# Fluctuations in the cosmic microwave background. I. Form factors and their calculation in the synchronous gauge

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It is shown that the fluctuation in the temperature of the cosmic microwave background in any direction may be evaluated as an integral involving scalar and dipole form factors, which incorporate all relevant information about acoustic oscillations before the time of last scattering. A companion paper gives asymptotic expressions for the multipole coefficient $C_l$ in terms of these form factors. Explicit expressions are given here for the form factors in a simplified hydrodynamic model for the evolution of perturbations.

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## I. INTRODUCTION

The purpose of this paper is, first, to exhibit a general formalism, expressing the observed fluctuations in the cosmic microwave background temperature in terms of a pair of form factors, and then to carry out an illustrative approximate analytic calculation of these form factors. A companion paper [1] gives general asymptotic formulas for the coefficient $C_l$ of the $l$th multipole term in the temperature correlation function for arbitrary form factors, and also uses these formulas to calculate $C_l$ for the form factors found in the present paper.

In Sec. II we show that under very general assumptions the fractional variation from the mean of the cosmic microwave background temperature observed in a direction $\hat{n}$ takes the form

$$\frac{\Delta T(\hat{n})}{T} = \int d^3k \, \epsilon_{\mathbf{k}} e^{i d_A \hat{n}\cdot\mathbf{k}} [F(k) + i\, \hat{n}\cdot\hat{k}\, G(k)]. \qquad (1)$$

Here $d_A$ is the angular diameter distance of the surface of last scattering,[1] and $\mathbf{k}^2\epsilon_{\mathbf{k}}$ is proportional (with a $\mathbf{k}$-independent proportionality coefficient) to the Fourier transform of the fractional perturbation in the total energy density at early times. [There are additional terms in $\Delta T/T$ that arise from times near the present, and chiefly effect the multipole coefficients $C_l$ for small $l$, especially $l=0$ and $l=1$. These will be discussed in Sec. IV and in the Appendix. Effects from a changing gravitational field soon after the time of last scattering are included in Eq. (1).]

One advantage of this formalism is that it provides a nice separation between the three different kinds of effect that influence the observed temperature fluctuation, that arise in three different eras: (i) at very early times, (ii) during the era of acoustic oscillations, and (iii) from the time of last scattering to the present.

(i) The $\mathbf{k}$-dependence of the unprocessed fluctuation amplitude $\epsilon_{\mathbf{k}}$ reflects the space-dependence of fluctuations in the energy density at very early times. The average of the product of two $\epsilon$'s is assumed to satisfy the conditions of statistical homogeneity and isotropy:

$$\langle \epsilon_{\mathbf{k}} \epsilon'_{\mathbf{k}} \rangle = \delta^3(\mathbf{k}+\mathbf{k}')\mathcal{P}(k) \qquad (2)$$

with $k \equiv |\mathbf{k}|$. Since the reality of the fluctuations in the energy density requires that $\epsilon^*_{\mathbf{k}} = \epsilon_{-\mathbf{k}}$, the power spectral function $\mathcal{P}(k)$ is real and positive. It is common to assume a "straight" spectrum

$$\mathcal{P}(k) \propto k^{n-4}. \qquad (3)$$

For instance, the "scale-invariant" $n=1$ form [2] suggested by theories of new inflation [3] is

$$\mathcal{P}(k) = B\, k^{-3}, \qquad (4)$$

with $B$ a constant that must be taken from observations of the cosmic microwave background or condensed object mass distributions, or from detailed theories of inflation.

(ii) The form factors $F(k)$ and $G(k)$ characterize acoustic oscillations, with $F(k)$ arising from the Sachs-Wolfe effect and intrinsic temperature fluctuations, and $G(k)$ arising from the Doppler effect.

(iii) Taking account of a constant vacuum energy density as well as cold matter in the time after last scattering, it is easy to calculate the angular diameter distance of the surface of last scattering:

$$d_A = \frac{1}{\Omega_C^{1/2} H_0 (1+z_L)} \times \sinh\left[ \Omega_C^{1/2} \int_{1/(1+z_L)}^{1} \frac{dx}{\sqrt{\Omega_\Lambda x^4 + \Omega_C x^2 + \Omega_M x}} \right], \qquad (5)$$

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[1]Note that in speaking of a surface of last scattering, we are not necessarily assuming that the transition from opacity to transparency takes place instantaneously. The physical wave number vector $\mathbf{k}$ varies with time as $1/a(t)$ [where $a(t)$ is the Robertson-Walker scale factor], while for large redshifts $d_A$ varies as $a(t)$, so the product $d_A\mathbf{k}$ is nearly independent of what we choose as a nominal time of last scattering.

 

where $z_L \simeq 1100$ is the redshift of last scattering, $\Omega_C \equiv 1 - \Omega_\Lambda - \Omega_M$, and $\Omega_\Lambda$ and $\Omega_M$ are as usual the present ratios of the energy densities of the vacuum and matter to the critical density $3H_0^2/8\pi G$.

We note in particular that $F(k)$ and $G(k)$ depend on $\Omega_M h^2$ and on the baryon density parameter $\Omega_B h^2$ (where $h$ is the Hubble constant in units of 100 km/sec/Mpc), but since the curvature and vacuum energy were negligible at and before last scattering, $F(k)$ and $G(k)$ are essentially independent of the present curvature and of $\Omega_\Lambda$. The exponent $n$ in $\mathcal{P}(k)$ is expected to be independent of all these parameters. On the other hand, $d_A$ is affected by whatever governed the paths of light rays since the time of last scattering, so it depends on $\Omega_M$, $\Omega_\Lambda$, and the curvature parameter $\Omega_C$, but it is essentially independent of quantities like the baryon density parameter $\Omega_B$ that effect acoustic oscillations before the time of last scattering. In quintessence theories $d_A$ would be given by a different formula, but $\mathcal{P}(k)$ and the form factors would be essentially unchanged as long as the quintessence energy density is a small part of the total energy density at and before the time of last scattering.

Another advantage of this formalism is that, although $C_l$ must be calculated by a numerical integration, it is possible to give approximate analytic expressions for the form factors in terms of elementary functions. The detailed confrontation of observation and theory must necessarily be done using computer codes that take into account all relevant astrophysical and observational effects [4]. Nevertheless, there is some value in also having an analytic treatment that, though not as accurate as possible, is as simple as possible while still capturing the main features of what is going on. The point is not to compete with the computer codes, but rather to gain some feeling for what is going on, in order to help us judge how predictions for the cosmic microwave background fluctuations may change with alterations in the underlying assumptions.

Analytic treatments of fluctuations in the cosmic microwave background already exist in the literature [5]. Our main purpose in going over the same ground here is not to give a more accurate or comprehensive treatment of acoustic oscillations, but to obtain simple expressions for the form factors as examples to which to apply the asymptotic formulas for $C_l$ derived in [1]. To derive analytic expressions for the temperature fluctuation it is necessary to neglect the contribution of radiation and neutrinos to the gravitational field, which should be a fair approximation near the first Doppler peak but not much beyond that. We employ a purely hydrodynamic treatment, relying on the Boltzmann equation only implicitly in the values used for the shear viscosity and heat conduction coefficients; the effects of viscosity and heat conduction are included from the beginning, not just by inserting damping factors; and "Landau" damping due to the finite duration of the era of last scattering is included along with "Silk" damping due to shear viscosity and heat conduction. As far as I know, this is the first work to obtain *explicit* analytic expressions for the temperature fluctuations that are correct within these approximations.

Section III presents an analytic calculation of the evolution of perturbations in the synchronous gauge up to the time of last scattering, which is then used in Sec. IV to calculate the form factors. For very small wave numbers the form factors are found to be

$$F(k) \to 1 - 3k^2 t_L^2/2 - 3[-\xi^{-1} + \xi^{-2}\ln(1+\xi)]k^4 t_L^4/4 + \dots, \tag{6}$$

$$G(k) \to 3kt_L - 3k^3 t_L^3/2(1+\xi) + \dots, \tag{7}$$

while for wave numbers large enough to allow the use of the WKB approximation, i.e., $kt_L \gg \xi$, the form factors are [6]

$$F(k) = (1 + 2\xi/k^2 t_L^2)^{-1}[-3\xi + 2\xi/k^2 t_L^2 + (1+\xi)^{-1/4} e^{-k^2 d_D^2}\cos(kd_H)], \tag{8}$$

and

$$G(k) = \sqrt{3}(1+\xi)^{-3/4}(1 + 2\xi/k^2 t_L^2)^{-1} e^{-k^2 d_D^2}\sin(kd_H). \tag{9}$$

Here $t_L$ is the time of last scattering; $\xi$ is 3/4 the ratio of the baryon to photon energy densities at this time:

$$\xi = \left(\frac{3\rho_B}{4\rho_\gamma}\right)_{t=t_L} \simeq 27\,\Omega_B h^2; \tag{10}$$

$d_H$ is the acoustic horizon size at this time, given by Eq. (75), and $d_D$ is a damping length, given by Eq. (89).

## II. FORM FACTORS

We first justify Eq. (1) under very general assumptions, not limited to those of Sec. III. At and before the time of last scattering the spatial curvature was negligible, so small perturbations in the cosmic metric and in all particle distributions at these times may conveniently be expressed as Fourier transforms of functions of a co-moving wave number vector $\mathbf{q}$ and the time $t$. Effects like pressure forces that involve spatial gradients are important for a given $\mathbf{q}$ only when the physical wave number $q/a(t)$ is at least as large as the cosmic expansion rate, which is of order $1/t$. Since $a(t)$ vanishes for $t \to 0$ no more rapidly than $\sqrt{t}$, the ratio $qt/a(t)$ vanishes as $t \to 0$, so whatever the value of $q$, there will always be some time early enough so that pressure forces and other effects of spatial gradients are negligible. At such early times, perturbations grow or decay with powers of time. Generically there is one most rapidly growing mode, and this is the one that eventually grows into the perturbations seen at the time of last scattering. Since the equations for the time dependence of the perturbations are linear, the Fourier transforms of all perturbations to the metric and particle distributions during the era of last scattering will then be proportional to the Fourier transform $e_{\mathbf{q}}$ of any one of these perturbations at any sufficiently early time. For definiteness, we can take $e_{\mathbf{q}}$ to be $q^{-2}$ times the Fourier transform of the fractional perturbation to the total energy density at some very early time, a choice that will prove to be convenient in Sec. IV.

Since the fractional change in the observed microwave background temperature seen in a direction $\hat{n}$ is linear in the perturbations to the metric and photon and matter distributions at various times during the era of last scattering, it can be written as

$$\frac{\Delta T(\hat{n})}{T} = \int dt \int d^3q\, e^{i\mathbf{q}\cdot\hat{n}r(t)} e_{\mathbf{q}} J(q,\hat{q}\cdot\hat{n},t), \tag{11}$$

where $r(t)$ is the co-moving radial coordinate of a source scattering light at time $t$ that would be received at the present. Note that the quantity $J$ can depend on $\mathbf{q}$ only through the scalars $q$ and $\hat{q}\cdot\hat{n}$, because the differential equations governing the growth of perturbations are rotationally invariant, even though the initial fluctuation amplitude $e_{\mathbf{q}}$ is not.

