbacks. An unfortunate fact is that there is no *a priori* guarantee that the approximation works well. One has to choose a trial action and a set of gap equations, and hope that with appropriate choices the approximation works well. In our case, we will be able to justify our choices *a posteriori* by showing that we get good agreement with black hole thermodynamics over a certain temperature range. Another way to justify the approximation is to compute higher-order terms in the expansion (6) and show that they are small. We have not attempted this for the full 0-brane problem, although toy models show promising behavior [12].

Although the approximation respects all symmetries which act linearly on the fields, it breaks symmetries that act non-linearly. As there is no superspace formulation of theories with 16 supercharges, we can only realize a subgroup of the supersymmetries (in our case $\mathcal{N}=2$) as acting linearly on the fields. This is sufficient, for example, to make our approximation to the vacuum energy vanish as $\beta\to\infty$. However, the remaining supersymmetries and R-symmetries are broken by the approximation. Another important symmetry which acts non-linearly on fields, and is therefore broken by our approximation, is gauge invariance. More precisely, our quadratic trial action is not invariant under Becchi-Rouet-Stora-Tyutin (BRST) transformations. As we shall show in Sec. III C, this difficulty can be largely overcome by an appropriate gauge choice.

[2]A "Fermi liquid" approach to black hole physics.

Before presenting the details of the approximation, let us note that the techniques we are using have a long history. They are closely related to variational methods [14] and self-consistent Hartree-Fock approximations, and also go by the name of modified perturbation theory [15]. They are equivalent to the effective action formalism developed in [16]. Similar techniques have been applied to QCD [17], and related techniques are used to study finite-temperature field theory [18]. Our own work on the subject began with [12], where we were motivated by the 0-brane problem to apply these techniques to several toy problems in supersymmetric quantum mechanics. Related techniques have been applied to $(0+0)$ dimensional Yang-Mills integrals in [19], and have also been used to study Wilson loops in $\mathcal{N}=4$ gauge theory in [20].

A. The 0-brane action in $\mathcal{N}=2$ superspace

We begin by formulating the 0-brane action in $\mathcal{N}=2$ superspace. For more details see appendix A of [12].

$\mathcal{N}=2$ supersymmetry means that we have an $SO(2)$ R-symmetry, with spinor indices $\alpha,\beta=1,2$ and vector indices $i,j=1,2$. The $SO(2)_R$ Dirac matrices $\gamma^i_{\alpha\beta}$ are real, symmetric, and traceless. Given two spinors ψ and χ, there are two invariants one can make, which we denote by

$$\psi_\alpha\chi_\alpha \quad\text{and}\quad \psi^\alpha\chi_\alpha \equiv \frac{i}{2}\epsilon_{\alpha\beta}\psi_\alpha\chi_\beta .$$

$\mathcal{N}=2$ superspace has coordinates (t,θ_α), where θ_α is a collection of real Grassmann variables that transform as a spinor of $SO(2)_R$. The simplest representation of supersymmetry is a real scalar superfield

$$\Phi=\phi+i\psi_\alpha\theta_\alpha+f\theta^2.$$

It contains a physical real boson ϕ and a physical real fermion ψ_α, along with a real auxiliary field f. To describe gauge theory we introduce a real spinor connection on superspace Γ_α, with component expansion

$$\Gamma_\alpha=\chi_\alpha+A_0\theta_\alpha+X^i\gamma^i_{\alpha\beta}\theta_\beta+d\epsilon_{\alpha\beta}\theta_\beta+2\epsilon_{\alpha\beta}\lambda_\beta\theta^2.$$

The fields X^i are physical scalars, while λ_α are their superpartners, d is an auxiliary boson, χ_α are auxiliary fermions, and A_0 is the $(0+1)$-dimensional gauge field.

To write a Lagrangian we introduce a supercovariant derivative

$$D_\alpha=\frac{\partial}{\partial\theta_\alpha}-i\theta_\alpha\frac{\partial}{\partial t} \tag{7}$$

and its gauge-covariant extension

$$\nabla_\alpha=D_\alpha+\Gamma_\alpha . \tag{8}$$

The 0-brane action is built from a collection of seven adjoint scalar multiplets Φ_a that transform in the **7** of a $G_2\subset SO(9)$ global symmetry, coupled to a $U(N)$ gauge multiplet Γ_α. The action reads

$$S_{SYM}=\frac{1}{g_{YM}^2}\int dtd^2\theta \mathrm{Tr}\bigg(-\frac{1}{4}\nabla^\alpha \mathcal{F}_i \nabla_\alpha \mathcal{F}_i - \frac{1}{2}\nabla^\alpha \Phi_a \nabla_\alpha \Phi_a - \frac{i}{3} f_{abc}\Phi_a[\Phi_b,\Phi_c]\bigg). \tag{9}$$

Here $\mathcal{F}_i=\frac{1}{4}\gamma^i_{\alpha\beta}\{\nabla_\alpha,\nabla_\beta\}$ is the field strength constructed from Γ_α, and f_{abc} is a totally antisymmetric G_2-invariant tensor, normalized to satisfy[3]

$$f_{abc}f_{abd}=\frac{3}{2}\delta_{cd}. \tag{10}$$

Strictly speaking, the relative positions of the N 0-branes are governed by an $SU(N)/\mathbb{Z}_N$ gauge theory, but in the large-N limit we can approximate this by $U(N)$.

We are interested in the finite temperature properties of the action (9). We work in Euclidean space, setting

$$S_E=-iS_M, \quad \tau=it, \quad A_{0\,E}=-iA_{0\,M}, \quad f_E=-if_M.$$

Note that we must Wick-rotate the auxiliary fields, to get a Euclidean action that is bounded below. As usual we compactify the Euclidean time direction on a circle of circumference β, which is identified with the inverse temperature. Bosons are periodic while fermions are antiperiodic; for example we write the mode expansions

$$X^i(\tau)=\frac{1}{\sqrt{\beta}}\sum_{l\in\mathbb{Z}} X^i_l e^{i2\pi l\tau/\beta}$$

$$\chi_\alpha(\tau)=\frac{1}{\sqrt{\beta}}\sum_{r\in\mathbb{Z}+1/2} \chi_{\alpha\,r}\, e^{i2\pi r\tau/\beta}.$$

B. Gauge fixing

Our approximation is based on resumming an infinite class of Feynman diagrams to obtain an approximation for the two-point functions at strong coupling. To make this procedure well-defined, we must fix a choice of gauge. For reasons we will explain, it is extremely advantageous to work in the gauge

$$D^\alpha \Gamma_\alpha=0. \tag{11}$$

The first advantage of this gauge is that, since Eq. (11) is a condition on superfields, our gauge choice preserves manifest supersymmetry. In terms of component fields, it sets

$$\partial_t A_0=0, \quad d=0, \quad \lambda_\alpha=\frac{1}{2}\partial_t\chi_\alpha. \tag{12}$$

This is a complete gauge fixing; i.e., having made this choice there is no residual freedom to make additional gauge transformations.

[3]This corrects a normalization error in [12].

A second advantage is that our gauge choice is well-defined at finite temperature. To see this, note that the zero mode of the gauge field, which we denote A_{00}, survives as a physical degree of freedom. This is important, because at finite temperature the gauge theory acquires an additional dynamical degree of freedom, namely the value of a Wilson-Polyakov loop around the Euclidean time direction $U=Pe^{i\oint d\tau A_0}$. In our gauge, this physical degree of freedom is parametrized by the zero mode

$$U=e^{i\sqrt{\beta}A_{00}}. \tag{13}$$

Note that at finite temperature

$$A_{00}\sim A_{00}+2\pi/\sqrt{\beta} \tag{14}$$

is a periodic variable.

Corresponding to our choice of gauge we must introduce a ghost action (but no gauge fixing term)

$$S_{ghost}=\frac{1}{g_{YM}^2}\int dtd^2\theta \mathrm{Tr}(D^\alpha \bar{C}\nabla_\alpha C).$$

For the ghost multiplet we adopt the component expansion

$$C=\alpha+\beta_\alpha\theta_\alpha+\gamma\theta^2$$

where α and γ are complex Grassmann fields and β_α is a complex boson. At finite temperature α and γ are periodic, while β is antiperiodic.

C. Slavnov-Taylor identities

Mean-field methods usually have a difficult time dealing with gauge symmetry. The problem is that the Slavnov-Taylor identities are typically violated by the approximation. After gauge fixing, Slavnov-Taylor identities arise from BRST invariance of the gauge fixed action. BRST transformations act non-linearly on fields, but the sort of mean-field approximation we wish to use is based on a trial action that is quadratic in the fundamental fields. Such a trial action cannot respect a symmetry that acts non-linearly. Thus mean-field techniques typically break BRST invariance, and hence violate Slavnov-Taylor identities.

