Adiabatic and isocurvature perturbations from inflation: Power spectra and consistency relations

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We study adiabatic and isocurvature perturbations produced during a period of cosmological inflation. We compute the power spectra and cross spectra of the curvature and isocurvature modes, as well as the tensor perturbation spectrum in terms of the slow-roll parameters. We provide two consistency relations for the amplitudes and spectral indices of the corresponding power spectra. These relations represent a definite prediction and a test of inflationary models which should be adopted when studying cosmological perturbations through the cosmic microwave background in forthcoming satellite experiments.

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I. INTRODUCTION

Inflation is the standard scenario for the generation of cosmological perturbations in the universe which are the seeds for the large scale structure formation and the cosmic microwave background (CMB) anisotropies. Many inflationary models have been proposed so far since the original proposal by Guth [1]. The simplest possibility is to assume the presence of a single scalar field ϕ with a potential $V(\phi)$, undergoing a slow-rolling phase [2]. The dynamics of the inflationary stage can then be studied introducing a set of slow-roll parameters [3,4] which are obtained from $V(\phi)$ and its derivatives $V', V'', \ldots, V^{(n)}$ with respect to the inflaton field ϕ. The physical observables can be expressed in terms of these parameters. The scalar perturbations are generally expected to be adiabatic, nearly Gaussian distributed and (almost) scale-free (i.e., with power-spectra $\propto k^n$). Furthermore, the tensor modes (gravitational waves) are Gaussian and scale-free. The scalar and tensor spectra can be parametrized as

$$A_S^2(k)=A_S^2(k_0)\left(\frac{k}{k_0}\right)^{n_S-1}, \quad A_T^2(k)=A_T^2(k_0)\left(\frac{k}{k_0}\right)^{n_T}, \tag{1}$$

where k_0^{-1} is a typical length scale probed by CMB experiments. The main observables are four: the two amplitudes and the spectral indices n_T and n_S. They can be expressed in terms of the slow-roll parameters

$$\epsilon=\frac{m_{Pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \qquad \eta=\frac{m_{Pl}^2}{8\pi}\frac{V''}{V} \tag{2}$$

(with $\epsilon,\eta \ll 1$ during slow-roll) via the relations $n_S=1-6\epsilon+2\eta$, $n_T=-2\epsilon$ and $A_T^2/A_S^2=\epsilon$. For single-field models,

$$n_T=-2\epsilon, \quad \frac{A_T^2}{A_S^2}=\epsilon \quad \Rightarrow \quad n_T=-2\frac{A_T^2}{A_S^2}. \tag{3}$$

The so called *consistency* relation $n_T=-2A_T^2/A_S^2$ reduces the number of independent observables to n_S, the relative amplitude of the two spectra and the scalar perturbation amplitude [which might be determined by normalizing to Cosmic Background Explorer (COBE) data].

Analyses of the observed CMB anisotropies have so far assumed this kind of power-spectra as far as the primordial perturbations are concerned (see, for example, [5]). One should emphasize, however, that the theoretical predictions for the initial cosmological perturbations should be at the same level of accuracy as the observations in order to constrain the cosmological parameters (Ω_{tot},Ω_b,h, etc.). The forthcoming set of data on the CMB anisotropies provided by the Microwave Anisotropy Probe (MAP) [6] and Planck [7] satellites are expected to reduce the errors on the determination of the cosmological parameters to a few percent [8]. This implies that the assumption that inflation has been driven by a single scalar field may turn out to be an oversimplification and that it would be useful to consider alternative possibilities to the simplest single-field models of inflation. For instance, adiabaticity and/or Gaussianity may not hold [9–11]. Isocurvature perturbations can be produced during a period of inflation if more than one scalar field is present. It could be the case of inflation driven by several scalar fields (the so called "multiple inflation"), or one where inflation is driven by a single scalar field (the inflaton), with other scalar fields whose energy densities are subdominant, but whose fluctuations must be taken into account too [12]. We will use ϕ and χ_I ($I=1,\ldots,K$) to indicate all the scalar fields, keeping in mind that, if the case, ϕ plays the role of the inflaton, and χ_I of the extra degrees of freedom. It is likely that in the early universe there were several scalar fields; moreover, from the particle physics point of view, the presence of different scalar fields is quite natural. An example is given by the supergravity and (super)string models where there are a large number of the so-called moduli fields. Another example is the theories of extra-dimensions where an infinite tower of spin-0 graviscalar Kaluza-Klein excitations appear [13].

