Inflationary perturbations from a potential with a step

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We use a numerical code to compute the density perturbations generated during an inflationary epoch which includes a spontaneous symmetry breaking phase transition. A sharp step in the inflaton potential generates k dependent oscillations in the spectrum of primordial density perturbations. The amplitude and extent in wave number of these oscillations depends on both the magnitude and gradient of the step in the inflaton potential. We show that observations of the cosmic microwave background anisotropy place strong constraints on the step parameters.

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I. INTRODUCTION

A period of inflation in the primordial universe provides a causal explanation for the existence of the large scale structure observed during the present epoch (reviewed in Ref. [1]). The simplest and most natural form of the scalar density perturbation spectrum is the scale invariant case, or $\mathcal{P}_{\mathcal{R}} \propto k^{n-1}$ with $n=1$, where $\mathcal{R}$ is the curvature perturbation.

In principle, inflationary models driven by a continuously evolving scalar field have a scale dependent spectral index, which can be calculated using the "slow roll" approximation [2]. This expresses n as a function of the inflaton potential and its derivatives at the instant a mode leaves the horizon during inflation. The inflaton evolves slowly, so only a small piece of the potential is "sampled" by the large scale structure in the present universe, ensuring that, if the underlying potential is smooth, n is not strongly scale dependent.

Potentials with a "feature" at the value of the inflaton, when perturbations corresponding to astrophysical scales in the present universe left the horizon, can produce a primordial perturbation spectrum with significant scale dependence. However, the inflaton moves slowly, and fine tuning is needed to put the feature in exactly the right part of the potential. Thus, while it is possible to construct inflationary models with a scale dependent spectrum, these models are often somewhat contrived.

However, arguing that using a feature in the inflaton potential to generate a complicated spectrum requires fine tuning assumes that the potential has just one feature, but is otherwise smooth. Adding a large number of features to the potential makes it far more likely that a randomly chosen piece of the perturbation spectrum will exhibit a considerable scale dependence. In particular, Adams *et al.* [3] showed that a class of models derived from supergravity theories naturally gives rise to inflaton potentials having a large number of sudden (downward) steps. Each step corresponds to a symmetry breaking phase transition in a field coupled to the inflaton, since the mass changes suddenly when each transition occurs. In the scenario studied by Adams *et al.*, a spectral feature is expected every 10–15 e folds, so if this model drove inflation it is likely that one of these features would be visible in the spectrum extracted from observations of large scale structure (LSS) and the cosmic microwave background (CMB).

Motivated by the existence of models which naturally and generically lead to scale dependent spectra, this paper carefully examines the consequences of introducing a step in the inflaton potential. We focus on spectral features which may be observable in the large-scale structure or cosmic microwave background anisotropy, and therefore had their origin around 50 e folds before the end of inflation.

We model the step by assuming the potential

$$V(\phi)=\frac{1}{2}m^2\phi^2\left[1+c\tanh\left(\frac{\phi-\phi_{\text{step}}}{d}\right)\right] \tag{1}$$

for the inflaton field ϕ. This potential has a step at $\phi=\phi_{\text{step}}$ with a size and gradient governed by c and d respectively. For physically realistic models, inflation is not interrupted, but the effect on the density perturbations is still significant. If inflation is actually interrupted the effect on the perturbation spectrum is severe enough to rule out models where this occurs during the interval of inflation corresponding to observable scales. In order to evaluate the spectrum accurately, we find that we must evolve the evolution equations numerically, rather than relying on the slow roll approximation.

Inflationary models with scale dependent spectral indices were examined in several previous investigations [4–9]. In particular, two recent papers, the first by Leach and Liddle [8] and the second by Leach *et al.* [9], relied, as we do, on numerical evaluations of the mode equation to compute the density perturbation spectrum. Our analysis focuses on small features in the potential. Conversely, Refs. [8] and [9] examined potentials which produce very abrupt changes in the inflationary dynamics, including the temporary cessation of inflationary expansion, so the spectra discussed in Refs. [8] and [9] are changed for all values of k larger than some critical value. In contrast, the spectra we consider here are essentially unchanged from their form at small k once the oscillations have died away. Moreover, most of the models discussed in Refs. [8] and [9] would need to be carefully tuned in order to produce observable effects in the spectrum,[1] whereas the mechanism described by Adams *et al.* can alter the observable spectrum without fine tuning.