We can make a great simplification in Eq. (11) by taking advantage of the fact that the Robertson-Walker radial coordinate $r(t)$ is nearly constant during the era of last scattering. Using equilibrium statistical mechanics to calculate the hydrogen ionization, and simple Thomson scattering to calculate scattering probabilities, one finds that for $\Omega_B/\Omega_M = 0.2$, the probability that a photon will never again be scattered rises from 2% at 3360 K to 98% at 2780 K. (This depends on the assumed value of $\Omega_B/\Omega_M$, but very weakly; for instance, for $\Omega_B/\Omega_M = 0.12$, the probability of no future scattering rises from 2% to 98% as the temperature drops from 3400 K to 2810 K.) For definiteness, we will round off these temperatures, taking the era of last scattering to extend from a temperature of 3400 K down to 2800 K, corresponding to a drop in redshift $z$ from 1220 to 1010. The radial coordinate can be expressed in terms of $z$ by the well-known formula

$$r(t_z) = \frac{1}{\Omega_C^{1/2} H_0 a(t_0)} \times \sinh\left[\Omega_C^{1/2} \int_{1/(1+z)}^{1} \frac{dx}{\sqrt{\Omega_\Lambda x^4 + \Omega_C x^2 + \Omega_M x}}\right], \tag{12}$$

where $\Omega_C \equiv 1 - \Omega_\Lambda - \Omega_M$, $a(t)$ is the Robertson-Walker scale factor, $t_z$ is the time corresponding to redshift $z$, and $t_0$ is the present. This approaches a constant limit for $z \to \infty$, and therefore varies very little in the range from $z = 1010$ to $z = 1220$ (or, for that matter, even in the range from $z = 1010$ to $z \to \infty$). For instance, if we take the popular values $\Omega_M = 0.3$ and $\Omega_V = 0.7$, then the fractional change in $r(t_z)$ as $z$ drops from 1220 to 1010 is 0.0034. We can therefore take the exponential in Eq. (11) outside the time integral, so that

$$\frac{\Delta T(\hat{n})}{T} = \int d^3q\, e^{i\mathbf{q}\cdot\hat{n}\, r(t_L)} e_{\mathbf{q}} \int dt\, J(q,\hat{q}\cdot\hat{n},t), \tag{13}$$

where $t_L$ is any conveniently chosen time during the era of last scattering, say at a redshift $z_L = 1100$.

It is convenient to replace the co-moving wave number vector $\mathbf{q}$ with the physical wave number at last scattering:

$$\mathbf{k} \equiv \mathbf{q}/a(t_L). \tag{14}$$

Equation (13) may then be written

$$\frac{\Delta T(\hat{n})}{T} = \int d^3k\, e^{i\mathbf{k}\cdot\hat{n} d_A} \boldsymbol{\epsilon}_{\mathbf{k}} \int dt\, \mathcal{J}(k,\hat{k}\cdot\hat{n},t), \tag{15}$$

where $\mathcal{J}(k,\hat{k}\cdot\hat{n},t) \equiv a(t_L)^3 J(q,\hat{q}\cdot\hat{n},t)$, $\boldsymbol{\epsilon}_{\mathbf{k}} \equiv e_{\mathbf{q}}$, and $d_A \equiv r(t_L)a(t_L)$ is the angular diameter distance of the surface of last scattering.

Part of the observed temperature fluctuations arise from perturbations in scalar quantities, like the gravitational potential and the intrinsic temperature, and therefore make a contribution to $\mathcal{J}$ that is independent of $\hat{n}$. Another part arises from fluctuations in a vector, the velocity of the baryon-electron plasma, and therefore makes a contribution that is linear in $\hat{n}$. Leaving aside other effects like gravitational radiation, the function $\mathcal{J}$ therefore takes the form

$$\mathcal{J}(k,\hat{k}\cdot\hat{n},t) = \mathcal{F}(k,t) + i\hat{k}\cdot\hat{n}\, \mathcal{G}(k,t), \tag{16}$$

with $\mathcal{F}(k,t)$ arising from the Sachs-Wolfe effect and intrinsic temperature fluctuations, and $\mathcal{G}(k,t)$ arising from the Doppler effect. Using this in Eq. (15) then gives

$$\frac{\Delta T(\hat{n})}{T} = \int d^3k\, e^{i\mathbf{k}\cdot\hat{n} d_A} [F(k) + i\hat{k}\cdot\hat{n}\, G(k)] \boldsymbol{\epsilon}_{\mathbf{k}}, \tag{17}$$

which is the same as Eq. (1), with the form factors identified as time integrals

$$F(k) \equiv \int dt\, \mathcal{F}(k,t), \quad G(k) \equiv \int dt\, \mathcal{G}(k,t). \tag{18}$$

This time integration introduces a damping of the oscillatory part of the form factors, but this will be less important than the effects of heat conduction and viscosity in the time interval between recombination and last scattering.

## III. EVOLUTION OF PERTURBATIONS IN THE SYNCHRONOUS GAUGE

We now turn to the approximate analytic calculation of the form factors.

### A. General approximations

We make two assumptions that will allow great simplifications in this calculation:

(i) The contents of the universe up to the time of last scattering are taken to consist of collisionless cold dark matter, collisionless neutrinos, a baryon-electron plasma treated as a perfect fluid, and a photon gas coupled to the plasma by Thomson scattering, with a short but non-negligible photon mean free time. The finite duration of the era of last scattering, when the mean free time becomes too large to allow a hydrodynamic treatment, will be taken into account by the time integrals in Eq. (18).

(ii) It is assumed that only the cold dark matter contributes to the expansion rate of the universe before the time of last scattering and to perturbations in the metric. This is not a very good approximation, but it is the price that has to be paid to get analytic expressions for the observed temperature fluctuation. To minimize errors introduced by the incorrect treatment of acoustic oscillations before the cross-over time $t_C$ when the photon energy density equaled the dark matter energy density, it is necessary to restrict the wave number to be less than an upper bound given in Sec. V.

### B. Gravitational field

We begin by reminding the reader of the equations that govern perturbations in the metric and fluid properties before the time of last scattering. The perturbed metric is taken as

$$g_{\mu\nu\,\text{total}}(\mathbf{x},t)=g_{\mu\nu}(t)+h_{\mu\nu}(\mathbf{x},t), \tag{19}$$

where $g_{\mu\nu}$ is the Robertson-Walker metric in co-moving coordinates $\mathbf{x}$ with spatial curvature neglected:

$$g_{00}=-1,\quad g_{0i}=0,\quad g_{ij}=a^2(t)\,\delta_{ij}, \tag{20}$$

and $h_{\mu\nu}(\mathbf{x},t)$ is a small perturbation. We work in the synchronous gauge, defined by the conditions

$$h_{0i}=h_{00}=0, \tag{21}$$

and by the requirement that the cold dark matter particles have time-independent spatial coordinates. These conditions leave an unbroken residual gauge invariance, under the transformation

$$h_{ij}\rightarrow h_{ij}+a^2\left(\frac{\partial e_i}{\partial x_j}+\frac{\partial e_j}{\partial x_i}\right), \tag{22}$$

with $e_i$ an arbitrary function of $\mathbf{x}$ but independent of $t$. As we will see, in the synchronous gauge the evolution of the compressional modes that concern us here depends on the gravitational field only through a quantity that is invariant under these transformations,

$$\psi\equiv\frac{\partial}{\partial t}\left(\frac{h_{kk}}{2a^2}\right). \tag{23}$$

The spatial curvature is negligible at and before the time of last scattering, so it will be convenient to express $\psi(\mathbf{x},t)$ as a Fourier transform:

$$\psi(\mathbf{x},t)=\int d^3q\; e^{i\mathbf{q}\cdot\mathbf{x}}\psi_{\mathbf{q}}(t). \tag{24}$$

Likewise, the total proper energy density of each of the constituents of the universe (labeled $f=D,B,\gamma$ for dark matter, the baryon-electron plasma, and photons, respectively) is written

$$\varrho_{f\,\text{total}}(\mathbf{x},t)=\varrho_f(t)+\delta\rho_f(\mathbf{x},t),$$

$$\delta\rho_f(\mathbf{x},t)=\int d^3q\;\varrho_{f\mathbf{q}}(t)e^{i\mathbf{q}\cdot\mathbf{x}}, \tag{25}$$

with quantities carrying a subscript $\mathbf{q}$ denoting small perturbations. Under our second assumption, the gravitational field equation in the synchronous gauge reads [7]

$$\frac{d}{dt}(a^2\psi_{\mathbf{q}})=-4\pi G a^2\rho_{D\mathbf{q}}. \tag{26}$$

### C. Dark matter perturbations

The dark matter particles are assumed to ride on the expanding coordinate mesh, with negligible peculiar velocities. (This is not affected by perturbations to the gravitational field, because in the synchronous gauge these perturbations leave $\Gamma^i_{00}$ zero.) Hence their energy-momentum tensor has only a 00-component, $T^{00}_D=\varrho_{D\,\text{total}}$. For the metric (19)–(21), the energy conservation equation $T^{0\mu}_{D\;;\mu}=0$ then reads

$$\frac{d\rho_{D\mathbf{q}}}{dt}+\frac{3\dot a}{a}\rho_{D\mathbf{q}}+\psi_{\mathbf{q}}\rho_D=0$$

or in other words

$$\frac{d\,\delta_{D\mathbf{q}}}{dt}=-\psi_{\mathbf{q}}, \tag{27}$$

where $\delta_{D\mathbf{q}}$ is the fractional dark matter density perturbation:

$$\delta_{D\mathbf{q}}\equiv\rho_{D\mathbf{q}}/\rho_D. \tag{28}$$

Combining Eqs. (26) and (27) gives

$$\frac{d}{dt}\left(a^2\frac{d\,\delta_{D\mathbf{q}}}{dt}\right)=4\pi G a^2\rho_D\delta_{D\mathbf{q}}. \tag{29}$$

During the dark matter dominated era, $a\propto t^{2/3}$ and $4\pi G\rho_D=2/3t^2$, so Eq. (29) can be written

$$\frac{d}{dt}\left(t^{4/3}\frac{d\,\delta_{D\mathbf{q}}}{dt}\right)=\frac{2}{3}t^{-2/3}\delta_{D\mathbf{q}}. \tag{30}$$

As is well known, the two solutions go as $t^{-1}$ and $t^{2/3}$. If these two modes have comparable strengths for very small $t$, then the relevant solution is the one that is most rapidly growing, which we shall write as

$$\delta_{D\mathbf{q}}=N_{\mathbf{q}}t^{2/3},\quad \psi_{\mathbf{q}}=-\frac{2}{3}N_{\mathbf{q}}t^{-1/3}. \tag{31}$$

(The normalization constant $N_{\mathbf{q}}$ will play a role in this section similar to that of the constant $e_{\mathbf{q}}$ in Sec. II.)