On the other hand, isocurvature perturbations, once generated *during* inflation, could not survive *after* inflation ends

[9,14–16]. If during reheating all the scalar fields decay into the same species, the only remaining perturbations will be of adiabatic type.

In the case of adiabatic plus isocurvature fluctuations, an interesting issue is the possible *correlation* between the two modes of perturbation. In fact, until recently, only independent mixtures of adiabatic and isocurvature modes were considered [17]. In Ref. [18] the effects of the correlation on the CMB anisotropies and on the mass power spectrum has been considered. It has been found that several peculiar imprints on the CMB spectrum arise. In that case the correlation has been put *by hand* as an additional parameter for structure formation at the beginnig of the radiation dominated era. In Ref. [19], instead, a specific realization of a double inflationary model with two noninteracting scalar fields was studied as an example for the origin of the correlation during inflation. A clear formalism was introduced in Ref. [20] to study the adiabatic and the isocurvature modes and their cross correlation in the case of several scalar fields interacting through a generic potential $V(\phi,\chi_I)$. In a previous paper [21] we have shown that, in the presence of several scalar fields, it is natural to expect a mixing and an oscillation mechanism between the fluctuations of the scalar fields ϕ and χ_I, in a manner similar to neutrino oscillations. This can happen even if the energy density of the scalar fields χ_I is much smaller than the energy density of the field ϕ. The correlation between the adiabatic and the isocurvature perturbations can be read as a result of this oscillation mechanism.

The aim of this paper is to express the spectra for the adiabatic and isocurvature modes and their cross spectrum in terms of the slow-roll parameters. We will show that, as for the standard single-field case, the physical observables are not independent, but there exist specific consistency relations which are predicted theoretically. Analyses of the present CMB anisotropies data coming from the BOOMERang and MAXIMA-1 experiments have been recently made [22] and used to constrain adiabatic and isocurvature perturbations; a study of the impact of isocurvature perturbation modes in our ability to accurately constrain cosmological parameters with the forthcoming MAP and Planck measurements has been made in Ref. [23]. However, in all these studies the physical observables (i.e., the different amplitudes and spectral indices) have been considered as independent parameters. Our findings, instead, indicate that the interplay between the cosmological perturbations generated during the inflationary epoch imposes some consistency relations among the physical obervables which could be tested in the future. The plan of the paper is as follows. In Sec. II we briefly recall the basic definitions of isocurvature and adiabatic perturbations, and define the correlation spectrum. In Sec. III we discuss the generation of the correlation during an inflationary period where two scalar fields are present, making an expansion of the solutions in slow-roll parameters. In Sec. IV, we derive the expressions of the spectra soon after inflation and from these we calculate the amplitude ratios and the spectral indices to give the consistency relations between them. Finally, Sec. V contains some concluding remarks.

II. BASIC DEFINITIONS

Let us consider a system composed by N components. These could be the N scalar fields during inflation or the different species which are present deep in the radiation era after inflation. Adiabatic perturbations are perturbations in the total energy density of the system, while isocurvature (or entropic) perturbations leave the total energy density unperturbed by a relative fluctuation between the different components of the system. Thus adiabatic perturbations are characterized by a perturbation in the intrinsic spatial curvature, while the isocurvature perturbations do not perturb the curvature. In order to have isocurvature perturbations it is necessary to have more than one component and at least one nonzero entropic perturbation $S_{\alpha\beta}$ [24]:

$$S_{\alpha\beta}\equiv\frac{\delta_\alpha}{1+w_\alpha}-\frac{\delta_\beta}{1+w_\beta}\neq 0, \tag{4}$$

where $\delta_\alpha=\delta\rho_\alpha/\rho_\alpha$, $w_\alpha=p_\alpha/\rho_\alpha$ (the ratio of the pressure to the energy density), and α and β stand for any two components of the system. $S_{\alpha\beta}$ is a gauge-invariant quantity and measures the relative fluctuations between the different components. Adiabatic perturbations are characterized by having $S_{\alpha\beta}=0$ for all of the components. Thus in general there will be one adiabatic perturbation mode and $N-1$ independent isocurvature modes and one must consider adiabatic plus isocurvature perturbations.