A major advantage of our gauge choice (11) is that many of the Slavnov-Taylor identities become trivially satisfied, so that even a quadratic trial action can respect many of the consequences of gauge invariance. To illustrate this, we consider a simplified model, which can be obtained from the full 0-brane quantum mechanics by discarding all fermion and auxiliary fields. That is, we study bosonic Yang-Mills quantum mechanics, with the following gauge-fixed Euclidean action:

$$S=\frac{1}{g_{YM}^2}\int d\tau \mathrm{Tr}\bigg\{\frac{1}{2}D_\tau X^i D_\tau X^i-\frac{1}{4}[X^i,X^j][X^i,X^j] + \frac{1}{2\xi}(\partial_\tau A_0)^2+\partial_\tau\bar{\alpha}D_\tau\alpha\bigg\}. \tag{15}$$

Here A_0 is a $U(N)$ gauge field, with $D_\tau = \partial_\tau + i[A_0,\cdot]$. The fields X^i are adjoint scalars, and α is a ghost field. One subtle point is that the antighost zero mode $\bar{\alpha}_{l=0}$ does not appear in the action, and therefore should not be regarded as a true degree of freedom. It is completely decoupled, and any correlators involving $\bar{\alpha}_0$ vanish.

To illustrate the difficulties with gauge invariance we have adopted a general class of gauges parametrized by ξ. The action is obtained by gauge fixing $\partial_\tau A_0 = f$ and then functionally integrating over f, with the weight $\exp(-\int d\tau f^2/2g_{YM}^2\xi)$. Our preferred gauge condition $\partial_\tau A_0 = 0$ is recovered in the limit $\xi \to 0$. Expanding the fields in Fourier modes, the action (15) is invariant under BRST transformations:

$$\begin{aligned}
\delta_\eta A_{0\,l} &= -\eta\left(\frac{2\pi l}{\beta}\alpha_l + \frac{1}{\sqrt{\beta}}\sum_{m+n=l}[A_{0\,m},\alpha_n]\right)\\
\delta_\eta X^i_l &= -\eta\frac{1}{\sqrt{\beta}}\sum_{m+n=l}[X^i_m,\alpha_n]\\
\delta_\eta \alpha_l &= \eta\frac{1}{\sqrt{\beta}}\sum_{m+n=l}\alpha_m\alpha_n\\
\delta_\eta \bar{\alpha}_l &= \frac{\eta}{\xi}\frac{2\pi l}{\beta}A_{0,-l}
\end{aligned} \tag{16}$$

where η is a Grassmann parameter. Note that the decoupled antighost zero mode $\bar{\alpha}_0$ is indeed invariant under BRST transformations.

We can use this BRST symmetry to derive Slavnov-Taylor identities in the standard way, from the fact that the expectation value of any BRST-exact quantity vanishes. For example, we must have

$$\langle \delta_\eta(\bar{\alpha}_l A_{0l})\rangle = 0. \tag{17}$$

This gives us the following relation among Green's functions.

$$\left\langle \frac{1}{\xi}\frac{2\pi l}{\beta}A_{0,-l}A_{0l}\right\rangle + \left\langle \bar{\alpha}_l\left(\frac{2\pi l}{\beta}\alpha_l + \frac{1}{\sqrt{\beta}}\sum_{m+n=l}[A_{0m}\alpha_n]\right)\right\rangle = 0. \tag{18}$$

For $l=0$ this Slavnov-Taylor identity is trivially satisfied: the first term vanishes since $l=0$, while the second term vanishes since $\bar{\alpha}_0$ is decoupled. Here we assume the $A_{0,l}$ two-point function is finite at $l=0$. For $l\neq 0$ the second term can be simplified using the following Schwinger-Dyson equation (a consequence of the ghost equation of motion)

$$\left(\frac{2\pi l}{\beta}\right)^2\langle \mathrm{Tr}(\bar{\alpha}_l\alpha_l)\rangle + \frac{1}{\sqrt{\beta}}\frac{2\pi l}{\beta}\sum_{m+n=l}\langle \mathrm{Tr}(\bar{\alpha}_l[A_{0m},\alpha_n])\rangle = -g_{YM}^2N^2 \tag{19}$$

where for simplicity we have taken a trace to get rid of matrix indices. Thus, modulo the use of an equation of motion, the content of the identity (17) is the well-known fact that the gauge field propagator at non-zero frequency is given exactly by the gauge-fixing term in the classical action:

$$\langle \mathrm{Tr}(A_{0l}A_{0,-l})\rangle = \frac{g_{YM}^2N^2\xi}{(2\pi l/\beta)^2} \quad \text{for} \quad l\neq 0. \tag{20}$$

In the limit $\xi\to 0$ this Slavnov-Taylor identity implies that the modes of A_0 with non-zero frequency do not propagate. But this is an automatic consequence of adopting our gauge choice (11), which eliminates all non-zero modes of A_0. In fact, in the full 0-brane quantum mechanics, *all Slavnov-Taylor identities which just constrain two-point functions are automatically satisfied by working in the gauge (11).*

Next let us consider a Slavnov-Taylor identity on a 3-point function. We have the requirement

$$\langle \delta_\eta(\bar{\alpha}_l X^i_m X^j_n)\rangle = 0. \tag{21}$$

Using the transformations (16), this gives rise to a Slavnov-Taylor identity with the schematic form

$$\frac{1}{\xi}\langle \partial_\tau A_0 X^i X^j\rangle = \langle \bar{\alpha}\alpha X^i X^j\rangle. \tag{22}$$

If the gauge field carries zero frequency this identity turns out to be trivially satisfied [for the same reasons that (18) was trivially satisfied at $l=0$]. If the gauge field carries non-zero frequency then this Slavnov-Taylor identity is non-trivial. In particular, in the limit $\xi\to 0$, it states that the amplitude to emit a gauge boson with non-zero frequency is $\mathcal{O}(\xi)$. *But this property is automatically satisfied by working in the gauge (11), where the non-zero modes of the gauge field are eliminated.* Again, the content of the Slavnov-Taylor identity (21) is automatically taken into account just by working in the gauge (11).

This pattern is quite general. All non-trivial Slavnov-Taylor identities follow from the requirement that correlators of the form

$$\langle \delta_\eta(\bar{\alpha}_l\cdots)\rangle \tag{23}$$

vanish. (There must be at least one $\bar{\alpha}$, since δ_η increases the ghost number by one and you need zero ghost number to have a non-vanishing correlator.) If $l=0$ this Slavnov-Taylor identity is trivially satisfied. If $l\neq 0$ this Slavnov-Taylor identity becomes a constraint on correlators that either involve a gauge boson with non-zero frequency, or involve an antighost with non-zero frequency. Correlators with $A_{0,l\neq 0}$ must vanish in the limit $\xi\to 0$, and this property is guaranteed by working in the gauge (11). In fact it is not clear to us whether the Slavnov-Taylor identities have any non-trivial content in the gauge (11). In principle it seems that they could give constraints on correlators involving antighosts, but at the level of 2-point and 3-point functions, no constraints arise which are not already implied by the Schwinger-Dyson equations.

Does this issue of Slavnov-Taylor identities have any practical importance? After all, the approximation could work well even though it is not gauge invariant. But it turns out that in our case, the gauge choice (11) is crucial. We have used mean-field methods to study gauge theories [including Eqs. (9),(15)] in the more general R_ξ class of gauges, and have found that the system of one-loop truncated Schwinger-Dyson equations does not have solutions when the gauge theory is strongly coupled. We believe this breakdown can be related to the fact that the violation of Slavnov-Taylor identities gets worse as the coupling increases.

In any case, at least for 0-brane quantum mechanics, this difficulty can be avoided by working in the gauge (11). The vertices that appear in the gap equations receive no constraints that are not already implied by the Schwinger-Dyson equations (quartic vertices that appear in the gap equations will not involve a pair of ghosts). The Schwinger-Dyson equations themselves will be satisfied at the one-loop level, so the approximation is self-consistent.

D. Trial action and gap equations

In applying mean-field methods to 0-brane quantum mechanics, the first step is to choose a trial action. We will adopt the following trial action, which is written in terms of component fields expanded in Matsubara modes.

$$\begin{aligned}S_0=&-\frac{N}{\lambda}\mathrm{Tr}(U+U^\dagger)+\sum_l \frac{1}{2\sigma_l^2}\mathrm{Tr}(X_l^i X_{-l}^i)\\&-\sum_r \frac{1}{2a_r}\mathrm{Tr}(\chi_{\alpha r}\chi_{\alpha,-r})+\sum_l \frac{1}{2\Delta_l^2}\mathrm{Tr}(\phi_l^a\phi_{-l}^a)\\&-\sum_r \frac{1}{2g_r}\mathrm{Tr}(\psi_{\alpha r}^a\psi_{\alpha,-r}^a)+\sum_l \frac{1}{2\epsilon_l^2}\mathrm{Tr}(f_l^a f_{-l}^a)\\&-\sum_{l\neq 0}\frac{1}{s_l}\mathrm{Tr}(\overline{\alpha}_l\alpha_l)+\sum_r \frac{1}{t_r}\mathrm{Tr}(\overline{\beta}_{\alpha r}\beta_{\alpha r})-\sum_l \frac{1}{u_l}\mathrm{Tr}(\overline{\gamma}_l\gamma_l).\end{aligned}\tag{24}$$

Recall that $l,m\in\mathbb{Z}$ and $r,s\in\mathbb{Z}+\frac{1}{2}$ label Fourier modes, $\alpha,\beta=1,2$ are $SO(2)_R$ spinor indices, $i,j=1,2$ are $SO(2)_R$ vector indices, and $a,b=1,\ldots,7$ are indices in the **7** of G_2. The parameters $\lambda,\sigma_l^2,\ldots$ can be thought of as variational parameters, which we will fix by solving a set of one-loop gap equations.