II. FORMALISM

In this section we reproduce important equations governing the evolution of scalar curvature perturbations and gravitational waves during inflation. We use the formalism developed by Stewart and Lyth [10] where the quantities of interest are the curvature perturbation $\mathcal{R}$ and tensor perturbation ψ.

In the scalar case it is advantageous to define a gauge invariant potential

$$u=-z\mathcal{R}, \tag{2}$$

where $z\equiv a\dot{\phi}/H$. We use the standard notation, where a denotes the scale factor, H the Hubble parameter, ϕ the inflaton field, and a dot the derivative with respect to time t.

The equation of motion for the Fourier components, u_k, is

$$u_k''+\left(k^2-\frac{z''}{z}\right)u_k=0, \tag{3}$$

where the prime denotes differentiation with respect to conformal time and k is the modulus of the wave number [10–12]. The form of the solution depends on the relative sizes of k^2 and z''/z. In the limit $k^2\gg z''/z$, u_k tends to the free field solution

$$u_k\rightarrow\frac{1}{\sqrt{2k}}e^{-ik\tau}, \tag{4}$$

where the normalization is determined by the quantum origin of the perturbations (see Ref. [13] for a more detailed discussion). Conversely, in the limit $k^2\ll z''/z$ the growing mode is

$$u_k\propto z, \tag{5}$$

which means that the curvature perturbation

$$|\mathcal{R}_k|=|u_k/z| \tag{6}$$

is constant in this regime. The z''/z term can be written as $2a^2H^2$ plus terms that are small during slow roll inflation, so that the first regime applies to a mode well inside the horizon with $k\gg aH$, and the second to superhorizon scales when $k\ll aH$.

The spectrum $\mathcal{P}_{\mathcal{R}}(k)$ is defined in the usual way as

$$\langle\mathcal{R}_{k_1}\mathcal{R}^*_{k_2}\rangle=\frac{2\pi^2}{k^3}\mathcal{P}_{\mathcal{R}}\delta^3(k_1-k_2), \tag{7}$$

and is given by

$$\mathcal{P}_{\mathcal{R}}^{1/2}(k)=\sqrt{\frac{k^3}{2\pi^2}}\left|\frac{u_k}{z}\right|. \tag{8}$$

The mode equation for gravitational waves is

$$v_k''+\left(k^2-\frac{a''}{a}\right)v_k=0, \tag{9}$$

where $v_k=a\psi_k$. In slow roll inflation $a''/a\simeq 2a^2H^2$ and the behavior of v_k is again characterized by whether the mode is inside or outside the horizon:

$$v_k\rightarrow\frac{1}{\sqrt{2k}}e^{-ik\tau}\quad\text{as}\quad aH/k\rightarrow 0, \tag{10}$$

$$v_k\propto a\quad\text{for}\quad aH/k\gg 1. \tag{11}$$

The power spectrum of gravitational waves $\mathcal{P}_g(k)$, analogous to Eq. (8), is

$$\mathcal{P}_g^{1/2}(k)=\sqrt{\frac{k^3}{2\pi^2}}\left|\frac{v_k}{a}\right|. \tag{12}$$

III. NUMERICAL SOLUTION

Normally, the perturbation spectra of inflationary models driven by a continuously evolving, minimally coupled scalar field can be calculated using the slow roll approximation. However, when the potential has a sharp feature, its derivatives with respect to ϕ and the time derivatives of the field need not be small. Consequently, we evolve the full mode equation numerically, without any approximations other than those already implicit in the use of perturbation theory.

In Eq. (3), the mode function is expressed in terms of conformal time. The intrinsic time scale of the dynamics is not constant in conformal time, so we shift the independent variable to $\alpha=\log a$, facilitating the numerical integration. With this replacement, the system of equations we are to solve is

$$H_\alpha=-4\pi GH\phi_\alpha^2, \tag{13}$$

$$\phi_{\alpha\alpha}+\left(\frac{H_\alpha}{H}+3\right)\phi_\alpha+\frac{1}{H^2}\frac{dV}{d\phi}=0, \tag{14}$$

$$u_{\alpha\alpha}+\left(\frac{H_\alpha}{H}+1\right)u_\alpha+\left\{\frac{k^2}{e^{2\alpha}H^2}-\left[2-4\frac{H_\alpha}{H}\frac{\phi_{\alpha\alpha}}{\phi_\alpha}-2\left(\frac{H_\alpha}{H}\right)^2-5\frac{H_\alpha}{H}-\frac{1}{H^2}\frac{d^2V}{d\phi^2}\right]\right\}=0, \tag{15}$$

[1]An exception is the enhancement of the perturbations produced just before the end of inflation, which might lead to the formation of primordial black holes [8]. These have observable consequences which are distinct from observations of large scale structure.

where the subscript α denotes differentiation. To compute the spectrum, we repeat the integration for many values of k.