For a generic cosmological perturbation $\Delta(\mathbf{x})$, it is standard to define its dimensionless power spectrum $\mathcal{P}_\Delta$ as

$$\langle\Delta_{\mathbf{k}}\Delta_{\mathbf{k}'}\rangle=2\pi^2k^{-3}\mathcal{P}_\Delta(k)\,\delta(\mathbf{k}+\mathbf{k}'), \tag{5}$$

where the angular brackets denote ensemble averages and $\Delta_{\mathbf{k}}$ is the Fourier transform of $\Delta(\mathbf{x})$:

$$\Delta_{\mathbf{k}}=\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{x}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\Delta(\mathbf{x}). \tag{6}$$

Thus for two quantities $\Delta_1(\mathbf{x})$ and $\Delta_2(\mathbf{x})$ it can be defined a cross spectrum as

$$\langle\Delta_{1\mathbf{k}}\Delta_{2\mathbf{k}'}\rangle=2\pi^2k^{-3}\mathcal{C}_{\Delta_1\Delta_2}(k)\,\delta(\mathbf{k}+\mathbf{k}'). \tag{7}$$

III. ADIABATIC AND ISOCURVATURE PERTURBATIONS FROM INFLATION: A SLOW-ROLL FORMALISM

As already mentioned in Sec. I, adiabatic and isocurvature perturbations can be produced during a period of inflation in which more than one scalar field is present. One of the difficulties in studying mixtures of isocurvature and adiabatic perturbations produced during inflation is that, in general, one cannot trace back the adiabatic mode to the perturbations of some of these scalar fields only, and the entropic modes to the perturbations of the remaining scalar fields. Rather the fluctuations of all of the scalar fields contribute to the adiabatic and isocurvature modes. On the other hand, this is the reason why one must expect a correlation between them. In this respect the authors of Ref. [20] have provided a general formalism to better disentangling the adiabatic and isocurvature perturbation modes.

Let us now enter into the details. For simplicity we will restrict here to the case of two fields, ϕ and χ with a generic potential $V(\phi,\chi)$. In order to study the field perturbations $\delta\phi$ and $\delta\chi$, we can write the line element for scalar perturbations of the metric as

$$ds^2=-(1+2A)dt^2+2aB_{,i}dx^i dt+a^2[(1-2\psi)\delta_{ij}+2E_{,ij}]dx^i dx^j. \quad (8)$$

Thus the equation for the evolution of the perturbation $\delta\phi_I$ ($I=1,2$ and $\delta\phi_1=\delta\phi$, $\delta\phi_2=\delta\chi$) with comoving wave number $k=2\pi a/\lambda$ for a mode with physical wavelength λ is

$$\ddot{\delta}\phi_I+3H\dot{\delta}\phi_I+\frac{k^2}{a^2}\delta\phi_I+\sum_J V_{\phi_I\phi_J}\delta\phi_J =-2V_{\phi_I}A+\dot{\phi}_I\left[\dot{A}+3\dot{\psi}+\frac{k^2}{a^2}(a^2\dot{E}-aB)\right], \quad (9)$$

where the dots stand for time derivatives.

In the following we will recall the basic equations and results of Ref. [20]. It is possible to define the adiabatic and entropy fields (δA and δs respectively) in terms of the original ones $\delta\phi$, $\delta\chi$ as

$$\delta A=(\cos\beta)\delta\phi+(\sin\beta)\delta\chi \quad (10)$$

and

$$\delta s=(\cos\beta)\delta\chi-(\sin\beta)\delta\phi, \quad (11)$$

where

$$\cos\beta=\frac{\dot{\phi}}{\sqrt{\dot{\phi}^2+\dot{\chi}^2}}, \quad \sin\beta=\frac{\dot{\chi}}{\sqrt{\dot{\phi}^2+\dot{\chi}^2}}. \quad (12)$$

Introducing the gauge-invariant Sasaki-Mukhanov variables [26]

$$Q_I\equiv\delta\phi_I+\frac{\dot{\phi}_I}{H}\psi, \quad (13)$$

it can be checked that δA and δs can be rewritten as

$$Q_A=(\cos\beta)Q_\phi+(\sin\beta)Q_\chi, \quad (14)$$

$$\delta s=(\cos\beta)Q_\chi-(\sin\beta)Q_\phi. \quad (15)$$

Note that the entropy field δs is gauge-invariant.

