Fluctuations in the cosmic microwave background. II. C_l at large and small l

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General asymptotic formulas are given for the coefficient C_l of the term of multipole number l in the temperature correlation function of the cosmic microwave background, in terms of scalar and dipole form factors introduced in a companion paper. The formulas apply in two overlapping limits: for $l \gg 1$ and for $ld/d_A \ll 1$ (where d_A is the angular diameter distance of the surface of last scattering, and d is a length, of the order of the acoustic horizon at the time of last scattering, that characterizes acoustic oscillations before this time). The frequently used approximation that C_l receives its main contribution from wave numbers of order l/d_A is found to be less accurate for the contribution of the Doppler effect than for the Sachs-Wolfe effect and intrinsic temperature fluctuations. For $ld/d_A \ll 1$ and $l \geq 2$, the growth of C_l with l is shown to be affected by acoustic oscillation wave numbers of all scales. The asymptotic formulas are applied to a model of acoustic oscillations before the time of last scattering, with results in reasonable agreement with more elaborate computer calculations.

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I. INTRODUCTION

A companion paper [1] has shown how to express the temperature fluctuation in the cosmic microwave background in any direction as an integral involving scalar and dipole form factors $F(k)$ and $G(k)$, which characterize acoustic oscillations before the time of last scattering. In the present paper we derive asymptotic formulas for the strength C_l of fluctuations at multipole number l for form factors of arbitary functional form. After outlining our assumptions and reviewing some generalities in Sec. II, our general result in the limit of $l \gg 1$ [Eq. (26)] is derived in Sec. III. In this limit $l(l+1)C_l$ depends on l and the angular diameter distance d_A at the time of last scattering only through the ratio l/d_A. (This is why the heights of the Doppler peaks do not depend on parameters like the cosmological constant that affect d_A but not the form factors.) Our result in the limit $ld/d_A \ll 1$ [Eq. (43)] is derived in Sec. IV. (Here d is some length characterizing acoustic oscillations, such as the acoustic horizon distance d_H at the time of last scattering.) These ranges of l overlap because $d_A \gg d$.

Even without a detailed calculation of the form factors, these results have a moral for the physical interpretation of measurements of C_l. It is common to interpret these measurements by supposing that C_l arises mostly from fluctuations of wave number $k \simeq l/d_A$. Equation (27) shows that this is a fair approximation for the contribution of the scalar form factor $F(k)$, which represents the Sachs-Wolfe effect and intrinsic temperature fluctuations; C_l receives no contribution from $F(k)$ with $k < l/d_A$, and the contribution from $k \gg l/d_A$ is suppressed by a factor $\beta^{-2}(\beta^2-1)^{-1/2}$, where $\beta \equiv kd_A/l$. In particular, a peak in the magnitude of the scalar form factor $F(k)$ at some wave number k_1 (like the peak found in the simple model studied in Ref. [1] at $k=\pi/d_H$) will show up in $l(l+1)C_l$ at a value of l less than but close to $k_1 d_A$. For instance, we will see in Sec. V that the peak in $|F(k)|$ at $k=\pi/d_H$ produces a peak in $l(l+1)C_l$ at $l \simeq 2.6 d_A/d_H$ rather than at $\pi d_A/d_H$. But Eq. (27) also shows that this interpretation of C_l is much less useful for the contribution of the vector form factor $G(k)$, which arises from the Doppler effect; C_l also receives no contribution from $G(k)$ with $k < l/d_A$, but instead of the contribution from $k \simeq l/d_A$ being enhanced by a factor $(\beta^2-1)^{-1/2}$, it is *suppressed* by a factor $(\beta^2-1)^{1/2}$. Indeed, we will see in Sec. V that for sufficiently small baryon number the peak in $G(k)$ at $k=\pi/2d_H$ found in the simple model of Ref. [1] does show up as a peak in $l(l+1)C_l$, but at $l \simeq 0.45 d_A/d_H$, much less than $(\pi/2)d_A/d_H$. Furthermore, the behavior of $l(l+1)C_l$ for ld/d_A near zero depends on the values of $F(k)$ and $G(k)$ for all k. This points to the value of observations that can measure the correlation function of temperature fluctuations directly, as a supplement to measurements of C_l.

The results obtained in Secs. III and IV are used in Sec. V to calculate C_l for the approximate form factors calculated in Ref. [1]. In agreement with what is found in more accurate computer calculations, the position l_1 of the first Doppler peak is not a sensitive function of the baryon density parameter $\Omega_B h^2$. On the other hand, we find that the ratio of the value of $l(l+1)C_l$ at the first Doppler peak to its value at $l \ll d_A/d_H$ is a sensitive indicator of the value of $\Omega_B h^2$.