The action (24) is essentially the most general Gaussian trial action that is compatible with the linearly-realized bosonic symmetries of the problem.[4] Supersymmetry is broken at finite temperature, so we have not imposed supersymmetry on the action (24), although as we discuss below supersymmetry gets incorporated into our approximation in a natural way.

[4]Including the $\mathbb{Z}_2$ R-parity symmetry discussed in appendix A of [12].

There are a few subtle points to note about this action. One point is that, due to the periodicity (14), it is not appropriate to adopt a Gaussian trial action for A_{00}. Rather we have adopted the unitary one-plaquette model action [21]

$$S_\square=-\frac{N}{\lambda}\mathrm{Tr}(U+U^\dagger)$$

for the holonomy $U=e^{i\sqrt{\beta}A_{00}}$. This action undergoes a large-N phase transition when $\lambda=2$. As discussed in [12], such a transition is expected to separate the perturbative gauge theory regime from the supergravity regime, presuming couplings to other fields do not turn this into a smooth crossover. A second minor point is that, as discussed in Sec. III C, the antighost zero mode is not a physical degree of freedom. We have therefore suppressed the terms involving $\overline{\alpha}_0$ in Eq. (24).

Corresponding to the action (24) we have the 2-point correlators

$$\begin{aligned}&\langle A_{00}A_{00}\rangle_0\equiv\rho_0^2\quad \langle X_l^i X_m^j\rangle_0=\sigma_l^2\delta^{ij}\delta_{l+m}\\&\langle\chi_{\alpha r}\chi_{\beta s}\rangle_0=a_r\delta_{\alpha\beta}\delta_{r+s}\\&\langle\phi_l^a\phi_m^b\rangle_0=\Delta_l^2\delta^{ab}\delta_{l+m}\\&\langle\psi_{\alpha r}^a\psi_{\beta s}^b\rangle_0=g_r\delta^{ab}\delta_{\alpha\beta}\delta_{r+s}\quad \langle f_l^a f_m^b\rangle_0=\epsilon_l^2\delta^{ab}\delta_{l+m}\\&\langle\overline{\alpha}_l\alpha_m\rangle_0=s_l\delta_{lm}\quad \langle\overline{\beta}_{\alpha r}\beta_{\beta s}\rangle_0=t_r\delta_{\alpha\beta}\delta_{rs}\\&\langle\overline{\gamma}_l\gamma_m\rangle_0=u_l\delta_{lm}\end{aligned}\tag{25}$$

where $\langle\cdots\rangle_0$ denotes an expectation value computed using S_0, and where the two-point function of the gauge field zero mode is given by

$$\rho_0^2=\begin{cases}\dfrac{2}{\beta N}\left[\mathrm{li}_2\left(1-\dfrac{\lambda}{2}\right)+\left(1-\dfrac{2}{\lambda}\right)\log\left(1-\dfrac{\lambda}{2}\right)-1\right], & \lambda\leq 2,\\ \dfrac{1}{\beta N}\left(\dfrac{\pi^2}{3}-\dfrac{4}{\lambda}\right), & \lambda\geq 2,\end{cases}\tag{26}$$

involving a dilogarithm [12].

Next we need to choose a set of gap equations to fix the parameters that appear in our trial action. For most degrees of freedom we will adopt the one-loop gap equations discussed in [12]. These equations can be obtained by demanding that the quantity[5]

$$I_{\mathrm{eff}}=\beta F_0+\langle S_{II}+S_{IV}-S_0\rangle_0-\frac{1}{2}\langle(S_{III})^2\rangle_{\mathrm{C},0}\tag{27}$$

is stationary with respect to arbitrary variations of the 2-point functions (25), where S_{II},S_{III},S_{IV} refer to terms in the super Yang-Mills– (SYM-) plus-ghost action that are quadratic,

[5]This quantity can be identified with the two-loop 2PI effective action of Cornwall, Jackiw and Tomboulis [16].

cubic, quartic in the fundamental fields. The explicit expression for this effective action is given in the Appendix.

For the gauge field, however, we use a slightly different gap equation. The starting point is the Schwinger-Dyson equation for $\langle \mathrm{Tr} U\rangle$, which follows from demanding that

$$\langle U\rangle = \int dU d(\cdots)\, U e^{-S} \tag{28}$$

is invariant under an infinitesimal change of variables $U \to gU$ with $g = 1 + i\omega \in U(N)$. At leading order this implies

$$\langle \mathrm{Tr} U\rangle = -\frac{i}{\sqrt{\beta}}\left\langle \mathrm{Tr}\left(U \frac{\delta S}{\delta A_{00}}\right)\right\rangle \tag{29}$$

where we have dropped higher-order terms in δA_{00}, coming from the Campbell-Baker-Hausdorff lemma, which do not contribute at one loop in the mean field approximation. Evaluating Eq. (29) to one-loop order gives a relation between expectation values computed with respect to S_0.

$$\frac{1}{N}\langle \mathrm{Tr} U\rangle_{\square} = \frac{2}{\beta}\left(-\frac{i\sqrt{\beta}}{N}\langle \mathrm{Tr}(UA_{00})\rangle_{\square}\right) \times \frac{\partial}{\partial \rho_0^2}\left(\langle S_{IV}\rangle_0 - \frac{1}{2}\langle (S_{III})^2\rangle_{\mathrm{C},0}\right). \tag{30}$$

The relevant one-plaquette expectation values are [21]

$$\frac{1}{N}\langle \mathrm{Tr} U\rangle_{\square} = \begin{cases} 1-\lambda/4 & \lambda \leqslant 2 \\ 1/\lambda & \lambda \geqslant 2 \end{cases} \tag{31}$$

$$-\frac{i\sqrt{\beta}}{N}\langle \mathrm{Tr}(UA_{00})\rangle_{\square} = \begin{cases} \frac{1}{2}\left(1+\frac{\lambda}{4}\right) - \frac{1}{\lambda}\left(1-\frac{\lambda}{2}\right)^2 \log\left(1-\frac{\lambda}{2}\right) & \lambda \leqslant 2 \\ 1-\frac{1}{2\lambda} & \lambda \geqslant 2, \end{cases}$$

while the expressions for $\langle S_{IV}\rangle_0$ and $-\frac{1}{2}\langle (S_{III})^2\rangle_{\mathrm{C},0}$ are given in the Appendix. We adopt Eq. (30) as the gap equation that fixes the one-plaquette coupling λ.

Let us pause to note a few important features of this system of gap equations. First, in computing expectation values we have kept only planar contributions. This means 't Hooft large-N counting is automatic: the free energy will come with an overall factor of N^2, and the Yang-Mills coupling will only appear in the combination $g_{YM}^2 N$. We henceforth adopt units which set $g_{YM}^2 N = 1$, by rescaling all dimensionful quantities as in [12]. Second, since we have consistently worked to one-loop order while including all auxiliary fields, these gap equations respect supersymmetry. Of course supersymmetry gets broken at finite temperature, but in the zero-temperature limit these symmetry breaking effects go away.[6] Thus as $\beta \to \infty$ the bosonic and fermionic propagators will be related by supersymmetry Ward identities, and the vacuum energy will automatically vanish.

To summarize, the parameters appearing in our trial action are fixed by solving the following set of gap equations:

$$\begin{aligned}\frac{1}{N}\langle \mathrm{Tr} U\rangle_{\square} = \frac{2}{\beta}\left(-\frac{i\sqrt{\beta}}{N}\langle \mathrm{Tr}(UA_{00})\rangle_{\square}\right) &\times \left[\frac{2}{\beta}\sum_l \sigma_l^2 + \frac{5i}{2\beta}\sum_r \frac{2\pi r}{\beta} a_r + \frac{7}{\beta}\sum_l \Delta_l^2 \right. \\ &- \frac{4}{\beta}\sum_l \left(\frac{2\pi l}{\beta}\right)^2 (\sigma_l^2)^2 - \frac{4}{\beta}\sum_r \left(\frac{2\pi r}{\beta}\right)^4 (a_r)^2 \\ &- \frac{14}{\beta}\sum_l \left(\frac{2\pi l}{\beta}\right)^2 (\Delta_l^2)^2 - \frac{7}{\beta}\sum_r (g_r)^2 \\ &\left. + \frac{1}{\beta}\sum_l \left(\frac{2\pi l}{\beta}\right)^2 (s_l)^2 + \frac{1}{2\beta}\sum_r (t_r)^2 \right]\end{aligned} \tag{32}$$

$$\begin{aligned}\frac{1}{\sigma_l^2} = \left(\frac{2\pi l}{\beta}\right)^2 + \frac{2}{\beta}\sum_m \sigma_m^2 + \frac{3i}{\beta}\sum_r \frac{2\pi r}{\beta} a_r &+ \frac{14}{\beta}\sum_m \Delta_m^2 + \frac{2}{\beta}\rho_0^2 - \frac{8}{\beta}\left(\frac{2\pi l}{\beta}\right)^2 \sigma_l^2 \rho_0^2 \\ &- \frac{14}{\beta}\sum_{r+s+l=0} g_r g_s + \frac{1}{\beta}\sum_{r+s+l=0} t_r t_s\end{aligned} \tag{33}$$

