Coalescence of two spinning black holes: An effective one-body approach

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We generalize to the case of spinning black holes a recently introduced ''effective one-body'' approach to the general relativistic dynamics of binary systems. We show how to approximately map the conservative part of the third post-Newtonian (3PN) dynamics of two spinning black holes of masses m_1, m_2 and spins $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ onto the dynamics of a non-spinning particle of mass $\mu \equiv m_1 m_2/(m_1+m_2)$ in a certain effective metric $g^{\rm eff}_{\mu\nu}(x^\lambda; M, \nu, \boldsymbol{a})$ which can be viewed either as a spin deformation [with the deformation parameter $\boldsymbol{a} \equiv \boldsymbol{S}_{\rm eff}/M$] of the recently constructed 3PN effective metric $g^{\rm eff}_{\mu\nu}(x^\lambda; M, \nu)$, or as a ν deformation [with the comparable-mass deformation parameter $\nu \equiv m_1 m_2/(m_1+m_2)^2$] of a Kerr metric of mass $M \equiv m_1+m_2$ and (effective) spin $\boldsymbol{S}_{\rm eff} \equiv [1+3m_2/(4m_1)]\boldsymbol{S}_1+[1+3m_1/(4m_2)]\boldsymbol{S}_2$. The combination of the effective one-body approach, and of a Padé definition of the crucial effective radial functions, is shown to define a dynamics with much improved post-Newtonian convergence properties, even for black hole separations of the order of 6 GM/c^2. The complete (conservative) phase-space evolution equations of binary spinning black hole systems are written down and their exact and approximate first integrals are discussed. This leads to the approximate existence of a two-parameter family of ''spherical orbits'' (with constant radius), and of a corresponding one-parameter family of ''last stable spherical orbits'' (LSSO). These orbits are of special interest for forthcoming LIGO-VIRGO-GEO gravitational wave observations. The binding energy and total angular momentum of LSSO's are studied in some detail. It is argued that for most (but not all) of the parameter space of two spinning holes the approximate (leading-order) effective one-body approach introduced here gives a reliable analytical tool for describing the dynamics of the last orbits before coalescence. This tool predicts, in a quantitative way, how certain spin orientations increase the binding energy of the LSSO. This leads to a detection bias, in LIGO-VIRGO-GEO observations, favoring spinning black hole systems, and makes it urgent to complete the conservative effective one-body dynamics given here by adding (resummed) radiation reaction effects, and by constructing gravitational waveform templates that include spin effects. Finally, our approach predicts that the spin of the final hole formed by the coalescence of two arbitrarily spinning holes never approaches extremality.

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I. INTRODUCTION

The most promising candidate sources for the Laser Interferometric Gravitational Wave Observatory–(LIGO)-VIRGO-GEO network of ground-based gravitational wave (GW) interferometric detectors are coalescing binary systems made of massive (stellar) black holes [1–5]. Signal to noise ratio (SNR) estimates [5] suggest that the first detections will concern black hole binaries of total mass $\gtrsim 25 M_\odot$. Modeling the GW signal emitted by such systems poses a difficult theoretical problem because the observationally most ''useful'' part of the gravitational waveform is emitted in the last ~ 5 orbits of the inspiral, and during the ''plunge'' taking place after crossing the last stable circular orbit. The transition between the (adiabatic) inspiral and plunge takes place in a regime where the two bodies are moving at relativistic speeds ($v/c \sim 1/\sqrt{6} \sim 0.4$) and where their gravitational interaction becomes (nearly by definition) highly non-linear ($GM/c^2 r \sim 1/6$).

Several authors (notably [6,3]) have taken the view that the modelling of this crucial transition between inspiral and plunge is (in the general case of comparable-mass systems) beyond the reach of analytical tools and can only be tackled by (possibly special-purpose [3]) numerical simulations. By contrast, other authors [7–9,5,10,11] have introduced new ''resummation methods'' to improve the analytical description of the last few GW cycles near this transition and have argued that these resummed analytical results gave a reliable description of the gravitational physics near the transition. The purpose of the present paper is to further the latter approach by generalizing the (resummed) ''effective one-body'' (EOB) methods introduced in [8–10] to the case of binary systems of *spinning* black holes. Before doing this, we wish to clarify what is the rationale for arguing that the ''resummed'' analytical approach can describe the last stages of inspiral and the transition between inspiral and plunge.

Let us first recall that a lot of effort has been devoted in recent years to the analytical computing, by means of post-Newtonian (PN) expansions in powers of $v^2/c^2 \sim GM/c^2 r$, of the equations of motion, and the GW emission, of comparable-mass binary systems. The equations of motions have been computed to v^6/c^6 (3PN) accuracy by two separate groups, [12–15] and [16–18], and the two results have been shown to agree [19,20]. Until recently, there remained (in both approaches) an ambiguous parameter, ω_s, linked to the problem of regularizing some badly divergent integrals arising at the 3PN level. In a recent work, using an improved regularization method (dimensional continuation), the first group [21] has succeeded in determining without ambiguity the value of ω_s, namely $\omega_s = 0$. This unique determination of the 3PN equations of motion is consistent with an old argument of [22] showing that it should be possible to model

TABLE I. Binding energies $e \equiv (\mathcal{E}_{\rm real}/M) - 1$ for circular orbits of equal-mass (non-spinning) binary systems near the LSO. The invariant dimensionless l-radius $r_l \equiv R_l/GM$ is defined in the text. The binding energy is computed from the Padé-resummed effective-one-body Hamiltonian at three successive post-Newtonian approximations: 1PN, 2PN, and 3PN (with $\omega_s = 0$).

r_l	12	11	10	9	8	7	6
l	4	3.889	3.780	3.674	3.578	3.5	3.464
$100e_{\rm 1PN}$	−0.9482	−1.020	−1.101	−1.193	−1.291	−1.387	−1.440
$100e_{\rm 2PN}$	−0.9441	−1.015	−1.094	−1.183	−1.277	−1.366	−1.412
$100e^{(\omega_s=0)}_{\rm 3PN}$	−0.9412	−1.011	−1.088	−1.174	−1.264	−1.346	−1.388

black holes by point particles without ambiguity up to the 5PN level (excluded). The emission of GW is unambiguously known to v^5/c^5 (2.5PN) accuracy [23], and has recently been formally computed to v^7/c^7 (3.5PN) accuracy [24], modulo the appearance of several ambiguous parameters (ξ, κ, ζ) linked to the problem of regularizing some divergences arising at the 3PN level. Dimensional regularization is expected to determine without ambiguity the values of ξ, κ, and ζ, but has not (yet) been applied to the radiation problem.

We wish to emphasize that such high-order PN results are a *necessary*, but *not* by themselves *sufficient*, ingredient for computing with adequate accuracy the gravitational waveform of coalescing binaries. Indeed, it was emphasized long ago [6] that the PN series (written as a straightforward Taylor series in powers of some parameter $\varepsilon \sim v/c$) become slowly convergent in the late stages of binary inspiral. A first attempt was made in [25] to improve the convergence of the PN-expanded equations of motion so as to determine the (crucial) location of the last stable (circular) orbit (LSO) for comparable-mass systems. However, further work [26,27,7] has shown the unreliability (and coordinate dependence) of this attempt. There is, however, no reason of principle preventing the existence of gauge-invariant ''resummation methods'' able to give reliable results near the LSO. Indeed, as emphasized in [7] and [10] most coordinate-invariant functions (of some invariant quantity $x \sim v^2/c^2 \sim GM/c^2 r$) that one wishes to consider when discussing the dynamics and GW emission of circular orbits are expected to have a singularity only at the ''light ring'' (LR) value of x (the last possible *unstable* circular orbit). If we trust (for orders of magnitude considerations) the small mass-ratio limit $[\nu \equiv \mu/M \equiv m_1 m_2/(m_1+m_2)^2 \ll 1]$, we know that $x_{\rm LR} \simeq 1/3$ is smaller by a factor 2 than $x_{\rm LSO} \simeq 1/6$. If the functions $f(x)$ we are dealing with are meromorphic functions of x, the location of the expected closest singularity ($x_{\rm LR}$) determines their radius of convergence. Therefore, we expect that, for $x < x_{\rm LR}$, the Taylor expansion of $f(x)$ will converge and will behave essentially like $\Sigma_n (x/x_{\rm LR})^n$. In particular, one expects $f(x_{\rm LSO}) \sim \Sigma_n \ (x_{\rm LSO}/x_{\rm LR})^n \sim \Sigma 2^{-n}$. This heuristic argument suggests a rather slow convergence, but the crucial point is to have some convergence, so that the application of suitable *resummation methods* can be expected to accelerate the convergence and to lead to numerically accurate results from the knowledge of only a few terms in the Taylor expansion. There exist many types of resummation methods and none of them are of truly universal applicability. As a rule, one must know something about the structure of the functions $f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$ one is trying to resum to be able to devise an efficient resummation method. References [7–9], and [10] have studied in detail the various functions that might be used to discuss the GW flux and the dynamics of binary systems. This work has led to selecting some specific resummation methods, acting on some specific functions. For what concerns the GW flux we refer to Fig. 3 of [7] for evidence of the acceleration of convergence (near the LSO) provided by a specific resummation method combining a redefinition of the *GW flux* function with Padé approximants. We wish here, for the benefit of the skeptics, to exhibit some of the evidence for the acceleration of convergence (near the LSO) in the description of the two-body *dynamics* provided, at the 3PN level, by a resummation method defined by combining [8] and [10]. Specifically, we mean the combination of the effective-one-body (EOB) approach (further discussed below) and of a suitable Padé resummation of the effective radial potential at the nPN level: $A_{P_n}(u) = P^1_n[T_{n+1}(A(u))]$ (see below).

Let us consider a sequence of *circular* orbits, near the LSO and for two non-spinning black holes. In the EOB approach the circular orbits are obtained by minimizing a certain effective radial potential, defined by fixing the total orbital angular momentum L in the Hamiltonian. The most natural variable defining the one-parameter sequence of circular orbits is then simply the angular momentum L. It is therefore natural, for the purpose of this work, to measure the separation between the two holes (in a gauge-invariant and approximation-independent) way by *conventionally* defining an l radius $R_l \equiv GM r_l$, such that the (invariantly defined) total orbital angular momentum $L \equiv G\mu M l$ is given by $l^2 \equiv r_l^2/(r_l - 3)$, i.e., by the relation holding for a test particle in a Schwarzschild spacetime. [Here, and in the following, we shall often set $c=1$ and/or $G=1$, except in some (final) formulas where it might be illuminating to reestablish the dependence on c and/or G.] As the problem is to know whether the resummation method of the PN-expanded two-body dynamics is efficient, we compare in Table I the total energies $\mathcal{E}_{\rm real}$ of the binary system, computed using 1PN, 2PN, and 3PN information, for circular orbits at l radii $r_l = 12$, 11, 10, 9, 8, 7, and 6. We give the values of the binding energy per unit (total) mass, $e \equiv (\mathcal{E}/M) - 1$, for the equal-mass case $(m_1 = m_2;\ \nu = 1/4)$.

The numbers displayed in Table I illustrate the efficiency of the resummation method advocated in [8,10]. For $r_l = 12$

the fractional difference in binding energy between the 1PN approximation and the 3PN one is 0.74%, while even for $r_l = 6$ this difference is only 3.6%. These numbers indicate that, even near the LSO, the Padé-improved effective-one-body approach is a rationally sound way of computing the two-body dynamics. There are no signs of numerical unreliability, as there were in the calculations based on the straightforward coordinate-dependent, PN-expanded versions of the equations of motion [25] or of the Hamiltonian [26,27] [which gave results differing by $\mathcal{O}$ (100%) among themselves, and as one changed the PN order]. We shall see below that the robustness of the PN resummation exhibited in Table I extends to a large domain of the parameter space of spinning black holes.

As we do not know the exact result, and as current numerical simulations do not give reliable information about the late stages of the quasicircular orbital dynamics of two black holes (see below), the kind of internal consistency check exhibited in Table I is about the only evidence we can set forth at present. [Note that, from a logical point of view, the situation here is the same as for numerical simulations: in the absence of an exact solution (and of experimental data) one can only do internal convergence tests.] Ideally, it would be important to extend the checks of Table I to the 4PN level (to confirm the trend and see a real sign of convergence to a limit) but this seems to be a hopelessly difficult task with present analytical means. Finally, let us note that the fact that one can concoct many ''bad'' ways of using the PN-expanded information near the LSO (exhibiting as badly divergent results as wished) is not a valid argument against the reliability of the specific resummation technique used in [7,8,10] and here. An ambiguity problem would arise only if one could construct two different resummation methods, both exhibiting an internal ''convergence'' (as the PN order increases) as good as that illustrated in Table I, but yielding very different predictions for physical observables near the LSO. This is not the case at present because the comparative study of [10] (see Table I there) has shown that the EOB approach exhibited (when $\omega_s \neq -9$) significantly better PN convergence than a panel of other invariant resummation methods.

In the following we take for granted the soundness of the effective-one-body resummation approach and we show how to generalize it to the case of two (moderately) spinning black holes. Let us first recall that the basic idea of the EOB approach was first developed in the context of the electromagnetically interacting quantum two-body problem [28,29] (see also [30]). A first attempt to deal with the gravitationally interacting two-body problem (at the 1PN level) was made in [31] (see also [32]). A renewed EOB approach (which significantly differs from the general framework set up by Todorov and co-workers [29,31]) was introduced in [8]. The latter reference showed how to apply this method at the 2PN level. It was then used to study the transition between the inspiral and the plunge for comparable masses, and, in particular, to construct a complete waveform covering the inspiral, the plunge, and the final merger [9] (see [33] for the physical consequences of this waveform). More recently, Ref. [10] showed how to extend the EOB approach to the 3PN level (this required a non trivial generalization of the basic idea).

Before entering the details of our way of introducing the spin degrees of freedom in the EOB approach, let us state our general view of the usefulness of the EOB method in this context. As we shall discuss below, the present work (which incorporates spin effects at leading PN order) can only be expected to give physically reliable results in the case of moderate spins ($\hat{a}_p \lesssim 0.3$, see below). However, the EOB approach, far from being a rigid structure, is extremely flexible. One can modify the basic functions [such as $A(u)$] determining the EOB dynamics by introducing new parameters corresponding to (yet) uncalculated higher PN effects. [These terms become important only for orbits closer than $6GM$, and/or for fast-spinning holes.] Therefore, when either higher-accuracy analytical calculations are performed or numerical relativity becomes able to give physically relevant data about the interaction of (fast-spinning) black holes, we expect that it will be possible to complete the current EOB Hamiltonian so as to incorporate this information. As the parameter space of two spinning black holes (with arbitrarily oriented spins) is very large, numerical relativity will never be able, by itself, to cover it densely. We think, however, that a suitable ''numerically fitted'' (and, if possible, ''analytically extended'') EOB Hamiltonian should be able to fit the needs of upcoming GW detectors. The present work should be viewed as a first step in this direction.

The present paper is organized as follows. In Sec. II we show how to incorporate (in some approximation) the spin degrees of freedom of each black hole within a 3PN-level, resummed effective one-body approach. In Sec. III we study some of the predictions of our resummed dynamics, notably for what concerns the location of the transition between the inspiral and the plunge. Section IV contains our concluding remarks.

II. EFFECTIVE ONE-BODY APPROACH, EFFECTIVE SPIN AND A DEFORMED KERR METRIC

A. Effective one-body approach

Let us recall the basic set up of the effective one-body (EOB) approach. One starts from the (PN-expanded) two-body equations of motion, which depend on the dynamical variables of two particles. One separates the equations of motion in a ''conservative part,'' and a ''radiation reaction part.'' Though this separation is not well defined at the exact (general relativistic) level it is not ambiguous at the 3PN level (in the conservative part) which we shall consider here.[1] We shall henceforth consider only the conservative part of the dynamics. (We leave to future work the generalization to spinning black holes of the definition of *resummed* radiation reaction effects which was achieved in [9] for nonspinning black holes.) It has been explicitly shown that the

[1]We expect real ambiguities to arise only at the $v^{10}/c^{10} \sim 5$PN level, because this corresponds to the *square* of the leading, $v^5/c^5 \sim 2.5$PN, radiation reaction terms.

3PN[2] dynamics is Poincaré invariant [15,17]. The ten first integrals associated with the ten generators of the Poincaré group were constructed in [15] (see also [20]). In particular, we have the ''center of mass'' vectorial constant $\boldsymbol{K}=\boldsymbol{G}-t\boldsymbol{P}$. This constant allows one to define the center of mass frame, in which $\boldsymbol{K}=0$, which implies $\boldsymbol{P}=0$ and $\boldsymbol{G}=\boldsymbol{0}$. We can then reduce the PN-expanded two-body dynamics to a PN-expanded one-body dynamics by considering the relative motion in the center of mass frame. This reduction leads to a great simplification of the dynamics.