The curvature perturbation [25]

$$\mathcal{R}=H\sum_I\left(\frac{\dot{\varphi}_I}{\sum_{J=1}^{N}\dot{\varphi}_J^2}\right)Q_I \quad (16)$$

can be written in terms of the adiabatic field Q_A in the same way as in the single scalar field case:

$$\mathcal{R}=\frac{H}{\dot{A}}Q_A. \quad (17)$$

The master equations are the evolution equations for the quantities defined in Eqs. (14) and (15). They read

$$\ddot{\delta}s+3H\dot{\delta}s+\left(\frac{k^2}{a^2}+V_{ss}+3\dot{\beta}^2\right)\delta s=\frac{\dot{\beta}}{\dot{A}}\frac{k^2}{2\pi Ga^2}\Psi \quad (18)$$

and

$$\ddot{Q}_A+3H\dot{Q}_A+\left[\frac{k^2}{a^2}+V_{AA}-\dot{\beta}^2-\frac{8\pi G}{a^3}\left(\frac{a^3\dot{A}^2}{H}\right)^{\cdot}\right]Q_A =2(\dot{\beta}\delta s)^{\cdot}-2\left(\frac{V_A}{\dot{A}}+\frac{\dot{H}}{H}\right)\dot{\beta}\delta s, \quad (19)$$

where $V_{ss}=(\sin^2\beta)V_{\phi\phi}-(\sin 2\beta)V_{\phi\chi}+(\cos^2\beta)V_{\chi\chi}$, $\dot{A}=(\cos\beta)\dot{\phi}+(\sin\beta)\dot{\chi}$, $V_{AA}=(\sin^2\beta)V_{\chi\chi}+(\sin 2\beta)V_{\phi\chi}+(\cos^2\beta)V_{\phi\phi}$, $V_A=(\cos\beta)V_\phi+(\sin\beta)V_\chi$; $\psi=\Psi$ in the longitudinal gauge, and we use the notation $V_{\phi_I}=\partial V/\partial\phi_I$.

Following Ref. [20], let us take at horizon crossing during inflation:

$$Q_I|_{k=aH}\approx\frac{H_k}{\sqrt{2k^3}}e_I(\mathbf{k}), \quad (20)$$

where $I=\phi,\chi$, H_k is the Hubble parameter when the mode crosses the horizon (i.e., $a_kH_k=k$) and e_ϕ and e_χ are independent random variables satisfying

$$\langle e_I(\mathbf{k})\rangle=0, \quad \langle e_I(\mathbf{k})e_J^*(\mathbf{k}')\rangle=\delta_{IJ}\delta(\mathbf{k}-\mathbf{k}'). \quad (21)$$

These initial conditions are strictly valid only for modes well within the horizon. Indeed, as emphasized in Ref. [21], curvature and isocurvature perturbations become cross-correlated as soon as they leave the horizon when the oscillations between these two modes is resonantly amplified.

For super-horizon scales, $k\ll aH$, we can neglect all terms proportional to k^2/a^2 in Eqs. (18) and (19), and consider only the non-decreasing modes which amounts to neglecting the second time derivatives. Thus it follows

$$Q_A\simeq Af(t)+P(t), \quad (22)$$

$$\delta s\simeq Bg(t), \quad (23)$$

where $f(t)$ is the general solution for the homogeneous part of Eq. (19), $P(t)$ is a particular integral of the full Eq. (19), and $g(t)$ is the general solution of Eq. (18). The amplitudes $A(k)$ and $B(k)$ are given by

$$A(\mathbf{k})\approx\frac{H_k}{\sqrt{2k^3}}e_A(\mathbf{k}), \quad B(\mathbf{k})\approx\frac{H_k}{\sqrt{2k^3}}e_s(\mathbf{k}), \quad (24)$$