[6]Supersymmetry is unbroken at zero temperature in this model.

$$
\begin{aligned}
\frac{1}{a_r}=&-i\left(\frac{2\pi r}{\beta}\right)^3-\frac{3i}{\beta}\frac{2\pi r}{\beta}\sum_l \sigma_l^2\\
&+\frac{4}{\beta}\frac{2\pi r}{\beta}\sum_s \frac{2\pi s}{\beta}a_s-\frac{5i}{2\beta}\frac{2\pi r}{\beta}\rho_0^2\\
&-\frac{14i}{\beta}\frac{2\pi r}{\beta}\sum_l \Delta_l^2+\frac{8}{\beta}\left(\frac{2\pi r}{\beta}\right)^4 a_r\rho_0^2\\
&-\frac{14}{\beta}\sum_{r+s+l=0}\left(\frac{2\pi l}{\beta}\right)^2\Delta_l^2 g_s\\
&-\frac{14}{\beta}\sum_{r+s+l=0}\epsilon_l^2 g_s+\frac{1}{\beta}\sum_{r+s+l=0}t_s u_l\\
&-\frac{1}{\beta}\sum_{r+s+l=0}\left(\frac{2\pi l}{\beta}\right)^2 s_l t_s
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
\frac{1}{\Delta_l^2}=&\left(\frac{2\pi l}{\beta}\right)^2+\frac{4}{\beta}\sum_m \sigma_m^2+\frac{4i}{\beta}\sum_r \frac{2\pi r}{\beta}a_r+\frac{2}{\beta}\rho_0^2\\
&+\frac{12}{\beta}\sum_{m+n+l=0}\Delta_m^2\epsilon_n^2-\frac{12}{\beta}\sum_{r+s+l=0}g_r g_s\\
&+\frac{4}{\beta}\left(\frac{2\pi l}{\beta}\right)^2\sum_{r+s+l=0}a_r g_s-\frac{8}{\beta}\left(\frac{2\pi l}{\beta}\right)^2\\
&\times\Delta_l^2\rho_0^2
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
\frac{1}{g_r}=&\,i\frac{2\pi r}{\beta}+\frac{12}{\beta}\sum_{r+s+l=0}\Delta_l^2 g_s\\
&-\frac{2}{\beta}\sum_{r+s+l=0}\left(\frac{2\pi l}{\beta}\right)^2\Delta_l^2 a_s-\frac{2}{\beta}\sum_{r+s+l=0}\epsilon_l^2 a_s\\
&+\frac{4}{\beta}\sum_{r+s+l=0}\sigma_l^2 g_s+\frac{2}{\beta}g_r\rho_0^2
\end{aligned}
\tag{36}
$$

$$
\frac{1}{\epsilon_l^2}=1+\frac{6}{\beta}\sum_{m+n+l=0}\Delta_m^2\Delta_n^2+\frac{4}{\beta}\sum_{r+s+l=0}a_r g_s
\tag{37}
$$

$$
\begin{aligned}
\frac{1}{s_l}=&-\left(\frac{2\pi l}{\beta}\right)^2-\frac{1}{\beta}\left(\frac{2\pi l}{\beta}\right)^2\sum_{r+s+l=0}t_r a_s\\
&-\frac{2}{\beta}\left(\frac{2\pi l}{\beta}\right)^2 s_l\rho_0^2
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
\frac{1}{t_r}=&\,i\frac{2\pi r}{\beta}+\frac{1}{\beta}\sum_{r+s+l=0}\sigma_l^2 t_s-\frac{1}{2\beta}\sum_{r+s+l=0}a_s u_l\\
&+\frac{1}{2\beta}\sum_{r+s+l=0}\left(\frac{2\pi l}{\beta}\right)^2 s_l a_s+\frac{1}{2\beta}\rho_0^2 t_r
\end{aligned}
\tag{39}
$$

$$
\frac{1}{u_l}=1+\frac{1}{\beta}\sum_{r+s+l=0}t_r a_s .
\tag{40}
$$

E. Numerical methods

The gap equations we have described form an infinite set of coupled algebraic equations. We now outline the numerical methods that we used to solve these equations. For additional details see appendix B of [12].

The first step is to reduce the infinite set of Eqs. (32)–(40) down to a finite set. To do this we use the following asymptotic forms of the propagators, which are valid at large momenta:

$$
\begin{aligned}
&\sigma_l^2\approx\frac{1}{(2\pi l/\beta)^2+m_\sigma^2}\qquad a_r\approx\frac{i}{2\pi r/\beta}\frac{1}{(2\pi r/\beta)^2+m_a^2}\\
&\Delta_l^2\approx\frac{1}{(2\pi l/\beta)^2+m_\Delta^2}\qquad g_r\approx-\frac{i2\pi r/\beta}{(2\pi r/\beta)^2+m_g^2}\\
&\epsilon_l^2\approx\frac{(2\pi l/\beta)^2}{(2\pi l/\beta)^2+m_\epsilon^2}\\
&s_l\approx-\frac{1}{(2\pi l/\beta)^2+m_s^2}\qquad t_r\approx-\frac{i2\pi r/\beta}{(2\pi r/\beta)^2+m_t^2}\\
&u_l\approx\frac{(2\pi l/\beta)^2}{(2\pi l/\beta)^2+m_u^2}.
\end{aligned}
\tag{41}
$$

At leading order these are simply the tree-level propagators.[7] Demanding that these propagators satisfy the gap equations to the first subleading order of an expansion in 1/(momentum) fixes the asymptotic masses to be

$$
\begin{aligned}
m_\sigma^2&=\frac{2}{\beta}\sum_l \sigma_l^2+\frac{3i}{\beta}\sum_r\frac{2\pi r}{\beta}a_r+\frac{14}{\beta}\sum_l \Delta_l^2-\frac{6}{\beta}\rho_0^2\\
m_a^2&=\frac{3}{\beta}\sum_l \sigma_l^2+\frac{4i}{\beta}\sum_r\frac{2\pi r}{\beta}a_r+\frac{14}{\beta}\sum_l \Delta_l^2-\frac{11}{2\beta}\rho_0^2
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
m_\Delta^2&=\frac{4}{\beta}\sum_l \sigma_l^2+\frac{12}{\beta}\sum_l \Delta_l^2-\frac{6}{\beta}\rho_0^2\\
m_g^2&=\frac{4}{\beta}\sum_l \sigma_l^2+\frac{12}{\beta}\sum_l \Delta_l^2-\frac{2}{\beta}\rho_0^2\\
m_\epsilon^2&=-\frac{4i}{\beta}\sum_r\frac{2\pi r}{\beta}a_r+\frac{12}{\beta}\sum_l \Delta_l^2\\
m_s^2&=-\frac{i}{\beta}\sum_r\frac{2\pi r}{\beta}a_r-\frac{2}{\beta}\rho_0^2\\
m_t^2&=\frac{1}{\beta}\sum_l \sigma_l^2-\frac{1}{2\beta}\rho_0^2
\end{aligned}
$$

[7]This reflects the fact that the quantum mechanics is free in the ultraviolet.

$$m_u^2 = -\frac{i}{\beta}\sum_r \frac{2\pi r}{\beta} a_r .$$

The next step is to fix a mode cutoff N. For modes with $-N \leq l,\ r \leq N$ we regard the Fourier modes of the propagators themselves as the unknowns, while for modes outside this range we parametrize the propagators in terms of the eight unknown asymptotic masses appearing in Eq. (41). The propagators with $-N \leq l,\ r \leq N$ are to be found by directly solving the relevant gap equations (32)–(40), while the asymptotic masses are to be determined by solving the system of equations (42). Note that all these equations are coupled. For example, we evaluate the high-momentum parts of the loop sums that appear in Eqs. (32)–(40) and (42) analytically, in terms of the asymptotic masses.

This leaves us with a finite set of equations.[8] Our basic strategy is to start at high temperatures $\beta \ll 1$, where we have the following approximate solution to the gap equations.

$$\lambda \approx 0.418\beta^{3/2} \quad \sigma_0^2 \approx 0.209\beta^{1/2} \quad \Delta_0^2 \approx 0.282\beta^{1/2}$$

$$\epsilon_0^2 \approx 0.677 \quad u_0 \approx 1. \tag{43}$$

(All non-zero modes are approximately given by their free-field values.) Then we use the Newton-Raphson method [22] to solve the system of equations at a sequence of successively lower temperatures. Our numerical results were obtained starting at $\beta = 0.1$, with

$$\beta \to \min(1.2\beta, \beta + 0.25)$$

on each step, and with a mode cutoff $N = \max(3,5\beta)$.