Indeed, the full 3PN Hamiltonian in an arbitrary reference frame [15] contains $\mathcal{O}(100)$ terms, while its center-of-mass-reduced version contains only 24 terms. However, this simplification is, by itself, insufficient for helping in any way the crucial problem of the slow convergence of the PN expansion. One should also mention that the use of a Hamiltonian framework [like the Arnowitt-Deser-Misner (ADM) formalism used in [12,13,10,15,21]) is extremely convenient (much more so than an approach based on the harmonic-coordinates equations of motion, as in [16–18,20]). Indeed, on the one hand it simplifies very much the reduction to the center-of-mass relative dynamics (which is trivially obtained by setting $\boldsymbol{p}_{\rm rel}=\boldsymbol{p}_1=-\boldsymbol{p}_2$), and on the other hand it yields directly (without guesswork) an action principle for the dynamics.[3]. We shall find also below that a Hamiltonian approach is very convenient for dealing with the spin degrees of freedom.

Up to now, we only mentioned the dynamics of the orbital degrees of freedom, i.e., (in the order-reduced Hamiltonian formalism) the (ADM coordinate) positions and momenta $\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{p}_1,\boldsymbol{p}_2$ of the two black holes.[4] After reduction to the center-of-mass frame ($\boldsymbol{P}=\boldsymbol{p}_1+\boldsymbol{p}_2=0$), and to the relative dynamics ($\boldsymbol{x}\equiv\boldsymbol{x}_1-\boldsymbol{x}_2$, $\boldsymbol{p}\equiv\boldsymbol{p}_1=-\boldsymbol{p}_2$), one ends up with a canonical pair $\boldsymbol{x}$, $\boldsymbol{p}$ of phase-space variables.

The addition of spin degrees of freedom on each black hole is, *a priori*, a rather complicated matter. If one wished to have a *relativistically covariant* description of the dynamics of two spinning objects, one would need not only to add, in Einstein's equations, extra (covariant) source terms proportional to suitable derivatives of delta functions (spin dipoles), but also to enlarge the two-body action principle to incorporate the spin variables. The first task is doable, and has been performed (to the lowest order) in several works, such as Refs. [34–38]. For other works on the relativistic equations of motion of black holes or extended bodies (endowed with spin and higher multipole moments) see [39–41]. Recently, Ref. [42] has tackled the next-to-leading order contribution to spin-orbit effects. (We consider here only the case of interacting, comparable massive objects. The problem of a spinning *test* particle in an external field is simpler and has been dealt with by many authors, such as Mathisson [43], Papapetrou [44], etc . . .) On the other hand, the second task (incorporating the spin degrees of freedom in an action principle) is quite intricate. First, it has been found that, within a relativistically covariant set up for a spinning particle, the Lagrangian describing the *orbital* motion could not, even at lowest order in the spin, be taken as an ordinary Lagrangian $L(x,\cdot x)$, but needed to be a higher-order one $L(x,\dot{x},\ddot{x},\ldots)$ [37]. Second, a relativistically covariant treatment of the *spin* degrees of freedom is an intricate matter, involving all the subtleties of constrained dynamical systems, even in the simplest case of a free relativistic top [45]. Contrary to the case of the spin-independent EOB where it was easy to use the test-mass results to constrain the EOB Hamiltonian, the Mathisson-Papapetrou dynamics of spinning test masses in external gravitational fields is rather complicated and cannot easily be used to constrain the spin-dependent EOB Hamiltonian. (It might, however, be interesting to try to do so.) Fortunately, there is a technically much lighter approach which bypasses these problems and simplifies *both* the description of orbital degrees of freedom and that of spin degrees of freedom. This approach is *not* manifestly relativistically covariant. This lack of *manifest* Poincaré covariance is not (in principle) a problem at all for two reasons: (i) it does not prevent the expected global Poincaré covariance of the two-body dynamics to be realized as a phase-space symmetry (as was explicitly proven, at 3PN, for the orbital degrees of motion in Ref. [15]), and (ii) as we are, at this stage, mainly interested in the description of the relative motion in a specific (center-of-mass) Lorentz frame, there is no physical need to enforce any boost invariance. This non-covariant approach to the gravitational interaction of spinning objects stems from the classic work of Breit on the electromagnetic interaction of (quantum) spinning electrons (see, e.g., [46]) and has been developed in a sequence of papers [47–50]. (The change of variables needed to pass from the covariant, higher-order Lagrangian description to the non-covariant, ordinary Lagrangian has been discussed in several papers, e.g., [45,51,52].) In the present paper, we shall combine the non-manifestly covariant, ADM Hamiltonian treatment of the orbital degrees of freedom of [12,15], with the similarly non-covariant, but Hamiltonian, treatment of the spin degrees of freedom of [47,49]. Moreover, as is explained below, we shall improve upon [48–50] in using a direct Poisson-bracket treatment of the dynamical spin variables.

Finally, our starting point (for the effective one-body approach) is a *PN-expanded* Hamiltonian for the *relative* motion of two spinning objects of the form

$$H^{\rm PN}_{\rm real}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_1,\boldsymbol{S}_2)=H^{\rm PN}_{\rm orb}(\boldsymbol{x},\boldsymbol{p})+H^{\rm PN}_{S}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_1,\boldsymbol{S}_2)+H^{\rm PN}_{SS}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_1,\boldsymbol{S}_2)+H^{\rm PN}_{SSS}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_1,\boldsymbol{S}_2)+\cdots. \tag{2.1}$$

[2]Henceforth, ''3PN'' will mean the conservative 3PN dynamics, i.e., $N+1{\rm PN}+2{\rm PN}+3{\rm PN}$.

[3]Note that a subtlety arises at 3PN [12] in that the Hamilton action principle involves *derivatives* of the phase space variables. However, it was shown in [14] how to reduce the problem to an ordinary Hamiltonian dynamics by means of a suitable $\mathcal{O}(v^6/c^6)$ shift of phase-space variables. We henceforth assume that we work with the shifted variables defined in [14].

[4]We recall that the high-order perturbative, PN-expanded, calculations of the dynamics of two non-spinning compact objects model these objects by delta-function (monopole) sources. The supports of these delta functions define the coordinate ''positions'' of the compact objects. As explained in [22] these positions physically correspond to some ''centers of the gravitational field'' generated by the objects.

Here, $H^{\rm PN}_{\rm orb}$ denotes the PN-expanded *orbital* Hamiltonian, which is the sum of the free Hamiltonian $H_0=\sqrt{m_1^2c^4+\boldsymbol{p}_1^2c^2}+\sqrt{m_2^2c^4+\boldsymbol{p}_2^2c^2}$ and of the monopolar interaction Hamiltonian $H^m_{\rm int}$ generated by the source terms proportional to the masses. Before the reduction to the center-of-mass frame $H^m_{\rm int}$ has the symbolic structure: $H^m_{\rm int}\sim m_1m_2+m_1^2m_2+m_1m_2^2+m_1^3m_2+m_1^2m_2^2+m_1m_2^3+\cdots \sim m_1m_2[1+m_1+m_2+(m_1+m_2)^2+(m_1+m_2)^3+\cdots]$. It is explicitly known up to the 3PN level [i.e., up to velocity-independent terms $\propto m_1m_2(m_1+m_2)^3$]. After reduction to the center-of-mass frame the PN expansion of $H^{\rm PN}_{\rm orb}$ reads [with $M\equiv m_1+m_2$, $\mu\equiv m_1m_2/M$, $\nu\equiv\mu/M\equiv m_1m_2/(m_1+m_2)^2$, $r\equiv|\boldsymbol{x}|$, $\hat{\boldsymbol{p}}\equiv\boldsymbol{p}/\mu$, $\hat{\boldsymbol{x}}\equiv\boldsymbol{x}/GM$]

$$H^{\rm PN}_{\rm orb}(\boldsymbol{x},\boldsymbol{p})=Mc^2+H_N(\boldsymbol{x},\boldsymbol{p})+H_{\rm 1PN}(\boldsymbol{x},\boldsymbol{p})+H_{\rm 2PN}(\boldsymbol{x},\boldsymbol{p})+H_{\rm 3PN}(\boldsymbol{x},\boldsymbol{p}),\tag{2.2}$$

$$H_N(\boldsymbol{x},\boldsymbol{p})=\frac{\boldsymbol{p}^2}{2\mu}-\frac{GM\mu}{r}=\mu\left[\frac{1}{2}\hat{\boldsymbol{p}}^2-\frac{1}{\hat{r}}\right],\tag{2.3}$$

$$H_{\rm 1PN}(\boldsymbol{x},\boldsymbol{p})=\frac{\mu}{c^2}\left[\frac{1}{8}(3\nu-1)\hat{\boldsymbol{p}}^4-\frac{1}{2}[(3+\nu)\hat{\boldsymbol{p}}^2+\nu(\boldsymbol{n}\cdot\hat{\boldsymbol{p}})^2]\times\frac{1}{\hat{r}}+\frac{1}{2\hat{r}^2}\right],\tag{2.4}$$

$$H_{\rm 2PN}(\boldsymbol{x},\boldsymbol{p})=\frac{\mu}{c^4}\left[\frac{1}{16}(1-5\nu+5\nu^2)\hat{\boldsymbol{p}}^6+\cdots-\frac{1}{4}(1+3\nu)\frac{1}{\hat{r}^3}\right],\tag{2.5}$$

$$H_{\rm 3PN}(\boldsymbol{x},\boldsymbol{p})=\frac{\mu}{c^6}\left[\frac{1}{128}(-5+35\nu-70\nu^2+35\nu^3)\hat{\boldsymbol{p}}^8+\cdots+\left[\frac{1}{8}+\left(\frac{109}{12}-\frac{21}{32}\pi^2+\omega_s\right)\nu\right]\frac{1}{\hat{r}^4}\right].\tag{2.6}$$

We have exhibited (for illustration) in Eqs. (2.5) and (2.6) only the first and the last terms. We refer to [52] for the full (center-of-mass) 2PN Hamiltonian (seven terms in all), and to [12,14,15] for the full (center-of-mass) 3PN Hamiltonian (11 terms in all). Our effective one-body treatment will take into account the *full* 2PN and 3PN structures, but in a very streamlined way which will be explicitly displayed below.

The other terms in Eq. (2.1) denote the various spin-dependent contributions to the Hamiltonian: respectively, the terms linear ($H^{\rm PN}_S$), quadratic ($H^{\rm PN}_{SS}$), cubic ($H^{\rm PN}_{SSS}$), etc . . . . in the spins $\boldsymbol{S}_1$, $\boldsymbol{S}_2$. Before reduction to the center-of-mass frame they have the symbolic structure

$$H^{\rm PN}_S\sim S_1m_2[1+m_1+m_2+(m_1+m_2)^2+\cdots]+S_2m_1[1+m_1+m_2+(m_1+m_2)^2+\cdots],\tag{2.7}$$

$$H^{\rm PN}_{SS}\sim S_1^2m_2\left(\frac{1}{m_1}+1+m_1+m_2+\cdots\right)+S_1S_2(1+m_1+m_2+\cdots)+S_2^2m_1\left(\frac{1}{m_2}+1+m_1+m_2+\cdots\right),\tag{2.8}$$

etc . . . . We shall explain below the occurence of the terms quadratic in the spins and inversely proportional to a mass. In contradistinction with the case of the orbital Hamiltonian which has been worked out with a high PN accuracy, only the simplest spin-dependent terms have been explicitly derived, namely the lowest PN-order term in $H^{\rm PN}_S$, whose center-of-mass reduction reads [49]

$$H^{\rm PN}_S(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_1,\boldsymbol{S}_2)=\frac{2G}{c^2r^3}\left[\left(1+\frac{3}{4}\frac{m_2}{m_1}\right)\boldsymbol{S}_1+\left(1+\frac{3}{4}\frac{m_1}{m_2}\right)\boldsymbol{S}_2\right]\cdot(\boldsymbol{x}\times\boldsymbol{p})+\mathcal{O}\left(\frac{1}{c^4}\right),\tag{2.9}$$

and the lowest PN-order one-graviton exchange contribution to the bilinear term ($\propto S_1S_2$) in $H^{\rm PN}_{SS}$. Reference [42] contains some information about the $\mathcal{O}(1/c^4)$ corrections in Eq. (2.9), but, because of the use of different gauge and spin conditions, not in a form which can be directly used to derive these corrections. We shall discuss the spin-bilinear contribution ($\propto S_1S_2$) below, together with the leading spin-quadratic contributions proportional to $S_1^2m_2/m_1+S_2^2m_1/m_2$.

Before going further, let us make clear that, before and after any type of resummation, the dynamics entailed by the Hamiltonians we shall consider $H(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_1,\boldsymbol{S}_2)$ follow, for all degrees of freedom, from the basic Poisson brackets

$$\{x^i,p_j\}=\delta^i_j,\tag{2.10}$$

$$\{S_1^i,S_1^j\}=\varepsilon^{ijk}S_1^k,\tag{2.11}$$

$$\{S_2^i,S_2^j\}=\varepsilon^{ijk}S_2^k,\tag{2.12}$$

$$0=\{x^i,x^j\}=\{p_i,p_j\}=\{S_1^i,S_2^j\}=\{x^i,S_1^j\}=\{x^i,S_2^j\}=\{p_i,S_1^j\}=\{p_i,S_2^j\}.\tag{2.13}$$

The (real) time evolution of any dynamical quantity $f(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_1,\boldsymbol{S}_2)$ is given by

$$\frac{d}{dt}f(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_1,\boldsymbol{S}_2)=\{f,H_{\rm real}\},\tag{2.14}$$

where the Poisson bracket (PB) $\{f,H_{\rm real}\}$ is computed from the basic PB's (2.10)–(2.13) by using the standard PB properties (skew symmetry: $\{f,g\}=-\{g,f\}$; Leibniz rule: $\{f,gh\}=\{f,g\}h+g\{f,h\}$; and the Jacobi identity: $\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0$). The simplest way to prove the statements (2.10)–(2.14) is to consider our dynam-

ics as the classical limit of the quantum dynamics of a system of gravitationally interacting spinning particles. Surprisingly, though Refs. [47,48] derived (à la Breit) the spin-dependent contributions to the Hamiltonian by a quantum route, they never noticed that they could very simply derive the spin evolution equations by using the PB's (2.11),(2.12). They had to go back to a Lagrangian formalism and add some explicit spin kinetic energy terms $(\frac{1}{2}I_1\boldsymbol{\omega}_1^2+\frac{1}{2}I_2\boldsymbol{\omega}_2^2)$ to derive the spin evolution equations. Note also that we have kept the label ''real'' on the Hamiltonian in Eq. (2.14) to distinguish the evolution with respect to the real time (associated with the original two-body system) from the evolution generated by the effective Hamiltonian to be introduced below (which is associated with an auxiliary, effective time).

Before generalizing it, by including the spin degrees of freedom, let us recall the results of [8] (2PN level) and [10] (3PN level) concerning the effective one-body ''upgrading'' of the PN-expanded orbital Hamiltonian $H^{\rm PN}_{\rm orb}(\boldsymbol{x},\boldsymbol{p})$. Again, the simplest way to motivate it is to think of our dynamics as the classical limit of a quantum dynamics defined by some Hermitian Hamiltonian operator $H_{\rm orb}(\boldsymbol{x},-i\hbar\boldsymbol{\nabla})$. We are mainly interested in the bound states of $H_{\rm orb}$. It is crucial to note that the orbital Hamiltonian (2.2)–(2.6) is symmetric under an $O(3)$ group (corresponding to arbitrary rotations of the relative position $\boldsymbol{x}=\boldsymbol{x}_1-\boldsymbol{x}_2$, in the center-of-mass frame). Therefore the quantum (and classical) bound states will be labeled (besides parity) by only two quantum numbers: (i) the total orbital angular momentum $\boldsymbol{L}^2=L(L+\hbar)$, and (ii) some ''principal quantum number'' N [such that $(N-L)/\hbar$ counts the number of nodes in the radial relative wave function]. Both L and N are quantized in units of $\hbar$. The full list of two-body bound states is thereby encoded in the formula giving the bound state energy as a function of the two quantum numbers L and N: $E_{\rm real}=E_{\rm real}(L,N)=Mc^2-\frac{1}{2}\mu(Gm_1m_2)^2/N^2+E_{1\rm PN}(L,N)+E_{2\rm PN}(L,N)+E_{3\rm PN}(L,N)$. The basic idea of the effective one-body method is to map (in a one-to-one manner) the discrete set of real two-body bound states $E_{\rm real}(L,N)$ onto the discrete set of bound states of an auxiliary (''effective'') one-body Hamiltonian $H_{\rm eff}(\boldsymbol{x}_{\rm eff},\boldsymbol{p}_{\rm eff})$. Because of the special labeling by (only) two integer quantum numbers $L/\hbar, N/\hbar$ one is naturally led to imposing that (i) the EOB Hamiltonian be spherically symmetric,[5] and (ii) the integer-valued quantum numbers be identified in the two problems, i.e., $L/\hbar=L_{\rm eff}/\hbar$ and $N/\hbar=N_{\rm eff}/\hbar$. On the other hand, one can (and one *a priori* should) leave free a (one-to-one) continuous function f mapping the real energies onto the effective ones: $E_{\rm eff}(L,N)=f(E_{\rm real}(L,N))$. Evidently, for this method to be advantageous we wish the effective dynamics to be significantly simpler than the original $H_{\rm real}(\boldsymbol{x},\boldsymbol{p})$, and, in particular, to reduce, in some approximation, to the paradigm of the simplest gravitational one-body problem, namely the dynamics of an (effective) test particle in some (to be determined) effective metric $g^{\rm eff}_{\mu\nu}(x^\lambda_{\rm eff})$. Remarkably enough, it was found in [8] that such a mapping between the very complicated real two-body orbital 2PN Hamiltonian (2.2)–(2.5) and the usual (''geodesic'') dynamics of a test particle of mass $\mu=m_1m_2/(m_1+m_2)$ in a very simple (spherically symmetric) effective metric

$$ds^2_{\rm eff}=-A(r_{\rm eff})c^2dt^2_{\rm eff}+\frac{D(r_{\rm eff})}{A(r_{\rm eff})}dr^2_{\rm eff}+r^2_{\rm eff}(d\theta^2+\sin^2\theta d\varphi^2), \tag{2.15}$$

is possible, if and only if the energy mapping $E_{\rm eff}=f(E_{\rm real})$ is given by

$$\frac{E_{\rm eff}}{\mu c^2}=\frac{E^2_{\rm real}-m_1^2c^4-m_2^2c^4}{2m_1m_2c^4}. \tag{2.16}$$

Remarkably, the simple energy map (2.16) (which is here determined by our requirements) coincides with the energy map introduced in several other investigations [28,7] (and is simply related to the one defined *a priori* in [29,31,32]).