where $e_A(\mathbf{k})$ and $e_s(\mathbf{k})$ are random variables satisfing the same relations of Eq. (21). $P(t)$ can be written as $P(t) = B\tilde{P}(t)$. From Eqs. (22), (23) and (24) one gets the expression for Q_A and δs spectra and their cross-correlation during inflation:

$$\mathcal{P}_{Q_A} \simeq \left(\frac{H_k}{2\pi}\right)^2 [|f^2| + |\tilde{P}^2|], \tag{25}$$

$$\mathcal{P}_{\delta s} \simeq \left(\frac{H_k}{2\pi}\right)^2 |g^2|, \tag{26}$$

$$\mathcal{C}_{Q_A \delta s} \simeq \left(\frac{H_k}{2\pi}\right)^2 g\tilde{P}. \tag{27}$$

Slow-roll expansion

The most important comment on the previous formulas is that the correlation is nonzero when $\tilde{P}$ is nonzero (we are considering that, in general, in a multicomponent system $\delta s \neq 0$). On the other hand, $\tilde{P}$ is nonzero only when the source term on the right-hand side of Eq. (19) is nonzero. This happens when the time derivative of the angle β, defined in Eq. (12), is not vanishing. Note that this is also the condition for the evolution of Q_A and δs not to be independent, since in this case δs *feeds* the adiabatic part of perturbations on large scales, as observed in Ref. [20]. In the language of Ref. [21] this can be rephrased saying that the probability of oscillation between the perturbations of the scalar fields is resonantly amplified when perturbations cross the horizon and the perturbations in the inflaton field may disappear at horizon crossing giving rise to perturbations in scalar fields other than the inflaton. Adiabatic and isocurvature perturbations are therefore inevitably correlated at the end of inflation. Provided that $\delta s \neq 0$, we can conclude that the correlation will be present under the condition $\dot{\beta} \neq 0$. It is remarkable that only in some special cases this condition is not satisfied. As can be checked from Eq. (12), β is *exactly* constant in time if there are attractor-like solutions for the evolution of the two fields ϕ and χ of the kind $\dot{\chi} \propto \dot{\phi}$. For example, this is the case of the models of assisted inflation [27]. Therefore, if the entropic modes are not strongly supressed during inflation, the correlation between isocurvature and adiabatic perturbations is quite natural to arise.

Now let us introduce the following generalization of the slow-roll parameters [see Eq. (2)] in the case of two scalar fields:

$$\epsilon_I = \frac{m_{Pl}^2}{16\pi}\left(\frac{V_{\phi_I}}{V}\right)^2 \quad \text{and} \quad \eta_{IJ} = \frac{m_{Pl}^2}{8\pi}\frac{V_{\phi_I \phi_J}}{V}, \tag{28}$$

where $V_{\phi_I} = \partial V/\partial \phi_I$, and $\phi_I = \phi$ or χ.

We have expanded the master equations (18) and (19) to lowest order in these parameters, since during inflation ϵ_I and η_{IJ} are $\ll 1$. In the following we will quote only the main results. More technical details can be found in Appendix A.

For nondecreasing modes and $k \ll aH$ Eq. (18) can be written as

$$\dot{\delta s} = -\frac{1}{3H}(V_{ss} + 3\dot{\beta}^2)\delta s. \tag{29}$$

Note that $\mu_s^2 \equiv V_{ss} + 3\dot{\beta}^2$ is the effective mass for the entropy field. To lowest order it is given by

$$-\frac{\mu_s^2}{3H^2} = -\frac{\epsilon_\chi}{\epsilon_{tot}}\eta_{\phi\phi} + 2\frac{(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}{\epsilon_{tot}}\eta_{\phi\chi} - \frac{\epsilon_\phi}{\epsilon_{tot}}\eta_{\chi\chi}. \tag{30}$$

The sign $\pm$ stands for the cases $\dot{\phi}$ $(\dot{\chi}) > 0$ and < 0, respectively, and ϵ_{tot} stands for $(\epsilon_\phi + \epsilon_\chi)$.