### D. Plasma and photon perturbations

Next, let us consider the imperfect fluid formed by the baryon-electron plasma and the photons. It has a total velocity four-vector of the form

$$U^\mu_{\text{total}}(\mathbf{x},t)=U^\mu+\int d^3q\; U^\mu_{\mathbf{q}}(t)e^{i\mathbf{q}\cdot\mathbf{x}} \tag{32}$$

where $U^\mu$ is the unperturbed velocity four-vector

$$U^0=1, \quad U^i=0, \tag{33}$$

and $U^\mu_{\mathbf{q}}(t)$ is a small perturbation. The normalization condition $g_{\mu\nu\,\mathrm{total}}U^\mu_{\mathrm{total}}U^\nu_{\mathrm{total}}=-1$ tells us that the first-order perturbations are purely spatial,

$$U^0_{\mathbf{q}}(t)=0. \tag{34}$$

We will be considering only compressional modes, so we will assume that the spatial part of $U^\mu_{\mathrm{total}}$ is the gradient of a velocity potential $u$:

$$U^i_{\mathbf{q}}(t)=i\,q^i u_{\mathbf{q}}(t). \tag{35}$$

We will write the conservation laws for this fluid in terms of fractional perturbations to the baryon-electron plasma mass density and the photon energy density:

$$\delta_{B\mathbf{q}}\equiv\rho_{B\mathbf{q}}/\rho_B, \quad \delta_{\gamma\mathbf{q}}\equiv\rho_{\gamma\mathbf{q}}/\rho_\gamma. \tag{36}$$

The particle conservation equation [8] for the baryon-electron plasma mass density is then

$$\frac{d\delta_{B\mathbf{q}}}{dt}=q^2u_{\mathbf{q}}-\psi_{\mathbf{q}}. \tag{37}$$

The energy conservation equation [9] for the baryon-electron-photon fluid is

$$\frac{d}{dt}(\rho_B\delta_{B\mathbf{q}}+\rho_\gamma\delta_{\gamma\mathbf{q}})+\frac{3\dot{a}}{a}\left(\rho_B\delta_{B\mathbf{q}}+\frac{4}{3}\rho_\gamma\delta_{\gamma\mathbf{q}}\right)$$
$$=-\left(\rho_B+\frac{4}{3}\rho_\gamma\right)(\psi_{\mathbf{q}}-q^2u_{\mathbf{q}})-\chi q^2$$
$$\times\left(\dot{T}u_{\mathbf{q}}+\frac{T}{a^2}\frac{d(a^2u_{\mathbf{q}})}{dt}+\frac{T\delta_{\gamma\mathbf{q}}}{4a^2}\right), \tag{38}$$

where $T$ is the unperturbed photon temperature and $\chi$ is the coefficient of heat conduction caused by photon energy transport. Finally, the momentum conservation equation [10] is

$$\left[\frac{d}{dt}+16\pi G\eta\right]\left[-a^5\left(\rho_B+\frac{4}{3}\rho_\gamma-\chi\dot{T}\right)u_{\mathbf{q}}\right.$$
$$\left.+\chi Ta^3\left(\frac{\delta_{\gamma\mathbf{q}}}{4}+\frac{d}{dt}(a^2u_{\mathbf{q}})\right)\right]$$
$$=\frac{1}{3}a^3\rho_\gamma\delta_{\gamma\mathbf{q}}-\frac{4\eta a^3}{3}[-q^2u_{\mathbf{q}}+\psi_{\mathbf{q}}], \tag{39}$$

where $\eta$ is the coefficient of viscosity due to photon momentum transport. By using Eq. (37) and recalling that $\rho_B\propto a^{-3}$, $\rho_\gamma\propto a^{-4}$, and $T\propto a^{-1}$, we can simplify Eq. (38) to read

$$\frac{d}{dt}\left[\delta_{\gamma\mathbf{q}}-\frac{4}{3}\delta_{B\mathbf{q}}\right]=\frac{\chi T}{\rho_\gamma}\left[-\frac{1}{a}\frac{\partial}{\partial t}[a(\dot{\delta}_{B\mathbf{q}}+\psi)]-\frac{q^2\delta_{\gamma\mathbf{q}}}{4a^2}\right]. \tag{40}$$

Also, using Eq. (37) lets us write Eq. (39) as

$$\left[\frac{d}{dt}+16\pi G\eta\right]\left[a^5\left(\rho_B+\frac{4}{3}\rho_\gamma-\chi\dot{T}\right)(\dot{\delta}_{B\mathbf{q}}+\psi_{\mathbf{q}})\right.$$
$$\left.-\chi Ta^3\left(\frac{q^2\delta_{\gamma\mathbf{q}}}{4}+\frac{d}{dt}[a^2(\dot{\delta}_{B\mathbf{q}}+\psi_{\mathbf{q}})]\right)\right]$$
$$=-\frac{q^2a^3}{3}[\rho_\gamma\delta_{\gamma\mathbf{q}}+4\eta\dot{\delta}_{B\mathbf{q}}]. \tag{41}$$

Now, $\eta/\rho_\gamma$ and $\chi T/\rho_\gamma$ are of the order of the photon mean free time, which as long as hydrodynamics is applicable must be short compared with the cosmic age. Therefore we can neglect $\eta$ and $\chi$ everywhere, except where they are accompanied with a maximum number of space and/or time derivatives of $\delta_{B\mathbf{q}}$ or $\delta_{\gamma\mathbf{q}}$, in which case powers of a high wave number can compensate for the smallness of $\chi$ or $\eta$. Then Eqs. (40) and (41) simplify further to

$$\frac{d}{dt}\left[\delta_{\gamma\mathbf{q}}-\frac{4}{3}\delta_{B\mathbf{q}}\right]=\frac{\chi T}{\rho_\gamma}\left[-\ddot{\delta}_{B\mathbf{q}}-\frac{q^2\delta_{\gamma\mathbf{q}}}{4a^2}\right], \tag{42}$$

$$\frac{d}{dt}\left[a^5\left(\rho_B+\frac{4}{3}\rho_\gamma\right)(\dot{\delta}_{B\mathbf{q}}+\psi_{\mathbf{q}})\right]$$
$$-\chi Ta^3\left(\frac{q^2\dot{\delta}_\gamma}{4}+a^2\frac{d^3\delta_{B\mathbf{q}}}{dt^3}\right)$$
$$=-\frac{q^2a^3}{3}\rho_\gamma\delta_{\gamma\mathbf{q}}-\frac{4q^2\eta a^3}{3}\dot{\delta}_{B\mathbf{q}}. \tag{43}$$

We also neglect terms of second order in $\chi$ and/or $\eta$, so we can set $\dot{\delta}_{B\mathbf{q}}$ equal to $3\dot{\delta}_{\gamma\mathbf{q}}/4$ in the dissipative terms in Eqs. (42) and (43). Then using Eq. (42) to eliminate $\delta_{B\mathbf{q}}$ in Eq. (43) gives our differential equation for $\delta_{\gamma\mathbf{q}}$:

$$\frac{d}{dt}\left[a^5\left(\rho_B+\frac{4}{3}\rho_\gamma\right)\left(\frac{3}{4}\frac{d\delta_{\gamma\mathbf{q}}}{dt}+\psi_{\mathbf{q}}\right)\right]$$
$$+\frac{3a^5\chi T\rho_B}{4\rho_\gamma}\left(\frac{3}{4}\frac{d^3\delta_{\gamma\mathbf{q}}}{dt^3}+\frac{q^2}{4a^2}\frac{d\delta_{\gamma\mathbf{q}}}{dt}\right)$$
$$=-\frac{q^2a^3}{3}\rho_\gamma\delta_{\gamma\mathbf{q}}-\eta q^2a^3\frac{d\delta_{\gamma\mathbf{q}}}{dt}. \tag{44}$$

It will be convenient to multiply with $t^{4/3}/a^5\rho_\gamma$. Recalling that $\rho_\gamma\propto a^{-4}$, $\rho_B\propto a^{-3}$ and $a\propto t^{2/3}$, this gives finally

$$t^{2/3}\frac{d}{dt}\left[(1+R)t^{2/3}\frac{d\delta_{\gamma\mathbf{q}}}{dt}\right]+\frac{k^2t_L^{4/3}}{3}\delta_{\gamma\mathbf{q}}+\frac{\eta k^2t_L^{4/3}}{\rho_\gamma}\frac{d\delta_{\gamma\mathbf{q}}}{dt}$$
$$+\frac{R\chi T}{4\rho_\gamma}\left(3t^{4/3}\frac{d^3\delta_{\gamma\mathbf{q}}}{dt^3}+k^2t^{4/3}\frac{d\delta_{\gamma\mathbf{q}}}{dt}\right)$$
$$=-\frac{4t^{2/3}}{3}\frac{d}{dt}[(1+R)t^{2/3}\psi_\mathbf{q}]=\frac{8N_\mathbf{q}}{27}(1+3R), \quad (45)$$

where (as before) $t_L$ is some typical time of last scattering, $k\equiv q/a(t_L)=qt^{2/3}/t_L^{2/3}a(t)$, and

$$R\equiv 3\rho_B/4\rho_\gamma\propto a. \quad (46)$$

We next turn to two different ranges of wave number in which it is possible to find an analytic solution of this equation.

### E. Solution for large *k*

We consider first wave numbers that are large enough to allow the use of the WKB approximation. For this purpose, we introduce a new variable

$$\zeta\equiv(t/t_L)^{1/3}. \quad (47)$$

(This is the usual conformal time $\eta$, but with a different normalization.) Multiplying Eq. (45) with $9t_L^{2/3}$ then gives

$$\frac{d}{d\zeta}\left[(1+\xi\zeta^2)\frac{d\delta_{\gamma\mathbf{q}}}{d\zeta}\right]+3k^2t_L^2\delta_{\gamma\mathbf{q}}+\frac{3\eta k^2t_L}{\rho_\gamma\zeta^2}\frac{d\delta_{\gamma\mathbf{q}}}{d\zeta}$$
$$+\frac{\xi\chi T}{4\rho_\gamma}\left(\frac{1}{t_L}\frac{d^3\delta_{\gamma\mathbf{q}}}{d\zeta^3}+3k^2t_L\frac{d\delta_{\gamma\mathbf{q}}}{d\zeta}\right)$$
$$=\frac{8N_\mathbf{q}t_L^{2/3}}{3}(1+3\xi\zeta^2). \quad (48)$$

Here we have again assumed that dissipative terms are negligible except where a maximum number of derivatives (i.e., factors of $k$ and/or $\zeta$-derivatives) acts on $\delta_{\gamma\mathbf{q}}$. We have also used the fact that $R\propto a$ to set $R=\xi\zeta^2$, where $\xi$ is the ratio (46) at time $t_L$.

In the absence of dissipation, Eq. (48) would have the exact solution

$$\delta_{\gamma\mathbf{q}}=\frac{8N_\mathbf{q}t_L^{2/3}(1+3\xi\zeta^2)}{9(k^2t_L^2+2\xi)}. \quad (49)$$

(This is actually independent of our choice of $t_L$, because $\xi$ and $k^2t_L^2$ both scale as $t_L^{2/3}$.) The neglect of dissipation is justified in this solution, because the rate of change of this expression does not yield a factor of the large wave number $k$ that could compensate for the smallness of $\chi$ and $\eta$.

To this particular solution, we must add a suitable solution of the corresponding homogeneous equation. In the absence of damping we can find exact solutions of the form $P_\nu(i\sqrt{\xi}\zeta)$, where $P_\nu(z)$ is the usual Legendre function, and $\nu$ is either of the roots of the quadratic equation $\nu(\nu+1)=-3k^2t_L^2/\xi$. But this will not be useful in calculating the $C_l$ in our companion paper [1]. To get a more useful result, we must use the WKB approximation.

Under the assumption that

$$kt_L\gg\xi \quad (50)$$

we can find a pair of approximate solutions of the homogeneous equation

$$\delta_{\gamma\mathbf{q}}\propto\exp(\pm i\varphi) \quad (51)$$

with

$$\varphi=\sqrt{3}kt_L\int_0^\zeta\frac{d\zeta}{\sqrt{1+\xi\zeta^2}} \quad (52)$$

provided we neglect dissipative terms. Note that if $\xi$ as well as $\eta$ and $\chi$ were zero, then these homogeneous solutions would be exact. More generally, inspection of Eq. (48) shows that these are approximate solutions if the fractional rate of change of $1+\xi\zeta^2$ is small compared with the rate of change of the phase $\varphi$:

$$\frac{2\xi\zeta}{1+\xi\zeta^2}\ll\frac{kt_L\sqrt{3}}{\sqrt{1+\xi\zeta^2}},$$

which is true at all times if and only if it is satisfied at $\zeta=1$, i.e.,

$$kt_L\gg\frac{2\xi}{\sqrt{3(1+\xi)}}.$$

For plausible values of $\xi$ this condition is actually somewhat weaker than Eq. (50), but we will need the greater strength of Eq. (50) later, when we calculate the plasma velocity potential.