In general, u has two distinct solutions since it is a second order linear differential equation, and we must choose the combination which guarantees that the mode equation has the limiting form of Eq. (4). We impose the initial conditions when the mode is far inside the horizon assuming that the conformal time τ is zero, which amounts to an irrelevant choice of phase. Consequently,

$$u|_{\tau=0}=\frac{1}{\sqrt{2k}}, \tag{16}$$

$$\left.\frac{du}{d\alpha}\right|_{\tau=0}=-i\sqrt{\frac{k}{2}}\left.\frac{1}{e^{\alpha}H}\right|_{\tau=0}. \tag{17}$$

Rather than work with complex coefficents in the numerical code, we define two orthogonal solutions u_k^1 and u_k^2, such that

$$u_k^1|_{\tau=0}=1, \tag{18}$$

$$\left.\frac{du_k^1}{d\alpha}\right|_{\tau=0}=0, \tag{19}$$

$$u_k^2|_{\tau=0}=0, \tag{20}$$

$$\left.\frac{du_k^2}{d\alpha}\right|_{\tau=0}=1. \tag{21}$$

At any subsequent time, u_k is thus

$$u_k=\frac{1}{\sqrt{2k}}u_k^1-i\sqrt{\frac{k}{2}}\left.\frac{1}{e^{\alpha}H}\right|_{\tau=0}u_k^2. \tag{22}$$

We start the evolution by evolving the two background equations until any initial transient solution has died away but the mode is still well inside the horizon. We then identify the two orthogonal solutions that contribute to u_k, and extract the coefficients in Eq. (22). This ensures that an initial transient contribution to the background dynamics cannot contaminate the initial values of u and u_α. Finally, to compute the spectrum, we need the asymptotic value of $|u/z|$, and we find this by continuing the integration until the mode is far outside the horizon and this value is effectively constant. The numerical integrations are carried out using the Bulirsch-Stoer algorithm [14], and we check our calculations by ensuring that the results are independent of the distance inside the horizon where we apply the normalization, and the distance beyond the horizon where we evaluate the asymptotic value of $|u/z|$.

IV. INFLATIONARY POTENTIAL WITH A STEP

Figure 1 shows the power spectrum for the potential of Eq. (1) with $c=0.002$, or a 0.4% change in the amplitude of the potential. The most striking aspect of the scalar spectrum is the scale dependent oscillations. Even with this small change in the amplitude of the inflaton potential the oscillations last for two decades of k and, at their peak, change the amplitude of the spectrum by a factor of 3. We have set the position of the step so that the scale where the oscillations begin, k_{low}, is probed by observations of the galaxy correlation function and the anisotropy in the cosmic microwave background. Before we look at the origin of the oscillations in the scalar spectrum it will be helpful to have a picture of how inflation proceeds when there is a step in the potential. A general, qualitative analysis of the spectrum produced by a ''feature'' in the potential is given by Starobinsky [15]. For the specific model we are considering here, we can understand the numerical results as follows. Energy conservation requires that the change in the inflaton kinetic energy term cannot exceed the change in the potential energy so, if we are originally well inside the vacuum-dominated regime, a small change in the amplitude of the inflaton potential cannot suspend inflation. The evolution of $\ddot{a}$ in Fig. 2 clearly shows that the expansion is always accelerating. However the z''/z term, also shown in Fig. 2, determines the growth of the scalar perturbations and is very different from $2a^2H^2$. It first grows

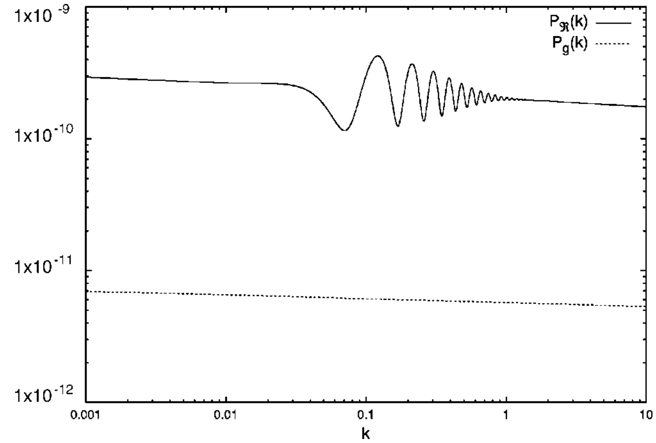

FIG. 1. The scalar and tensor power spectrum for $c=0.002$ and $d=0.01$. The z''/z term for these parameters is shown in Fig. 2.