II. GENERALITIES

The companion paper [1] shows that, in very general models (but assuming only compressional normal modes, with no gravitational radiation), the fractional variation from the mean of the cosmic microwave background temperature observed in a direction $\hat{n}$ takes the general form

$$\frac{\Delta T(\hat{n})}{T} = \int d^3k\, \epsilon_{\mathbf{k}} e^{id_A \hat{n}\cdot\mathbf{k}}[F(k) + i\hat{n}\cdot\hat{k}G(k)], \qquad (1)$$

aside from effects arising from late times, which chiefly af-

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fect the coefficients C_l for relatively small l. Here d_A is the angular diameter distance of the surface of last scattering,

$$d_A=\frac{1}{\Omega_C^{1/2}H_0(1+z_L)} \times\sinh\left[\Omega_C^{1/2}\int_{1/(1+z_L)}^{1}\frac{dx}{\sqrt{\Omega_\Lambda x^4+\Omega_C x^2+\Omega_M x}}\right], \quad (2)$$

where $\Omega_C\equiv 1-\Omega_\Lambda-\Omega_M$, and Ω_Λ and Ω_M are the present ratios of the energy densities of the vacuum and matter to the critical density $3H_0^2/8\pi G$. [If the vacuum energy were to change with time, as in theories of quintessence, then the formula for d_A would need modification, but there would be essentially no change in the other ingredients in Eq. (1), as long as the quintessence energy density makes a negligible contribution to the total energy density at and before the time of last scattering.] Also, $\mathbf{k}^2\epsilon_{\mathbf{k}}$ is proportional (with a **k**-independent proportionality coefficient) to the Fourier transform of the fractional perturbation in the energy density early in the radiation-dominated era. The average[1] of the product of two ϵ's is assumed to satisfy the conditions of statistical homogeneity and isotropy:

$$\langle\epsilon_{\mathbf{k}}\epsilon'_{\mathbf{k}}\rangle=\delta^3(\mathbf{k}+\mathbf{k}')\mathcal{P}(k) \quad (3)$$

with $k\equiv|\mathbf{k}|$. The power spectral function $\mathcal{P}(k)$ is real and positive. Where a specific expression for $\mathcal{P}(k)$ is needed, we will use the "scale-invariant" (or $n=1$) Harrison-Zel'dovich form suggested by theories of new inflation:

$$\mathcal{P}(k)=Bk^{-3}, \quad (4)$$

with B a constant that must be taken from observations of the cosmic microwave background or condensed object mass distributions, or from detailed theories of inflation.

The form factors $F(k)$ and $G(k)$ characterize acoustic oscillations, with $F(k)$ arising from the Sachs-Wolfe effect and intrinsic temperature fluctuations, and $G(k)$ arising from the Doppler effect. For instance, they are calculated in Ref. [1] in the approximation that perturbations in the gravitational field at and before the time of last scattering arise entirely from perturbations in the density of cold dark matter. For very small wave numbers the form factors are

$$F(k)\rightarrow 1-3k^2t_L^2/2-3[-\xi^{-1}+\xi^{-2}\ln(1+\xi)]k^4t_L^4/4+\cdots, \quad (5)$$

$$G(k)\rightarrow 3kt_L-3k^3t_L^3/2(1+\xi)+\cdots, \quad (6)$$

while for wave numbers large enough to allow the use of the WKB approximation, i.e.,

$$kt_L>\xi \quad (7)$$

the form factors are

$$F(k)=(1-2\xi/k^2t_L^2)^{-1}[-3\xi+2\xi/k^2t_L^2 +(1+\xi)^{-1/4}e^{-k^2d_D}\cos(kd_H)], \quad (8)$$

and

$$G(k)=\sqrt{3}(1-2\xi/k^2t_L^2)^{-1}(1+\xi)^{-3/4}e^{-k^2d_D}\sin(kd_H). \quad (9)$$

Here t_L is the time of last scattering; ξ is 3/4 the ratio of the baryon to photon energy densities at this time:

$$\xi=\left(\frac{3\rho_B}{4\rho_\gamma}\right)_{t=t_L}=27\Omega_B h^2; \quad (10)$$

d_H is the acoustic horizon size at this time, and d_D is a damping length, given by Eq. (48). These formulas for the form factors are mentioned at this point only for illustration; we will be working here with general form factors $F(k)$ and $G(k)$, and will not make use of the specific formulas (5)–(10) until Sec. V. But we will assume throughout that any lengths d that [like d_H and d_D in Eqs. (8) and (9)] characterize the k-dependence of the form factors are much smaller than the angular diameter distance d_A of last scattering. This is a good approximation: for instance, if the ratios of matter and vacuum energy densities to the critical density have the present values $\Omega_M=0.3$ and $\Omega_\Lambda=0.7$, then d_A/d_H runs from 91.7 to 79.7 for values of $\Omega_B h^2$ running from zero to 0.03, and d_D is smaller than d_H, independent of the value of H_0.