Recently, the problem of mapping the extremely complicated real two-body 3PN Hamiltonian (2.6) onto an effective one-body dynamics has been solved [10]. Again the result is remarkably simple, though less simple than at the 2PN level. Indeed, it was found[6] that the effective one-body dynamics was given by an Hamilton-Jacobi equation of the form (with $p^{\rm eff}_\alpha=\partial S/\partial x^\alpha_{\rm eff}$)

$$0=\mu^2+g^{\alpha\beta}_{\rm eff}p^{\rm eff}_\alpha p^{\rm eff}_\beta+Q_4(p^{\rm eff}), \tag{2.17}$$

with a simple (spherically symmetric) effective metric of the form (2.15) and some additional quartic-in-momenta contribution $Q_4(p)$. Remarkably, it was found that, at the 3PN level, the energy mapping is again uniquely determined to be the simple relation (2.16). As for the metric coefficients of the (covariant) effective metric $g^{\rm eff}_{\alpha\beta}$, and the quartic terms $Q_4(p)$, they were found to be

$$A(r)=1-2\hat{u}+2\nu\hat{u}^3+a_4(\nu)\hat{u}^4, \tag{2.18}$$

$$D(r)=1-6\nu\hat{u}^2+2(3\nu-26)\nu\hat{u}^3, \tag{2.19}$$

$$\frac{Q_4(p)}{\mu^2}=2(4-3\nu)\nu\hat{u}^2(\boldsymbol{n}\cdot\hat{\boldsymbol{p}})^4, \tag{2.20}$$

[5]We are making this very explicit because some people, when they hear about the EOB approach, think that the effective metric describing the one-body dynamics should, at some level of approximation, include some Kerr-like features to model the velocity-dependent two-body interactions. This is not true for the orbital dynamics, whatever the PN accuracy level is. On the other hand, we shall see that we need Kerr-like features to accomodate the intrinsic spin effects.

[6]In fact, [10] found that it was possible to map the real dynamics onto the geodesic dynamics of a test particle. However, both the effective metric and the modified energy map needed for this representation are rather complicated. It was felt that it is more convincing to keep a simple effective metric, and a simple energy map, but to relax the constraint of *geodesic* motion.

where $\hat{u} \equiv GM/r$, $\hat{\boldsymbol{p}} = \boldsymbol{p}/\mu$, and

$$a_4(\nu) = \left(\frac{94}{3} - \frac{41}{32}\pi^2 + 2\omega_s \right) \nu. \tag{2.21}$$

As we said above, the correct value of ω_s has been recently found [21] to be simply $\omega_s = 0$. We shall find it convenient, however, to keep ω_s as a free parameter in order to assess the quantitative importance of 3PN effects.

Let us emphasize again the streamlined nature of the effective one-body description of the orbital dynamics. Successively, as the PN order increases, one can say that (i) the 6 terms of the Newtonian plus the first post-Newtonian relative Hamiltonian (2.3),(2.4) can be mapped [via Eq. (2.16) and a canonical transformation of $(\boldsymbol{x},\boldsymbol{p})$] onto geodesic motion in a Schwarzschild spacetime of mass M (i.e., $A_{1\mathrm{PN}} = 1 - 2\hat{u}$, $D_{1\mathrm{PN}} = 1$); (ii) to take into account the 7 additional terms entering $H_{2\mathrm{PN}}$, Eq. (2.5), it is enough to add $+2\nu\hat{u}^3$ to $A(r)$ and $-6\nu\hat{u}^2$ to $D(r)$; and (iii) to take into account the 11 additional terms entering $H_{3\mathrm{PN}}$, it is enough to further add $+a_4(\nu)\hat{u}^4$ to $A(r)$ and $+2(3\nu - 26)\nu\hat{u}^3$ to $D(r)$, and to add the simple quartic term (2.20) to the mass-shell condition (2.17). Note that the effective one-body dynamics is a ''deformation'' of a geodesic dynamics for a particle of mass μ in a Schwarzschild spacetime of mass M, with the symmetric mass ratio $\nu = \mu/M = m_1 m_2/(m_1 + m_2)^2$ as deformation parameter ($0 < \nu \leqslant \frac{1}{4}$, with ν being small if $m_1 \ll m_2$ or $m_2 \ll m_1$, and reaching its maximal value of $\frac{1}{4}$ when $m_1 = m_2$). Note also that, at this stage, we have not yet introduced any particular resummation technique. The effective quantities (2.18)–(2.20) are still given as straightforward PN expansions in powers of $\hat{u} = GM/c^2 r$. However, this already means an appreciable gain over the original PN expansions, Eqs. (2.3)–(2.6). Indeed, there are far less terms in the ''effective'' PN expansions and they generically have significantly smaller coefficients (which are now all multiplied by $\nu < 1/4$). For instance, the ''radial'' potential determining the circular orbits is now fully encoded in the simple function $g_{00}^{\mathrm{eff}} = -A(r_{\mathrm{eff}})$ which differs from the test-mass (Schwarzschild) result, $A_{(\nu=0)} = 1 - 2\hat{u}$, only by two numerically smallish terms when one is above the last stable orbit. Indeed, when $r_{\mathrm{eff}} > 6GM/c^2$, $2\nu\hat{u}^3 < 0.23\%$, and $a_4(\nu)\hat{u}^4 < 0.36\%$ (for $\omega_s = 0$). When working at the 2PN level it was, in fact, found unnecessary [8] to further ''resum'' the effective PN expansions of $A(r)$ and $D(r)$. As they stand, they led to small deviations from the test-mass ($\nu \to 0$) results. However, it was found in [10] that the situation is not quite as good at the 3PN level. Because of the largish coefficient in Eq. (2.21), $\frac{94}{3} - \frac{41}{32}\pi^2 \simeq 18.688$, the additional term $+a_4(\nu)\hat{u}^4$ in Eq. (2.18) significantly modifies the qualitative behavior of the metric coefficient $A(r)$ for $r_{\mathrm{eff}} < 6GM$. In particular, when $\omega_s = 0$, the straightforward PN-expanded function $A(r)$ no longer features a zero near $r_{\mathrm{eff}} = 2GM$ for all possible values of the deformation parameter $0 < \nu \leqslant \frac{1}{4}$. As this zero (which corresponds to the Schwarzschild horizon) is a crucial qualitative feature of the $\nu = 0$ limit, it was argued in [10] that one should Padé resum the PN expansion of $A(r)$ so as to ensure the stable presence of a similar ''horizon'' when $0 < \nu \leqslant \frac{1}{4}$. We shall also do so here, but only after having introduced the spin effects, which modify the radial function which is the analog of $A(r)$.

B. Effective spin

Let us now tackle the central task of this work: to introduce spin effects in the effective one-body approach. Let us first emphasize that the ambition of the present work is somewhat limited. Our main goal is to derive a spin-dependent EOB Hamiltonian which is physically reliable for small and moderate spins. We shall proceed toward this goal in successive steps. First, we consider only (at the lowest PN order where they enter) the interactions which are *linear* in the spins. Then, we shall incorporate (to some approximation) the interactions which are quadratic in the spins. Contrary to the case of the spin-independent interactions where many years of work have yielded high-PN-accuracy information which allowed one to refine and test the EOB approach, one does not have in hand enough information for gauging the reliability of the spin-dependent interactions in the case of fast spins. As a consequence, we will be able to trust the presently introduced EOB Hamiltonian only when spin effects are not too large. As we said in the Introduction, one will need new information (either from numerical relativity, or from improved analytical methods) to find a reliable form of the EOB Hamiltonian for large spins.

We have written down in Eq. (2.9) the contribution to the real, PN-expanded, two-body Hamiltonian which is linear in the two spins. Our first proposal is to map this contribution to the spin-orbit coupling of a (spinless) effective particle moving in a suitably ''spinning'' effective metric, i.e. some type of generalized Kerr metric. If we formally consider $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ as deformation parameters (on top of the basic *orbital* deformation parameter ν), the effective dynamics we are looking for should be a ''spin deformation'' of the currently most accurate orbital dynamics, as described by the 3PN effective dynamics Eqs. (2.15)–(2.21). In particular, we should keep the non-geodesic Hamilton-Jacobi equation (2.17). Note that the spin effects [notably Eq. (2.9)] break the $O(3)$ symmetry of the orbital interaction. At the quantum level, this means that spin interactions lift the degeneracy (lack of dependence on the ''magnetic'' quantum number $L_z/\hbar$) of the orbital energy states. This shows that the effective (co)metric $g_{\mathrm{eff}}^{\alpha\beta}$ entering Eq. (2.17) should no longer be spherically symmetric, but should contain special directions linked to the spins. For generality, let us first consider an arbitrary (time-independent) effective cometric

$$g_{\mathrm{eff}}^{\alpha\beta} p_\alpha p_\beta = g_{\mathrm{eff}}^{00} p_0^2 + 2 g_{\mathrm{eff}}^{0i} p_0 p_i + g_{\mathrm{eff}}^{ij} p_i p_j . \tag{2.22}$$

Let us define

$$\alpha \equiv (-g_{\mathrm{eff}}^{00})^{-1/2}, \quad \beta^i \equiv \frac{g_{\mathrm{eff}}^{0i}}{g_{\mathrm{eff}}^{00}}, \quad \gamma^{ij} \equiv g_{\mathrm{eff}}^{ij} - \frac{g_{\mathrm{eff}}^{0i} g_{\mathrm{eff}}^{0j}}{g_{\mathrm{eff}}^{00}}, \tag{2.23}$$

i.e.,

$$g^{00}_{\rm eff}=-\frac{1}{\alpha^2},\quad g^{0i}_{\rm eff}=-\frac{\beta^i}{\alpha^2},\quad g^{ij}_{\rm eff}=\gamma^{ij}-\frac{\beta^i\beta^j}{\alpha^2}. \quad (2.24)$$

The effective energy $E_{\rm eff}\equiv -p_0^{\rm eff}$ is conserved (because of the assumed stationarity of $g^{\alpha\beta}_{\rm eff}$). Using the parametrization (2.23), Eq. (2.17) reads

$$(E_{\rm eff}-\beta^i p_i)^2=\alpha^2[\mu^2+\gamma^{ij}p_ip_j+Q_4(p)]. \quad (2.25)$$

Solving Eq. (2.25) for $E_{\rm eff}$ [using the fact that $Q_4(p)$, Eq. (2.20), depends only on the p_i's] yields the effective Hamiltonian

$$E_{\rm eff}=H_{\rm eff}(\boldsymbol{x},\boldsymbol{p},\ldots)=\beta^i p_i+\alpha\sqrt{\mu^2+\gamma^{ij}p_ip_j+Q_4(p_i)}, \quad (2.26)$$

where we have suppressed, for readability, the labels ''eff'' on the orbital phase space variables $\boldsymbol{x}^{\rm eff}$, $\boldsymbol{p}^{\rm eff}$. The ellipsis in the arguments of $H_{\rm eff}$ are added to remind us that $H_{\rm eff}$ will ultimately also depend on the spin variables $\boldsymbol{S}_1$, $\boldsymbol{S}_2$, which enter the metric coefficients α, β^i, γ^{ij} as parameters.

We also assume that Eq. (2.16) (which was found to hold at 1PN, 2PN, and 3PN) still holds. Solving it for the real energy $E_{\rm real}$ in terms of the effective one finally yields the real Hamiltonian

$$E_{\rm real}=H_{\rm real}(\boldsymbol{x},\boldsymbol{p},\ldots)=M\sqrt{1+2\nu\left(\frac{H_{\rm eff}-\mu}{\mu}\right)}. \quad (2.27)$$

We recall that, at the linearized level and at the lowest PN order, the addition of a spin $\boldsymbol{S}_{\rm eff}$ onto an initially spherical symmetric metric leads to an off-diagonal term in the metric:

$$\beta^i\simeq -g^{0i}\simeq -g_{0i}\simeq +\frac{2G}{r^3}\varepsilon^{ijk}S^j_{\rm eff}x^k. \quad (2.28)$$

Inserting this term in Eq. (2.26), and expanding Eq. (2.27) in a PN series yields, as leading spin-orbit coupling [linearized in $S_{\rm eff}$ and taken to formal order $\mathcal{O}(1/c^2)$] in $H_{\rm real}$, the term

$$\delta_{S_{\rm eff}}H_{\rm real}\simeq\beta^i p_i\simeq\frac{2G}{c^2r^3}\varepsilon^{ijk}p_iS^j_{\rm eff}x^k. \quad (2.29)$$

This term can exactly reproduce the leading[7] two-body spin-orbit coupling (2.9) if we define

$$\boldsymbol{S}^{\rm eff}\equiv\sigma_1\boldsymbol{S}_1+\sigma_2\boldsymbol{S}_2, \quad (2.30)$$

with

$$\sigma_1\equiv 1+\frac{3}{4}\frac{m_2}{m_1},\quad \sigma_2\equiv 1+\frac{3}{4}\frac{m_1}{m_2}. \quad (2.31)$$

[7]We use here the fact that the real phase-space coordinates $\boldsymbol{x}^{\rm real}$, $\boldsymbol{p}^{\rm real}$ differ only by $\mathcal{O}(1/c^2)$ from the effective ones entering $H_{\rm eff}$ [8].

C. A deformed Kerr metric

Remembering that the main message of the effective one-body method is that the orbital dynamics of two comparable-mass black holes can be described in terms of a slightly deformed (with deformation parameter ν) Schwarzschild metric, we expect that the orbital-plus-spin dynamics of two black holes can be described in terms of some deformation of the Kerr metric. In other words, we are expecting that not only the effects linear in the spins, such as Eq. (2.9), but also the spin-dependent non-linear effects, can be described in terms of some deformed, effective Kerr metric. At this stage it is therefore very natural to construct a suitable ''deformed Kerr metric'' which combines the orbital deformations (2.18),(2.19) with the full spin effects linked to the ''effective spin'' (2.30). After constructing this deformed Kerr metric, we shall *a posteriori* check that it approximately incorporates the expected two-body interactions which are *quadratic* in the spins.