Starting from Eq. (23) the resulting solution for δs will be

$$\begin{aligned} \delta s &\simeq B(k) g(t) \\ &= B(k)\exp\Bigg[\Bigg(-\frac{\epsilon_\chi}{\epsilon_{tot}}\eta_{\phi\phi} + 2\frac{(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}{\epsilon_{tot}} \\ &\quad \times \eta_{\phi\chi} - \frac{\epsilon_\phi}{\epsilon_{tot}}\eta_{\chi\chi}\Bigg)[N_k - N(t)]\Bigg], \end{aligned} \tag{31}$$

where $N_k - N(t) = \int_{t_k}^{t} H dt$. $N(t) = \int_t^{t_f} H dt$, with t_f the time inflation ends, is the number of e-folds between the end of inflation and a generic instant t, $N_k = \int_{t_k}^{t_f} H dt = \ln(a_f/a_k)$ is the number of e-folds between the time t_k the mode crosses the horizon and the end of inflation. Typically, $N_k \simeq 60$ as far as large scale CMB anisotropies are concerned.

In order to write Eq. (31), we have neglected the time dependence of the term that appears as a combination of the slow-roll parameters, since its time derivative is $\mathcal{O}(\epsilon^2, \eta^2)$,[1] and so we have extracted this term out of the integral $N_k - N(t)$. Since it can be treated as a constant, it can be evaluated at horizon crossing, $k = aH$. At the end of inflation δs will be

$$\begin{aligned} \delta s|_{t_f} = B(k)\exp\Bigg[\Bigg(&-\frac{\epsilon_\chi}{\epsilon_{tot}}\eta_{\phi\phi} + 2\frac{(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}{\epsilon_{tot}} \\ &\times \eta_{\phi\chi} - \frac{\epsilon_\phi}{\epsilon_{tot}}\eta_{\chi\chi}\Bigg)N_k\Bigg]. \end{aligned} \tag{32}$$

As for the adiabatic mode, Eq. (19) can be written as

$$\begin{aligned} \dot{Q}_A = &-\frac{1}{3H}\left[V_{AA} - \dot{\beta}^2 - \frac{8\pi G}{a^3}\left(\frac{a^3\dot{A}^2}{H}\right)^{\cdot}\right]Q_A + \frac{2}{3H}\Bigg[(\dot{\beta}\delta s)^{\cdot} \\ &- \left(\frac{V_A}{\dot{A}} + \frac{\dot{H}}{H}\right)\dot{\beta}\delta s\Bigg]. \end{aligned} \tag{33}$$

[1] With $\mathcal{O}(\epsilon, \eta)$ and $\mathcal{O}(\epsilon^2, \eta^2)$ we indicate general combinations of the slow-roll parameters of lowest order or next order, respectively.

Putting the entropic solution (31) into Eq. (33), and following the same procedure of expansion in the slow-roll parameters, we find the adiabatic solution (22):

$$f(t)|_{t_f}=\exp\left[\left(-\frac{\epsilon_\chi}{\epsilon_{tot}}\eta_{\chi\chi}-\frac{\epsilon_\phi}{\epsilon_{tot}}\eta_{\phi\phi}-2\frac{(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}{\epsilon_{tot}}\eta_{\phi\chi}+2\epsilon_{tot}\right)N_k\right] \tag{34}$$

and

$$\tilde{P}(t)|_{t_f}=2\left[\frac{\dot{\beta}}{H}\right]_{l.o.}g(t)|_{t_f}\frac{1}{C}(e^{CN_k}-1), \tag{35}$$

where $[\dot{\beta}/H]_{l.o.}$ is the expression of $\dot{\beta}/H$ to lowest order:

$$\left[\frac{\dot{\beta}}{H}\right]_{l.o.}=\frac{\epsilon_\phi-\epsilon_\chi}{\epsilon_{tot}}\eta_{\phi\chi}+\frac{(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}{\epsilon_{tot}}(\eta_{\phi\phi}-\eta_{\chi\chi}) \tag{36}$$

and C is given by

$$C=\frac{\epsilon_\phi-\epsilon_\chi}{\epsilon_{tot}}\eta_{\chi\chi}+\frac{\epsilon_\chi-\epsilon_\phi}{\epsilon_{tot}}\eta_{\phi\phi}-4\frac{(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}{\epsilon_{tot}}\eta_{\phi\chi}+2\epsilon_{tot}. \tag{37}$$

Again $[\dot{\beta}/H]_{l.o.}$ and C, which are $\mathcal{O}(\epsilon,\eta)$, have been treated as constant and can be taken at horizon crossing when $k=aH$.