It is usual to employ the well-known expansion of a plane wave in Legendre polynomials, and write Eq. (1) as

$$\frac{\Delta T(\hat{n})}{T}=\sum_{l=0}^{\infty}(2l+1)i^l\int d^3k\,\epsilon_{\mathbf{k}}P_l(\hat{n}\cdot\hat{k})[j_l(kd_A)F(k) +j_l'(kd_A)G(k)]. \quad (11)$$

Using Eq. (3) and the orthogonality property of Legendre polynomials

$$\int d\Omega_{\hat{k}}P_l(\hat{n}\cdot\hat{k})P_{l'}(\hat{n}'\cdot\hat{k})=\left(\frac{4\pi}{2l+1}\right)\delta_{ll'}P_l(\hat{n}\cdot\hat{n}'), \quad (12)$$

one finds that

$$\left\langle\frac{\Delta T(\hat{n})}{T}\frac{\Delta T(\hat{n}')}{T}\right\rangle=\sum_{l=0}^{\infty}\left(\frac{2l+1}{4\pi}\right)C_lP_l(\hat{n}\cdot\hat{n}'), \quad (13)$$

with the conventional coefficient C_l taking the value

[1]The average here is over an ensemble of possible fluctuations. Using Eq. (3) to analyze the particular element of this sample observed in our universe relies on ergodic arguments, which are not exact except in the limit $l\rightarrow\infty$. However, corrections are manageable [2] even for small l.

$$C_l = 16\pi^2 \int_0^\infty \mathcal{P}(k) k^2 dk [j_l(kd_A)F(k) + j_l'(kd_A)G(k)]^2. \tag{14}$$

This familiar formula is adequate for numerical calculation of C_l, but it hides the essential qualitative aspects of the dependence of C_l on l: that C_l for $l \gg 1$ depends on the ratio l/d_A, and that $l(l+1)C_l$ approaches a constant for sufficiently small values of this ratio, whether l itself is large or small. To obtain these results, we must now distinguish between the two cases $l \gg 1$ and $l \ll d_A/d$ (but $l \geq 2$), where d is a typical length characteristic of the form-factors $F(k)$ and $G(k)$. These two cases overlap because, as remarked above, d_A is much larger than d.

III. LARGE l

The usual way of obtaining the contribution of the scalar form factor to C_l for large l is to note that the integral (14) receives its largest contribution when the argument of the spherical Bessel function is of order l, in which case we can use the approximation that, for $l \to \infty$,

$$j_l(z) \to \begin{cases} 0, & z < \nu, \\ z^{-1/2}(z^2-\nu^2)^{-1/4}\cos\left(\sqrt{z^2-\nu^2} - \nu\arccos(\nu/z) - \dfrac{\pi}{4}\right), & z > \nu, \end{cases} \tag{15}$$

where z/ν is held fixed at a value $\neq 1$, with $\nu \equiv l + 1/2$. The procedure is straight forward for the F^2 terms in Eq. (14), but for the FG and G^2 terms involving the Doppler effect we run into a difficulty: differentiating the factor $(z^2-\nu^2)^{-1/4}$ in Eq. (15) yields larger negative powers of $z^2-\nu^2$ that introduce divergences from the part of the integral in Eq. (14) near the lower bound $k = \nu/d_A$. These infrared divergences are spurious, because the asymptotic formula (15) breaks down if we let z and ν go to infinity in such a way that $z/\nu \to 1$. This problem can be dealt with by switching to a different asymptotic limit [3] for k near ν/d_A. Here we will use a different method [4] which avoids the delicate problem of the asymptotic behavior of $j_l(z)$ and $j_l'(z)$ for z near ν.

We return to Eq. (1), and use Eq. (3) to put the correlation function of observed temperature fluctuations in the form

$$\left\langle \frac{\Delta T(\hat{n})}{T}\frac{\Delta T(\hat{n}')}{T}\right\rangle = \int d^3k \mathcal{P}(k) \exp(i d_A \mathbf{k}\cdot(\hat{n}-\hat{n}')) \times [F^2(k) + i\hat{k}\cdot(\hat{n}-\hat{n}')F(k)G(k) + (\hat{k}\cdot\hat{n})(\hat{k}\cdot\hat{n}')G^2(k)]. \tag{16}$$

The integral over the direction of $\mathbf{k}$ is easy, and gives the correlation function

$$\left\langle \frac{\Delta T(\hat{n})}{T}\frac{\Delta T(\hat{n}')}{T}\right\rangle = 4\pi \int_0^\infty k^2 dk \mathcal{P}(k) \left\{ F^2(k) + F(k)G(k) \times \frac{\partial}{\partial(d_A k)} + \frac{1}{2}G^2(k)\left[1 + \frac{\theta^4}{4} + \left(\frac{1}{\theta^2} - \frac{1}{2} + \frac{3\theta^2}{4}\right)\frac{\partial^2}{\partial(d_A k)^2}\right]\right\} \times \frac{\sin(d_A k\theta)}{d_A k\theta}, \tag{17}$$

where $\theta \equiv |\hat{n} - \hat{n}'|$. (This formula may prove useful in analyzing observations that give the correlation function directly, rather than in terms of C_l.) The amplitude C_l is defined as the integral