Finally, after solving all the gap equations, we wish to compute the free energy. This has the expansion given in Eq. (6), which we truncate to

$$\beta F \approx \beta F_0 + \langle S - S_0 \rangle_0 - \frac{1}{2}\langle (S_{III})^2 \rangle_{C,0}. \tag{44}$$

That is, our approximation to βF is simply the effective action $I_{\rm eff}$ of Eq. (27).[9] The explicit expression for $I_{\rm eff}$ is given in the Appendix. To calculate $I_{\rm eff}$ numerically we must make use of the asymptotic forms (41). For example we define the following renormalized sum:

$$-\sum_l \log \sigma_l^2 = -\sum_{-N}^{N} \log\{\sigma_l^2[(2\pi l/\beta)^2 + m_\sigma^2]\} + 2\log(2\sinh(\beta m_\sigma/2)). \tag{45}$$

[8] $8N+13$ equations, to be precise, where we have taken advantage of the fact that time-reversal invariance makes the (bosonic, fermionic) propagators (even, odd) functions of their momenta.

[9] The 0-brane action also has 6-point couplings, but it turns out that $\langle S_{VI} \rangle_0 = 0$ so these terms do not contribute to our approximation for the free energy.

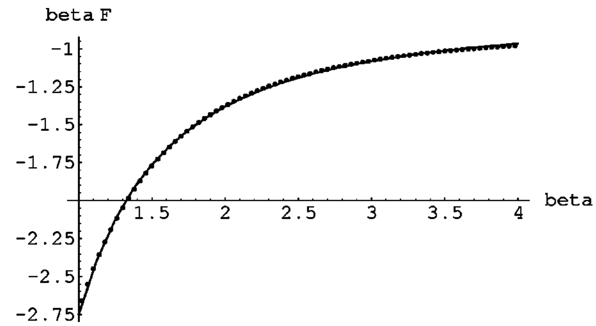

FIG. 1. The solid curve is the power law fit (47) for βF. The data points are calculated from numerical solutions to the gap equations.

III. MEAN-FIELD RESULTS FOR THERMODYNAMIC QUANTITIES

In principle the trial action we have constructed contains a great deal of information about correlation functions in the quantum mechanics. But in this section we will just present numerical results for the behavior of three basic quantities: the free energy, the Wilson loop, and the mean size of the state.

A. Free energy

At high temperatures, where the gauge theory is weakly coupled, we find that the free energy of the system is

$$\beta F = 6 \log \beta + \mathcal{O}(1). \tag{46}$$

This result can be obtained analytically: the gap equations are dominated by the bosonic zero modes, and the free energy is dominated by βF_0.

In general, for a weakly-coupled theory in $0+1$ dimensions, one would expect the free energy to behave like $\log \beta$. But note that, even though the gauge theory is weakly coupled at high temperature, the perturbation series is afflicted with IR divergences. Thus, to determine the coefficient of the logarithm (which depends on the value of the dynamically generated IR cutoff) one must re-sum part of the perturbation series. This is a well-known phenomenon in finite temperature field theory [18]. In any case, we expect *a priori* that mean-field methods give good results in the high temperature regime.

As the temperature is lowered the behavior of the free energy changes: at $\beta \approx 0.7$ we find that it begins to roll over and fall off as a non-trivial power of the temperature. In the range $1 < \beta < 4$ the numerical results for the free energy are well fit by

$$\beta F \approx -0.79 - 2.0\beta^{-1.7}. \tag{47}$$

This fit to the numerical results is illustrated in Fig. 1. Note that supersymmetry is crucial in making such power-law behavior possible. Without supersymmetry the free energy would behave as $\beta F \approx \beta E_0$ in the low temperature regime $(\beta > 1)$, where E_0 is the ground state energy of the system.

We obtained Eq. (47) by performing a Levenberg-Marquardt nonlinear least-squares fit [22] to 75 numerical

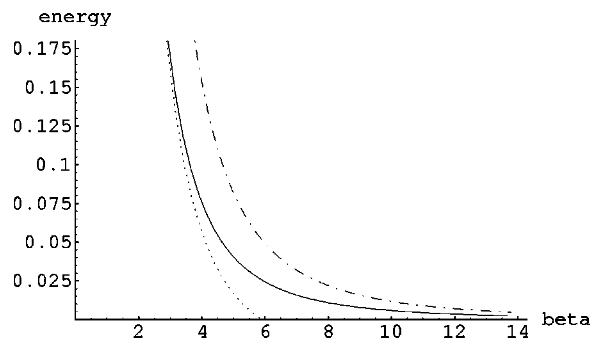

FIG. 2. Energy vs β. For $\beta>2.5$ fixing λ by fitting βF to a power law leads to the solid middle line, while the Schwinger-Dyson gap equation for lambda leads to the lower dashed line. The upper dot-dashed line is the semiclassical energy of the black hole.

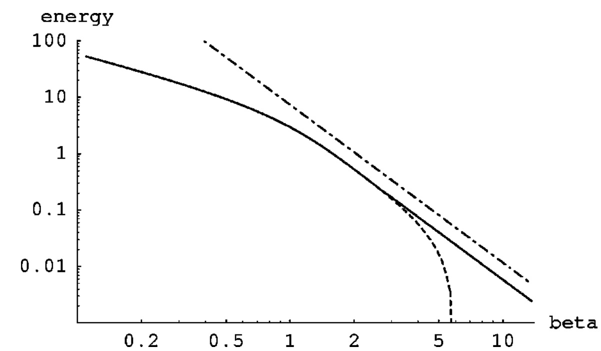

FIG. 3. Energy vs β. Same as Fig. 2, but plotted on a log-log scale.

calculations of the free energy, carried out in the temperature range $1\leq\beta\leq4$. To estimate the uncertainty in the best fit parameters we varied the window of β over which the fit was performed (fitting over the ranges $2<\beta<4$ and $1<\beta<3$), which leads to: -0.79 ± 0.06, -2.0 ± 0.1, and -1.7 ± 0.2.

It is quite remarkable that the power law (47) is in excellent agreement with the semiclassical black hole prediction [10,11]

$$\beta F=-4.12\beta^{-1.80}. \tag{48}$$

The exponents differ by 6% while the coefficients of the power-law differ by a factor of 2. (An additive constant appears in the mean-field approximation for the free energy. We will generally ignore this "ground state degeneracy," since it seems to be an artifact of the approximation when applied to systems with a continuous spectrum. Similar behavior was noted in [12].) In a toy model studied in [12] it was noted that higher order terms in the expansion of the free energy (6) appear with approximately the same power law dependence on temperature as the leading term. Thus by computing higher-order corrections one might hope for better agreement of the overall coefficient, with the power law essentially unchanged.

As we go to still lower temperatures, we find that the energy $\partial(\beta F)/\partial\beta$ calculated in the mean-field approximation begins to drop below the energy of the black hole. In fact the mean-field energy becomes negative around $\beta=5.8$. Ultimately, as $\beta\to\infty$, the mean-field energy does asymptote to zero, as required by the $\mathcal{N}=2$ supersymmetry which is manifest in the approximation. But a negative energy clearly reflects some problem with the approximation.

Fortunately, we can be rather precise about exactly where the approximation is going wrong: the difficulty is with the Schwinger-Dyson gap equation we have been using to fix the value of the one-plaquette coupling λ. Although we do not know how to write down a better gap equation for λ, we can give a *prescription* for fixing λ, that will allow us to obtain reasonable results at much lower values of the temperature. This may be regarded either as a check on our understanding of why the approximation is breaking down, or as a way of building a model for the black hole that can be used at lower temperatures. Our prescription for fixing λ is simply that, when $\beta>2.5$ (the midpoint of our range $1\leq\beta\leq4$), we choose λ so that the free energy is given by Eq. (47). The energy $E=\partial(\beta F)/\partial\beta$ calculated with this prescription is shown in Figs. 2 and 3.

B. Wilson loop

In our approximation the expectation value of the timelike Wilson loop $U=Pe^{i\oint d\tau A_0}$ is controlled by the one-plaquette coupling λ, as in Eq. (31). A key feature of the one-plaquette action is that a large-N phase transition occurs at $\lambda=2$ [21]. At this value of the coupling the eigenvalues of U spread out around a circle, and become sensitive to the fact that the gauge field is a periodic variable. It has been argued that just such a phase transition is expected to occur in 0-brane quantum mechanics, as the system moves from weak coupling into the supergravity regime [12].

Our mean-field results for λ are shown in Fig. 4. We present the results for λ that are obtained by solving the Schwinger-Dyson equation (30), as well as the results that are obtained from our prescription of fitting βF to a power law.

Note that in both cases, λ increases monotonically with β. The Gross-Witten phase transition takes place when $\lambda=2$; with the prescription of fitting βF to a power law this value is reached at $\beta=7.8$. Thus, as expected, a phase transition takes place as the system moves into the supergravity regime [12]. By adopting the prescription of fitting βF to a power law, we cannot say anything about the order of the phase transition. But if one takes the Schwinger-Dyson result for λ seriously, then the Gross-Witten transition occurs at $\beta=14.2$, and is weakly second order (the second derivative of the free energy drops by 0.01 in crossing the transition).