Let us start from the simplest form of the Kerr cometric, underlying its separability properties [53]

$$g^{\alpha\beta}_{\rm Kerr}p_\alpha p_\beta=\frac{1}{r^2+a^2\cos^2\theta}\left[\Delta_K(r)p_r^2+p_\theta^2+\frac{1}{\sin^2\theta}\times(p_\varphi+a\sin^2\theta p_t)^2-\frac{1}{\Delta_K(r)}\times[(r^2+a^2)p_t+ap_\varphi]^2\right], \quad (2.32)$$

with $\Delta_K(r)=r^2-2Mr+a^2$. In the non-spinning limit ($a\to 0$) the coefficients of p_r^2 and p_t^2 become, respectively, $\Delta_K(r)/r^2$ and $-r^2/\Delta_K(r)$. However, we know that in this limit we should get [from (2.15)] $A(r)/D(r)$ and $-1/A(r)$, respectively. It is therefore very natural to generalize the Kerr metric (2.32) (while still keeping its separability properties) by assuming that the coefficients of the first and last terms in the large square brackets of Eq. (2.32) involve two different functions of r, say $\Delta_r(r)$ and $-1/\Delta_t(r)$, whose product reduces to $-1/D(r)$ when $a\to 0$. This reasoning leads us, as the simplest[8] possibility for combining spin effects with orbital effects, to postulating that the effective metric entering (2.17) has the form

$$g^{\alpha\beta}_{\rm eff}p_\alpha p_\beta=\frac{1}{r^2+a^2\cos^2\theta}\left[\Delta_r(r)p_r^2+p_\theta^2+\frac{1}{\sin^2\theta}\times(p_\varphi+a\sin^2\theta p_t)^2-\frac{1}{\Delta_t(r)}\times((r^2+a^2)p_t+ap_\varphi)^2\right], \quad (2.33)$$

[8]We leave untouched the dependence on a to ensure that, when $GM\to 0$ with a being fixed, the metric $g^{\alpha\beta}_{\rm eff}$ be Minkowski in disguise.

with

$$\Delta_r^{\rm PN}(r)=\frac{r^2A^{\rm PN}(r)+a^2}{D^{\rm PN}(r)},\quad \Delta_t^{\rm PN}(r)=r^2A^{\rm PN}(r)+a^2. \tag{2.34}$$

Here, the superscripts ''PN'' indicate that, at this stage, we are only comparing PN expansions. We already know from the 3PN study of [10] that this is unsatisfactory because it tends to change the qualitative behavior of the radial functions and, in particular, the presence of a horizon in the metric (2.33). To get a regular horizon in Eq. (2.33) we need the two functions $\Delta_t(r)$ and $\Delta_r(r)$ to have a zero at the same value of r. The simplest (and most robust) way of ensuring this is (as discussed in [10]) to define them as

$$\Delta_t(r)\equiv r^2P_3^1\left[A^{\rm PN}(r)+\frac{a^2}{r^2}\right],\quad \Delta_r(r)\equiv\Delta_t(r)\left(\frac{1}{D(r)}\right)^{\rm PN}. \tag{2.35}$$

Here, $P_m^n[f^{\rm PN}(u)]$, with $u\equiv 1/r$, denotes the $N_n(u)/D_m(u)$ Padé of a certain PN-expanded function $f^{\rm PN}(u)=c_0+c_1u+c_2u^2+\cdots+c_{n+m}u^{n+m}$ [$N_n(u)$ and $D_m(u)$ being polynomials in u of degrees n and m, respectively]. We do not write down the (uniquely defined) explicit expression of

$$\begin{aligned}\overline{A}(u)&\equiv P_3^1[A^{\rm PN}(u)+a^2u^2]\\&=P_3^1[1-2\hat{u}+\hat{a}^2\hat{u}^2+2\nu\hat{u}^3+a_4(\nu)\hat{u}^4]\\&=\frac{1+n_1\hat{u}}{1+d_1\hat{u}+d_2\hat{u}^2+d_3\hat{u}^3}\end{aligned}$$

(where $\hat{u}=GM/r$, $\hat{a}=a/GM$) because (i) it is rather complicated and not illuminating, and (ii) modern algebraic manipulators compute it directly from its Padé definition.

In the definition of $\Delta_r(r)$ [which is less important than that of $\Delta_t(r)$] we have factorized the Padé resummed $\Delta_t(r)$ and assumed that it was enough to work with the non-resummed PN expansion of the inverse of the D function, i.e., [from Eq. (2.19)]

$$[D^{-1}(r)]^{\rm PN}\equiv 1+6\nu\hat{u}^2+2(26-3\nu)\nu\hat{u}^3. \tag{2.36}$$

If the need arises, it would be easy to define improved (resummed) versions of $D^{-1}(r)$. Because of the positive coefficients in Eq. (2.36) the present definition does not interfere (as would the consideration of $[D^{\rm PN}(r)^{-1}]$ with the desired feature of having a simple zero in $\Delta_r(r)$ located at the same value as that in $\Delta_t(r)$.

Finally, we shall see later that there are some advantages in defining the quartic-in-momenta contribution $Q_4(p)$ in the following (deformed) way:

$$Q_4(p)=\frac{1}{\mu^2}2(4-3\nu)\nu\frac{(GM)^2}{r^2+a^2\cos^2\theta}(\boldsymbol{n}\cdot\boldsymbol{p})^4. \tag{2.37}$$

Equation (2.33) defines $g_{\rm eff}^{\alpha\beta}$ only with respect to some (instantaneous) polar coordinate system with the z-axis aligned with the effective spin (2.30). Such a coordinate system cannot be used for describing the evolution of two gravitationally interacting spinning black holes. Indeed, we expect (and shall check below) that the total real Hamiltonian imposes some type of precession motion for $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ and therefore $\boldsymbol{S}_{\rm eff}$. To get the full dynamics of the system we need to rewrite $g_{\rm eff}^{\alpha\beta}$ in a general, Cartesian-like coordinate system. This is achieved by explicitly introducing, besides $n^i\equiv x^i/r$, the quantities $[S_{\rm eff}\equiv(\delta_{ab}S_{\rm eff}^aS_{\rm eff}^b)^{1/2}]$

$$s^i\equiv\frac{S_{\rm eff}^i}{S_{\rm eff}},\quad a\equiv\frac{S_{\rm eff}}{M},\quad \cos\theta\equiv n^is^i,\quad \rho^2\equiv r^2+a^2\cos^2\theta. \tag{2.38}$$

This leads to the following, Cartesian-like, effective metric

$$-\rho^2g_{\rm eff}^{00}=\frac{(r^2+a^2)^2-a^2\Delta_t\sin^2\theta}{\Delta_t}, \tag{2.39}$$

$$\rho^2g_{\rm eff}^{0i}=-\frac{a[r^2+a^2-\Delta_t]}{\Delta_t}(\boldsymbol{s}\times\boldsymbol{x})^i, \tag{2.40}$$

$$\rho^2g_{\rm eff}^{ij}=\Delta_rn^in^j+r^2(\delta^{ij}-n^in^j)-\frac{a^2}{\Delta_t}(\boldsymbol{s}\times\boldsymbol{x})^i(\boldsymbol{s}\times\boldsymbol{x})^j, \tag{2.41}$$

from which follows

$$\alpha=\sqrt{\frac{\rho^2\Delta_t}{(r^2+a^2)^2-a^2\Delta_t\sin^2\theta}}, \tag{2.42}$$

$$\beta^i=\frac{a[r^2+a^2-\Delta_t](\boldsymbol{s}\times\boldsymbol{x})^i}{(r^2+a^2)^2-a^2\Delta_t\sin^2\theta}, \tag{2.43}$$

$$\gamma^{ij}=g_{\rm eff}^{ij}+\frac{\beta^i\beta^j}{\alpha^2}. \tag{2.44}$$

Note that near the ''horizon,'' i.e., as $\Delta_t\to 0$, the quantity α tends to zero like $\sqrt{\Delta_t}$, while β^i and γ^{ij} have finite limits. [The singular last term on the right-hand side of Eq. (2.41) is cancelled near the horizon by the contribution $+\beta^i\beta^j/\alpha^2$ to γ^{ij}, Eq. (2.44).]

Finally, the spin-dependent, real two-body Hamiltonian $H_{\rm real}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_1,\boldsymbol{S}_2)$ is defined by ($\hat{p}_i\equiv p_i/\mu$, $\hat{u}_\rho\equiv GM/\sqrt{\rho^2}$, $n^i\equiv x^i/r$)

$$H_{\rm real}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_1,\boldsymbol{S}_2)\equiv M\sqrt{1+2\nu[\beta^i\hat{p}_i+\alpha\sqrt{1+\gamma^{ij}\hat{p}_i\hat{p}_j+2(4-3\nu)\nu\hat{u}_\rho^2(n^i\hat{p}_i)^4}-1]}, \tag{2.45}$$

where we recall that the basic effective Kerr spin vector is defined by

$$Mas^i \equiv S^i_{\rm eff} \equiv \sigma_1 S^i_1 + \sigma_2 S^i_2, \tag{2.46}$$

with σ_1 and σ_2 defined in Eq. (2.31). The phase-space coordinates appearing in this Hamiltonian are the effective ones $(x^i_{\rm eff}, p^{\rm eff}_i)$. They differ [8,10] by $\mathcal{O}(1/c^2)$ terms from the coordinates used in usual PN calculations, such as ADM ones. The evolution equations defined by the Hamiltonian (2.45) are obtained by the Poisson bracket equations (2.10)–(2.13). Before discussing them let us show how the Hamiltonian (2.45) contains spin-quadratic effects of the good sign and magnitude.

D. Effects quadratic in the spins

Note first that if we introduce the ''non-relativistic'' effective Hamiltonian $H^{\rm NR}_{\rm eff} \equiv H_{\rm eff} - \mu c^2$, and similarly $H^{\rm NR}_{\rm real} \equiv H_{\rm real} - Mc^2$, one has

$$H^{\rm NR}_{\rm eff} = H^{\rm NR}_{\rm real}\left(1 + \frac{1}{2}\frac{H^{\rm NR}_{\rm real}}{Mc^2}\right). \tag{2.47}$$

Therefore, if we are interested in the leading PN approximation to any additional term in $H^{\rm PN}_{\rm real}$, one can neglect the [$\mathcal{O}(1/c^2)$ smaller] difference between $H_{\rm eff}$ and $H_{\rm real}$. By this argument, the leading PN approximation to the term linear in the spins is

$$H_{{\rm real}\,S} \simeq H_{{\rm eff}S} \simeq \beta^i p_i = \frac{a[r^2 + a^2 - \Delta_t]}{(r^2+a^2)^2 - a^2\Delta_t \sin^2\theta}(\boldsymbol{s}\times\boldsymbol{x})^i p_i\,. \tag{2.48}$$

We write it explicitly in the form in which it appears in our Hamiltonian for the reader to see how the term (2.29) is generated. (The important feature here is that $r^2 + a^2 - \Delta_t \simeq 2GMr$ at the leading PN approximation.)

Let us now consider the interaction terms in $H_{\rm real}$ or $H_{\rm eff}$ which are quadratic in the spins, and therefore quadratic in the Kerr-like parameter a, Eq. (2.46). First, one should remember that most terms of order a^2, as they appear in the effective metric (2.39)–(2.41), do not directly correspond to physical effects proportional to $\boldsymbol{S}^2_{\rm eff}$. Indeed, we are using here Boyer-Lindquist-type coordinates which differ, even in the flat space limit $GM \to 0$, from usual (flat-space, Cartesian-like) coordinates by terms of order $\mathcal{O}(a^2)$. As we are interested in the leading PN effects quadratic in the spins, we can view the Kerr-like metric (2.39)–(2.41) as a deformation, by the a-dependent terms, of the Schwarzschild metric (which is the leading PN version of the orbital effective metric). We then expect that the leading physical effects quadratic a will be those linked to the a-dependent quadrupole moment deformation of the Schwarzschild metric. The quadrupole moment of the Kerr metric (which coincides with our metric when we neglect additional 2PN fractional corrections) has been determined to be [54]

$$Q_{ij} = -Ma^2 s_i s_j + \frac{1}{3}Ma^2\delta_{ij}\,. \tag{2.49}$$

This corresponds to an additional term in the interaction Hamiltonian equal to

$$H^{\rm real}_Q \simeq H^{\rm eff}_Q \simeq -\frac{1}{2}\mu Q_{ij}\partial_{ij}\frac{1}{r} = +\frac{1}{2}\mu M a^2 s^i s^j \partial_{ij}\frac{1}{r}. \tag{2.50}$$

In terms of the effective spin this reads (in standard units)

$$H^{\rm real}_Q \simeq +\frac{1}{2}\frac{G}{c^2}\frac{\mu}{M}S^i_{\rm eff}S^j_{\rm eff}\partial_{ij}\frac{1}{r}. \tag{2.51}$$

Such is the prediction from our Hamiltonian. Let us now compare it to the expected real two-body, spin-quadratic effects. As sketched in Eq. (2.8) there are several sources of spin-quadratic effects. At leading PN order, it is enough to consider: (i) the term proportional to $m_2 S_1^2/m_1$ which arises because of the interaction of the monopole m_2 with the spin-induced quadrupole moment of the spinning black hole of mass m_1, (ii) the term proportional to $m_1 S_2^2/m_2$ obtained by exchanging $1 \to 2$, and (iii) the term proportional to $S_1 S_2$ coming from the direct, one-graviton interaction between the two spin dipoles. The first term is obtained by relabeling the result (2.51) by $\mu \to m_2$, $M \to m_1$, $S^i_{\rm eff} \to S^i_1$. Therefore the sum of (i) and (ii) reads

$$H_{S_1S_1} + H_{S_2S_2} \simeq +\frac{1}{2}\frac{G}{c^2}\left(\frac{m_2}{m_1}S^i_1 S^j_1 + \frac{m_1}{m_2}S^i_2 S^j_2\right)\partial_{ij}\frac{1}{r}. \tag{2.52}$$

The term (iii) has been computed in [47,48] and reads

$$H_{S_1S_2} \simeq +\frac{G}{c^2 r^3}S^i_1 S^j_2(3n^i n^j - \delta^{ij}) = +\frac{G}{c^2}S^i_1 S^j_2 \partial_{ij}\frac{1}{r}. \tag{2.53}$$

It is easily checked that the sum of Eqs. (2.52) and (2.53), say $H_{SS} \equiv H_{S_1S_1} + H_{S_2S_2} + H_{S_1S_2}$, can be written as

$$H_{SS} \simeq \frac{1}{2}\frac{G}{c^2}\frac{\mu}{M}S^i_0 S^j_0 \partial_{ij}\frac{1}{r}, \tag{2.54}$$

with $S^i_0/M \equiv a^i_0 \equiv a^i_1 + a^i_2 \equiv S^i_1/m_1 + S^i_2/m_2$, i.e., explicitly,

$$S^i_0 \equiv \left(1 + \frac{m_2}{m_1}\right)S^i_1 + \left(1 + \frac{m_1}{m_2}\right)S^i_2. \tag{2.55}$$

The result (2.54),(2.55) is remarkably similar to the prediction (2.51) [with Eqs. (2.30) and (2.31)]. The only discrepancy is a 25% difference in the coefficient of the mass ratios in the definition of the effective spin. Though there might be physical situations where this smallish difference might play a significant role, we think that in most cases where one will be entitled to trust the approximate spin-dependent EOB Hamiltonian introduced here this difference will not matter.

Indeed, because of the partially *ad hoc* way in which we constructed our deformed Kerr metric, we cannot trust our predictions beyond the domain where spin effects are moderate corrections to orbital effects. However, it is useful to incorporate in a qualitatively correct manner the non-linear spin effects. This is what our prescription achieves. For instance: (i) in the limit where, say, $m_2 \ll m_1$ (and $|\boldsymbol{S}_2| \leqslant m_2^2$) Eqs. (2.51) and (2.54) become equivalent, or (ii) in the case where $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are parallel (in the same direction), Eqs. (2.51) and (2.54) differ only by a numerical factor which is near one for all mass ratios.

It might, however, be useful to define another Hamiltonian, say $H'_{\rm real}$, which (a) reduces (like $H_{\rm real}$) to the Kerr one in the test-mass (and test-spin) limit, (b) contains (like $H_{\rm real}$) the spin-orbit terms (2.9), and (c) contains the exact spin-spin terms (2.54) [instead of their "25%" approximation (2.51) contained in $H_{\rm real}$]. A simple way to do that is to define $H'_{\rm real} \equiv M\sqrt{1+2\nu(H'_{\rm eff}-\mu)/\mu}$ with a modified effective Hamiltonian defined as the sum of Eq. (2.26), written with the replacement $\boldsymbol{S}_{\rm eff} \to \boldsymbol{S}_0$, and of an additional spin-orbit interaction term, $\Delta\beta^i p^i$, with $\Delta\beta^i$ proportional to the difference $\sigma^i \equiv S^i_{\rm eff} - S^i_0$:

$$H'_{\rm eff}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_1,\boldsymbol{S}_2) = H_{\rm eff}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_0) + \Delta H_{\rm SO}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_0,\boldsymbol{\sigma}), \tag{2.56}$$

where (denoting $\boldsymbol{a}_0 \equiv \boldsymbol{S}_0/M$, $\cos\theta_0 \equiv n^i S^i_0/|\boldsymbol{S}_0|$)

$$\Delta H_{\rm SO}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_0,\boldsymbol{\sigma}) \equiv \frac{r^2+a_0^2-\Delta_t(a_0)}{(r^2+a_0^2)^2-a_0^2\Delta_t(a_0)\sin^2\theta_0}\,\frac{\varepsilon^{ijk}p_i\sigma^j x^k}{M}, \tag{2.57}$$

with

$$\sigma^i \equiv S^i_{\rm eff} - S^i_0 \equiv -\frac{1}{4}\left(\frac{m_2}{m_1}S^i_1 + \frac{m_1}{m_2}S^i_2\right). \tag{2.58}$$

The consideration of the new Hamiltonian $H'_{\rm real}$ would considerably complicate (even at the qualitative level) the discussion of the following section. As we are not sure that this complication really entails a better *quantitative* description of spin effects, when these become important, we shall, in the following, content ourselves with studying the consequences of the simpler (though slightly less "accurate") Hamiltonian $H_{\rm real}$, Eq. (2.45) with Eq. (2.46). However, we mention that it might be useful to consider simultaneously $H_{\rm real}$ and $H'_{\rm real}$, and to trust their predictions only in the cases where they differ only by a slight amount. This gives a useful measure of the domain of validity of the present spin-dependent effective-one-body approach.