$$C_l = 2\pi \int_{-1}^{+1} P_l(\mu) \left\langle \frac{\Delta T(\hat{n})}{T}\frac{\Delta T(\hat{n}')}{T}\right\rangle d\mu, \tag{18}$$

where $\mu \equiv \hat{n}\cdot\hat{n}' = 1 - \theta^2/2$. For large l the Legendre polynomial $P_l(\mu)$ oscillates rapidly for $\theta \gg 1/l$, so the integral is dominated by values of θ of order $1/l$, in which case we can use the well-known limiting expression $P_l(\mu) \to J_0(l\theta)$, and write

$$C_l \to 8\pi^2 \int_0^\infty k^2 dk \mathcal{P}(k) \int_0^2 J_0(l\theta)\theta d\theta \left\{ F^2(k) + F(k)G(k) \times \frac{\partial}{\partial(d_A k)} + \frac{1}{2}G^2(k)\left[1 + \frac{\theta^4}{4} + \left(\frac{1}{\theta^2} - \frac{1}{2} + \frac{3\theta^2}{4}\right) \times \frac{\partial^2}{\partial(d_A k)^2}\right]\right\} \frac{\sin(d_A k\theta)}{d_A k\theta}. \tag{19}$$

The integral over k is dominated by values for which $k d_A \theta$ is of order unity, so the derivative $\partial/\partial(d_A k)$ is effectively of order $\theta \approx 1/l$. Thus to leading order in $1/l$, Eq. (19) may be simplified to

$$C_l \to 8\pi^2 \int_0^\infty k^2 dk \mathcal{P}(k) \int_0^2 J_0(l\theta)\theta d\theta \left[F^2(k) + \frac{1}{2}G^2(k) \times \left(1 + \frac{1}{\theta^2}\frac{\partial^2}{\partial(d_A k)^2}\right)\right] \frac{\sin(d_A k\theta)}{d_A k\theta}. \tag{20}$$

Introducing a new variable $s \equiv l\theta$ and changing the upper limit on the s-integral from $2l$ to infinity, we may write this as

$$C_l \to \frac{8\pi^2}{l^2}\int_0^\infty k^2 dk \mathcal{P}(k)\int_0^\infty J_0(s) s\, ds\left[F^2(k)+\frac{1}{2}G^2(k)\right.$$
$$\left.\times\left(1+\frac{\partial^2}{\partial(d_A ks/l)^2}\right)\right]\frac{\sin(d_A ks/l)}{(d_A ks/l)}. \tag{21}$$

The integral over s is easy for the F^2 term; we need only use the formula [5]:

$$\int_0^\infty J_0(s)\sin(\beta s)ds=\begin{cases}0, & \beta<1\\ (\beta^2-1)^{-1/2}, & \beta>1,\end{cases} \tag{22}$$

where here $\beta = d_A k/l$. The integral of the G^2 term takes a little more work. We use the formula $(1+d^2/dx^2)\sin x/x = -(2/x)d/dx(\sin x/x)$ and do the remaining integral by parts, so that

$$\int_0^\infty J_0(s)s\left[1+\frac{\partial^2}{\partial(\beta s)^2}\right]\frac{\sin(\beta s)}{\beta s}ds$$
$$=-\frac{2}{\beta^2}\int_0^\infty J_0(s)\frac{\partial}{\partial s}\frac{\sin(\beta s)}{s}ds$$
$$=\frac{2}{\beta^2}-\frac{1}{\beta^3}\int_0^\infty[J_2(s)+J_0(s)]\sin(\beta s)ds. \tag{23}$$

Here we also need the formula [5]

$$\int_0^\infty J_2(s)\sin(\beta s)ds$$
$$=\begin{cases}2\beta, & \beta<1,\\ -(\beta^2-1)^{-1/2}(\beta+\sqrt{\beta^2-1})^{-1}, & \beta>1,\end{cases} \tag{24}$$

so that

$$\int_0^\infty J_0(s)s\left[1+\frac{\partial^2}{\partial(\beta s)^2}\right]\frac{\sin(\beta s)}{\beta s}ds$$
$$=\begin{cases}0, & \beta<1,\\ 2\beta^{-3}\sqrt{\beta^2-1}, & \beta>1.\end{cases} \tag{25}$$

Using Eqs. (22) and (25) in Eq. (21) then gives our final general formula for C_l at large l:

$$C_l \to \frac{8\pi^2 l}{d_A^3}\int_1^\infty d\beta \mathcal{P}(l\beta/d_A)\left[\frac{\beta F^2(l\beta/d_A)}{\sqrt{\beta^2-1}}\right.$$
$$\left.+\frac{\sqrt{\beta^2-1}G^2(l\beta/d_A)}{\beta}\right]. \tag{26}$$

Note that l^2C_l depends on l and d_A only through its dependence on the ratio l/d_A.