We can do better than Eq. (51), and include the effects of viscosity and heat conduction, by seeking solutions of the form $\delta_{\gamma\mathbf{q}}=A\exp(\pm i\varphi)$, with $A$ a slowly varying real amplitude. By calculating the rate of change of the Wronskian of these two solutions [and replacing $d^3\delta_{\gamma\mathbf{q}}/d\zeta^3$ with $-3k^3t_L^2(1+\xi\zeta^2)^{-1}d\delta_{\gamma\mathbf{q}}/d\zeta$ in the dissipative term], we easily find the WKB solutions of the homogenous equation:

$$\delta_{\gamma\mathbf{q}}\propto(1+\xi\zeta^2)^{-1/4}\exp[\pm i\varphi-k^2\mathcal{D}^2], \quad (53)$$

where

$$\mathcal{D}^2=3t_L\int_0^\zeta\left[\frac{\eta}{2\rho_\gamma(1+\xi\zeta^2)}+\frac{\chi T\xi^2\zeta^4}{8\rho_\gamma(1+\xi\zeta^2)^2}\right]\zeta^2d\zeta. \quad (54)$$

The viscosity and heat conduction coefficients are given by [11]

$$\eta=\frac{16}{45}\rho_\gamma\tau_\gamma,\quad \chi T=\frac{4}{3}\rho_\gamma\tau_\gamma. \quad (55)$$

Here $\tau_\gamma$ is the photon mean free time

$$\tau_\gamma = \frac{1}{\sigma_T n_e c} = \frac{t_L(6\pi G m_p)^{1/2}(\Omega_M/\Omega_B)^{1/2}\zeta^{9/2}}{\sigma_T c(2\pi m_e k_B T_0/h^2)^{3/4}(1+z_L)^{3/4}(1-Y/2)^{1/2}} \times \exp\left(\frac{\zeta^2\Delta}{2k_B T_L}\right), \tag{56}$$

where $\sigma_T$ is the Thomson scattering cross section, $h$ is (only here) the original Planck constant, $T_0=2.738$ K is the present microwave background temperature, $T_L=3100$ K is the temperature at last scattering, $k_B$ is Boltzmann's constant, $Y\simeq .23$ is the primordial helium abundance, and $\Delta = 13.6$ eV is the hydrogen ionization energy.

The relevant solution is again the one that increases most rapidly at early times, which we can find by requiring that $\delta_{\gamma\mathbf{q}}\to 0$ as $\zeta\to 0$. In the limit $\zeta\to 0$ the phase $\varphi$ vanishes as $O(\zeta)$, while $\mathcal{D}^2$ vanishes more rapidly because the mean free time of photons is very small at early times. Hence the linear combination of the particular inhomogeneous solution (49) and the homogeneous solutions (53) that grows most rapidly at early times is

$$\delta_{\gamma\mathbf{q}} = \frac{8N_\mathbf{q} t_L^{2/3}}{9(k^2t_L^2+2\xi)}[1+3\xi\zeta^2-(1+\xi\zeta^2)^{-1/4}e^{-k^2\mathcal{D}^2}\cos\varphi]. \tag{57}$$

[We would be able to neglect the term $2\xi$ in the denominator only under the condition $kt_L \gg \sqrt{2\xi}$, which for plausible values of $\xi$ is stronger than our assumption (50).]

To calculate the velocity potential of the plasma-photon fluid for large wave numbers, we will also need the rate of change of $\delta_{B\mathbf{q}}$. At times of order $t_L$, the time derivatives of $\xi\zeta^2$, $\varphi$, and $\mathcal{D}^2$ are of the orders of $\xi/t_L$, $k$, and $\tau_L$, respectively, where $\tau_L$ is the photon mean free time $\tau\approx\eta/\rho_\gamma \approx \chi T/\rho_\gamma$ at time $t_L$. We are assuming that $kt_L \gg \xi$, so the time derivative of $\varphi$ is larger than the time derivative of $\xi\zeta^2$. Equation (56) shows that damping becomes important if $k^2 t_L\tau_L \gg 1$, but even for such large values of $k$ we can still limit ourselves to the case

$$k\tau_L \ll 1, \tag{58}$$

in which case the time derivative of $\varphi$ is also larger than the time derivative of $k^2\mathcal{D}^2$. Hence for wave numbers $k$ in the range defined by Eqs. (50) and (58), we have

$$\frac{d\delta_{\gamma\mathbf{q}}}{dt} \simeq \frac{8N_\mathbf{q} t_L^{2/3} k e^{-k^2\mathcal{D}^2}\sin\varphi}{9\sqrt{3}(1+\xi\zeta^2)^{3/4}(k^2t_L^2+2\xi)\zeta^2}. \tag{59}$$

The dissipative terms in Eq. (42) are smaller than this by a factor $k\tau$, so here we can take $\dot\delta_{B\mathbf{q}} \simeq 3\dot\delta_{\gamma\mathbf{q}}/4$, and Eqs. (37), (31) and (59) then give the velocity potential

$$u_\mathbf{q} = \frac{2N_\mathbf{q}}{3k^2a^2(t_L)t_L^{1/3}\zeta}\left[-1+\frac{kt_L e^{-k^2\mathcal{D}^2}\sin\varphi}{\sqrt{3}(1+\xi\zeta^2)^{3/4}(k^2t_L^2+2\xi)\zeta}\right]. \tag{60}$$

### F. Solution for small $k$

Here we can neglect viscosity and heat conduction. For $k=0$, Eq. (48) has an obvious solution

$$\delta_{\gamma\mathbf{q}} = 4N_\mathbf{q} t_L^{2/3}\zeta^2/3 = 4\delta_{D\mathbf{q}}/3. \tag{61}$$

To this we can add any linear combination of the two solutions of the corresponding homogeneous solution, for which $\delta_{\gamma\mathbf{q}}$ is respectively time-independent or proportional to

$$\int_0^\zeta \frac{d\zeta}{1+\xi\zeta^2},$$

which near the beginning of the dark-matter dominated era goes as $\zeta$. As $\zeta$ increases these homogeneous solutions become negligible compared with the inhomogeneous solution (61), so at later times the solution for $k=0$ is given by Eq. (61).

To get the term in $\delta_{\gamma\mathbf{q}}$ of first order in $k^2$, we can use the solution (61) in the terms in Eq. (48) proportional to $k^2$, so that

$$\frac{d}{d\zeta}\left[(1+\xi\zeta^2)\frac{d}{d\zeta}\left(\frac{3}{4}\delta_{\gamma\mathbf{q}}-\delta_{D\mathbf{q}}\right)\right] = -3k^2t_L^2\delta_{D\mathbf{q}}. \tag{62}$$

Discarding a homogeneous term for the same reason as before, we have

$$\frac{d}{d\zeta}\left(\frac{3}{4}\delta_{\gamma\mathbf{q}}-\delta_{D\mathbf{q}}\right) = -\frac{k^2t_L^2\zeta\,\delta_{D\mathbf{q}}}{1+\xi\zeta^2} \tag{63}$$

which gives

$$\delta_{\gamma\mathbf{q}} = \frac{4N_\mathbf{q}t_L^{2/3}}{3}\zeta^2\left[1-\frac{k^2t_L^2}{2}\left(\frac{1}{\xi}-\frac{1}{\zeta^2\xi^2}\ln(1+\xi\zeta^2)\right)+\cdots\right]. \tag{64}$$

Also, Eqs. (37), (27), (63) and (31) give the plasma velocity potential for $k\to 0$ as

$$u_\mathbf{q} = \frac{1}{q^2}\frac{d}{dt}(\delta_{B\mathbf{q}}-\delta_{D\mathbf{q}}) \to -\frac{N_\mathbf{q}t_L^{5/3}\zeta}{3a^2(t_L)(1+\xi\zeta^2)}. \tag{65}$$

As we will see, this provides a small correction to the Doppler shift, which for small $k$ will turn out to be mostly due to perturbations in the gravitational field.

## IV. OBSERVED TEMPERATURE FLUCTUATIONS

There are three separate sources of the observed temperature fluctuation in the cosmic microwave background: the Sachs-Wolfe effect due to perturbations in the gravitational potential, the Doppler effect due to plasma peculiar velocities, and the intrinsic temperature fluctuations themselves. We will consider each of these in turn, and then put the results together. In calculating the Sachs-Wolfe and Doppler contributions, we will use a non-relativistic approach, taking the effect of the gravitational field perturbations on the observed photon temperature to consist entirely of the time dilation caused by a Newtonian gravitational potential plus the Doppler shift caused by the gravitational acceleration of the source and receiver. This approach has the virtue of getting useful results quickly, but the results obtained in this need to be justified by a thoroughly relativistic treatment of the Sachs-Wolfe and Doppler effects, which will be given in the Appendix.

### A. Sachs-Wolfe effect

We can define a Newtonian gravitational potential $\phi$ as the solution of the Poisson equation

$$a^{-2}(t)\nabla^2\phi(\mathbf{x},t)=4\pi G\,\delta\rho_D(\mathbf{x},t) \tag{66}$$

with the factor $a^{-2}$ inserted to take account of the difference between the Robertson-Walker co-moving coordinate vector $\mathbf{x}$ used here and the coordinate vector $a(t)\mathbf{x}$ that measures proper distances at time $t$. Using Eqs. (31) and (25), this gives

$$\phi(\mathbf{x},t)=-4\pi G\rho_D(t)t^{2/3}a^2(t)\int d^3q\,q^{-2}e^{i\mathbf{q}\cdot\mathbf{x}}N_{\mathbf{q}}$$

$$=-\frac{2a^2(t)}{3t^{4/3}}\int d^3q\,q^{-2}e^{i\mathbf{q}\cdot\mathbf{x}}N_{\mathbf{q}}. \tag{67}$$

It is important to note that this is time-independent during the dark matter era, when $a(t)\propto t^{2/3}$.

This potential makes two separate contributions to the Sachs-Wolfe effect. There is a gravitational redshift, yielding a fractional fluctuation in the observed temperature in a direction $\hat{n}$ equal to $\phi(r_L\hat{n})-\phi(0)$, where $r_L$ is the Robertson-Walker radial coordinate of the surface of last scattering. There is also a time-delay; if the unperturbed cosmic temperature reaches the value $T_L\simeq 3000$ K of last scattering at a time $t_L$, then the gravitational potential causes the cosmic temperature in a direction $\hat{n}$ to reach the value $T_L$ at a time $[1+\phi(r_L\hat{n})-\phi(0)]t_L$, so that the redshifted temperature seen now is changed by a fractional amount [12]:

$$-[t_L\dot{a}(t_L)/a(t_L)][\phi(r_L\hat{n})-\phi(0)]=-\frac{2}{3}[\phi(r_L\hat{n})-\phi(0)].$$

(This argument is valid only because $\phi$ is time-independent; otherwise we would have to consider the complete gravitationally delayed time-history of the cosmic temperature, as done in the Appendix.) Combining the two effects, the net fractional change in observed temperature is

$$\left(\frac{\Delta T(\mathbf{n})}{T}\right)_{\text{Sachs-Wolfe}}=\frac{1}{3}[\phi(r_L\hat{n})-\phi(0)]. \tag{68}$$

As we shall see in the Appendix, this formula can be derived using the formalism of general relativity, which in the synchronous gauge gives the famous factor of 1/3 directly, without having to consider separately the gravitational redshift and expansion time delay.