III. DYNAMICS OF TWO SPINNING BLACK HOLES

A. Equations of motion and exact or approximate first integrals

In the previous section we have explicitly constructed a Hamiltonian $H_{\rm real}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_1,\boldsymbol{S}_2)$ describing (to some approximation) the (conservative part of the) gravitational interaction of two spinning black holes in the center-of-mass frame of the binary system. In the present section we shall describe some consequences of this Hamiltonian. Let us start by writing down explicitly the evolution equations for all the dynamical variables. From the basic PB's (2.10)–(2.14) we get

$$\frac{dx^i}{dt} = \{x^i, H_{\rm real}\} = +\frac{\partial H_{\rm real}}{\partial p_i}, \tag{3.1}$$

$$\frac{dp_i}{dt} = \{p_i, H_{\rm real}\} = -\frac{\partial H_{\rm real}}{\partial x^i}, \tag{3.2}$$

$$\frac{dS^i_1}{dt} = \{S^i_1, H_{\rm real}\} = \varepsilon^{ijk}\frac{\partial H_{\rm real}}{\partial S^j_1}S^k_1, \tag{3.3}$$

$$\frac{dS^i_2}{dt} = \{S^i_2, H_{\rm real}\} = \varepsilon^{ijk}\frac{\partial H_{\rm real}}{\partial S^j_2}S^k_2. \tag{3.4}$$

In vectorial notation, the spin evolution equations read (e.g. for the first spin)

$$\frac{d\boldsymbol{S}_1}{dt} = \boldsymbol{\Omega}_1 \times \boldsymbol{S}_1, \quad \boldsymbol{\Omega}_1 \equiv \frac{\partial H_{\rm real}}{\partial \boldsymbol{S}_1}. \tag{3.5}$$

A first consequence of these results is that the magnitudes of the two spins are exactly conserved:

$$\boldsymbol{S}_1^2 = \text{const}, \quad \boldsymbol{S}_2^2 = \text{const}. \tag{3.6}$$

Another general consequence is the exact conservation of the total angular momentum

$$\boldsymbol{J} \equiv \boldsymbol{L} + \boldsymbol{S}_1 + \boldsymbol{S}_2, \tag{3.7}$$

where $L^i \equiv \varepsilon^{ijk}x^j p_k$. Indeed, it is easily checked that J^i generates, by Poisson brackets, global rotations of all the vectorial dynamical quantities: $\{J^i, V^j\} = \varepsilon^{ijk}V^k$ for $\boldsymbol{V} = \boldsymbol{x}$, $\boldsymbol{p}$, $\boldsymbol{S}_1$ or $\boldsymbol{S}_2$. As the Hamiltonian is a scalar constructed out of $\boldsymbol{x}$, $\boldsymbol{p}$, $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$, we have

$$\frac{d}{dt}J^i = \{J^i, H_{\rm real}\} = 0. \tag{3.8}$$

Therefore

$$J^i = \text{const and, in particular, } \boldsymbol{J}^2 = \text{const}. \tag{3.9}$$

Evidently, we have also the conservation of the total energy

$$\frac{dH_{\rm real}}{dt} = \{H_{\rm real}, H_{\rm real}\} = 0 \Rightarrow H_{\rm real} = \text{const}. \tag{3.10}$$

This closes the list of generic first integrals of the evolution. It should be noted that, in general, quantities such as $\boldsymbol{L}^2$ or $\boldsymbol{S}_1 \cdot \boldsymbol{S}_2$ are not conserved in time. This means, in particular, that the magnitude of the effective spin, $a^2 = M^{-2}\boldsymbol{S}^2_{\rm eff}$, will not stay constant during the evolution.

Evidently, in particular situations, more quantities might be approximately conserved. An interesting case is that in which the spins are small enough for one to retain only the terms linear in them. In this approximation

$$H_{\rm eff}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_1,\boldsymbol{S}_2)\simeq H_0(\boldsymbol{x},\boldsymbol{p})+\frac{[r^2-\Delta_t^{(a=0)}]}{Mr^4}\boldsymbol{L}\cdot\boldsymbol{S}_{\rm eff}, \tag{3.11}$$

where $H_0(\boldsymbol{x},\boldsymbol{p})$ is spherically symmetric.

Let us, more generally,[9] assume that $H_{\rm eff}$, as well as $H_{\rm real}$, are spherically symmetric, functions of $\boldsymbol{x}$ and $\boldsymbol{p}$ except for a dependence on the combination $\boldsymbol{L}\cdot\boldsymbol{S}_{\rm eff}$:

$$H_{\rm real}=H_{\rm real}(r,p_r,\boldsymbol{L}^2,\boldsymbol{L}\cdot\boldsymbol{S}_{\rm eff}), \tag{3.12}$$

where $p_r\equiv n^i p_i$ is canonically conjugated to $r(\{r,p_r\}=1)$. Under the assumption (3.12) the angular momenta evolution equations become [with $({\rm LS})\equiv\boldsymbol{L}\cdot\boldsymbol{S}_{\rm eff}$]

$$\frac{d\boldsymbol{S}_1}{dt}=\frac{\partial H_{\rm real}}{\partial({\rm LS})}\boldsymbol{\sigma}_1\boldsymbol{L}\times\boldsymbol{S}_1, \tag{3.13}$$

$$\frac{d\boldsymbol{S}_2}{dt}=\frac{\partial H_{\rm real}}{\partial({\rm LS})}\boldsymbol{\sigma}_2\boldsymbol{L}\times\boldsymbol{S}_2, \tag{3.14}$$

$$\frac{d\boldsymbol{L}}{dt}=\frac{\partial H_{\rm real}}{\partial({\rm LS})}\boldsymbol{S}_{\rm eff}\times\boldsymbol{L}. \tag{3.15}$$

These evolution equations imply not only (as in the general case) the conservation of $\boldsymbol{J}=\boldsymbol{L}+\boldsymbol{S}_1+\boldsymbol{S}_2$, and of $\boldsymbol{S}_1^2$ and $\boldsymbol{S}_2^2$, but also that of

$$\boldsymbol{L}^2={\rm const},\quad \boldsymbol{L}\cdot\boldsymbol{S}_{\rm eff}={\rm const}. \tag{3.16}$$

Note, however, that $\boldsymbol{S}_{\rm eff}^2$ is not conserved. Moreover, the radial motion is governed by the equations

$$\dot{r}=\frac{\partial H_{\rm real}(r,p_r,\boldsymbol{L}^2,\boldsymbol{L}\cdot\boldsymbol{S}_{\rm eff})}{\partial p_r}, \tag{3.17}$$

$$\dot{p}_r=-\frac{\partial H_{\rm real}(r,p_r,\boldsymbol{L}^2,\boldsymbol{L}\cdot\boldsymbol{S}_{\rm eff})}{\partial r}. \tag{3.18}$$

In view of the constancy of $\boldsymbol{L}^2=C_0$ and $\boldsymbol{L}\cdot\boldsymbol{S}_{\rm eff}=C_1$, we see from these equations that the function of r and p_r, $H_{\rm rad}(r,p_r)=H_{\rm real}(r,p_r,C_0,C_1)$, defines a reduced Hamiltonian describing the radial motion, separately from the angular degrees of freedom. In particular, we see (using the fact that p_r enters at least quadratically in $H_{\rm real}$) that, under our current (approximate) assumption (3.12), there exists a class of *spherical orbits*, i.e. of orbits satisfying

$$r={\rm const},\quad p_r=0,\quad \frac{\partial H_{\rm real}(r,p_r=0,\boldsymbol{L}^2,\boldsymbol{L}\cdot\boldsymbol{S}_{\rm eff})}{\partial r}=0. \tag{3.19}$$

Because of the (possibly non-linear) spin-orbit coupling, i.e., the dependence of $H_{\rm real}$ on $\boldsymbol{L}\cdot\boldsymbol{S}_{\rm eff}$, the orbital plane of these "spherical" orbits is not fixed in space. But the radial coordinate r being constant, these orbits trace a complicated path on a sphere (hence the name). These orbits are the analogs, in our two-body problem, and in the approximation (3.12), of similar exact "spherical" orbits for the geodesic motion of test particles in a Kerr spacetime [55]. Their existence (under some approximation) in the two-body problem is interesting for the following reason. One expects most black hole binary sources of interest for the LIGO-VIRGO-GEO network to have had the time to relax, under radiation reaction, to circular orbits. When the two black holes get closer, these circular orbits will adiabatically shrink until they come close enough for feeling the effect of the spin-orbit coupling (which varies proportionally to r^{-3}). In some intermediate domain where the spin-orbit coupling is significant, but couplings quadratic in the spins are still small, the initially circular orbit will evolve into an adiabatic sequence of "spherical" orbits of the type just discussed. (We are here adding by hand the effect of radiation reaction, treated as an adiabatic perturbation of the conservative dynamics discussed in this paper.) These considerations indicate that, in first approximation, the total amount of gravitational radiation emitted by coalescing spinning black holes will be determined by the binding energy of the last stable spherical orbit (LSSO), i.e., the last stable solution of Eqs. (3.19), which will satisfy

$$\frac{\partial H_{\rm real}}{\partial r}(r,p_r=0,\boldsymbol{L}^2,\boldsymbol{L}\cdot\boldsymbol{S}_{\rm eff})=0,$$

$$\frac{\partial^2 H_{\rm real}}{\partial r^2}(r,p_r=0,\boldsymbol{L}^2,\boldsymbol{L}\cdot\boldsymbol{S}_{\rm eff})=0. \tag{3.20}$$

Before studying the energetics of these LSSO's let us mention the existence of other approximate first integrals in the dynamics of binary spinning black holes. Let us keep all the terms non-linear in $\boldsymbol{S}_{\rm eff}$, i.e., the full expression of $H_{\rm real}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{S}_1,\boldsymbol{S}_2)$, but let us try to approximately decouple the orbital motion from the spin degrees of freedom by considering that the two spin vectors evolve adiabatically (i.e., slowly on the orbital time scale), through Eqs. (3.3) and (3.4). In this adiabatic-spin approximation, the orbital motion is described by the Hamilton-Jacobi equation (2.17), with an adiabatically fixed effective metric $g_{\rm eff}^{\alpha\beta}$. With the definition (2.37) of the quadratic-in-momenta term $Q_4(p)$, one can check that, in this approximation, there will exist a two-body analog of the Carter constant for geodesic motion in Kerr [53]. Indeed, we have constructed our deformed Kerr metric (2.33) so as to respect its separability properties. Let us work in an (adiabatic) Boyer-Lindquist-type coordinate system (t,r,θ,φ), as in Eq. (2.33). We find that the separability of the effective Hamilton-Jacobi equation yields the following first integrals (of the effective Hamiltonian):

$$p_t=-E_{\rm eff},\quad p_\varphi=L_z, \tag{3.21}$$

$$p_\theta^2+\frac{(L_z-aE_{\rm eff}\sin^2\theta)^2}{\sin^2\theta}+\mu^2a^2\cos^2\theta$$

[9] For instance, we can assume Eq. (3.11) for $H_{\rm eff}$, but we make no further approximation in computing $H_{\rm real}=f(H_{\rm eff})$.

$$= \mathcal{K} \equiv \mathcal{Q} + (L_z - aE_{\rm eff})^2, \tag{3.22}$$

$$p_\theta^2 + \cos^2\theta \left[\frac{L_z^2}{\sin^2\theta} + a^2(\mu^2 - E_{\rm eff}^2) \right] = \mathcal{Q} \equiv \mathcal{K} - (L_z - aE_{\rm eff})^2. \tag{3.23}$$

The last two equations are equivalent to each other, but, depending on the context, one can be more convenient than the other. Let us note the connection of the first integrals (3.21)–(3.23) with the above analysis of the first integrals of the Hamiltonian depending only on the combination $\boldsymbol{L}\cdot\boldsymbol{S}_{\rm eff}$. The conservation of L_z, Eq. (3.21), corresponds to the conservation of $\boldsymbol{L}\cdot\boldsymbol{S}_{\rm eff}$, Eq. (3.16), while the conservation of $\mathcal{K}$ or $\mathcal{Q}$ corresponds to the conservation of $\boldsymbol{L}^2$. Indeed, if we neglect the terms proportional to a^2 in Eq. (3.23) we get

$$\mathcal{Q} \simeq p_\theta^2 + L_z^2 \frac{1-\sin^2\theta}{\sin^2\theta} = \boldsymbol{L}^2 - L_z^2. \tag{3.24}$$

This suggests that, even beyond the adiabatic-spin approximation, the quantities, now defined in an arbitrary frame as

$$L_z \equiv \boldsymbol{L}\cdot\boldsymbol{s}, \quad \mathcal{Q} \equiv \boldsymbol{L}^2 - (\boldsymbol{L}\cdot\boldsymbol{s})^2 + a^2(\boldsymbol{n}\cdot\boldsymbol{s})^2(\mu^2 - E_{\rm eff}^2), \tag{3.25}$$

will be [as well as $\boldsymbol{S}_{\rm eff}^2$ and $\mathcal{K} \equiv \mathcal{Q} + (L_z - aE_{\rm eff})^2$] conserved to a good approximation. We are mentionning here these approximate conservation laws because they could be helpful in qualitatively understanding the full two-body dynamics.

B. Spherical orbits and last stable spherical orbits

We discussed above the existence of spherical orbits under the assumption (or the approximation) that $H_{\rm real}$ depends only on the "spin-orbit" combination $\boldsymbol{L}\cdot\boldsymbol{S}_{\rm eff}$ (as it does at the linear-in-spin level). More generally, we have seen that if we treat the evolution of the spins as being adiabatic, we have the (approximate) first integrals (3.25). If we use (as a heuristic mean of studying the main features of the orbital dynamics) this adiabatic approximation, we can define a family of spherical orbits by drawing on the conservation of the quantities (3.25). Indeed, inserting the definitions (3.25) into Eq. (2.17) we get an equation controlling the radial motion:

$$\Delta_r p_r^2 + 2(4-3\nu)\nu(GM)^2 p_r^4/\mu^2 = \frac{1}{\Delta_t(r)}[(r^2+a^2)E_{\rm eff} - aL_z]^2 - [\mu^2 r^2 + \mathcal{Q} + (L_z - aE_{\rm eff})^2]. \tag{3.26}$$

The right-hand side of Eq. (3.26) defines (when $a^2 = \boldsymbol{S}_{\rm eff}^2/M^2$, L_z, and $\mathcal{Q}$ are considered as adiabatic constants) a radial potential whose local minima, in r, determines (adiabatic) spherical orbits. The last stable spherical orbit is obtained when this radial potential has an inflection point. More precisely let us define

$$R(r, E_{\rm eff}, L_z, \mathcal{Q}) \equiv [(r^2+a^2)E_{\rm eff} - aL_z]^2 - \Delta_t(r)[\mu^2 r^2 + \mathcal{Q} + (L_z - aE)^2]. \tag{3.27}$$

The spherical orbits are the solutions of

$$R(r, E_{\rm eff}, L_z, \mathcal{Q}) = \frac{\partial}{\partial r} R(r, E_{\rm eff}, L_z, \mathcal{Q}) = 0. \tag{3.28}$$

The solutions of Eq. (3.28) yield a two-parameter family of solutions, along which, for instance, r and $E_{\rm eff}$ are functions of L_z and $\mathcal{Q}$. The last stable spherical orbit (LSSO) along such a family of solutions must satisfy the three equations

$$R(r, E_{\rm eff}, L_z, \mathcal{Q}) = \frac{\partial}{\partial r} R(r, E_{\rm eff}, L_z, \mathcal{Q}) = \frac{\partial^2}{\partial r^2} R(r, E_{\rm eff}, L_z, \mathcal{Q}) = 0. \tag{3.29}$$

There is a one-parameter family of LSSO's. For instance, one can take as a free parameter the dimensionless ratio $\mathcal{Q}/L_z^2$ which is a measure of the maximum angle between the orbital plane and the equatorial plane defined by $\boldsymbol{S}_{\rm eff}$. [Note that $\mathcal{Q}=0$ for an orbit in the equatorial plane.] For each value of this angle, and for each value of the effective spin parameter a, there will be some LSSO, with particular values of r, $E_{\rm eff}$, and L_z.