Our prescription for choosing λ by fitting βF to a power law begins to break down around $\beta=14$, as we find that λ rapidly diverges as β approaches 14.[10] By itself, this is not necessarily a problem: infinite λ simply means that the Wilson loop is uniformly distributed over $U(N)$. But unfortunately, we do not have a good prescription for continuing

[10]The Schwinger-Dyson gap equation for λ has solutions at all temperatures.

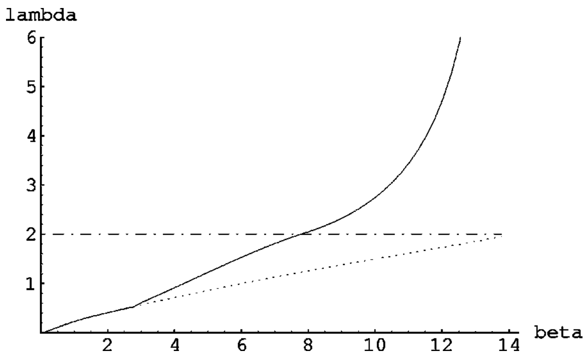

FIG. 4. The one-plaquette coupling λ vs β. The Gross-Witten transition occurs when $\lambda=2$. For $\beta<2.5$ we use the Schwinger-Dyson gap equation to determine λ. For $\beta>2.5$ the Schwinger-Dyson gap equation gives the dashed line, while fitting βF to a power law gives the solid line.

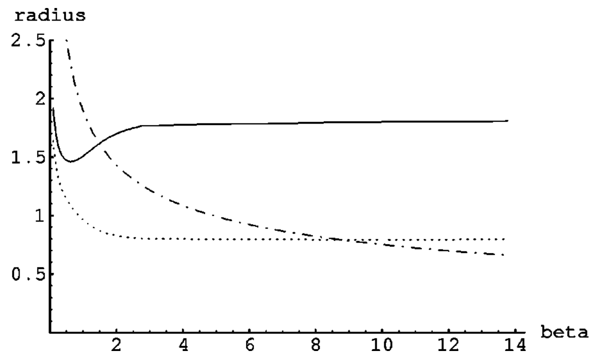

FIG. 5. Range of eigenvalues (radius of the Wigner semi-circle) vs β. The upper solid curve is for the scalar fields in the scalar multiplets; the lower dotted curve is for the scalar fields in the gauge multiplet. The dashed curve is the Schwarzschild radius of the black hole. These results were calculated with βF fit to a power law for $\beta>2.5$.

past this temperature. Evidently some of the other gap equations (not just the gap equation for λ) start to break down at this point. Note that this breakdown does not occur until well into the strong coupling regime, as an inverse temperature $\beta=14$ corresponds to an effective gauge coupling $g_{\rm eff}^2=\beta^3\approx 3\times 10^3$.

C. Mean size

Finally, let us comment on the average "size" of the state. In our approximation the scalar fields $X^i(\tau)$ and $\phi^a(\tau)$ are Gaussian random matrices, and their eigenvalues obey a Wigner semi-circle distribution. We can define the size of the state in terms of the quantities

$$R^2_{\rm gauge}=\frac{1}{N}\langle {\rm Tr}[X^i(\tau)]^2\rangle_0, \qquad R^2_{\rm scalar}=\frac{1}{N}\langle {\rm Tr}[\phi^a(\tau)]^2\rangle_0. \tag{49}$$

The radius of the Wigner semi-circle, given by $2\sqrt{R^2}$, is shown in Fig. 5. Note that the radius stays fairly constant in the region corresponding to the black hole. However, because the superfield formalism we are using does not respect the full $SO(9)$ invariance, the radius measured in the scalar multiplet directions is not the same as the radius measured in the gauge multiplet directions. At $\beta=14$ we find

$$2R_{\rm scalar}=1.81 \qquad 2R_{\rm gauge}=0.80.$$

This shows that, as expected, the trial action does not respect the underlying $SO(9)$ invariance. Nonetheless, the trial action may provide a useful approximate description of the black hole density matrix in the supergravity regime.

In Fig. 5 we have also plotted the Schwarzschild radius of the black hole

$$U_0/2\pi=1.89\beta^{-2/5}. \tag{50}$$

Note that, as the temperature decreases, the Schwarzschild radius becomes much smaller than the radius of the eigenvalue distributions. It seems appropriate to identify the radius of the eigenvalue distributions with the size of the region $U\ll(g_{\rm YM}^2N)^{1/3}$ in which 10-dimensional supergravity is valid [10].

This brings up a subtle issue. The Higgs fields of the gauge theory are expected to correspond to spatial coordinates in the supergravity geometry (2). But one is always free to reparametrize the radial coordinate in supergravity. In Eq. (50) we have implicitly made use of the naive identification $X=U/2\pi$, where X is a Higgs field and U is the supergravity coordinate appearing in Eq. (2). This can be justified at zero temperature, because supersymmetry fixes the mass of a Bogomol'nyi-Prasad-Sommerfield (BPS) stretched string in the gauge theory to be given by the tree-level formula $m_W=X$, while in supergravity one has $m_W=U/2\pi$ [10]. However this particular identification is not appropriate at finite temperature. A proposal for relating the two radial coordinates has been presented in [23].

An unambiguous way to fix the relation is to use the fact that $m_W=(U-U_0)/2\pi$ in the non-extremal black hole geometry. By computing m_W in the gauge theory, the mapping between the Higgs field X and the supergravity coordinate U can be fixed. However one must first take account of the fact that one has a continuous distribution of masses in the quantum mechanics. The spacetime geometry will only correspond to the lightest of these states as we discuss in Sec. V. This procedure will be studied further in [24].

IV. PROPAGATORS AND SPECTRAL WEIGHTS

Important information about spacetime geometry is encoded in the spectrum of single-string excitations in the quantum mechanics. To extract this information from the Euclidean propagators we introduce a spectral representation for the 2-point functions. By inserting complete sets of states, one can show that at finite temperature the analog of the Lehmann spectral representation takes the form

$$\langle\phi(\tau)\phi(0)\rangle=\int_0^\infty d\omega\rho(\omega)\frac{\cosh\omega(\tau-\beta/2)}{2\omega\sinh(\beta\omega/2)} \quad 0\leq\tau\leq\beta \tag{51}$$

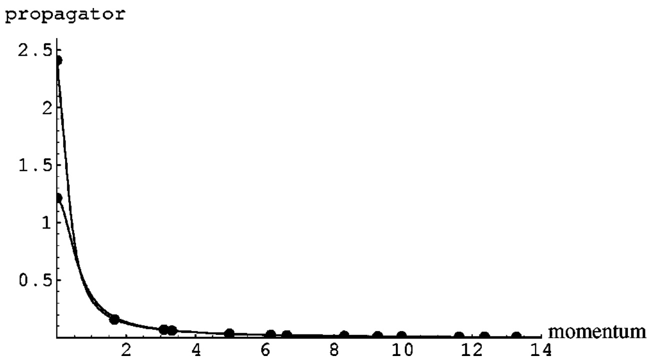

FIG. 6. The 2-point function of the scalar fields in the scalar multiplets, calculated at Matsubara frequencies. The solid curve is a fit to the propagator with a twin peak ansatz for the density of states. The upper curve is for $\beta=3.78$, the lower curve for $\beta=2.03$.

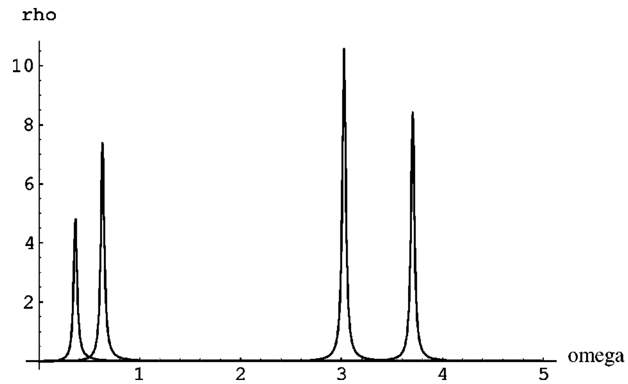

FIG. 7. The effective density of states arising from the scalar multiplet. We show the result for two different temperatures: the first and third peaks are for $\beta=3.78$, the second and fourth for $\beta=2.03$.

where the spectral weight $\rho(\omega)$ is defined as the thermal average

$$\rho(\omega)=\frac{1}{Z}\sum_{m} e^{-\beta E_m}\sum_{n>m} |\langle n|\phi|m\rangle|^2 2\omega(1-e^{-\beta\omega}) \times\delta(\omega-E_n+E_m).$$

We can interpret $\rho(\omega)d\omega$ as the effective number of single-string microstates with a mass between ω and $\omega+d\omega$. We will apply this to the scalar fields in the scalar multiplets, setting

$$\Delta_l^2=\int_0^\infty d\omega\rho(\omega)\frac{1}{(2\pi l/\beta)^2+\omega^2}.$$

In general, solving the inverse problem to extract the density of states from Δ_l^2 is a difficult numerical problem, which we analyze in detail in [24]. However some gross features of the spectral density can be easily seen. Consider the Δ^2 propagator which is plotted in Fig. 6. At large frequency the behavior of the propagator is controlled by the asymptotic mass m_Δ^2 (42). This mass is of order 1 in ’t Hooft units; for example $m_\Delta=2.8$ at $\beta=3.78$. This indicates that states with a mass of order 1 are present in the spectrum. In addition, note that a clear enhancement of the propagator at small frequency compared to its asymptotic form can be seen in Fig. 6. This suggests the presence of light states in the spectrum, with a mass of order the temperature. These light states make the dominant contribution to the entropy.