It will be convenient to rewrite Eq. (67) in terms of the physical wave number at the time of last scattering, $k\equiv q/a(t_L)$, so that Eq. (68) gives

$$\left(\frac{\Delta T(\mathbf{n})}{T}\right)_{\text{Sachs-Wolfe}}=\int d^3k[e^{i\mathbf{k}\cdot\hat{n}d_A}-1]\epsilon_{\mathbf{k}}, \tag{69}$$

where $\epsilon_{\mathbf{k}}$ is an amplitude for fluctuations not processed by acoustic oscillations, defined by

$$\epsilon_{\mathbf{k}}d^3k\equiv-\frac{2N_{\mathbf{q}}a^2(t_L)}{9q^2t_L^{4/3}}d^3q, \tag{70}$$

and $d_A=r_La(t_L)$ is the angular diameter distance of the surface of last scattering.

### B. Doppler shifts

The plasma velocity potential $u_{\mathbf{q}}$ calculated in Sec. III yields a pressure-induced plasma velocity perturbation

$$\mathbf{v}_{\text{pressure}}(\mathbf{x},t)=a(t)\boldsymbol{\nabla}\int d^3q\,e^{i\mathbf{q}\cdot\mathbf{x}}u_{\mathbf{q}}(t). \tag{71}$$

[The factor $a(t)$ enters because it is the velocity in co-moving coordinates that is given by the co-moving gradient of the velocity potential.] This yields a Doppler shift of the temperature of the cosmic microwave background seen in a direction $\hat{n}$:

$$\left(\frac{\Delta T(\hat{n})}{T}\right)_{\text{pressure Doppler}}=-ia(t_L)\int d^3q\,\hat{n}\cdot\mathbf{q}\,u_{\mathbf{q}}(t_L)e^{i\mathbf{q}\cdot\hat{n}r_L}$$

$$=-i\int d^3k\,\hat{n}\cdot\hat{k}\,\epsilon_{\mathbf{k}}g(k)e^{i\mathbf{k}\cdot\hat{n}d_A}, \tag{72}$$

with the form factor $g(k)$ given by Eq. (65) for small $k$ as

$$g(k)=\frac{3k^3t_L^3}{2(1+\xi)} \tag{73}$$

and for large $k$ by Eq. (60) as

$$g(k)=3kt_L-\sqrt{3}(1+\xi)^{-3/4}(1+2\xi/k^2t_L^2)^{-1}e^{-k^2d_D^2}\sin(kd_H) \tag{74}$$

where $\xi$ as before is 3/4 the ratio of baryon and photon energy densities at the time of last scattering, $\mathcal{D}_L$ is the damping length $\mathcal{D}$ given in Eq. (56), evaluated at $\zeta=1$ (actually, as discussed in the next section, at $\zeta$ a little less than unity), and $d_H$ is the acoustic horizon at the time of last scattering:

$$d_H=\sqrt{3}t_L\int_0^1\frac{d\zeta}{\sqrt{1+\xi\zeta^2}}=\frac{\sqrt{3}t_L}{\sqrt{\xi}}\ln(\sqrt{\xi}+\sqrt{1+\xi}). \tag{75}$$

In the non-relativistic approach used here, there is also an additional velocity perturbation induced by the gravitational potential $\phi(\mathbf{x})$. The proper peculiar velocity $\mathbf{v}_{\rm grav}$ produced in this way is given by the equation of motion [13]

$$\frac{\partial}{\partial t}\mathbf{v}_{\rm grav}(\mathbf{x},t)+\frac{\dot{a}(t)}{a(t)}\mathbf{v}_{\rm grav}(\mathbf{x},t)=-\frac{1}{a(t)}\nabla\phi(\mathbf{x}). \tag{76}$$

Because the gravitational potential $\phi$ is time-independent, this has the simple solution

$$\mathbf{v}_{\rm grav}(\mathbf{x},t)=-a^{-1}(t)t\,\nabla\,\phi(\mathbf{x})=\frac{2ia(t)}{3\,t^{1/3}}\int d^3q\,q^{-2}\mathbf{q}\,e^{i\mathbf{q}\cdot\mathbf{x}}N_{\mathbf{q}}. \tag{77}$$

This contributes a fractional temperature shift seen in a direction $\hat{n}$:

$$\left(\frac{\Delta T(\hat{n})}{T}\right)_{\text{gravity Doppler}}=-\hat{n}\cdot[\mathbf{v}_{\rm grav}(\hat{n}r_L,t_L)-\mathbf{v}_{\rm grav}(0,t_0)]$$
$$=3i\int d^3k(\hat{k}\cdot\hat{n})kt_L\epsilon_{\mathbf{k}}\times\left[e^{i\mathbf{k}\cdot\hat{n}d_A}-\frac{t_L^{1/3}a(t_0)}{t_0^{1/3}a(t_L)}\right], \tag{78}$$

where $t_0$ is the present time. (A general relativistic derivation of this result in the synchronous gauge is given in the Appendix.)

### C. Intrinsic temperature fluctuations

The fractional change in the photon temperature is one-fourth the fractional change in the photon energy density. The contribution of intrinsic density fluctuations at the time $t_L$ of last scattering to the fractional change of temperature seen coming from a direction $\hat{n}$ is, therefore,[2]

$$\left(\frac{\Delta T(\hat{n})}{T}\right)_{\text{intrinsic}}=\frac{\delta\rho_\gamma(\hat{n}r_L,t_L)}{4\rho_\gamma(t_L)}=\frac{1}{4}\int d^3q\,e^{i\mathbf{q}\cdot\hat{n}r_L}\delta_{\gamma\mathbf{q}}(t_L). \tag{79}$$

Equations (57), (64) and (70) then give

$$\left(\frac{\Delta T(\hat{n})}{T}\right)_{\text{intrinsic}}=\int d^3k\,\epsilon_{\mathbf{k}}f(k)e^{i\mathbf{k}\cdot\hat{n}d_A}, \tag{80}$$

with the partial form factor $f$ given by

$$f(k)=\begin{cases}-3k^2t_L^2/2+3[\xi^{-1}-\xi^{-2}\ln(1+\xi)]k^4t_L^4/4+\cdots, & k\to 0\\ (1+2\xi/k^2t_L^2)^{-1}[-1-3\xi+(1+\xi)^{-1/4}e^{-k^2\mathcal{D}_L^2}\cos(kd_H)], & k\text{ large.}\end{cases} \tag{81}$$

### D. Total temperature fluctuations

We now put together the fractional temperature fluctuations given by Eqs. (69), (72), (78), and (80), and obtain the total fractional temperature fluctuation

$$\left(\frac{\Delta T(\hat{n})}{T}\right)=\int d^3k\,\epsilon_{\mathbf{k}}\left\{[F(k)+i\hat{k}\cdot\hat{n}G(k)]e^{i\mathbf{k}\cdot\hat{n}d_A}-1-3i\mathbf{k}\cdot\hat{n}\frac{t_L^{4/3}a(t_0)}{t_0^{1/3}a(t_L)}\right\}, \tag{82}$$

where $F(k)$ is the total scalar form factor, given by Eqs. (69), (80), and (81) as

$$F(k)=1+f(k)$$
$$=\begin{cases}1-3k^2t_L^2/2-3[-\xi^{-1}+\xi^{-2}\ln(1+\xi)]k^4t_L^4/4+\ \dots, & k\to 0,\\ (1+2\xi/k^2t_L^2)^{-1}[-3\xi+2\xi/k^2t_L^2+(1+\xi)^{-1/4}e^{-k^2\mathcal{D}_L^2}\cos(kd_H)], & k\text{ large}\end{cases} \tag{83}$$

and $G(k)$ is the total dipole form factor, given by Eqs. (72), (73), (74), and (78) as

[2]There is a subtlety here. To the extent that the opacity drops sharply from 100% to zero, last scattering occurs at a fixed value of the *perturbed* temperature $T+\delta T$, near 3000 K, rather than at a fixed value of the unperturbed temperature or the time. The effect of the intrinsic temperature fluctuation $\delta T(t)$ is thus to change the time of last scattering, in such a way as to produce a change $-\delta T$ in the value of the *unperturbed* temperature $T(t)$ at this time. Since $T(t)\propto 1/a(t)$, we then have $\delta a/a=+\delta T/T$ at the time of last scattering, so that the observed temperature is shifted by the change in the cosmological redshift by a fractional amount $\Delta T/T=\delta a/a=+\delta T/T$.

$$G(k)=3kt_L-g(k)=\begin{cases}3kt_L-3k^3t_L^3/2(1+\xi)+\ \dots, & k\to 0\\ \sqrt{3}(1+\xi)^{-3/4}(1+2\xi/k^2t_L^2)^{-1}e^{-k^2\mathcal{D}_L^2}\sin(kd_H), & k \text{ large.}\end{cases} \tag{84}$$

The last two terms in the curly brackets in Eq. (82) contribute only to the multipole coefficients $C_l$ for $l=0$ and $l=1$ [15], and may therefore be dropped [as they are in Eq. (1)] in considering the higher multipoles.

We see that the WKB solution for large $k$ gives a poor picture of what happens for $k\to 0$, except in the case $\xi\ll 1$, where $d_H=\sqrt{3}t_L$, in which case the above pairs of expressions for $F(k)$ and $G(k)$ agree for small $k$.

As discussed in Sec. II, it still remains to average over the time of last scattering. The effect of this averaging on the damping factor $\exp -k^2\mathcal{D}^2$ is small [14]. Otherwise, the averaging over $t$ chiefly affects the $\sin kd_H$ and $\cos kd_H$ factors in Eqs. (83) and (84), which oscillate rapidly with the time of last scattering when $k$ is large. We will approximate the probability distribution of the actual time of last scattering $t$ as a Gaussian of the form $(1/\pi\,\Delta t)\exp(-(t-t_L)^2/\Delta t^2)$, where $t_L$ is a nominal time of last scattering. Replacing $t_L$ in the sines and cosines in Eqs. (83) and (84) with $t$, multiplying with this probability distribution, and integrating over $t$ then gives the same result for the form factors for large $k$, but with an additional term now added to $\mathcal{D}_L^2$:

$$\Delta\mathcal{D}_L^2=d_H^2\left(\frac{\Delta t}{2t_L}\right)^2. \tag{85}$$

This is a sort of "Landau damping," except that the damping arises from a spread in the time at which the temperature of the medium is observed rather than from a spread in wave numbers. As we will see in the next section, this term makes a smaller but not insignificant contribution to the total damping.

## V. DISCUSSION

In a companion paper [1] we show how to use the formula (82) for the total temperature fluctuation to derive expressions for the coefficient $C_l$ of the term of multipole number $l$ in the temperature fluctuation correlation function for general form factors $F(k)$ and $G(k)$. As we will see there, the contribution of the scalar form factor $F(k)$ to $C_l$ arises mostly from wave numbers of order $l/d_A$ (where $d_A$ is the angular diameter distance of the surface of last scattering), while this approximation is much worse for the contribution of the dipole form factor $G(k)$.