To study the values of the (effective and real) binding energy, and of the orbital angular momentum along this one-parameter family of LSSO's, it is convenient to work with slightly different variables. Let us introduce

$$\bar{L}_z \equiv L_z - aE_{\rm eff}, \quad \mathcal{K} \equiv \mathcal{Q} + \bar{L}_z^2. \tag{3.30}$$

Let us also work with the radial variable $u \equiv 1/r$ and denote

$$\bar{A}(u) \equiv \frac{\Delta_t(r)}{r^2} \equiv P_3^1[A_{\rm orb}^{\rm PN}(u) + a^2u^2]. \tag{3.31}$$

We have

$$r^{-4}R(r) \equiv U(u) \equiv (E_{\rm eff} - a\bar{L}_z u^2)^2 - \bar{A}(u)(\mu^2 + \mathcal{K}u^2). \tag{3.32}$$

The equation $U(u)=0$ [i.e., $R(r)=0$] is now solved as

$$E_{\rm eff} = W_a(u, \bar{L}_z, \mathcal{K}) \equiv a\bar{L}_z u^2 + \sqrt{\bar{A}(u)(\mu^2 + \mathcal{K}u^2)}. \tag{3.33}$$

The two-parameter family of spherical orbits is now obtained (as functions of the parameters $\bar{L}_z$ and $\mathcal{K}$) by solving $\partial W/\partial u = 0$, while the one-parameter family of LSSO's is obtained by solving $\partial W/\partial u = \partial^2 W/\partial u^2 = 0$. The advantage of this formulation is that it exhibits in the simplest way the analogy with the effective radial potential discussed in [8,10] for the pure (3PN) orbital motion (without spin), namely,

$$W_0(u, L) \equiv \sqrt{A(u)(\mu^2 + L^2u^2)}, \tag{3.34}$$

with $A(u)\equiv P^1_3[A^{\rm PN}_{\rm orb}(u)]$. Apart from the replacement $L^2\to\mathcal{K}$, the only two differences between the spinning case [Eq. (3.33)] and the spinless one [Eq. (3.34)] is the addition of the spin-orbit energy term $+a\overline{L_z}u^2$, and the additional a^2u^2 term in the PN expansion of $\overline{A}(u)$. [Note that $\overline{A}(u)\neq\overline{A}(u)+a^2u^2$ because the Padé resumming is done after the addition of a^2u^2.] We have chosen to parametrize $W_a(u)$ in terms of $\overline{L_z}$ and $\mathcal{K}$ because it simplifies very much its expression and thereby renders more transparent the new physics incorporated in our effective one-body approach. The fact that $\overline{L_z}$ depends both on L_z and $E_{\rm eff}$ is not a problem for solving Eq. (3.32) for $E_{\rm eff}$. Indeed, we are discussing a continuous family of solutions and it is essentially indifferent to parametrize them in terms of L_z or $\overline{L_z}$. We could have introduced another effective potential $W'_a(r,L_z,\mathcal{Q})$ by solving $R(r,E_{\rm eff},L_z,\mathcal{Q})=0$, with Eq. (3.27), which would be more complicated, but which would describe the same physics. [Note that $W'_a(r)$ would directly exhibit the correct fact that the spin-orbit energy, for given L_z, decreases like r^{-3}, while this fact is hidden in $W_a(u)$ which assumes that $\overline{L_z}=L_z-aE_{\rm eff}$ is given.]

C. Binding energy of last stable spherical orbits

To get a first idea of the physical consequences of our effective one-body description of coalescing spinning black holes we have numerically investigated the properties of the one-parameter family of LSSO's. The most important quantity we are interested in is the binding energy at the last stable spherical orbit because it is the prime quantity determining the detectability of the GW emitted during the inspiral. We recall that the real, two-body energy is related to the effective energy entering the equations of Sec. III B through

$$E_{\rm real}=M\sqrt{1+2\nu\left(\frac{E_{\rm eff}}{\mu}-1\right)}. \tag{3.35}$$

We are mostly interested in the (dimensionless) binding energy per unit total mass, say

$$e\equiv\frac{E_{\rm real}-M}{M}=\sqrt{1+2\nu\left(\frac{E_{\rm eff}}{\mu}-1\right)}-1. \tag{3.36}$$

The value of e at the LSSO depends on three dimensionless parameters

$$\nu_4\equiv 4\frac{\mu}{M}\equiv 4\nu,\quad \hat{a}\equiv\frac{a}{M}\equiv\frac{|\boldsymbol{S}_{\rm eff}|}{M^2},\quad \cos\theta_{\rm LS}\equiv\frac{\overline{L_z}}{\sqrt{\mathcal{K}}}. \tag{3.37}$$

Here, the parameter ν_4 (renormalized so that $0<\nu_4\leqslant 1$) determines the effect of having comparable masses ($\nu_4\simeq 1$) rather than a large mass hierarchy ($\nu_4\ll 1$). The dependence of $e^{\rm LSO}$ on ν_4 in the absence of spins was studied in [8,10]. It was found that the ratio $e^{\rm LSO}/\nu_4$ was essentially linear in ν_4 (even for ν_4 as large as 1, corresponding to the equal-mass case)

$$e^{\rm LSO}_{S_{\rm eff}=0}\simeq -0.014298\nu_4(1+c_1\nu_4). \tag{3.38}$$

Here, the numerical value $-1.4298\%=\frac{1}{4}(\sqrt{\frac{8}{9}}-1)$ is one-fourth the specific binding LSO energy of a test particle in the Schwarzschild spacetime. The numerical coefficient c_1 which condenses the effect of resummed PN interactions was found to have a value $c_1^{\rm 2PN}\simeq 0.048$ at 2PN and $c_1^{\rm 3PN}(\omega_s=0)\simeq 0.168$ at 3PN, and for $\omega_s=0$. [The dependence of $c_1^{\rm 3PN}$ on ω_s is also roughly linear: $c_1^{\rm 3PN}(\omega_s)\simeq 0.168+0.0126\omega_s$, at least when $-10\lesssim\omega_s\lesssim 0$.]

We expect that the dependence on ν_4 of the spin-dependent effects will also be roughly linear [after factorization of an overall factor ν_4 which comes from expanding the square root in Eq. (3.36)]. In the following we shall generally consider (in our numerical investigations) the case $\nu_4=1$, and concentrate on the dependence on the other parameters.

Let us clarify the meaning of the parameters $\hat{a}$ and $\cos\theta_{\rm LS}$ introduced in Eq. (3.37). The quantities $\boldsymbol{S}_{\rm eff}$, $\overline{L_z}$ and $\mathcal{K}$ entering these definitions are all supposed to be computed at the last stable spherical orbit of an adiabatic sequence of spherical orbits (in the sense discussed above). Physically, we have in mind the sequence of inspiralling orbits driven by radiation reaction. Technically, we define $e(\nu_4,\hat{a},\cos\theta_{\rm LS})$ by solving the effective radial potential problem defined in Sec. III B. We are aware of the fact that we cannot really attach to $\cos\theta_{\rm LS}$ the meaning of being the cosinus between $\boldsymbol{L}$ and $\boldsymbol{S}_{\rm eff}$ (as the name would suggest), but this is not important. What is important is that there is indeed a physical degree of freedom related to the misalignment between $\boldsymbol{L}$ and $\boldsymbol{S}_{\rm eff}$ at the LSSO and that we measure it by a parameter normalized so that $\cos\theta_{\rm LS}=1$ (or -1) when all angular momenta are aligned [in this limit the concept of the last stable circular equatorial orbit is meaningful and coincides with the $\cos\theta_{\rm LS}=1$ (or -1) limit of our formal definitions].

Note that, with the definitions (3.37) and the additional definition $\overline{l}\equiv(\mathcal{K})^{1/2}/GM\mu$, the effective radial potential (3.33) yields (in dimensionless form, $\hat{u}\equiv GM/r$)

$$\frac{E_{\rm eff}}{\mu}=\hat{W}_{\hat{a}}(\hat{u},\cos\theta_{\rm LS},\overline{l})$$
$$=\hat{a}\cos\theta_{\rm LS}\overline{l}\hat{u}^2+\sqrt{\overline{A}(\hat{u},\hat{a}^2)(1+\overline{l}^2\hat{u}^2)}. \tag{3.39}$$

This form makes it clear that $E^{\rm LSSO}_{\rm eff}/\mu$, and therefore $e^{\rm LSSO}$, Eq. (3.36), will depend primarily on the combination ("projected value of $\hat{a}$")

$$\hat{a}_p\equiv\hat{a}\cos\theta_{\rm LS}\simeq\frac{\boldsymbol{k}\cdot\boldsymbol{S}_{\rm eff}}{M^2} \tag{3.40}$$

where $\boldsymbol{k}=\boldsymbol{L}/|\boldsymbol{L}|$ (at the LSSO). For smallish spins, the combination $\hat{a}_p$ is the only one entering the problem *linearly*. As recalled by the notation in Eq. (3.39) the non-projected value of $\hat{a}$ enters only quadratically in $\overline{A}(\hat{u},\hat{a}^2)$.

Let us consider more closely the crucial quantity

$$\hat{a}_p=\frac{1}{(m_1+m_2)^2}\left[\left(1+\frac{3}{4}\frac{m_2}{m_1}\right)\boldsymbol{k}\cdot\boldsymbol{S}_1+\left(1+\frac{3}{4}\frac{m_1}{m_2}\right)\boldsymbol{k}\cdot\boldsymbol{S}_2\right]$$
$$=\left(X_1^2+\frac{3}{4}X_1X_2\right)\boldsymbol{k}\cdot\hat{\boldsymbol{a}}_1+\left(X_2^2+\frac{3}{4}X_1X_2\right)\boldsymbol{k}\cdot\hat{\boldsymbol{a}}_2. \quad (3.41)$$

In the second form, we have defined $X_1\equiv m_1/M$, $X_2\equiv m_2/M$ $(X_1+X_2=1,\ X_1X_2=\nu)$, and $\hat{\boldsymbol{a}}_1\equiv\boldsymbol{S}_1/m_1^2$, $\hat{\boldsymbol{a}}_2\equiv\boldsymbol{S}_2/m_2^2$. (We recall that a maximally spinning hole would have $|\hat{\boldsymbol{a}}_1|=1$.)

An important question (for the relevance of the present work) is the following: what are the plausible values of $\hat{a}_p$ in the sources that will be detected by LIGO-VIRGO-GEO? Present astrophysical ideas about the formation of binary black holes [1,4] suggest neither that the holes be typically maximally spinning, nor that there be any correlation between the spin and angular momenta, i.e. between the directions of $\boldsymbol{k}$, $\boldsymbol{S}_1$, and $\boldsymbol{S}_2$. Not much is known either about the probable value of the mass ratio. To have an idea of the plausible values of $\hat{a}_p$ (which is an algebraic quantity which can take positive or negative values) let us consider the random mean square (rms) value of $\hat{a}_p$ under the assumption of random, uncorrelated directions $\boldsymbol{k}$, $\boldsymbol{S}_1$, and $\boldsymbol{S}_2$ (so that $\langle\hat{a}_p\rangle=0$). Let us assume (for simplicity) that $m_1=m_2$, i.e., $\nu_4=1$, which is the most favorable case because $e^{\rm LSSO}\propto\nu_4$. We assume also that $\langle\hat{\boldsymbol{a}}_1^2\rangle=\langle\hat{\boldsymbol{a}}_2^2\rangle\equiv(a_1^{\rm rms})^2$ is some given quantity (to be determined by astrophysical models). This yields for $\hat{a}_p^{\rm rms}\equiv\sqrt{\langle\hat{a}_p^2\rangle}$

$$\hat{a}_p^{\rm rms}=\frac{7}{16}\frac{\sqrt{2}}{\sqrt{3}}a_1^{\rm rms}=0.357a_1^{\rm rms}. \quad (3.42)$$

Even if $\hat{a}_1^{\rm rms}=1$ (which would mean that all black holes are maximally spinning) we get $\hat{a}_p^{\rm rms}=0.357$. However, we find it highly plausible that $\hat{a}_1^{\rm rms}$ will be significantly smaller than 1. For instance, if we optimistically assume a uniform distribution of spin kinetic energy between 0 and the maximal value we would get $\hat{a}_1^{\rm rms}=1/\sqrt{2}$ and therefore $\hat{a}_p^{\rm rms}=7/(16\sqrt{3})=0.253$. In view of these arguments, we find plausible that most LIGO-VIRGO binary black hole sources will have $|\hat{a}_p|\lesssim 0.3$. This consideration is important because we shall see later that for such smallish values of $\hat{a}_p$ the simple analytical approach advocated here seems to be quite reliable. However, one should also be able to compute physically reliable (or, at least, sufficiently flexible) templates for fast spinning binary black holes. As we said in the Introduction, we think that the EOB approach can be an essential tool for this purpose, in conjunction with numerical data, by feeding the (necessarily sparse) numerical data into some multiparameter version of the EOB Hamiltonian.

This statistical estimate of the plausible value of a_p suggests that a typical value of $\cos\theta_{\rm LS}\simeq\boldsymbol{k}\cdot\boldsymbol{S}_{\rm eff}/|\boldsymbol{S}_{\rm eff}|$ is around $\pm1/\sqrt{3}$. In our numerical estimates of $e^{\rm LSSO}$ we have used this value, as well as the (implausible) value $\cos\theta_{\rm LS}=\pm1$ corresponding to perfect alignment. As a first step towards exploring the ''parametric flexibility'' of the EOB approach, we have studied the dependence of $e^{\rm LSSO}$ on the value of the parameter ω_s, as it appears in Eq. (2.21). We have done numerical simulations for three fiducial values: $\omega_s=0\equiv\omega^{\rm DJS}$ [21], $\omega_s=-1987/840\equiv\omega_s^{\rm BF}$ [16,18],[10] and also for $\omega_s=-\frac{1}{2}(\frac{94}{3}-41\pi^2/32)\equiv\omega_s^*$. Note the numerical values: $\omega_s^{\rm BF}\simeq-2.3655$, $\omega_s^*\simeq-9.3439$. The original motivation (when writing this paper, before the completion of the work [21] which determined the correct value of ω_s) was to study the sensitivity of our results to the ''3PN ambiguity.'' We kept it here as an interesting case study of the sensitivity of EOB results to modifications of the various coefficients entering the EOB Hamiltonian. The change from $\omega_s=0$ to $\omega_s=\omega_s^{\rm BF}$ corresponds to a change of the coefficient $a_4\equiv a_4(\nu)/\nu$ from 18.688 to 13.957, i.e., a fractional change of -25.32%. The value $\omega_s=\omega_s^*$ has the effect of completely cancelling the 3PN contribution to the radial functions $A(u)$ and $\bar{A}(u)$. Therefore, choosing $\omega_s=\omega_s^*$ gives for the LSSO quantities the same results as the 2PN effective-one-body Hamiltonian [8]. Its consideration is useful for exhibiting the difference between the 2PN-based results and the 3PN-based ones. Our results are displayed in Table II and Fig. 1.

The most important conclusion we wish to draw from these results is that, when $\hat{a}\lesssim0.3$ (which, as we argued above, covers a large domain of the physically relevant cases), the binding energy at the LSSO seems to be reliably describable by the simple analytical EOB Hamiltonian defined above. Indeed, the differences between (i) the nonspinning case and the spinning ones, and (ii) the 2PN orbital approximation and the 3PN one, are all quite moderate (which indicates that the effective one-body approach is effective in resumming PN interactions near the LSO). Furthermore, the difference between (iii) the spinning 3PN case with $\omega_s=0=\omega_s^{\rm DJS}$ [10], and the same case with $\omega_s=\omega_s^{\rm BF}$ [18] is rather small. This is a testimony of the robustness of the EOB approach. A change of its 3PN coefficient by 25% does not affect much the physical predictions. This robustness at the 3PN level is indicative of some robustness against the addition of higher PN effects. Note also (from Table II) the confirmation that when $|\hat{a}_p|\lesssim0.2$, the binding energy at the LSSO depends nearly only on the projected effective spin parameter $\hat{a}_p=\hat{a}\cos\theta_{\rm LS}$, with a very weak dependence on the value of $\cos\theta_{\rm LS}$.

On the other hand, it must be admitted that when, say, $\hat{a}\gtrsim0.4$ the differences between the three cases (i), (ii), (iii) become so large, and the radius of the LSSO becomes so small, that the present spin-dependent EOB predictions cannot be quantitatively trusted. (However, as discussed in more detail below, we think that they remain qualitatively correct.)

[10]This value corresponds to taking $\lambda=0$ in $\omega_s=-11\lambda/3-1987/840$. Here, λ denotes the natural ambiguity parameter entering the Blanchet-Faye framework. Note that the authors of Refs. [16–18] do not claim that $\lambda=0$ is a preferred value. However, as λ is expected to be of order unity we use $\lambda=0$, i.e., $\omega_s=-1987/840$ as a fiducial deviation from $\omega_s=0$.

TABLE II. Binding energies $e \equiv (\mathcal{E}_{\rm real}/M) - 1$ and (effective Boyer-Lindquist) radii of last stable spherical orbits (LSSO) for equal-mass spinning binary systems. The LSSO's depend on two independent parameters: $|\hat{\boldsymbol{a}}| \equiv |\boldsymbol{S}_{\rm eff}|/M^2$ and $\cos\theta_{\rm LS} \equiv \overline{L_z}/\sqrt{\mathcal{K}}$ (which is, morally, the cosine of the angle between the orbital angular momentum and the effective spin). The combined parameter $\hat{a}_p \equiv |\hat{\boldsymbol{a}}| \cos\theta_{\rm LS}$ (projected spin) plays a primary role for moderate spins. The algebraic quantity $\hat{a}$ is defined as $+|\hat{\boldsymbol{a}}|$ if $\cos\theta_{\rm LS} > 0$ and $-|\hat{\boldsymbol{a}}|$ if $\cos\theta_{\rm LS} < 0$. All quantities are computed from the 3PN-level Padé-resummed effective-one-body Hamiltonian (2.45) with $\omega_s = 0$.