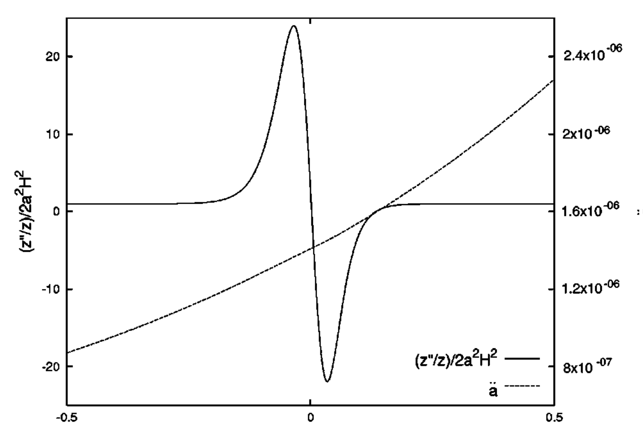

FIG. 2. Evolution of z''/z and $\ddot{a}$ for $c=0.002$ and $d=0.01$ with the number of e folds of inflation, N. We have set $N=0$ at the step in the potential.

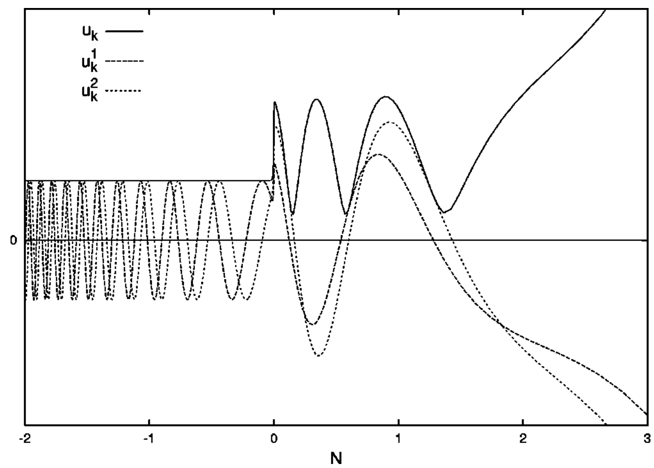

FIG. 3. Evolution of the independent modes u_k^1 and u_k^2 [with initial conditions for u_k^1 and u_k^2 given in Eqs. (18)–(21)] and the linear combination of their amplitude [Eq. (22)] for $k=0.3$.

in magnitude as the inflaton field accelerates and then drops to a large negative amplitude as the field slows. However, the tensor power spectrum is unaffected since a''/a remains constant throughout the step.

To understand the scalar power spectrum we begin by considering the evolution of a particular scalar mode. The evolution is governed by the competition between the k^2 and z''/z terms. A step in the potential of the magnitude we are interested in only has a lasting effect on k modes within the horizon, and not on modes which are already well outside the horizon. That is, the lowest wave number affected is approximately given by $k_{\mathrm{low}} \sim aH|_{\mathrm{step}}$. Moreover, from the form of the $k^2 - z''/z$ term in the mode equation, we can see that the range of k affected by the step will scale roughly with the square root of the maximum value of z''/z in the region of the step.

In Fig. 3 we show the evolution of u_k^1, u_k^2, and u_k for an intermediate wave number in the range of k affected. The rise in z''/z introduces a brief interlude of growing mode behavior into the oscillatory regime. The subsequent interval where z''/z is negative causes the amplitude of u_k to briefly resume its oscillatory behavior. Finally, when the inflaton field resumes slow rolling, the oscillations leave the horizon with an altered phase and increased amplitude. Both of the two initially independent solutions are affected similarly, as they now have the same phase, and the amplitude of their linear combination oscillates. In other words, the presence of the step introduces a boundary condition which selects a solution with an oscillating envelope in contrast to the unconstrained plane wave solution with constant envelope seen at small k.