Now we are able to give the expressions for the spectra (25), (26) and (27):

$$\mathcal{P}_{Q_A}=\left(\frac{H_k}{2\pi}\right)^2|f^2(t)|_{t_f}\left[1+4\left[\frac{\dot{\beta}}{H}\right]^2_{l.o.}\frac{1}{C^2}(1-e^{-CN_k})^2\right], \tag{38}$$

$$\mathcal{P}_{\delta s}=\left(\frac{H_k}{2\pi}\right)^2\exp\left[\left(-2\frac{\epsilon_\chi}{\epsilon_{tot}}\eta_{\phi\phi}+4\frac{(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}{\epsilon_{tot}}\eta_{\phi\chi}-2\frac{\epsilon_\phi}{\epsilon_{tot}}\eta_{\chi\chi}\right)N_k\right] \tag{39}$$

and

$$\mathcal{C}_{Q_A\delta s}=\left(\frac{H_k}{2\pi}\right)^2 2\left[\frac{\dot{\beta}}{H}\right]_{l.o.}g^2(t)|_{t_f}\frac{1}{C}(e^{CN_k}-1). \tag{40}$$

Since the isocurvature perturbation δs is a source for the adiabatic one, the curvature perturbation spectrum, which characterizes the adiabatic mode, does not remain constant during inflation in general, unlike the single field case (see, for example, Ref. [28]). This is the reason why we have evaluated all the previous expressions at the end of inflation. In the next section we will specify the initial conditions in the post inflationary epoch.

A few comments are in order here. As can be seen in Eq. (40) the cross correlation is proportional to $\dot{\beta}$, as already mentioned at the beginning of this section. Moreover, it depends on the factor e^{CN_k}, which is the ratio between f and g. In other words, C is $(\mu_s^2-\mu_A^2)/3H^2$ to lowest order, where μ_s^2 and μ_A^2 are the effective masses for the entropic and adiabatic perturbations [the terms proportional to δs and Q_A in Eqs. (29), (33)]. This means that, in order to have a strong correlation, what is important is just the *relative* magnitude of the adiabatic and the entropic masses, even if they are both $\mathcal{O}(\epsilon)$. This result is in agreement with our previous findings [21], where we have demonstrated that the correlation emerges as soon as there is a mixing between the original fields ϕ and χ and that this mixing can be large even if the masses of the scalar fields are all $\mathcal{O}(\epsilon_I,\eta_{IJ})$.

IV. INITIAL CONDITIONS IN THE POST INFLATIONARY EPOCH

In the following we will assume that the mixing between the scalar fields is negligible after inflation and that, for example, the field ϕ decays into ‘‘ordinary’’ matter (present-day photons, neutrinos and baryons), and the scalar field χ decays only into cold dark matter. The field χ could also not decay, as it happens in axion models. In fact, if during reheating all the scalar fields decay into the same species, the perturbations will be only of adiabatic type deep in the radiation era: no relative fluctuation $S_{\alpha\beta}$ is generated. In the present case a CDM-isocurvature mode will survive after inflation. Using the notation of Sec. II and Ref. [19], we can write

$$\delta_{CDM}=S_{CDM-\mathrm{rest}}+\delta_A, \quad \delta_A=\frac{3}{4}\delta_\gamma=\frac{3}{4}\delta_\nu=\delta_b \tag{41}$$

where δ_A specifies the amplitude of the adiabatic mode of perturbations, and ‘‘rest’’ stands for ordinary matter.

In order to set the initial conditions for the evolution of cosmological perturbations, and which can be used in some numerical codes calculating the CMB anisotropies, we must link the two relevant quantities $S_{CDM-rest}$ and $\mathcal{R}$ deep in the radiation era to the inflationary quantities δs and Q_A.

For the adiabatic perturbations this is immediate from Eq. (17),

$$\mathcal{R}_{rad}=\frac{H}{\dot{A}}Q_A \tag{42}$$

where the right-hand side of this equation is evaluated at the end of inflation. As far as $S_{CDM-rest}$ is concerned, it is useful to introduce the following quantity:

$$\delta_{\chi\phi}\equiv\frac{\delta\chi}{\dot{\chi}}-\frac{\delta\phi}{\dot{\phi}}. \tag{43}$$

For the scalar fields ϕ and χ the isocurvature perturbation $S_{\chi\phi}$, Eq. (4), results $S_{\chi\phi}=a^3 d(\delta_{\chi\phi}/a^3)/dt$ [29].