For the present, we will content ourselves with noting that if we tentatively use the WKB approximation, neglect damping effects, and drop the terms in the second line of Eq. (83) proportional to $\xi/k^2t_L^2$, then for $\xi$ less than 0.311 [that is, for $3\xi<(1+\xi)^{-1/4}$] the squared scalar form factor $F^2(k)$ has peaks at the wave numbers

$$k_n=n\pi/d_H \tag{86}$$

(with $n=1,2,\dots$), with higher peaks for odd $n$ [where the two terms in $F(k)$ have the same sign] than for even $n$. The minima are at the zeros of $F(k)$. For $\xi>.311$ the only peaks are those for $n$ odd, and the minima are at $n$ even. This suggests that there should be peaks in $C_l$ near $l_n=(2n-1)\pi d_A/d_H$ and either lower peaks or dips near $2n\pi d_A/d_H$, depending on the value of $\xi$. These peaks are known as the Doppler peaks [though Eq. (84) shows that the contribution of the Doppler shift is very small at all the wave numbers $k_n$]. These results depend critically on the negative sign of the term $-3\xi$ in the second line of Eq. (83); if this term had turned out to be positive then for $\xi>.311$ the positions of the peaks and dips would be interchanged. Despite what is sometimes said [16], there is no way without detailed calculations to see that the first Doppler peak should be at $l\simeq\pi d_A/d_H$, rather than at a multipole number twice as large.

We can now check whether the WKB approximation used in Sec. III E is valid at the first Doppler peak. According to Eqs. (86) and (75), we have

$$k_1t_L=\frac{\pi\sqrt{\xi}}{\sqrt{3}\ln(\sqrt{\xi}+\sqrt{1+\xi})}, \tag{87}$$

so the ratio of the wave number at the first Doppler peak to the mininum wave number $k_{\min}$ allowed by the inequality (50) is

$$\frac{k_1}{k_{\min}}=\frac{\pi}{\sqrt{3\xi}\ln(\sqrt{\xi}+\sqrt{1+\xi})}. \tag{88}$$

The WKB approximation is valid at wave numbers down to the first Doppler peak if this ratio is sufficiently larger than unity. For instance, for $\Omega_Bh^2=0.03$ we have $\xi=0.81$, so Eq. (88) gives $k_1/k_{\min}=2.5$, making the WKB approximation fairly good at the first Doppler peak. The WKB approximation is somewhat better at $k_1$ for smaller values of $\Omega_Bh^2$, though it still breaks down at smaller wave numbers unless $\Omega_Bh^2=0$. For all plausible values of $\xi$ the WKB approximation is excellent at the higher Doppler peaks.

Next, let us consider the importance of damping. It might seem that we should calculate the damping length $\mathcal{D}_L$ by integrating in Eq. (54) up to the time of last scattering, corresponding to $\zeta=1$. But at the nominal time of last scattering (defined so that the probability of any future scattering is 50%), the photon collision rate $1/\tau_\gamma$ given by Eq. (56) is $0.2\sqrt{\Omega_B/\Omega_M}/t_L$, which is already considerably smaller than the expansion rate $2/3t_L$, so that we cannot trust the hydrodynamic calculations used to obtain Eq. (54). We will instead integrate in Eq. (54) only up to a value $\zeta_{\max}$ of $\zeta$ at which the photon collision rate becomes equal to the expansion rate, and set

$$\tau_\gamma \simeq \frac{3t_L}{2}\left(\frac{\zeta}{\zeta_{\max}}\right)^{9/2}\exp\left(-\frac{\Delta}{2k_BT_L}(\zeta_{\max}^2-\zeta^2)\right).$$

The exponential factor (with $\Delta/2k_BT_L=25.5$) is so sharply peaked at $\zeta=\zeta_{\max}$ that we can approximate $\zeta_{\max}^2-\zeta^2 \simeq 2\zeta_{\max}(\zeta_{\max}-\zeta)$ in the exponent and set $\zeta$ equal to $\zeta_{\max}$ everywhere else in the integral, giving

$$\mathcal{D}_L^2 \simeq \frac{3t_L^2}{2\zeta_{\max}^3}\left(\frac{8}{15(1+\xi\zeta_{\max}^2)}+\frac{\xi^2\zeta_{\max}^4}{2(1+\xi\zeta_{\max}^2)^2}\right)\left(\frac{k_BT_L}{\Delta}\right).$$

Furthermore, $\zeta_{\max}$ is very close to unity. (For instance, for $\Omega_M/\Omega_B=7.5$, we have $\zeta_{\max}=0.96$. That is, we carry the damping integral down to a temperature $T_L/\zeta_{\max}^2 \simeq 3360$ K instead of 3100 K.) Hence in this result we may as well replace $\zeta_{\max}$ with unity, so that

$$\mathcal{D}_L^2 \simeq \frac{3t_L^2}{2}\left(\frac{8}{15(1+\xi)}+\frac{\xi^2}{2(1+\xi)^2}\right)\left(\frac{k_BT_L}{\Delta}\right).$$

This approximation leads to the additional simplification that the damping length is independent of most of the parameters appearing in Eq. (56), including the ratio $\Omega_B/\Omega_M$.

There is a smaller additional contribution from the averaging over oscillatory terms, given by Eq. (85). To evaluate this Landau damping term, we will need the ratio $\Delta t/t_L$. We noted in Sec. II that the probability that a photon will not be scattered again rises from 2% at about 3400 K to 98% at about 2800 K, with very little dependence on any cosmological parameters. Matching this to the probabilities calculated from the approximation that the probability of scattering in a time interval from $t$ to $t+\Delta t$ is a Gaussian $(dt/\pi\,\Delta t)\exp(-(t-t_L)^2/\Delta t^2)$, and using the relation $T \propto t^{-2/3}$, we find $\Delta t/t_L=0.10$, so the contribution to $\mathcal{D}_L^2$ in Eq. (85) has a value $0.0025 d_H^2$. Adding this to the quantity we have calculated from the integral (54) gives the total squared damping length

$$d_D^2 \equiv \mathcal{D}_L^2+\Delta\mathcal{D}_L^2 \simeq 0.029\,t_L^2\left(\frac{8}{15(1+\xi)}+\frac{\xi^2}{2(1+\xi)^2}\right) + 0.0025\,d_H^2. \tag{89}$$

For instance, for $\Omega_Bh^2=0.02$ (so that $\xi=0.54$) Eq. (75) gives $d_H=1.61\,t_L$, so $d_D^2=0.0071\,d_H^2$. Hence at the first Doppler peak the argument of the damping exponential is $d_D^2k_1^2 \simeq 0.07$. (This depends very little on $\xi$.) We see that damping is not important at the first Dopper peak, in agreement with more accurate computer calculations [17], but is quite significant at the second Doppler peak. One effect of damping is to shift the second and higher Doppler peaks to lower values of $k$ and $l$.

In deriving the wave numbers (86) of the Doppler peaks we also neglected the terms proportional to $\xi/k^2t_L^2$ in the second line of Eq. (83). At the first Doppler peak this quantity is given by Eq. (75) as

$$\frac{\xi}{k_1^2t_L^2}=\frac{3}{\pi^2}[\ln(\sqrt{\xi}+\sqrt{1+\xi})]^2.$$

This is 0.20 for $\Omega_Bh^2=0.03$, for which $\xi=0.81$, and less for smaller values of $\Omega_Bh^2$. This approximation is thus fair at the first Doppler peak, and becomes excellent at the higher Doppler peaks.

Finally, we must ask what values of $k$ are small enough so that we can ignore acoustic oscillations during the era when the photon energy density exceeded the dark matter plus baryon density, during which our analysis does not apply. During this era the Robertson-Walker scale factor $a(t)$ went as $t^{1/2}$, and the speed of sound was $1/\sqrt{3}$, so the phase change of acoustic oscillations up to the time $t_C$ of the crossover from radiation dominance to matter dominance was

$$\Delta\varphi = q\int_0^{t_C}\frac{dt}{\sqrt{3}a(t)}=\frac{2qt_C}{\sqrt{3}a(t_C)}=\frac{2kt_L}{\sqrt{3}}\left(\frac{t_Ca(t_L)}{t_La(t_C)}\right).$$

The redshift $z_C$ at the crossover is given by $1+z_C=\Omega_M/\Omega_\gamma=4\times10^4\Omega_Mh^2$. During the period from this crossover to the present the scale factor $a(t)$ went as $t^{2/3}$, so the ratio in parentheses is

$$\frac{t_Ca(t_L)}{t_La(t_C)}=\sqrt{\frac{1+z_L}{1+z_C}}=\frac{1}{6.0\sqrt{\Omega_Mh^2}}.$$

Using this and Eq. (75) gives

$$\Delta\varphi \simeq \frac{0.35}{\sqrt{\Omega_Mh^2}}\left(\frac{k}{k_1}\right)\frac{\sqrt{\xi}}{\ln(\sqrt{\xi}+\sqrt{1+\xi})}. \tag{90}$$

For instance, if we take $\Omega_Mh^2=0.15$ and $\Omega_Bh^2=0.03$, then $\Delta\varphi \simeq 1$ at the first Doppler peak, indicating that oscillations in the radiation-dominated era are becoming important at the first Doppler peak. This is not to say that we are making an error of order unity in the argument $\varphi$ of the sines and cosines in Eqs. (83) and (84), but rather that the evolution of the perturbations during this much of their oscillations has not been reliably calculated. This source of error is mitigated in Ref. [1] by including the effects of photon and neutrino energies on $a(t)$ in calculating the horizon distance.

Our formula (84) for the dipole form factor $G(k)$ raises the possibility of a maximum in $G(k)$ at $kd_H=\pi/2$, yielding a "zeroth Doppler peak," produced (as the first Doppler peak is not) by the Doppler effect. For $\Omega_Bh^2=0.03$ the wave number at this supposed peak is too small for us to trust the WKB approximation used to derive Eq. (84) at this peak, but the peak in $G(k)$ at $kd_H=\pi/2$ would definitely be there for much smaller values of $\Omega_Bh^2$. In particular, the calculations of Ref. [1] show such a zeroth Doppler peak in $C_l$ at $l \simeq 0.45 d_A/d_H$ for $\Omega_B=0$.

*Note added in proof.* There was a numerical error in the calculation of the equilibrium hydrogen ionization at various temperatures used in Secs. II and V. However, the quoted results for the range of redshifts in which last scattering occurs happen to agree well with the range of redshifts for last scattering calculated in more exact non-equilibrium studies of recombination, so this error has little effect on the results of this paper.

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## APPENDIX: RELATIVISTIC CALCULATION OF THE SACHS-WOLFE AND DOPPLER EFFECTS

In Sec. IV we gave a derivation of the Sachs-Wolfe and Doppler effects, using heuristic arguments to supplement relativistic results. For completeness, this appendix will present a thoroughly relativistic derivation in the synchronous gauge, taking into account the possible presence of a vacuum energy, which may or may not be constant. This goes over familiar ground, first considered by Sachs and Wolfe [18], but as far as I know there is no published treatment of the ''integrated Sachs-Wolfe effect'' in the synchronous gauge that goes explicitly and analytically into the details presented here, including the possibility of a varying vacuum energy.