$\cos\theta_{\rm LA} = \pm 1/\sqrt{3}$				$\cos\theta_{\rm LS} = \pm 1$			
$\hat{a}_p$	$\hat{a}$	$e^{\rm LSSO}$	$\hat{r}^{\rm LSSO}$	$\hat{a}_p$	$\hat{a}$	$e^{\rm LSSO}$	$\hat{r}^{\rm LSSO}$
−0.6	−1.039	−0.01319	6.298	−0.6	−0.6	−0.01150	7.344
−0.5	−0.8660	−0.01337	6.220	−0.5	−0.5	−0.01207	6.989
−0.4	−0.6928	−0.01364	6.109	−0.4	−0.4	−0.01271	6.623
−0.3	−0.5196	−0.01405	5.940	−0.3	−0.3	−0.01345	6.242
−0.2	−0.3464	−0.01463	5.700	−0.2	−0.2	−0.01433	5.841
−0.1	−0.1732	−0.01547	5.377	−0.1	−0.1	−0.01538	5.415
0.	0.	−0.01670	4.954	0.	0.	−0.01670	4.954
+0.1	+0.1732	−0.01859	4.391	+0.1	+0.1	−0.01842	4.439
+0.2	+0.3464	−0.02203	3.580	+0.2	+0.2	−0.02091	3.833
+0.3	+0.5196	−0.05146	1.344	+0.3	+0.3	−0.02529	3.005
+0.4	+0.6928	−0.1790	0.7752	+0.4	+0.4	−0.04930	1.538
				+0.5	+0.5	−0.1048	1.194
				+0.6	+0.6	−0.1474	0.9792

If the orbital dynamics were well described by the 2PN-level orbital EOB metric (i.e. if ω_s had been near -9; see upper curve in Fig. 1), the binding energy, even in such extreme cases, would differ only moderately from the non-spinning case, and we could trust the EOB-plus-spin predictions. However, as ω_s is zero the 3PN EOB + spin predictions become, for $\cos\theta_{\rm LS} \simeq 1$ and $a \gtrsim 0.4$, very different from the 2PN ones and quite sensitive to the numerical values of the expansion coefficients entering the EOB potentials. Let us note, however, that in all cases (even the most extremely spinning ones) if the spin parameter $\hat{a}_p$ is *negative* (i.e., if the effective spin vector, whatever its magnitude may be has a negative projection on the orbital angular momentum) the EOB predictions become extremely reliable because all the differences between the cases (i), (ii), (iii) become quite small.

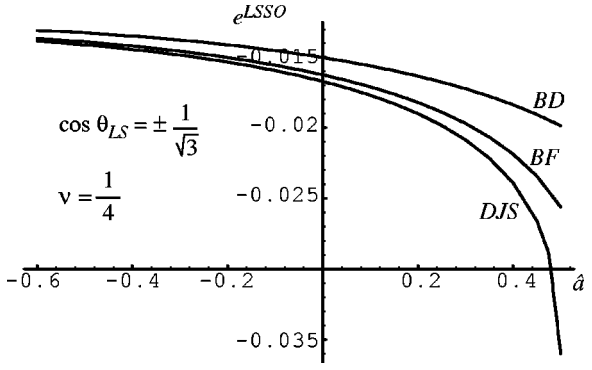

FIG. 1. Dependence of the binding energy $e \equiv (\mathcal{E}_{\rm real}/M) - 1$ of the LSSO on the effective spin parameter $\hat{a} \equiv S_{\rm eff}/M^2$ (taken with the sign of $\cos\theta_{\rm LS}$). We consider an equal-mass system with a typical misalignment angle $\cos\theta_{\rm LS} = \pm 1/\sqrt{3}$. All three curves used the Padé-resummed effective-one-body approach. The lower curves use a 3PN-level Hamiltonian: the lowest one uses $\omega_s = 0 = \omega_s^{\rm DJS}$ [10], while the middle one uses $\omega_s = -2.365 = \omega_s^{\rm BF}$ [17]. The upper curve uses $\omega_s = -9.344 = \omega_s^*$, which is (essentially) equivalent to using a 2PN-level Hamiltonian [8].

All these results are easy to interpret physically. This can be seen from the basic equations of the EOB approach which simplify so much the description of the physical interactions by representing them as slightly deformed versions of the well-known gravitational physics of test particles in Schwarzschild or Kerr geometries. Indeed, the basic equation of the EOB approach determining the binding of the LSO is Eq. (3.39) which differs from its well-known[11] Kerr limit (i.e., $\nu \to 0$) only by the change

$$A_K(\hat{u}, \hat{a}^2) = 1 - 2\hat{u} + \hat{a}^2\hat{u}^2 \to \overline{A}(\hat{u}, \hat{a}^2) = P_3^1[1 - 2\hat{u} + \hat{a}^2\hat{u}^2 + 2\nu\hat{u}^3 + a_4(\nu)\hat{u}^4]. \tag{3.43}$$

The crucial point (which is, finally, the most important new information obtained by the 2PN and 3PN orbital calculations) is that the 2PN and 3PN additional terms to the radial function $A^{\rm PN}_{a=0}(\hat{u})$ have both *positive* coefficients. This means that, even before the addition of the effect of spin [which

[11] Actually, as far as we know, the Kerr limit of Eq. (3.39) has never been written down before. Usual treatments [55] use the more complicated effective radial potential $W'_a(r, L_z, \mathcal{Q})$. Anyway, the physics is the same, but it is more cleanly presented in Eq. (3.39).

leads to a $+\hat{a}^2\hat{u}^2$ additional term in $A_K(\hat{u})$, corresponding to the famous $+a^2$ term in $\Delta_K(r)=r^2-2Mr+a^2$] the main effect of non-linear orbital interactions for comparable masses is *"repulsive"*, i.e., it corresponds to a partial screening of the basic Schwarzschild attractive term $1-2\hat{u}=1-2GM/c^2r$ by the addition of repulsive terms $\propto+\nu/r^3$ and $+\nu/r^4$. Now, paradoxically, the addition of a repulsive term leads to a more tightly bound LSSO because the *less attractive*, but still attractive,[12] radial function $A(\hat{u},\hat{a}^2=0)$ will be able to "hold" a particle in spherical orbit down to a *lower* orbit. In other words, when a radial potential becomes less attractive, its LSSO gets closer to the horizon, and the binding energy of the LSSO becomes more negative. This being said, one understands immediately the additional effects due to the spin interaction. There are basically two such effects: (a) a linear "spin-orbit" effect linked to the $+\hat{a}_p\overline{l}\hat{u}^2$ term in Eq. (3.39) (with $\hat{a}_p\equiv\hat{a}\cos\theta_{\rm LS}$), and (b) a non-linear spin-quadratic modification of the metric coefficient, i.e., the additional $+\hat{a}^2\hat{u}^2$ term in $\overline{A}(\hat{u},\hat{a}^2)$ [or in $A_K(\hat{u})=1-2\hat{u}+\hat{a}^2\hat{u}^2$]. The crucial points are that (1) when $\hat{a}_p<0$, i.e., $\cos\theta_{\rm LS}<0$ (coarse antialignment of angular momenta) the dominant linear spin-orbit coupling is *attractive* and therefore pushes the LSSO *upwards*, towards a less bound orbit, while (2) when $\hat{a}_p>0$, i.e., $\cos\theta_{\rm LS}>0$ (coarse alignment of angular momenta) *both* the linear spin-orbit coupling $+\hat{a}_p\overline{l}\hat{u}^2$ and the spin-quadratic additional term $+\hat{a}^2\hat{u}^2$ are *repulsive* and tend to draw the LSSO *downwards*, i.e., closer to the horizon, in a more bound orbit. Therefore we see that, when $\hat{a}_p>0$, all the new effects (the ν-dependent non-linear orbital interactions and the spin effects) tend in the same direction: towards a closer, more bound orbit. As the existence of a LSSO is due to a delicate balance between the attractive gravitational effects and the usual repulsive ("centrifugal") effect of the orbital angular momentum [i.e., the term $+\overline{l}^2\hat{u}^2\propto+L^2/r^2$ in Eq. (3.39)], when several attractive effects combine their action, they start having a large effect on the binding of the LSSO. This is well known to be the case for circular, equatorial, corotating ($\hat{a}_p=+\hat{a}$) orbits of a test particle in Kerr, which feature, in the case of an extreme Kerr ($\hat{a}=1$) an LSO at $\hat{r}=1$, with μ-fractional binding $(E_{\rm eff}-\mu)/\mu=1/\sqrt{3}-1=-0.42265$ [corresponding to $e\simeq\nu(E_{\rm eff}-\mu)/\mu\simeq-0.10566\nu_4$]. It is also well known that, again for extreme Kerr, a counterrotating ($\hat{a}_p=-\hat{a}$) circular, equatorial orbit in extreme Kerr has a LSO at $\hat{r}=9$, with μ-fractional binding $(E_{\rm eff}-\mu)/\mu=5/(3\sqrt{3})-1=-0.037\,750$ (corresponding to $e\simeq-0.009\,437\,4\,\nu_4$). What is less well known is that the extreme binding of the circular, equatorial, corotating LSO around an extreme Kerr is not representative of the binding of typical LSSO's around typical (or even extreme) Kerr holes. Indeed, when $\cos\theta_{\rm LS}\neq\pm1$ (i.e., in more invariant language, when $\mathcal{Q}\neq0$) and when $\hat{a}<1$, the LSSO is, in general, moderately perturbed away from the Schwarzschild value $\hat{r}_{\rm LSO}=6$ and its binding is correspondingly moderately different from its Schwarzschild limit $e^{\rm LSO}_{\rm Schw}\simeq-0.014\,298\nu_4$. The present work has shown that the location of the LSSO's for binary spinning holes can be rather simply obtained, in the EOB approach, by balancing in the specific way of Eq. (3.39) the centrifugal effect of the orbital angular momentum against the overall attractive effect of gravity, but with the critical addition of the 2PN and 3PN repulsive terms, of the spin-quadratic repulsive term, and of the indefinite-spin effect of the spin-orbit interaction.

D. Expected spin of the hole formed by the coalescence of two spinning holes

The last topic we wish to discuss concerns the expected result of the coalescence of two holes. In particular, we are interested in estimating the maximal spin that the final hole, resulting from the coalescence of two spinning holes, might have. It was estimated in [9,11] (by using the EOB approach) that the coalescence of two non-spinning holes of the same mass $m_1=m_2=M/2$ leads (after taking into account the effect of gravitational radiation on the orbital evolution and on the loss of energy and angular momentum) to the formation of a rotating black hole of mass $M_{\rm BH}\simeq(1-\varepsilon_{\rm rd})0.976M$ and spin parameter $\hat{a}_{\rm BH}\simeq0.80$ (we have included a factor $1-\varepsilon_{\rm rd}$ in $M_{\rm BH}$ to take into account the energy loss during the ring-down. Ref. [11] found $\varepsilon_{\rm rd}\simeq0.7\%$). The fractional energy $0.976-1=-0.024$ roughly corresponds to the (adiabatically estimated) LSO binding energy (-0.015 in the 2PN-based estimate of [8]) minus the energy per unit mass radiated during the plunge (~-0.007 [11]). We shall leave to future work a similar estimate, for the 3PN-plus-spin case, of the amount of energy emitted in GW. We wish here to focus on the issue of the spin of the final hole. The above value $\hat{a}_{\rm BH}\simeq0.80$ is rather close to the maximal value $\hat{a}_{\rm max}=1$ and there arises the question of whether an EOB treatment of the coalescence of two spinning holes might not formally predict a final value of $\hat{a}_{\rm BH}$ larger than one. By "EOB treatment" we mean here a completed version of the EOB approach (as in [9] at the 2PN, non-spinning level) obtained by (i) adding a resummed radiation force to the "conservative" EOB dynamics, and (ii) pushing the calculation of the EOB evolution down to its point of unreliability (near the last *unstable* orbit) where it is matched to a perturbed-single-black-hole description. A zeroth approximation to this completed EOB approach is the one we study in this paper: an adiabatic sequence of solutions of the conservative dynamics, terminated at the LSO. In this approach one entirely neglects the losses of energy and angular momentum during the plunge phase following the crossing of the LSO. The numbers recalled above show that the energy loss during the plunge (and the ring-down) is not negligible compared to the binding energy at the LSO. However, for the present question this is not a problem. What is important is that the

[12]Remember that we Padé resum $A(\hat{u})$ and $\overline{A}(\hat{u})$ to ensure that these functions qualitatively behave like $1-2\hat{u}$ or $1-2\hat{u}+\hat{a}^2\hat{u}^2$ (for $\hat{a}^2<1$), i.e., (generically) have a simple zero near $\hat{r}=\hat{u}^{-1}=1+\sqrt{1-\hat{a}^2}$, which means that the effective metric becomes "infinitely attractive" at some deformed horizon.

angular momentum loss during the plunge is a very small fraction (a percent or so) of the angular momentum at the LSO, and that the final mass of the black hole is nearly equal to $M=m_1+m_2$. This leads us to the following zeroth order estimate of the spin parameter of the final hole:

$$\hat{a}_{\rm BH}\simeq \frac{|\boldsymbol{J}|^{\rm LSSO}}{(E^{\rm LSSO}_{\rm real})^2}\simeq \frac{|\boldsymbol{J}|^{\rm LSSO}}{M^2}. \tag{3.44}$$

In view of the exact conservation of $\boldsymbol{J}$ in our conservative EOB (real) dynamics, it is clear that it is $|\boldsymbol{J}|^{\rm LSSO}$ which is a good measure of the total angular momentum of the final spacetime, i.e. of the final black hole.

We are facing here a potential consistency problem of this simple-minded EOB treatment: when computing Eq. (3.44) for spinning configurations does one always get $\hat{a}_{\rm BH}<1$? One might worry that, starting with a value of $\hat{a}_{\rm BH}\simeq 0.80$ for non-spinning holes, the addition of large spins on the holes might quickly exceed the extremal limit. It is plausible that the most dangerous situation is the ''aligned case,'' where all the angular momenta, $\boldsymbol{L}$, $\boldsymbol{S}_1$, and $\boldsymbol{S}_2$ are parallel (or antiparallel). In this case the numerator of Eq. (3.44) reads

$$J^{\rm LSSO}=L_z^{\rm LSSO}+S_1+S_2, \tag{3.45}$$

while the spin parameter of the effective metric reads

$$\hat{a}=\hat{a}_p=\frac{\boldsymbol{k}\cdot\boldsymbol{S}_{\rm eff}}{M^2}=\left(X_1^2+\frac{3}{4}\nu\right)\hat{a}_1+\left(X_2^2+\frac{3}{4}\nu\right)\hat{a}_2. \tag{3.46}$$

Here, we consider S_1, S_2, and $\hat{a}_1\equiv S_1/m_1^2$, $\hat{a}_2\equiv S_2/m_2^2$ as algebraic numbers (positive or negative). This allows us to investigate also the case where the spins might be antiparallel to $\boldsymbol{k}$. For simplicity, we shall only study the symmetric case where $m_1=m_2$ and $S_1=S_2$. For this case

$$\hat{a}=\hat{a}_p=\frac{7}{8}\hat{a}_1, \tag{3.47}$$

and

$$\hat{a}_{\rm BH}=\frac{J^{\rm LSSO}}{M^2}=\frac{1}{4}\hat{L}_z^{\rm LSSO}+\frac{1}{2}\hat{a}_1=\frac{1}{4}\hat{L}_z^{\rm LSSO}+\frac{4}{7}\hat{a}_p \tag{3.48}$$

where the dimensionless orbital angular momentum $\hat{L}_z\equiv L_z/\mu M$ is related to the dimensionless quantity (when $\cos\theta_{\rm LS}=1$) $\bar{l}\equiv\sqrt{\bar{\mathcal{K}}}/\mu M=\bar{L}_z/\mu M$ appearing in Eq. (3.39) through

$$\hat{L}_z=\bar{l}+\hat{a}\frac{E_{\rm eff}}{\mu}. \tag{3.49}$$

It is interesting to note that, even in the case where both holes are extreme ($\hat{a}_1=\hat{a}_2=1$) the maximum value of the effective spin parameter is $\hat{a}_{\rm max}=\frac{7}{8}<1$. We have numerically investigated the quantity $\hat{a}_{\rm BH}$, Eq. (3.48), as a function of the effective $\hat{a}=\hat{a}_p$. The result is plotted in Fig. 2 for different values of the 3PN parameter ω_s.

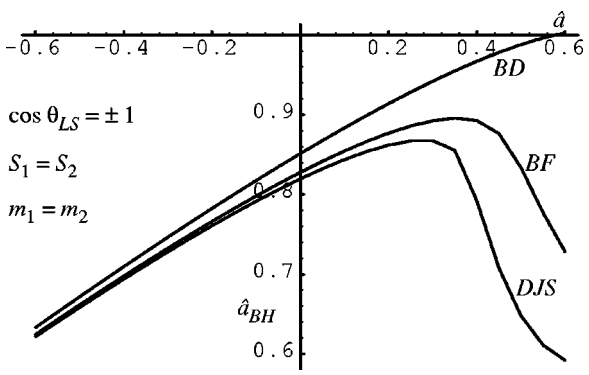

FIG. 2. Approximate prediction for the spin parameter $\hat{a}_{\rm BH}\simeq|\boldsymbol{J}|^{\rm LSSO}/M^2$ of the black hole formed by the coalescence of two identical spinning holes (with spins parallel or antiparallel to the orbital angular momentum). The horizontal axis is the effective spin parameter $\hat{a}=\frac{7}{8}\hat{a}_1=\frac{7}{8}\hat{a}_2$. The three curves correspond to the three cases plotted in Fig. 1. Note the prediction (robust under changing the 3PN contribution to the effective potential by 25%) that the final spin parameter is always subextremal, and reaches a maximum $\hat{a}_{\rm BH}\simeq 0.87$ for $\hat{a}\simeq+0.3$.