For instance, if we take the power spectral function to have the scale-invariant form $\mathcal{P}(k)=Bk^{-3}$, then for $l \gg 1$

$$l(l+1)C_l \to 8\pi^2 B\int_1^\infty d\beta\left[\frac{F^2(l\beta/d_A)}{\beta^2\sqrt{\beta^2-1}}\right.$$
$$\left.+\frac{\sqrt{\beta^2-1}G^2(l\beta/d_A)}{\beta^4}\right]. \tag{27}$$

[We have taken advantage of the fact that here we are considering $l \gg 1$ to change a factor l^2 to $l(l+1)$, in order to facilitate comparison with the results of the next section.] The rapid fall-off of the coefficient of F^2 for $\beta>1$ suggests that the contribution of the scalar form factor F to C_l is dominated by wave numbers close to d_A/l, as is usually assumed. On the other hand, the contribution of the dipole form factor $G(k)$ for wave numbers immediately above d_A/l is actually suppressed by the factor $\sqrt{\beta^2-1}$ in the second term of Eq. (27).

IV. SMALL ld/d_A

Here we will adopt the "$n=1$" scale-invariant spectrum $\mathcal{P}(k)\simeq Bk^{-3}$ from the beginning, so that the general formula Eq. (14) becomes

$$C_l=16\pi^2 B\int_0^\infty\left[j_l(s)F\left(\frac{s}{d_A}\right)+j_l'(s)G\left(\frac{s}{d_A}\right)\right]^2\frac{ds}{s}. \tag{28}$$

To generate a series for $l(l+1)C_l$ in powers of l/d_A we expand the form factors in power series:

$$F(k)=F_0+F_2k^2+\cdots,\qquad G(k)=G_1k+G_3k^3+\cdots. \tag{29}$$

[The power series for F and G must be respectively even and odd in k, in order that the integrand in the temperature fluctuation (1) should be analytic in the three-vector $\mathbf{k}$ at $\mathbf{k}=0$.] The leading term in C_l is well known; using a standard formula [6]:

$$\int_0^\infty j_l^2(s)s^{m-1}ds=\frac{2^{m-3}\pi\Gamma(2-m)\Gamma\left(l+\frac{m}{2}\right)}{\Gamma^2\left(\frac{3-m}{2}\right)\Gamma\left(l+2-\frac{m}{2}\right)}, \tag{30}$$

we find the term in Eq. (28) of zeroth order in $1/d_A$:

$$C_l^{(0)}=\frac{8\pi^2 BF_0^2}{l(l+1)}. \tag{31}$$

There is no difficulty in also calculating the term in Eq. (28) of first order in $1/d_A$:

$$C_l^{(1)} = \left(\frac{32\pi^2 BF_0 G_1}{d_A}\right)\int_0^\infty j_l(s) j_l'(s)\,ds$$
$$= \left(\frac{16\pi^2 BF_0 G_1}{d_A}\right)[j_l^2(s)]_0^\infty = 0. \tag{32}$$

But we run into trouble in calculating the term of second order in $1/d_A$. The second derivative of C_l with respect to $1/d_A$ is

$$\frac{d^2 C_l}{d(1/d_A)^2} = 16\pi^2 B\int_0^\infty \{ j_l^2(s) F^{2\prime\prime}(s/d_A) + j'^2_l(s) G^{2\prime\prime}(s/d_A)$$
$$+ 2 j_l(s) j_l'(s)[F(s/d_A)G(s/d_A)]''\} s\, ds. \tag{33}$$

The $j_l j_l'$ term does not contribute to the part of C_l of second order in $1/d_A$, because $F(k)G(k)$ contains only odd powers of k. To calculate the contribution of the j'^2_l term, we need to supplement Eq. (30) with the additional formula[2]

$$\int_0^\infty j'^2_l(s) s^{m-1} ds$$
$$= \frac{2^{m-3}\pi\Gamma(2-m)\Gamma\left(l+\frac{m}{2}\right)}{\Gamma^2\left(\frac{3-m}{2}\right)\Gamma\left(l+2-\frac{m}{2}\right)}$$
$$\times\left[1 + \frac{(m-3)(m-2)[(m-2)(m-3)-2l(l+1)]}{2(3-m)^2\left(l+\frac{m}{2}-1\right)\left(l-\frac{m}{2}+2\right)}\right]. \tag{34}$$

The second derivative (33) is divergent at $1/d_A = 0$, as shown by the factors $\Gamma(2-m)$ in Eqs. (30) and (34), which become infinite for $m=2$. Of course, there is no infinity in C_l; it is simply not analytic in $1/d_A$ at $1/d_A = 0$.