To express this in a more quantitative way, we will make an ansatz for the form of $\rho(\omega)$. The ansatz will allow us to estimate the spectral density without performing a complete analysis of the inverse problem. Our ansatz is motivated by the results of [24], where we considered a 0-brane probe of the black hole background. The full analysis of [24] shows that the density of states consists of two narrow peaks (i.e. narrower than a scale of order the temperature), one centered at frequency of order the temperature, the other at frequency of order the ’t Hooft coupling. Motivated by this result, we introduce the following ansatz

$$\rho(\omega)=\frac{\omega z_1}{(\omega-m_1)^2+\gamma^2}+\frac{\omega z_2}{(\omega-m_2)^2+\gamma^2}+(\omega\rightarrow-\omega), \tag{52}$$

where we fix the width $\gamma=0.02$ (any number much smaller than T will do). The parameters in Eq. (52) are determined by fitting to the Δ_l^2 propagator. Note that the small value of γ is motivated by [24]; we could not hope to extract such a small width by fitting the ansatz just at Matsubara frequencies. The resulting densities of states are shown in Fig. 7. As can be seen from the plot, we obtain a very good fit to the propagators at the Matsubara frequencies. The fit determines the parameters z_i and m_i with uncertainties at the level of a few percent.

This result clearly indicates the presence of both the high and low frequency states mentioned above. It also points to a clear separation between the two sets of states—a separation which will play an important role in the next section.

V. RESOLVING SPACETIME GEOMETRY

In our mean-field approximation, we have modelled the cloud of 0-branes that make up the black hole using Gaussian random matrices. As we discussed in Sec. III C, the eigenvalues of these matrices have very large quantum fluctuations. Within the gauge theory, we find that the scale of these fluctuations is set by the ’t Hooft coupling.

$$R^2=\frac{1}{N}\langle \mathrm{Tr}\,\phi(\tau)^2\rangle_0\sim(g_{YM}^2N)^{2/3}. \tag{53}$$

In terms of supergravity, this means that the positions of the 0-branes that make up the black hole have very large quantum fluctuations. Indeed they fluctuate over roughly the entire region [of size $U\sim(g_{YM}^2N)^{1/3}$ [10]] in which supergravity is valid. One might suspect that these large fluctuations are an artifact of our approximation, but it has been argued that the scaling (53) is an intrinsic feature of 0-brane quantum mechanics [25].

This raises a very interesting question. How can we recover local spacetime physics from the quantum mechanics?

In particular, given the large fluctuations (53), how can we resolve the horizon of the black hole?

The answer is that local spacetime physics only arises as a *low energy* approximation to the quantum mechanics. To recover local spacetime physics from the quantum mechanics we must introduce a resolving time, and integrate out high frequency degrees of freedom.[11] The point is that most of the N^2 degrees of freedom in the quantum mechanics have a very large frequency, set by the 't Hooft coupling

$$\omega \sim (g_{YM}^2 N)^{1/3}.$$

From the supergravity point of view, this energy scale corresponds to the energy of a string that stretches across the entire region in which supergravity is valid. A low energy observer within supergravity cannot resolve such high-frequency fluctuations. Therefore, to recover local spacetime physics from the quantum mechanics, we must first introduce a resolving time ϵ, and integrate out all modes with frequencies larger than $1/\epsilon$. With an appropriate choice of resolving time, we should recover the expected result, that the 0-branes only fluctuate over a region whose size is set by the horizon of the black hole.

We begin by discussing a single harmonic oscillator. At finite temperature the fluctuation in the oscillator position coordinate is

$$\langle x^2 \rangle = \frac{1}{2\omega \tanh(\beta\omega/2)}.$$

We introduce a resolving time, by smearing the Heisenberg picture operators over a Lorentzian time interval ϵ.

$$\bar{x} = \int_{-\infty}^{\infty} \frac{dt}{\epsilon\sqrt{\pi}} e^{-t^2/\epsilon^2} x(t).$$

The fluctuations in the smeared operators are suppressed when $\omega > 1/\epsilon$.

$$\langle \bar{x}^2 \rangle = \frac{e^{-\omega^2\epsilon^2/2}}{2\omega \tanh(\beta\omega/2)}.$$

To take this over into 0-brane quantum mechanics, we use the spectral representation (51). The fluctuations in the field are given by

$$\langle \phi^2 \rangle = \int_0^{\infty} d\omega \rho(\omega) \frac{1}{2\omega \tanh(\beta\omega/2)}. \tag{54}$$

Following our treatment of the harmonic oscillator, we can introduce a resolving time by setting

$$\langle \bar{\phi}^2 \rangle = \int_0^{\infty} d\omega \rho(\omega) \frac{e^{-\omega^2\epsilon^2/2}}{2\omega \tanh(\beta\omega/2)}. \tag{55}$$

[11]We are grateful to Leonard Susskind and Emil Martinec for discussions on this topic.

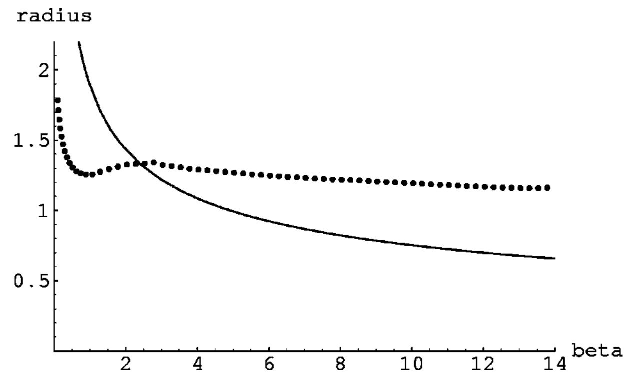

FIG. 8. The dotted curve is the smeared radius of the Wigner semi-circle. The solid curve is the Schwarzschild radius of the black hole, as measured in the U coordinate.

One might worry that this definition of horizon radius is ambiguous, since it seems to depend on the choice of resolving time. Fortunately, from Fig. 7, we see that the spectral density consists of two well-separated peaks. The low-frequency peak corresponds to states with an energy of order the temperature; these states can be thermally excited and should be included in the fluctuations which make up the horizon. The high-frequency peak corresponds to states with an energy of order the 't Hooft coupling, states which should be integrated out to see agreement with supergravity. Thus any resolving time which keeps the low-frequency peak and integrates out the high-frequency peak is acceptable, and will produce the same horizon radius.

For reasonable values of ϵ we can easily estimate $\langle \bar{\phi}^2 \rangle$. Rather than use the Gaussian cutoff (55), it turns out to be more convenient to define a time-averaged size by introducing a factor $1/\cosh(\beta\omega/2)$ into the integrand of Eq. (54)

$$\langle \bar{\phi}^2 \rangle \equiv \int_0^{\infty} d\omega \rho(\omega) \frac{1}{2\omega \sinh(\beta\omega/2)}. \tag{56}$$

The extra factor has the effect of cutting off the integral at $\omega \approx 1/\beta$. This corresponds to a reasonable choice of resolving time, $\epsilon \approx \beta$. Defined in this way, our estimate for the time-averaged fluctuations in the 0-brane positions is simply given by a Euclidean Green's function, cf. Eq. (51)

$$\langle \bar{\phi}^2 \rangle = \langle \phi(\beta/2) \phi(0) \rangle.$$

This is easily calculated as a Fourier transform of our momentum-space propagators. In Fig. 8 we plot the resulting smeared radius of the Wigner semicircle

$$\bar{R} = 2\sqrt{(7\bar{R}^2_{\rm scalar} + 2\bar{R}^2_{\rm gauge})/9}$$

as a function of β (we average over the scalar fields in the gauge and scalar multiplets).[12] The time-averaged fluctuations in the 0-brane positions go down with temperature.

[12]The result for $\bar{R}$ is dominated by the contribution of the zero-frequency Matsubara mode.

This is the expected behavior for the size of these black holes. The result for $\overline{R}$ is in rough agreement with the Schwarzschild radius U_0 of the black hole (50), which we show in the same plot.

VI. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper we have developed an approximation scheme for 0-brane quantum mechanics at strong coupling and finite temperature. We presented an ansatz for a trial action which captures some of the behavior of the large-N quantum mechanics. The parameters appearing in the trial action are chosen according to a set of gap equations which resum an infinite set of planar diagrams. The approximation automatically respects 't Hooft large-N counting, and also partially respects the supersymmetries and R-symmetries of the quantum mechanics.