We see that the final spin parameter reaches a maximum for a positive value of $\hat{a}_p$, i.e., for parallel (rather than antiparallel) spins. For the correct 3PN value $\omega_s=0$ the maximum value of $\hat{a}_{\rm BH}$ is comfortably below 1: namely, $\hat{a}_{\rm BH}^{\rm max}\simeq 0.87$, reached for $\hat{a}_p\simeq+0.3$. This is not much larger than the value $\hat{a}_{\rm BH}\simeq 0.82$ obtained for $\hat{a}_p=0$. We find that this is a nice sign of the consistency of the EOB approach. This consistency was not *a priori* evident. In fact for $\omega_s\leqslant -9$ one gets a maximum value of $\hat{a}_{\rm BH}$ slightly larger than 1. In particular, note that the 2PN treatment of the orbital dynamics (obtained for $\omega_s=\omega_s^*\simeq -9.3439$; upper curve in Fig. 2) formally leads to problematic over-extreme values of $\hat{a}_{\rm BH}$. This may be interpreted as a confirmation of the need of ''repulsive'' 3PN effects (i.e., $\omega_s+9\gg 1$). It is (*a posteriori*!) easy to understand physically why $\hat{a}_{\rm BH}$, after reaching a maximum, then decreases when one adds more spin on the two black holes. Indeed, there is here a competition between two effects: adding spin on the holes (i.e., increasing $\hat{a}_p$) on the one hand directly contributes to augmenting $\hat{a}_{\rm BH}$ through the second term of the right-hand side (RHS) of Eq. (3.48), but, on the other hand, indirectly contributes to reducing the total $J^{\rm LSSO}$ by reducing $\hat{L}_z^{\rm LSSO}$ (indeed, as we explained above, positive spin leads to a LSO orbit closer to the horizon, and therefore with less orbital angular momentum). The first effect wins for smallish spins, while the second (more nonlinear) effect wins for larger spins.

IV. CONCLUSIONS

We started by recalling the need of techniques for accelerating the convergence of the post-Newtonian (PN) expan-

sions in the last stages of the inspiral of binary systems. We summarized the evidence (Table I) showing the remarkable convergence properties of the best current resummation technique: the effective one-body (EOB) approach of Refs. [8,10]. We showed how to generalize the EOB approach to the case of two spinning black holes with comparable masses ($\nu=\mu/M\sim 1/4$). As a first step towards computing the spin-dependent EOB Hamiltonian we constructed an effective metric, which can be viewed either as a ν-deformation of the Kerr metric or as a spin-deformation of the ν-deformed effective metric. The effective spin entering this deformed Kerr metric is $M\boldsymbol{a}\equiv \boldsymbol{S}_{\rm eff}\equiv[1+\frac{3}{4}(m_2/m_1)]\boldsymbol{S}_1+[1+\frac{3}{4}(m_1/m_2)]\boldsymbol{S}_2$. The introduction of this effective $\boldsymbol{a}$ allows one to combine in a simple manner all (PN leading) spin-orbit coupling effects, and most of the spin-spin ones, with the rather complex but important 3PN effects, which have been incorporated only recently in the EOB approach [10].

We have also constructed a more complicated modified effective Hamiltonian, Eq. (2.56), which separately depends on two (effective) spin vectors, $M\boldsymbol{a}_0\equiv \boldsymbol{S}_0\equiv(1+m_2/m_1)\boldsymbol{S}_1+(1+m_1/m_2)\boldsymbol{S}_2$, and $\boldsymbol{\sigma}\equiv \boldsymbol{S}_{\rm eff}-\boldsymbol{S}_0\equiv -\frac{1}{4}[(m_2/m_1)\boldsymbol{S}_1+(m_1/m_2)\ \boldsymbol{S}_2]$, and which allows a (hopefully) more accurate representation of spin-spin effects. We recommend the simultaneous consideration of $H_{\rm eff}$ and $H'_{\rm eff}$ to determine the domain of trustability of the presently constructed spin-dependent EOB Hamiltonian. Namely, when $H_{\rm real}=\overline{f}(H_{\rm eff})$ and $H'_{\rm real}=\overline{f}(H'_{\rm eff})$ lead to numerically very similar evolutions, one is entitled to trust them both; while a significant difference in their predictions signals a breakdown of the trustability of the simple EOB Hamiltonian proposed here.

The present paper has only investigated a few aspects of the physics predicted by our spin-generalized EOB approach. In particular, as a first cut toward understanding the relevance of our construction for gravitational wave (GW) observations we have discussed the approximate existence of ''spherical orbits'' (orbits with fixed radial coordinate, as in the Kerr metric) and we studied the binding energy of the last stable spherical orbits (LSSO). A message of this study is that, for most physically relevant cases (in the parameter space where one randomly varies all angles and all spin values), the results are only weakly dependent on the exact numerical values of the 3PN coefficients. Moreover, they exhibit moderate deviations from the non-spinning case (see Fig. 1 and Table II). To give a numerical flavor of the effects of spin we note that, when the projected spin parameter $\hat{a}_p=\boldsymbol{k}\cdot\boldsymbol{a}/M$, Eq. (3.41), is smaller than about $+0.2$, its effect on the fractional binding energy $[e\equiv(E_{\rm real}-M)/M]$ of the LSSO is, approximately,

$$100e^{\rm LSSO}\simeq -1.43\nu_4(1+0.168\nu_4) -0.806\nu_4(1+0.888\nu_4)\hat{a}_p, \qquad (4.1)$$

where $\nu_4\equiv 4\nu\equiv 4m_1m_2/(m_1+m_2)^2\leqslant 1$. As in most cases (random angles, random spin-kinetic energies) it is plausible that $|\hat{a}_p|\lesssim a_p^{\rm rms}\sim 0.25$, we expect that spin effects will only modify the energy emitted as gravitational waves up to the LSSO by less than about $0.6\%\,\nu_4 M$.

Such an increase, though modest, is still a significant fractional modification of the corresponding energy loss predicted for non-spinning systems ($e_0=-1.67\%M$ for $\nu_4=1$). In fact, this effect might cause an important *bias* in the first observations. If the intrinsic spins of the holes can (at all) take large values, the highest signal-to-noise-ratio events in the first years of LIGO observations might select binary systems with rather large and rather aligned spins. It is therefore important to include spin effects in the data analysis of coalescing black holes. We have argued that, in most cases, the simple-minded generalized EOB approach presented here should be a reliable analytical tool for describing the dynamics of two spinning holes and for computing a catalogue of gravitational waveforms, to be used as matched filters in the detection of GW's. However, it must be admitted that, in the cases where the effective spin vector is coarsely (positively) aligned with the orbital angular momentum, and where the spins are so large that $\hat{a}\gtrsim 0.4$, the predictions from the above-introduced EOB Hamiltonian start predicting LSSO radii so near the ''effective horizon'' where $\Delta_t(r)=0$, that they cannot be quantitatively relied upon (though I would still argue that they can be qualitatively trusted, in view of the simple physics they use; see Sec. III C.] We give some examples of that in Table II. In such cases the EOB approach does predict much larger energy losses, possibly larger than $10\%M$. In these cases, the uncertainty in the waveform may be so large that one may need the type of non-linear filtering search algorithm advocated in Ref. [2]. We wish, however, to emphasize the differences between our treatment and conclusions and those of Flanagan and Hughes. These authors defined the ''merger'' phase as (essentially) what comes after the binary system crosses the non-spinning LSO (around 6GM), and they assumed that the signal from the ''merger'' phase can only be obtained from numerical relativity. Moreover, they optimistically assumed that (in all cases) $10\%M$ are emitted in GW energy during the merger phase, and 3% during the subsequent ring-down phase. By contrast, our treatment is based on the idea that a suitable resummed version of the PN-expanded dynamics, namely the EOB-plus-Padé approach, can, in most cases, give an analytical handle on the computation of the inspiral signal down to the spin-modified LSSO (and even during the subsequent plunge, as discussed for the spinless case in [9]). We have argued in several ways that the simple EOB Hamiltonian (2.45) gives reliable answers in most cases, and allows one to analytically control the possible amplification (or deamplification, when $\hat{a}_p<0$) in GW energy loss due to spin effects. Moreover, it is only in rather extreme cases that we could agree with [2] in predicting $\gtrsim 10\%M$ energy losses. In most other cases, we think that the EOB method provides a reliable basis for computing families of waveforms that will be useful templates for the detection of GW's. Another difference with [2] is that we have argued, on the basis of definite computations, that the spin of the final hole will never become nearly extremal (even if the initial spins are extremal). This is important for the data analysis of the ring-down signal, because the decay time of the least damped quasi-normal-mode starts becoming large only for near extremal holes.

Let us emphasize that the present work is only a first step toward an improved analytical understanding of the last stages of inspiral motion of two spinning compact object.[13] The explicit spin-dependent Hamiltonian (2.45) (or, better, $H'_{\rm real}$ defined at the end of Sec. II) has only taken into account the leading effects (in a PN expansion) of the spin-dependent interactions. More work is needed to analytically determine more accurate versions of the EOB Hamiltonian. In particular, it would be interesting to explicitly derive the next-to-leading (1PN) corrections to the EOB spin-orbit (and spin-spin) interactions. Furthermore, we have only provided a resummation of the conservative part of the dynamics. There remains the important complementary task of resumming the radiation reaction part. This was done in [9], using previous results of [7], only for the spinless case.

Once this is done, we expect, as in our previous study [9], that the presence of a LSSO along the sequence of adiabatic orbits will be blurred and will be replaced by a continuous transition between inspiral and plunge. There remains also the task of studying the effects of spin-dependent interactions on the gravitational waveform emitted during the last stages of inspiral and during the plunge (that we have not considered here). In other words, one needs to redo, by combining the EOB approach with resummed versions of radiation reaction, the studies, valid far from the LSO, which were based on straightforward PN-expanded results [56,57]. Note that our result above about the primary importance of the single parameter $\hat{a}_p$, combined with the understanding [5] that the number of ''useful'' cycles in the GW signal for massive binaries is rather small, suggests that a rather small number of ''spinning templates'' will be really needed in a matched filter data analysis. On the other hand, we recall that it was found in [5] that the plunge signal (but not the ring-down one, for stellar mass holes) plays a significant role in the data analysis.

It will also be interesting to see, within the EOB approach, the extent to which the non-linear spin-dependent interactions might, as has been recently suggested [58], lead to a chaotic dynamical evolution. We *a priori* suspect that two factors will diminish the significance of such chaotic evolutions: (1) they occur only in an improbably small region of phase space (involving, in particular, large spins), and (2) their effect on the crucial GW phasing is rather small.

It would be very useful to have independent means of testing the accuracy of the EOB approach. At this stage we see only three ways of doing that (beyond the performance of more internal checks of the robustness of the approach): (i) an analytical calculation of the 4PN interaction Hamiltonian, (ii) a comparison between numerical computations and the EOB results, and/or (iii) a comparison between the EOB predictions and the forthcoming GW observations. (i) would be important for assessing the convergence of the PN-resummed EOB Hamiltonian. In view of the extreme difficulties involved in the 3PN calculations [12–18] it would seem hopeless to even mention the 4PN level. But, in fact, the EOB approach itself suggests that the current methods used in PN calculations are highly inefficient, and unnecessarily complicated. Indeed, as emphasized in [10] the final, gauge-invariant content of the 3PN result is contained in only three quantities a_4, b_3, and z_3, and only one of them, $a_4(\nu)$, is really important for determining the dynamics of inspiralling quasi-circular binaries. If one could invent a new approximation scheme which computes directly a_4 (at 3PN), it might be possible to compute its 4PN counterpart, $a_5(\nu)$. (ii) is not yet possible (at least as a test of the EOB Hamiltonian) because numerical computations use as initial data geometrical configurations that do not take into account most of the crucial physics incorporated in PN calculations. Current numerical computations use somewhat *ad hoc* ''binary-black-hole-like'' data, often of the restricted spatially conformally flat type, without trying to match their initial data to the near LSO configurations predicted by (resummed) analytical approaches. On the other hand, let us stress that the value of the radial PN-expanded potential $A(\hat{u})=1-2\hat{u}+a_3(\nu)\hat{u}^3+a_4(\nu)\hat{u}^4+\cdots$ crucially depends on the non-linear gravitational interactions linked to the h^{TT}_{ij} part of the spatial metric, i.e. to its nonconformally flat part, and also to the non-linear interactions linked to the π^{ij}_{TT} part of the gravitational field momenta. For instance, already at the 2PN level, the truncation of the Einsteinian prediction for the two-body problem (driven into a close orbit by a long past interaction involving retarded GW interactions) corresponding to artificially assuming a conformally flat spatial metric changes the physically correct value $a_3^{\rm 2PN}(\nu)=2\nu$ into $a_3^{\rm conf.\ flat}(\nu)=\frac{1}{4}(18-5\nu)\nu$ [10]. [It also slightly changes the energy map f.] For equal-mass systems, this corresponds to multiplying the positive $a_3(\nu)$ by a factor $+2.09375$. As we discussed above, this (artificial) increase of the ''repulsive'' character of the non-linear gravitational interactions tends to artificially increase the binding of the LSO. As the 2PN coefficient is anyway too small to have a large impact on the LSO characteristics, this 2PN change does not, by itself, change much the LSO energy [10]. However, if this tendency to increase the ''repulsive'' character of the PN expansion (caused by the neglect of the h^{TT}_{ij}-, and π^{ij}_{TT}-dependent interactions) persists at the (numerically more important) 3PN level, this might explain the current discrepancy between analytical and numerical estimates of LSO characteristics. In fact, we note that the initial data taken by a recent attempt [59] at fulfilling the proposal of [8] to start a full numerical calculation only at the moment where it is really needed, i.e., after crossing the LSO, uses LSO initial data [60] with a binding energy $e_{\rm LSO}=E/(m_1+m_2)-1\simeq-2.3\%$ which is 38% larger than the value $e^{\rm 3PN}_{\rm LSO}\simeq-1.67\%$ obtained at 3PN (with $\omega_s=0$) by analytical estimates. Similarly, the LSO orbital period of the initial data of [59] is $T_{\rm LSO}\simeq 35(m_1+m_2)$ [60], which is twice smaller than the 3PN estimate $T^{\rm 3PN}_{\rm LSO}\simeq 71.2(m_1+m_2)$ [10]. These discrepancies between state-of-the-art numerical LSO initial data and state-of-the-art analytical estimates of LSO data are significantly larger than the

[13]Though, in most of the paper we only spoke of binary black holes, it should be clarified that our EOB Hamiltonian also applies to binary spinning neutron stars or to spinning neutron-star–black-hole systems, at least down to the stage where the quadrupole deformation of the neutron star becomes significant.

natural ''theoretical error bar'' on the (resummed) analytical estimates (derived, say, by comparing 2PN estimates to 3PN ones). (See, however, the new numerical approach of [61] whose LSO data agree well with the 3PN EOB estimates [62].) In our opinion, this makes it urgent for the numerical relativity community to develop ways of constructing initial data that correctly incorporate the crucial non-linear physics (linked to the h_{ij}^{TT} and π_{TT}^{ij} parts of the metric) which is taken into account in PN calculations. If a significant discrepancy remains after this is done, one will be entitled to blame the lack of convergence of the EOB-resummed PN calculations. If one finds agreement, this will be a confirmation of the claim made here that the Padé-improved EOB is a reliable description of the last orbits before coalescence. Once one succeeds in matching analytical and numerical results for non-spinning black holes, it will be very interesting to use numerical data on fast-spinning black holes to refine the EOB Hamiltonian by fitting the values of the extra parameters which can be introduced in the EOB Hamiltonian to represent higher PN effects. In the long term we expect that such a complementarity between numerical and analytical tools will be needed for defining a sufficiently dense set of GW templates. (In view of the large dimensionality of the parameter space of the two spinning hole system, it seems hopeless to use only numerical techniques to define a dense network of templates.)

Finally, even if no decisive progress is made on (i) or (ii) before the first sources are detected, there remains the possibility that the first observations might confirm the soundness of (or suggest specific modifications of) the EOB-based waveforms, and thereby facilitate further detections by narrowing the bank of templates. For instance, one might include a 4PN contribution $+a_5(\nu)u^5$ to $A(u)$, as a free parameter in constructing a bank of templates, and wait until LIGO-VIRGO-GEO get high signal-to-noise-ratio observations of massive coalescing binaries to determine its numerical value.

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