We can deal with this problem by a method similar to the dimensional regularization technique used in quantum field theory [7]. We treat m as a complex variable that approaches $m=2$. In this limit, Eqs. (30) and (34) give

$$\int_0^\infty j_l^2(s) s^{m-1} ds \to -\frac{1}{2}\left[\frac{1}{m-2} + \sum_{r=1}^{l}\frac{1}{r} - C + \ln 2 - D\right], \tag{35}$$

$$\int_0^\infty j'^2_l(s) s^{m-1} ds \to -\frac{1}{2}\left[\frac{1}{m-2} + \sum_{r=1}^{l}\frac{1}{r} - C\right.$$
$$\left. + \ln 2 - D + 1\right], \tag{36}$$

where C is the Euler constant $C \equiv -\Gamma'(1) = 0.57722$, and $D \equiv -\Gamma'(1/2)/\Gamma(1/2) = 1.96351$. The important point here is that the parts of the integrals (35) and (36) that are divergent at $m=2$ are independent of l, and thus so also is the part of C_l that is non-analytic in $1/d_A$ at $1/d_A = 0$. Using Eqs. (29), (35) and (36) in Eq. (33) thus gives the part of C_l that is of second order in $1/d_A$ as

$$C_l^{(2)} = -8\pi^2 B d_A^{-2}(2F_0F_2 + G_1^2)\sum_{r=1}^{l}\frac{1}{r}$$
$$+ l\text{-independent terms.} \tag{37}$$

We can check the consistency of these results and calculate the l-independent terms here by using our previous result (27) in the case where l is large *and* ld/d_A is small, where d is whatever length characterizes the k-dependence of the form factors. The term in Eq. (27) of zeroth order in ld/d_A is

$$l(l+1)C_l \to 8\pi^2 BF_0^2 \int_1^\infty \frac{d\beta}{\beta^2\sqrt{\beta^2-1}} = 8\pi^2 BF_0^2, \tag{38}$$

in agreement with Eq. (31). Also, Eq. (27) has no terms of first order in $1/d_A$, in agreement with Eq. (32). To calculate the terms in Eq. (27) of second order in $1/d_A$, we express $F^2(k)$ and $G^2(k)$ in terms of cosine transforms

$$F^2(k) = F_0^2 + \int_0^\infty da\, f(a)[1-\cos(ka)],$$

$$G^2(k) = \int_0^\infty da\, g(a)[1-\cos(ka)]. \tag{39}$$

Then for $l \gg 1$ *and* $ld/d_A \ll 1$, Eq. (27) gives

$$C_l^{(2)} \to -\frac{8\pi^2 B}{d_A^2}\left\{(2F_0F_2 + G_1^2)\left[\ln\left(\frac{l\bar{d}}{2d_A}\right) + C - \frac{3}{2}\right] + G_1^2\right\}, \tag{40}$$

where $\bar{d}$ is a typical value of the variable a in the cosine transforms (39):

$$\ln \bar{d} \equiv \frac{\int_0^\infty [f(a)+g(a)]\, a^2 \ln a\, da}{\int_0^\infty [f(a)+g(a)]\, a^2 da}. \tag{41}$$

Equation (40) agrees with the limit of Eq. (37) for large l, because in this limit $\Sigma_1^l 1/r \to \ln l + C$, and now fixes the l-independent terms in Eq. (37) so that, for any l with $ld/d_A \ll 1$,

[2]This formula was obtained by using the Bessel differential equation to show that $j'^2_l(z) = [1 - l(l+1)/z^2] j_l^2(z) + [z j_l^2(z)]''/2z$, and then using Eq. (30) with two integrations by parts.

$$C_l^{(2)} = -\frac{8\pi^2 B}{d_A^2}\left\{(2F_0F_2+G_1^2)\left[\ln\left(\frac{\bar{d}}{2d_A}\right)+\sum_{r=1}^{l}\frac{1}{r}-\frac{3}{2}\right]+G_1^2\right\}. \quad (42)$$

Putting together Eqs. (31), (32), and (42) gives our final formula for C_l in the case $ld/d_A \ll 1$ and $l \geqslant 2$:

$$l(l+1)C_l = 8\pi^2 B F_0^2\left(1-\frac{l(l+1)}{d_A^2}\left\{d^2\left[\ln\left(\frac{\bar{d}}{2d_A}\right)+\sum_{r=1}^{l}\frac{1}{r}\right]-d'^2\right\}+\cdots\right), \quad (43)$$

where now we introduce a pair of characteristic lengths:

$$d^2 \equiv \frac{2F_0F_2+G_1^2}{F_0^2}, \qquad d'^2 \equiv \frac{3F_0F_2+\frac{1}{2}G_1^2}{F_0^2}. \quad (44)$$

The logarithm in Eq. (43) is large and negative, so $l(l+1)C_l$ will increase or decrease with l for sufficiently small l accordingly as $d^2>0$ or $d^2<0$. [Taken literally, Eq. (43) would suggest that this behavior is reversed when the sum over r becomes large enough to cancel the logarithm, but this is at $l \simeq 2e^{-C}d_A/\bar{d}$, which is large enough to invalidate the approximations that led to Eq. (43).] Note that, while d and d' depend only on the behavior of the form factors near zero wave number, the length $\bar{d}$ given by Eq. (41) depends on the behavior of the form factors at all wave numbers. Consequently, although the *value* of C_l at low l depends only on the form factors at $k=0$, somewhat surprisingly the *growth* of C_l for small l depends on the form factors at all wave numbers.