A light ray travelling toward the center of the Robertson-Walker coordinate system from the direction $\hat{n}$ will have a co-moving radial coordinate $r$ related to $t$ by

$$0=g_{\mu\nu\,\text{total}}dx^\mu dx^\nu=-dt^2+[a^2(t)+h_{rr}(r\hat{n},t)]dr^2, \tag{A1}$$

or in other words

$$\frac{dr}{dt}=-(a^2+h_{rr})^{-1/2}\simeq-\frac{1}{a}+\frac{h_{rr}}{2a^3}. \tag{A2}$$

The first-order solution is

$$r(t)=s(t)+\frac{1}{2}\int_{t_L}^{t}\frac{dt'}{a^3(t')}h_{rr}(s(t')\hat{n},t'), \tag{A3}$$

where $s(t)$ is the zeroth order solution for the radial coordinate which has the value $r_L$ at $t=t_L$:

$$s(t)=r_L-\int_{t_L}^{t}\frac{dt'}{a(t')}. \tag{A4}$$

In particular, if the ray reaches $r=0$ at a time $t_0$, then

$$0=s(t_0)+\frac{1}{2}\int_{t_L}^{t_0}\frac{dt}{a^3(t)}h_{rr}(s(t)\hat{n},t). \tag{A5}$$

A time interval $\delta t_L$ between successive light wave crests at the time $t_L$ of last scattering produces a time interval $\delta t_0$ at $t_0$ given by the variation of Eq. (A5):

$$\begin{aligned}0=\delta t_L\Bigg[&\frac{1}{a(t_L)}-\frac{1}{2}\frac{h_{rr}(r_L\hat{n},t_L)}{a^3(t_L)}\\&+\frac{1}{2\,a(t_L)}\int_{t_L}^{t_0}\frac{dt}{a^3(t)}\left(\frac{\partial h_{rr}(r\hat{n},t)}{\partial r}\right)_{r=s(t)}\Bigg]\\&+\delta t_L\left(\frac{\partial u(r\hat{n},t_L)}{\partial r}\right)_{r=r_L}+\delta t_0\left[-\frac{1}{a(t_0)}+\frac{1}{2}\frac{h_{rr}(0,t_0)}{a^3(t_0)}\right].\end{aligned} \tag{A6}$$

[The velocity potential term on the right-hand side arises from the pressure-induced change with time of the radial coordinate $r_L$ of the light source in Eq. (A5).] The total rate of change of the quantity $h_{rr}(s(t)\hat{n},t)/a^2(t)$ in Eq. (A6) is

$$\begin{aligned}\frac{d}{dt}\frac{h_{rr}(s(t)\hat{n},t)}{a^2(t)}=&\left(\frac{\partial}{\partial t}\frac{h_{rr}(r\hat{n},t)}{a^2(t)}\right)_{r=s(t)}\\&-\frac{1}{a^3(t)}\left(\frac{\partial h_{rr}(r\hat{n},t)}{\partial r}\right)_{r=s(t)},\end{aligned}$$

so Eq. (A6) may be written

$$\begin{aligned}0=\delta t_L\Bigg[&\frac{1}{a(t_L)}-\frac{1}{2}\frac{h_{rr}(0,t_0)}{a^2(t_0)a(t_L)}\\&+\frac{1}{2\,a(t_L)}\int_{t_L}^{t_0}dt\left\{\frac{\partial}{\partial t}\left(\frac{h_{rr}(r\hat{n},t)}{a^2(t)}\right)\right\}_{r=s(t)}\Bigg]\\&+\delta t_L\left(\frac{\partial u(r\hat{n},t)}{\partial r}\right)_{r=r_L}+\delta t_0\left[-\frac{1}{a(t_0)}+\frac{1}{2}\frac{h_{rr}(0,t_0)}{a^3(t_0)}\right].\end{aligned} \tag{A7}$$

Hence to first order the ratio of the received and emitted frequencies is

$$\begin{aligned}\frac{\nu_0}{\nu_L}=\frac{\delta t_L}{\delta t_0}=\frac{a(t_L)}{a(t_0)}\Bigg[&1-\frac{1}{2}\int_{t_0}^{t_L}\left\{\frac{\partial}{\partial t}\left(\frac{h_{rr}(r\hat{n},t)}{a^2(t)}\right)\right\}_{r=s(t)}-a(t_L)\\&\times\left(\frac{\partial u(r\hat{n},t)}{\partial r}\right)_{r=r_L}\Bigg].\end{aligned} \tag{A8}$$

This gives a fractional shift in the radiation temperature observed at time $t_0$ coming from direction $\hat{n}$, from its unperturbed value: $T_0=T_La(t_L)/a(t_0)$:

$$\left(\frac{\Delta T(\hat{n})}{T}\right)_{\text{SW, Dop}}=\frac{\nu_0}{a(t_L)\nu_L/a(t_0)}-1$$
$$=-\int_{t_L}^{t_0}dt\left\{\frac{\partial}{\partial t}\left(\frac{h_{rr}(r\hat{n},t)}{2\,a^2(t)}\right)\right\}_{r=s(t)}$$
$$-a(t_L)\left(\frac{\partial u(r_L\hat{n},t_L)}{\partial r}\right)_{r=r_L}. \qquad \text{(A9)}$$

Now we have to think about how to relate the $rr$ component of the metric perturbation to the field $\psi$ appearing in Sec. III. In general, the metric perturbation may be written as

$$h_{ij}=A\delta_{ij}+\frac{\partial^2 B}{\partial x^i\partial x^j}. \qquad \text{(A10)}$$

The quantity entering into the integrand in Eq. (A9) is then

$$\frac{\partial}{\partial t}\left(\frac{h_{rr}(r\hat{n},t)}{a^2(t)}\right)=\alpha(r\hat{n},t)+\frac{\partial^2\beta(r\hat{n},t)}{\partial r^2}, \qquad \text{(A11)}$$

where

$$\alpha\equiv\frac{\partial}{\partial t}\left(\frac{A}{2a^2}\right),\quad \beta\equiv\frac{\partial}{\partial t}\left(\frac{B}{2a^2}\right). \qquad \text{(A12)}$$

The field $\psi$ defined by Eq. (23) is given by

$$\psi=3\,\alpha+\nabla^2\beta. \qquad \text{(A13)}$$

We also need a relation between $\alpha$ and $\beta$, which can be taken from the field equation for the full metric perturbation [19]:

$$\nabla^2 h_{ij}-\frac{\partial^2 h_{ik}}{\partial x^j\partial x^k}-\frac{\partial^2 h_{jk}}{\partial x^i\partial x^k}+\frac{\partial^2 h_{kk}}{\partial x^i\partial x^j}-a^2\ddot{h}_{ij}+a\dot{a}(\dot{h}_{ij}-\delta_{ij}\dot{h}_{kk})$$
$$+2\dot{a}^2\delta_{ij}h_{kk}+2a\ddot{a}h_{ij}=-8\pi G(\delta\varrho-\delta p)a^4\delta_{ij}. \qquad \text{(A14)}$$

(For simplicity we are here taking the universe to be spatially flat, which is certainly a good approximation at high redshifts, and seems to be a good approximation even at present.) The $\partial^2/\partial x^i\partial x^j$ terms in Eq. (A14) give

$$A=a^2\ddot{B}-a\dot{a}\dot{B}-2a\ddot{a}B=a\frac{\partial}{\partial t}\left(a^3\frac{\partial}{\partial t}(a^{-2}B)\right). \qquad \text{(A15)}$$

In terms of the quantities defined by Eq. (A12), this is

$$\alpha=\frac{\partial}{\partial t}\left(\frac{1}{a}\frac{\partial}{\partial t}(a^3\beta)\right). \qquad \text{(A16)}$$

Hence for a given gravitational potential $\psi$, we can calculate $\beta$ by solving Eq. (A13):

$$\frac{\partial}{\partial t}\left(\frac{1}{a}\frac{\partial}{\partial t}(a^3\beta)\right)+\nabla^2\beta=\psi \qquad \text{(A17)}$$

and then use Eq. (A16) to find $\alpha$.

Now we return to the fractional temperature shift (A9). Using Eqs. (A11) and (A16) let us write this as

$$\left(\frac{\Delta T(\hat{n})}{T}\right)_{\text{SW, Dop}}=-\int_{t_L}^{t_0}dt\left(\frac{\partial^2\beta(r\hat{n},t)}{\partial r^2}\right)_{r=s(t)}-a(t_L)$$
$$\times\left(\frac{\partial u(r_L\hat{n},t_L)}{\partial r}\right)_{r=r_L}$$
$$-\int_{t_L}^{t_0}\left[\frac{\partial}{\partial t}\left(\frac{1}{a}\frac{\partial}{\partial t}[a^3(t)\beta(r\hat{n},t)]\right)\right]_{r=s(t)}. \qquad \text{(A18)}$$

To do the first integral here we note that

$$\left(\frac{\partial^2\beta(r\hat{n},t)}{\partial r^2}\right)_{r=s(t)}=-\frac{d}{dt}\left[\left(a^2(t)\frac{\partial\beta(r\hat{n},t)}{\partial t}\right.\right.$$
$$+a(t)\dot{a}(t)\beta(r\hat{n},t)$$
$$\left.\left.+a(t)\frac{\partial\beta(r\hat{n},t)}{\partial r}\right)_{r=s(t)}\right]$$
$$+\left(a^2(t)\frac{\partial^2\beta(r\hat{n},t)}{\partial t^2}\right.$$
$$+3a(t)\dot{a}(t)\frac{\partial\beta(r\hat{n},t)}{\partial t}+[a(t)\ddot{a}(t)$$
$$\left.+\dot{a}^2(t)]\beta(r\hat{n},t)\right)_{r=s(t)}. \qquad \text{(A19)}$$

The fractional temperature fluctuation (A18) may therefore be written

$$\left(\frac{\Delta T(\hat{n})}{T}\right)_{\text{SW, Dop}}=\left(\frac{\Delta T(\hat{n})}{T}\right)_{\text{early}}+\left(\frac{\Delta T(\hat{n})}{T}\right)_{\text{late}}$$
$$+\left(\frac{\Delta T(\hat{n})}{T}\right)_{\text{integrated}}, \qquad \text{(A20)}$$

where

$$\left(\frac{\Delta T(\hat{n})}{T}\right)_{\text{early}} = -a^2(t_L)\left(\frac{\partial\beta(r_L\hat{n},t)}{\partial t}\right)_{t=t_L} - a(t_L)\dot{a}(t_L)\beta(r_L\hat{n},t_L) - a(t_L) \times \left(\frac{\partial\beta(r\hat{n},t_L)}{\partial r}\right)_{r=r_L} - a(t_L) \times \left(\frac{\partial u(r_L\hat{n},t_L)}{\partial r}\right)_{r=r_L} \tag{A21}$$

$$\left(\frac{\Delta T(\hat{n})}{T}\right)_{\text{late}} = a^2(t_0)\left(\frac{\partial\beta(0,t)}{\partial t}\right)_{t=t_0} + a(t_0)\dot{a}(t_0)\beta(0,t_0) + a(t_0) \times \left(\frac{\partial\beta(r\hat{n},t_0)}{\partial r}\right)_{r=0}. \tag{A22}$$

$$\left(\frac{\Delta T(\hat{n})}{T}\right)_{\text{integrated}} = -2\int_{t_0}^{t_L} dt \left( a^2(t)\frac{\partial^2\beta(r\hat{n},t)}{\partial t^2} + 4a(t)\dot{a}(t)\frac{\partial\beta(r\hat{n},t)}{\partial t} + 2[a(t)\ddot{a}(t) + \dot{a}^2(t)]\beta(r\hat{n},t)\right)_{r=s(t)}. \tag{A23}$$