As in the case of a featureless inflaton potential, u_k obtains a growing mode solution once it is outside the horizon. However the asymptotic limit reached by the curvature perturbation $|\mathcal{R}_k|$ depends on the oscillation phase of the mode at horizon exit, so that $|\mathcal{R}_k|$ oscillates, with maxima corresponding to the modes which exit at an extremum. The proper time interval between the step and when the mode with wave number k exits is approximately $\Delta\tau \sim 1/aH|_{\mathrm{step}} - 1/aH|_{\mathrm{exit}} = 1/k_{\mathrm{low}} - 1/k$ and in this time the amplitude of

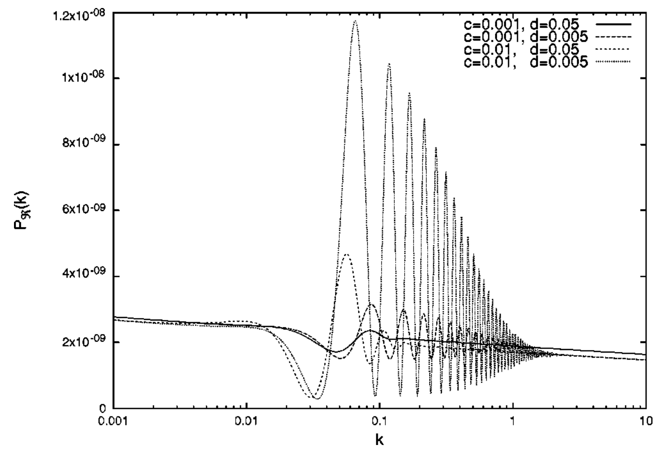

FIG. 4. The effect of changing c and d on the scalar perturbation spectrum.

the mode will have undergone $1/\pi(k/k_{\mathrm{low}} - 1)$ oscillations. Thus the period of the variation in $|\mathcal{R}_k|$ is approximately πk_{low}.

For higher wave numbers the effect of the z''/z term is smaller, the amplitude of their oscillation is not increased and the two modes are not set exactly in phase with each other. However their phases are still altered so that the linear combination u_k oscillates, but with a diminished amplitude compared to the lower k modes.

The magnitude of z''/z depends on both parameters c and d in the potential, and in a well motivated model these will be determined by particle physics. Alternatively, given accurate observations of the CMB and LSS, there may be possible cosmological constraints on the values of these parameters. In Sec. V we examine the observable consequences of a scale dependent primordial spectrum.

V. OBSERVABLE SPECTRA

Adams *et al.* [3] attempted to recover the primordial perturbation spectrum from the APM survey power spectrum using the relationship between the spectrum of mass fluctuations today and the primordial spectrum

$$\mathcal{P}_\delta \equiv \mathcal{P}_\mathcal{R} T^2(k) \left(\frac{k}{H_0}\right)^{3+n}, \tag{23}$$

where $T(k)$ is the matter transfer function that tracks the scale dependent rate of growth of linear perturbations and depends on the dark matter content of the Universe. Assuming a cold dark matter dominated universe the primordial $n(k)$ could be extracted from the three dimensional $P_{\mathrm{APM}}(k)$ inferred from the angular correlation function of galaxies in the APM survey [16]. A departure from scale invariance was found in the range $k \sim (0.05–0.6)h\ \ \mathrm{Mpc}^{-1}$. This feature was noted in Ref. [16] and in the power spectrum of Infrared Astronomy Satellite (IRAS) galaxies [17]. Adams *et al.* used the $n(k)$ they had extracted to predict the photon power spectrum and found that the height of the secondary acoustic peaks was suppressed by a factor of ~ 2.

Recently, a number of groups have revisited this analysis motivated by the recent Maxima and BOOMERanG obser-