V. APPLICATION

To illustrate the use of the asymptotic formulas obtained here, we will now apply them to the simplified model described in Ref. [1]: the universe before last scattering consisting of pressureless cold dark matter and a photon-nucleon-electron plasma; no gravitational radiation; and negligible contributions of the plasma and neutrinos to the gravitational field. In this case, the comparison of Eqs. (5) and (6) for the long wavelength limit of the form factors with Eq. (29) gives

$$F_0=1, \qquad F_2=-3t_L^2/2, \qquad G_1=3t_L, \quad (45)$$

so the lengths (44) are here

$$d^2=6t_L^2, \qquad d'^2=0. \quad (46)$$

Hence Eq. (43) then gives the behavior of C_l for $ld/d_A \ll 1$ and $l \geqslant 2$ as

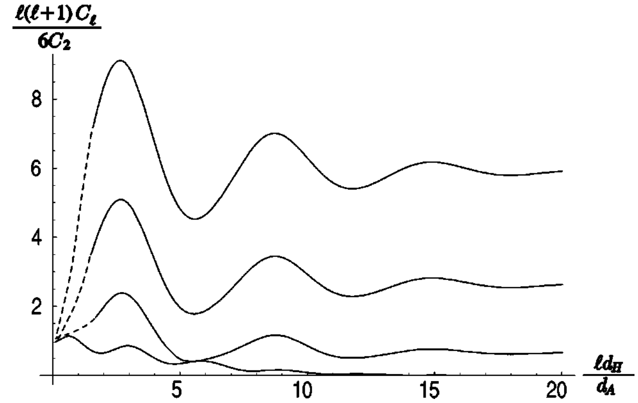

FIG. 1. Plots of the ratio of the multipole strength parameter $l(l+1)C_l$ to its value at small l, versus ld_H/d_A, where d_H is the horizon size at the time of last scattering and d_A is the angular diameter distance of the surface of last scattering. The curves are for $\Omega_B h^2$ ranging (from top to bottom) over the values 0.03, 0.02, 0.01, and 0, corresponding to ξ taking the values 0.81, 0.54, 0.27, and 0. The solid curves are calculated using the WKB approximation; dashed lines indicate an extrapolation to the known value at small ld_H/d_A. These results are independent of the parameters H_0, Ω_Λ, and Ω_M.

$$l(l+1)C_l = 8\pi^2 B\left\{1-\frac{6l(l+1)t_L^2}{d_A^2}\left[\ln\left(\frac{\bar{d}}{2d_A}\right)+\sum_{r=1}^{l}\frac{1}{r}\right]+\cdots\right\}. \quad (47)$$

Aside from its weak dependence on $\bar{d}$, the behavior of C_l for $ld/d_A \ll 1$ is independent of the baryon density, in agreement with more accurate computer calculations [8]. We cannot calculate the length $\bar{d}$ without a model that would give the form factors at all wave numbers, but $\bar{d}$ is expected to be roughly of order d_H, and since d_A/d_H is large the logarithm is not sensitive to the precise value of $\bar{d}$. If for instance we take $\bar{d}=\sqrt{3}t_L=d_A/58.5$ (the acoustic horizon at last scattering for $\Omega_M=0.4$, $\Omega_V=0.6$, and $\Omega_B=0$) then the quantity $l(l+1)C_l/8\pi^2 B$ rises from unity when extrapolated to $l=0$ to 1.044 at $l=5$, and to 1.118 at $l=10$, which is probably the highest value of l for which the approximations leading to Eq. (47) are reliable.

For l of the order of d_A/d_H the coefficients C_l can be calculated under the simplifying assumptions of this section by using the form factors given by Eqs. (8) and (9) in Eq. (27). The damping length is given in Ref. [1] as

$$d_D^2 \equiv \mathcal{D}_L^2+\Delta\mathcal{D}_L^2 \simeq 0.029\, t_L^2\left(\frac{8}{15(1+\xi)}+\frac{\xi^2}{2(1+\xi)^2}\right)+0.0025\, d_H^2. \quad (48)$$

Our results for C_l at and below the first Doppler peak are not sensitive to d_D. We will simplify our calculations here by dropping the terms in Eqs. (8) and (9) that are proportional to

TABLE I. Location $l_1 d_A/d_H$ of the first Doppler peak and height of the peak in $l(l+1)C_l$ relative to its value $8\pi^2 B \simeq 6C_2$ for l extrapolated to zero for various values of the baryon density parameter. These results, and the curves in Fig. 1, are independent of the values of H_0, Ω_Λ, and (within our approximations) Ω_M. The last two columns give the values of d_A/d_H and l_1 for $\Omega_M=0.3$, $\Omega_\Lambda=0.7$, with d_H calculated taking into account the contribution of photons and neutrinos to the expansion rate, and using $\Omega_M h^2 = 0.15$.