Our main result is that we find good agreement with black hole thermodynamics over the temperature range $1<\beta<4$. In addition, we studied the behavior of a Wilson loop, and found that as expected a large-N phase transition occurs as the system enters the supergravity regime. We also presented results on the mean size of the system, and argued that in the supergravity regime this mean size (as measured by quantum fluctuations) exceeds the Schwarzschild radius of the dual black hole. We could nonetheless recover the Schwarzschild radius from the quantum mechanics, by introducing a suitable resolving time; to make our prescription unambiguous it was important that the spectral density showed a clear separation between light and heavy degrees of freedom.

We would like to emphasize that in the temperature range $1<\beta<4$ our approximation applies strictly to the gauge theory, and makes no use of supergravity information. We presented a prescription for fixing the value of the Wilson loop, which allowed us to extend the agreement up to $\beta=14$. The prescription, however, relies on supergravity inputs.

Our results are based on several technical developments in the use of mean-field methods. Some of these developments were reported in our previous work [12]. In the present paper, the main new technical problem we faced was the difficulty of treating gauge theories using mean-field methods. By working in the gauge (11), many of the Slavnov-Taylor identities become trivially satisfied. This gauge choice was instrumental in enabling us to find a consistent set of gap equations, that could be solved in the strong-coupling regime.

There are two perspectives that one could take on this subject. The "supergravity" perspective is that, since gauge theory is better understood than quantum gravity, we should try to study supergravity phenomena from the gauge theory point of view. The "field theory" perspective is to regard 0-brane quantum mechanics as an interesting laboratory for developing and testing methods to study field theories at strong coupling.

Depending on which perspective one adopts, there are several interesting possible directions for future work. From the field theory point of view, it would be interesting to apply mean-field methods to other models, to better understand the range of validity of these techniques. Let us mention one possibility. One can apply our techniques to pure $\mathcal{N}=2$ gauge theory, simply by dropping the scalar multiplets. The resulting system of gap equations does not have a solution in the low temperature regime $\beta>1$. Presumably this can be related to the fact that the pure gauge model breaks supersymmetry spontaneously [28]. It would be interesting to understand this connection in more detail.

It would also be interesting to have additional tests of our approximation scheme. The supergravity makes further predictions for the behavior of the gauge theory, which could be tested. For example, in [26] a set of predictions were made for the scaling exponents of two-point functions of certain operators at zero temperature. These were extracted by computing Green's functions in the extremal supergravity background, and taking a large time/low frequency limit. It was argued that these predictions follow from a generalized conformal symmetry that appears in the 't Hooft limit [27].

It would be interesting to test these predictions against our numerical results. Even at finite temperature, one would still expect to recover the scaling behavior for frequencies satisfying $T\ll\omega\ll(g_{YM}^2N)^{1/3}$. Unfortunately, the correlators which are predicted to have a scaling behavior involve composite operators whose two-point functions we have not yet computed. We hope to study this question further in the future.

Another interesting direction would be to better understand the duality between gravity and gauge theory. In particular, it would be interesting to understand better how the supergravity properties of spacetime locality and causality emerge from the gauge theory. One might hope to see that the horizon of the black hole is reflected in the dynamics of the gauge theory along the lines of [29]. Also, as we mentioned in Sec. III C, there is the subtle question of which radial coordinate in supergravity corresponds to the gauge theory Higgs fields. To address these sorts of issues, it is natural to introduce a 0-brane to probe the supergravity background. The probe has a dual description in terms of a spontaneously broken gauge theory. In [24], we use mean-field methods to study this problem.

Ultimately, one might hope to use mean-field methods to study non-equilibrium processes in the gauge theory at strong coupling, perhaps using some sort of thermofield formalism [30]. For example, it would be extremely interesting to study scattering of a graviton wave packet off a black hole. Could one see correlations in the outgoing Hawking radiation?

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APPENDIX: 2PI EFFECTIVE ACTION

The two-loop, 2PI effective action is defined by

$$I_{\rm eff}=\beta F_0+\langle S_{II}+S_{IV}-S_0\rangle_0-\frac{1}{2}\langle (S_{III})^2\rangle_{\rm C,0}. \qquad (A1)$$

We adopt units which effectively set $g_{YM}^2 N=1$, and suppress the overall factor of N^2 in the free energy. Then the free energy of the trial action is given by

$$\begin{aligned}\beta F_0=&\beta F_{\Box}(\lambda)-\sum_l \log\sigma_l^2+\sum_r \log a_r-\frac{7}{2}\sum_l \log\Delta_l^2\\&+7\sum_r \log g_r-\frac{7}{2}\sum_l \log\epsilon_l^2+\sum_{l\neq 0}\log s_l-2\sum_r \log t_r\\&+\sum_l \log u_l\end{aligned} \qquad (A2)$$

where the free energy of the one-plaquette model is

$$\beta F_{\Box}(\lambda)=\begin{cases}-\dfrac{2}{\lambda}-\dfrac{1}{2}\log\dfrac{\lambda}{2}+\dfrac{3}{4}, & \lambda\leqslant 2,\\ -1/\lambda^2, & \lambda\geqslant 2.\end{cases} \qquad (A3)$$

We also have

$$\begin{aligned}\langle S_{II}-S_0\rangle_0=&\frac{N}{\lambda}\langle \mathrm{Tr}(U+U^\dagger)\rangle_{\Box}+\sum_l\left[\left(\frac{2\pi l}{\beta}\right)^2\sigma_l^2-1\right]\\&+\sum_r\left[i\left(\frac{2\pi r}{\beta}\right)^3 a_r+1\right]\\&+\frac{7}{2}\sum_l\left[\left(\frac{2\pi l}{\beta}\right)^2\Delta_l^2-1\right]\\&+7\sum_r\left(-i\frac{2\pi r}{\beta}g_r+1\right)\\&+\frac{7}{2}\sum_l(\epsilon_l^2-1)+\sum_{l\neq 0}\left[\left(\frac{2\pi l}{\beta}\right)^2 s_l+1\right]\\&+2\sum_r\left(i\frac{2\pi r}{\beta}t_r-1\right)-\sum_l(u_l-1)\end{aligned} \qquad (A4)$$

where the one-plaquette model contribution is

$$\frac{N}{\lambda}\langle \mathrm{Tr}(U+U^\dagger)\rangle_{\Box}=\begin{cases}\dfrac{2}{\lambda}-\dfrac{1}{2} & \lambda\leqslant 2\\ 2/\lambda^2 & \lambda\geqslant 2.\end{cases} \qquad (A5)$$

We also have the contribution of the 4-point couplings,

$$\begin{aligned}\langle S_{IV}\rangle_0=&-\frac{2}{\beta}\sum_{r,s}\frac{2\pi r}{\beta}\frac{2\pi s}{\beta}a_r a_s+\frac{3i}{\beta}\sum_{l,r}\frac{2\pi r}{\beta}a_r\sigma_l^2\\&+\frac{1}{\beta}\sum_{l,m}\sigma_l^2\sigma_m^2+\frac{14}{\beta}\sum_{l,m}\Delta_l^2\sigma_m^2+\frac{14i}{\beta}\sum_{l,r}\Delta_l^2\frac{2\pi r}{\beta}a_r\\&+\frac{7}{\beta}\sum_l\Delta_l^2\rho_0^2+\frac{5i}{2\beta}\sum_r\frac{2\pi r}{\beta}a_r\rho_0^2+\frac{2}{\beta}\sum_l\sigma_l^2\rho_0^2\end{aligned} \qquad (A6)$$

where the two-point function of A_{00} is defined in Eq. (26). Finally, the contribution of the three-point couplings is given by

$$\begin{aligned}-\frac{1}{2}\langle (S_{III})^2\rangle_{\rm C,0}=&-\frac{4}{\beta}\sum_l\left(\frac{2\pi l}{\beta}\right)^2(\sigma_l^2)^2\rho_0^2-\frac{4}{\beta}\sum_r\left(\frac{2\pi r}{\beta}\right)^4(a_r)^2\rho_0^2+\frac{14}{\beta}\sum_{l+r+s=0}\left(\frac{2\pi l}{\beta}\right)^2\Delta_l^2 a_r g_s+\frac{14}{\beta}\sum_{l+r+s=0}\epsilon_l^2 a_r g_s\\&-\frac{14}{\beta}\sum_{l+r+s=0}\sigma_l^2 g_r g_s-\frac{14}{\beta}\sum_l\left(\frac{2\pi l}{\beta}\right)^2(\Delta_l^2)^2\rho_0^2-\frac{7}{\beta}\sum_r(g_r)^2\rho_0^2+\frac{21}{\beta}\sum_{l+m+n=0}\Delta_l^2\Delta_m^2\epsilon_n^2\\&-\frac{42}{\beta}\sum_{l+r+s=0}\Delta_l^2 g_r g_s+\frac{1}{\beta}\sum_{l+r+s=0}\left[\sigma_l^2 t_r t_s-u_l t_r a_s+\left(\frac{2\pi l}{\beta}\right)^2 s_l t_r a_s\right]+\frac{1}{2\beta}\sum_r(t_r)^2\rho_0^2\\&+\frac{1}{\beta}\sum_l\left(\frac{2\pi l}{\beta}\right)^2(s_l)^2\rho_0^2.\end{aligned} \qquad (A7)$$

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