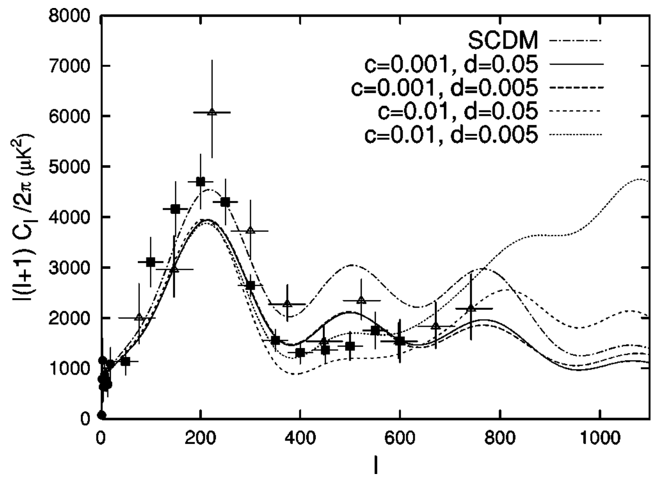

FIG. 5. The CMB angular power spectrum for the primordial spectra shown in Fig. 4. The normalization in each case is to COBE. The data are from COBE (circles), BOOMERanG (squares), and MAXIMA (triangles).

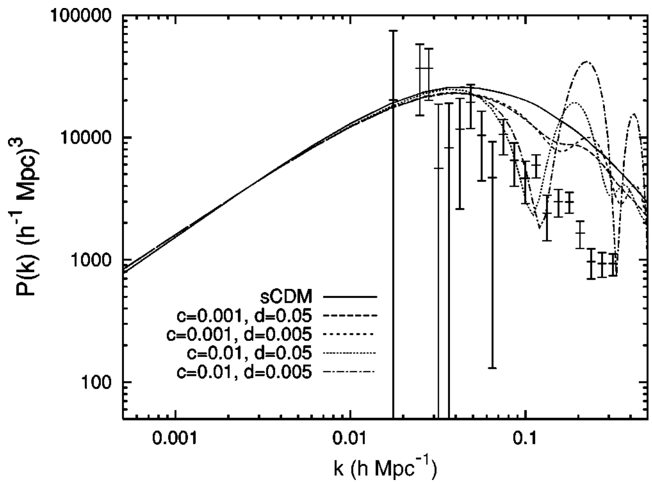

FIG. 6. Matter power spectrum for the primordial spectra shown in Fig. 4. The data points are from the PSCz catalog.

vations which show an anomalously low second peak. Barriga *et al.* performed a χ^2 analysis of the Cosmic Background Explorer (COBE) and BOOMERanG CMB data considering a simple step in the primordial perturbation spectrum (no oscillations). Griffiths *et al.* added a Gaussian bump to the primordial spectrum and performed a similar exercise. Both groups found support for a spectral feature [18].

We leave a χ^2 analysis for a forthcoming paper and include here for orientation the cosmic microwave background power spectrum and matter power spectrum predicted for the range of primordial spectra shown in Fig. 4. We use the Boltzmann code CMBFAST [19] to calculate the CMB angular power spectrum. We use the less fashionable sCDM as our background cosmology ($\Omega_{\rm CDM}=0.95, \Omega_B=0.05$, and $h=0.5$) as our motivation is to show the effect of the primordial density perturbation oscillations rather than find the best fit. The CMB angular power spectrum is shown in Fig. 5 along with observations from COBE [the uncorrelated COBE Differential Microwave Radiometer (DMR) points from [20]], BOOMERanG [21], and MAXIMA [22]. It is clear that a good fit to the data can be found, but that the amplitude of the step and its gradient are constrained to be small by the observations.

The matter power spectrum is shown in Fig. 6 along with the linear power spectrum generated from the PSCz catalog [23]. The theoretical spectra suffer from the sCDM problem of too much power on small scales; however, these spectra are given for indicative purposes and are by no means best-fit spectra.

VI. CONCLUSIONS

We use a numerical routine to accurately calculate the primordial density spectrum predicted by a *physically motivated* inflaton potential with steps in it due to symmetry breaking during inflation. The step in the potential induces oscillations in the density perturbation spectrum whose magnitude and extent is dependent on the amplitude and gradient of the step.

We have restricted our attention to a generic step, demonstrating that even a small feature in the potential can cause significant changes to the spectrum of large scale perturbations. Moreover, we have presented a detailed account of how a feature in the potential modifies the observable spectrum. In light of our calculations, we believe that tight cosmological constraints can be placed on the size of any feature in the potential, and thus on the particle physics model which produced it, and intend to return to this problem in future work. Conversely, if the density perturbation spectrum extracted from future measurements of large scale structure and the cosmic microwave background turns out to be incompatible with a smooth initial spectrum, the mechanism proposed by Adams *et al.* provides a natural mechanism for injecting significant scale dependence within the context of inflation [3].

ACKNOWLEDGMENTS

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