$\Omega_B h^2$	ξ	$l_1 d_H/d_A$	$l_1(l_1+1)C_{l_1}/6C_2$	d_A/d_H	l_1
0	0	2.83	0.863	91.7	260
0.01	0.27	2.65	2.34	87.1	231
0.02	0.54	2.60	5.09	83.6	217
0.03	0.81	2.58	9.115	79.7	206

the ratio $\xi/k^2 t_L^2$, on the grounds that these terms are not very different from corrections to the WKB approximation that are not included either. (At the first Doppler peak $\xi/k^2 t_L^2$ increases with ξ and hence with $\Omega_B h^2$, and for $\Omega_B h^2 = 0.03$ it has the value 0.20. But to be honest, the real reason for dropping these terms is that they spoil the agreement of our results for the height of the first Doppler peak with more accurate numerical calculations.) The results obtained now depend critically on the baryon density parameter $\xi \simeq 27\,\Omega_B h^2$, and are shown in Fig. 1 for values of $\Omega_B h^2$ ranging from zero to 0.03.

For $\Omega_B = 0$ [in which case the WKB approximation is not needed, so that Eq. (27) should give C_l down to values of l of order 2] the behavior of C_l is nothing like what is observed: $l(l+1)C_l/8\pi^2 B$ rises from unity to 1.1 at a "zeroth Doppler peak" at $l d_H/d_A \simeq 0.45$ [due to the maximum in the Doppler form factor $G(k)$ at $k d_H = \pi/2$], then dips to 0.7 at $l d_H/d_A \simeq 1.6$, and then rises again to a first Doppler peak at $l d_H/d_A \simeq 2.83$.

For $\Omega_B h^2 \geqslant 0.01$ the behavior of C_l within the range of validity of the WKB approximation is much more like what is observed: $l(l+1)C_l$ rises monotonically to a first Doppler peak at $l d_H/d_A$ very roughly of order π (though actually around 2.6). There is another clear peak at $l \simeq 8.7 d_H/d_A$, presumably arising from the peak in $F(k)$ at $k = 3\pi/d_H$. The weaker peaks in $l(l+1)C_l$ arising from peaks in $F(k)$ near even values of $k d_H/\pi$ are absent here, presumably because of our neglect of the contribution of radiation and neutrinos to the gravitational field. Another difference between the curves of Fig. 1 and more accurate computer calculations is that, again because we neglect the contribution of radiation and neutrinos to the gravitational field, our results do not show the fall-off of $l(l+1)C_l$ at large l associated with the fall-off of the familiar transfer function $T(k)$ at large k.

The values of the position $l_1 d_H/d_A$ of the first peak and the ratio of its height $l_1(l_1+1)C_{l_1}$ to the value $8\pi^2 B \simeq 6C_2$ for small l are given for various baryon densities in Table I. These results are independent of other parameters. In the last two columns of Table I we also give values of d_A/d_H for $\Omega_M = 0.3$ and $\Omega_\Lambda = 0.7$, and the corresponding results for the multipole number l_1 of the first Doppler peak. In calculating the horizon at last scattering d_H we have now (somewhat inconsistently) taken into account the effect of photons and three flavors of neutrinos and antineutrinos on the expansion rate, which gives

$$d_H \equiv \frac{a(t_L)}{\sqrt{3}} \int_0^{t_L} \frac{dt}{a(t)\sqrt{1+R(t)}}$$

$$= \frac{2}{H_0(1+z_L)^{3/2}\sqrt{3\xi\Omega_M}} \ln\left(\frac{\sqrt{1+\xi} + \sqrt{\xi(1+\lambda)}}{1+\sqrt{\xi\lambda}} \right), \tag{49}$$

where $\lambda = 0.047/\Omega_M h^2$ is the ratio of photon and neutrino energy density to dark matter energy density at the time of last scattering, and d_A is given by Eq. (2). In calculating the values of d_A/d_H in the table we have taken $\Omega_M h^2 = 0.15$.

We see from Table I that the position of the first Doppler peak does not depend strongly on $\Omega_B h^2$, while its height is a sensitive function of $\Omega_B h^2$. For $\Omega_B h^2$ between 0.02 and 0.03 the height and position are in fair agreement with what is observed, though of course the serious comparison of theory with observation relies on more accurate computer calculations. The qualitative results obtained here suggest that if one were to rely on a single feature of the plot of $l(l+1)C_l$ versus l to measure $\Omega_B h^2$, then the ratio of the the height of the first Doppler peak to the value for lower l values studied by the Cosmic Background explorer (COBE) satellite would be more useful than the ratio of the heights of the first and second Doppler peaks, which relies on less precise data, depends on complicated damping effects, and is more sensitive to other parameters, such as $\Omega_M h^2$ and the rate of change, if any, of the vacuum energy. Of course, for high precision one must use the whole plot of $l(l+1)C_l$ versus l to measure all these parameters together.

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