In evaluating these three contributions to the temperature fluctuation, it is helpful to note a relation between $\beta$ and the conventionally defined Newtonian potential $\phi$ that applies not only for a gravitational field dominated by cold dark matter, but also in the presence of a constant vacuum energy. Combining Eqs. (26) and (27) gives

$$\frac{\partial}{\partial t}\left(\frac{1}{4\pi G a^2 \rho_D}\frac{\partial}{\partial t}a^2\psi\right) = \psi. \tag{A24}$$

Taking into account the relation $\rho_D \propto a^{-3}$, an elementary manipulation then gives

$$\frac{\partial}{\partial t}\left(\frac{1}{a}\frac{\partial}{\partial t}a^3\psi\right) = \left[4\pi G\rho_D + \frac{d}{dt}\left(\frac{\dot{a}}{a}\right)\right]a^2\psi. \tag{A25}$$

The equations of the Friedmann model give

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = -4\pi G(\rho+p). \tag{A26}$$

A constant vacuum energy density $\rho_V$ is associated with a pressure $p_V = -\rho_V$, while cold dark matter by definition has zero pressure, so as long as the gravitational field is dominated by cold dark matter and a constant vacuum energy, the right-hand side of Eq. (A26) is $-4\pi G\rho_V$, and Eq. (A25) then gives

$$\frac{\partial}{\partial t}\left(\frac{1}{a}\frac{\partial}{\partial t}a^3\psi\right) = 0. \tag{A27}$$

Comparing Eq. (A16) with Eq. (A20), we now see that Eq. (A17) has the solution

$$\nabla^2\beta = \psi. \tag{A28}$$

More specifically, if we define a Newtonian gravitational potential $\phi$ by Poisson's equation

$$a^{-2}\nabla^2\phi = 4\pi G\,\delta\varrho_D, \tag{A29}$$

then Eqs. (26) and (A28) show that the Newtonian potential is

$$\phi = -\frac{\partial}{\partial t}(a^2\beta). \tag{A30}$$

This result is not applicable if the gravitational field receives significant contributions from a varying vacuum energy, but even in quintessence theories it is reasonable to assume that a vacuum energy density of any sort is negligible at and near the time of last scattering. (It certainly must be much less than the radiation energy density at the time of cosmological nucleosynthesis, in order to avoid the production of too much helium.) We have also been relying here on the approximation that the radiation energy density is much less than the dark matter density at around the time of last scattering. Therefore the early-time contribution (A21) to the temperature fluctuation can be calculated using the relation (A28) and $\psi \propto t^{-1/3}$, which give $\beta \propto t^{-1/3}$. Since here $a \propto t^{2/3}$, Eq. (A30) then gives $\beta = -t\phi/a^2$, with $\phi$ time-independent. The early-time contribution (A21) to the temperature fluctuation may therefore be expressed as

$$\left(\frac{\Delta T(\hat{n})}{T}\right)_{\text{early}} = \frac{1}{3}\phi(r\hat{n}) + \frac{t_L}{a(t_L)}\left(\frac{\partial\phi(r\hat{n})}{\partial r}\right)_{r=r_L} - a(t_L) \times \left(\frac{\partial u(r_L\hat{n},t_L)}{\partial r}\right)_{r=r_L}. \tag{A31}$$

This yields the Sachs-Wolfe temperature shift (68) and the gravitationally induced Doppler shift (77) (aside from the terms arising from $r=0$, about which we will say more later), as well as the pressure-induced Doppler shift (72). The famous factor 1/3 in the first term on the right-hand side arises in the "Newtonian gauge" as the sum of a gravitational redshift equal to $\phi$, and a term in the intrinsic temperature fluctuation equal to $-2\phi/3$, while in the synchronous gauge used here this term is due entirely to the metric perturbation. It is a curious feature of the synchronous gauge that what we have called the gravitationally induced Doppler shift also arises from the metric perturbation.

It is not appropriate to neglect the vacuum energy at $t = t_0$, so it cannot be ignored in the early-time contribution (A22) to the temperature fluctuation. Therefore in general this contribution is *not* the same as the $r=0$ terms in Eqs. (68) and (77). Nevertheless, the terms in the early-time contribution to the temperature fluctuation are only of zeroth and first order in $\hat{n}$ [like the $r=0$ terms in Eqs. (68) and (77)] so these terms can only affect the multipole coefficients for $l=0$ and $l=1$.

This leaves the integrated term (A23) as the only correction to the results of Sec. IV for $l\geqslant 2$. The integrand vanishes if we ignore the vacuum energy and radiation energy, in which case $a\propto t^{2/3}$ and $\beta\propto t^{-1/3}$, so the integral receives a contribution only for $t$ near $t_0$, and is therefore expected to be a small correction [20]. Furthermore, although this integral is fairly complicated, it has a simple dependence on $\hat{n}$. In the presence of a vacuum energy, $\psi(\mathbf{x},t)$ can have a fairly complicated dependence on time, but, without pressure forces acting on the dark matter, its $\mathbf{x}$ dependence is the same as we found in the absence of vacuum energy, given by Eqs. (31) and (70) as

$$\psi(\mathbf{x},t)=f(t)\int d^3k\, e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{k}^2\epsilon_{\mathbf{k}} \tag{A32}$$

with $f(t)$ not proportional to $t^{-1/3}$ where the vacuum energy is appreciable. For a constant vacuum energy, $\beta$ is then given by Eq. (A28) as

$$\beta(\mathbf{x},t)=-f(t)\int d^3k\, e^{i\mathbf{k}\cdot\mathbf{x}}\epsilon_{\mathbf{k}}. \tag{A33}$$

The "integrated" contribution (A23) to the temperature fluctuation then takes the form

$$\left(\frac{\Delta T(\hat{n})}{T}\right)_{\text{integrated}}=2\int d^3k\int_{t_0}^{t_L}dt\, e^{i\mathbf{k}\cdot\hat{n}s(t)}\epsilon_{\mathbf{k}}\{a^2(t)\ddot{f}(t)+4a(t)\dot{a}(t)\dot{f}(t)+2[a(t)\ddot{a}(t)+\dot{a}^2(t)]f(t)\}. \tag{A34}$$

It can be shown that this makes an additive contribution to $l(l+1)C_l$ that for large $l$ goes as $1/l$, with no interference between this contribution to the temperature fluctuation and the other contributions [21]. For a time-varying (but spatially constant) vacuum energy the function $\beta(\mathbf{x},t)$ does not satisfy the relations (A28) and (A33), but Eq. (A17) shows that its spatial Fourier transform is nevertheless just proportional to $\epsilon_{\mathbf{k}}$ for large $k$, so the integrated term still makes a contribution to $l(l+1)C_l$ that is proportional to $1/l$ for large $l$.


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[6] Several authors have given approximate formulas for the temperature fluctuation that fit the general form of Eq. (1). In particular, the results given in Eqs. (8) and (9) can be obtained from Eq. (1) of Naselsky and Novikov, [5], by applying some corrections: the factor $(1+\nu)$ should be omitted in their definition of $\xi$ (in their notation, $\epsilon$); the factor $(1+z)^{-1}$ should be omitted in their definition of their parameter $\omega$, so that their $\omega$ equals $\xi$; a factor $\omega$ should be included in the last term of the numerator in the argument of the logarithm of their Eq. (2); and the damping factor $\exp(-k^2d_D^2)$ and terms proportional to $\xi/k^2t_L^2$ should be inserted. This paper did not give the derivation of their Eq. (1), but quoted Doroshkevich [5], despite the fact that their result was quite different from that of Doroshkevich. There was an obvious misprint in the formula given by Doroshkevich, but this was corrected in the later paper by Atrio-Barandela, Doroshkevich, and Klypin [5]. The formula in this paper includes the terms proportional to $\xi/k^2t_L^2$, which had been omitted by Naselsky and Novikov, but omitted the factors of $(1+\xi)$ that had been included by Naselsky and Novikov, and also included a spurious term in the analog of $G(k)$ [the 1 in the numerator of the second line of their Eq. (4)]. This paper gave differential equations for the time development of the perturbations, but did not explain how they were used to calculate the temperature fluctuation. A few years later the formula given by Naselsky and Novikov was repeated by Jørgensen, Kotok, Naselsky, and Novikov [5], and the differential equations on which the formula was based were given. However, once again the derivation of the formula from these equations was not explained, and an overly restrictive lower bound on $k$ was given for the validity of the formula, that $k$ must be larger than the inverse conformal time. If this condition were really necessary, then the formula would not be applicable at the first Doppler peak. We will see in Sec. III that the lower bound on $k$ is actually less restrictive, and in particular disappears for $\xi\ll 1$.

[7] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), Eq. (15.10.13). In conformity with common present no-

tation, the symbol $R(t)$ used in this book for the Robertson-Walker scale factor has been replaced in the present paper with $a(t)$.

[8] Weinberg [7], Eq. (15.10.52).

[9] Weinberg [7], Eq. (15.10.51).

[10] Weinberg [7], Eq. (15.10.53). A factor $T$ was missing in the second term in the curly brackets in Eq. (15.10.53), and has been supplied here.

[11] These formulas are obtained by comparing the acoustic damping rate calculated by N. Kaiser, Mon. Not. R. Astron. Soc. **202**, 1169 (1983) with the damping rate calculated for arbitrary values of $\eta$ and $\chi$ by S. Weinberg, Astrophys. J. **168**, 175 (1971), Eq. (4.15); the latter article also gives values for $\chi$ and $\eta$, repeated in Ref. [7]: it gives the same value of $\chi$ and a value for $\eta$ that is 3/4 the value quoted in Eq. (55), but these results were based on calculations of L.H. Thomas, Quarterly J. Math. **1**, 239 (1930), that assumed isotropic scattering and ignored photon polarization. [The same value for $\eta$ had been given by C. Misner, Astrophys. J. **151**, 431 (1968).]. Kaiser's results are calculated using the correct differential cross section for Thomson scattering and take photon polarization into account, and therefore supersede the earlier value quoted for $\eta$. As late as 1995, the wrong value of the damping rate was still being used, for instance by Hu and Sugiyama [5], but the correct rate was used by Hu and White, Astrophys. J. **479**, 568 (1997).

[12] J. A. Peacock, *Cosmological Physics* (Cambridge University Press, Cambridge, England, 1999), p. 591.

[13] The derivation is given, for instance, in Weinberg [7], Eq. (15.9.13). The presence of the second term on the left-hand side has as a consequence the well known decay $\propto 1/a(t)$ of the peculiar velocities of nonrelativistic free particles. The factor $1/a(t)$ multiplying the gradient of the potential enters again to convert a derivative with respect to comoving coordinates into a derivative with respect to coordinates that measure proper distances.

[14] Hu and Sugiyama [5]; Hu and White [11].

[15] A. Dimitropoulos and L.P. Grishchuk, gr-qc/0010087.

[16] See, e.g., P.H. Frampton, Y.J. Ng, and R. Rohm, Mod. Phys. Lett. A **13**, 2541 (1998).

[17] Hu and White [11].

[18] R.K. Sachs and A.M. Wolfe, Astrophys. J. **1**, 73 (1967).

[19] Weinberg [7], Eqs. (15.10.29) and (15.1.19). [A misprint has been corrected here: the equals sign in the first line of Eq. (15.10.29) has been changed to a minus sign.]

[20] L.A. Kofman and A.A. Starobinskii, Sov. Astron. Lett. **11**, 271 (1985).

[21] The $1/l$ dependence was found in Ref. [20], but without consideration of a possible interference between this effect and the Doppler shift and intrinsic temperature shift.