On the other hand,

$$\delta s=\frac{\dot{\chi}\dot{\phi}}{\sqrt{\dot{\chi}^2+\dot{\phi}^2}}\,\delta_{\chi\phi}. \tag{44}$$

Then, to lowest order in the slow-roll parameters, one finds

$$S_{\chi\phi}=-3\frac{\sqrt{4\pi}}{m_{Pl}}\,\frac{\sqrt{\epsilon_{tot}}}{(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}\,\delta s. \tag{45}$$

To match to the radiation epoch we take $S_{CDM-\mathrm{rest}}=S_{\chi\phi}$ at the end of inflation.

A. Observables: Amplitudes and spectral indices

In this subsection we will give the explicit expressions for the power spectra of the adiabatic and isocurvature modes, and their cross correlation. To lowest order, they can be written as power laws $\propto k^n$, in a way analogous to the single scalar field models [cf. Eq. (1)]. This means that there will be three amplitudes and three spectral indices. Moreover, we have taken into account also the tensor perturbation (gravitational-wave) spectrum, yielding a total of four amplitudes and four spectral indices. Indeed, we must consider the normalization that fixes one amplitude and will bring to *seven* observables.

On the other hand, the reader should call that we have introduced *five* slow-roll parameters. In the single field case there are three observables (the tensor to scalar amplitude ratio A_T^2/A_S^2, n_S and n_T), and one finds one consistency relation between A_T^2/A_S^2 and n_T [see Eq. (3)]. Thus in the present case we expect to find *two* consistency relations between the observables. To fit the CMB anisotropies one must consider the initial fluctuation spectra with their amplitudes and spectral indices. The existence of such consistency relations means that not all the amplitudes and spectral indices must be considered as independent.

For the curvature perturbation $\mathcal{R}$, it results from Eqs. (25) and (42):

$$\mathcal{P}_{\mathcal{R}}=\frac{4\pi}{m_{Pl}^2}\left(\frac{H_k}{2\pi}\right)^2\frac{1}{\epsilon_{tot}}[|f^2|+|\tilde{P}^2|]\bigg|_{t_f}, \tag{46}$$

where we have used the fact that $(\dot{A}/H)^2=m_{Pl}^2/4\pi\,\epsilon_{tot}$.

If not written otherwise, we intend this and all the subsequent expressions evaluated at the end of inflation for the reasons explained at the end of Sec. III A. For the isocurvature perturbation S, we can write from Eqs. (26) and (45):

$$\mathcal{P}_S=9\frac{4\pi}{m_{Pl}^2}\left(\frac{H_k}{2\pi}\right)^2\frac{\epsilon_{tot}}{\epsilon_\phi\epsilon_\chi}|g^2|\bigg|_{t_f}. \tag{47}$$

Finally, for the cross-spectrum $\mathcal{P}_{\mathcal{C}}$ we find, from Eqs. (27) and (40),

$$\mathcal{P}_{\mathcal{C}}=-6\frac{4\pi}{m_{Pl}^2}\left[\frac{\dot{\beta}}{H}\right]_{l.o.}\left(\frac{H_k}{2\pi}\right)^2\frac{1}{C}(e^{CN_k}-1)\times\frac{1}{(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}g^2\bigg|_{t_f}. \tag{48}$$

Now we can calculate the spectral indices to lowest order. They are defined as[2]

$$n-1\equiv\frac{d\ln\mathcal{P}}{d\ln k}. \tag{49}$$

The dependence of the above expressions on the comoving wave number k comes from H_k, N_k, and those slow-roll parameters which are evaluated at horizon crossing, and which are contained in f, g, $[\dot{\beta}/H]_{l.o.}$ and C. Therefore, in order to calculate n to lowest order, we have made use of the following formula:

$$\frac{d\ln\mathcal{P}}{d\ln k}=\frac{d\ln\mathcal{P}}{d\ln(aH)}\bigg|_{aH=k}=(1+\epsilon_{tot})\frac{d\ln\mathcal{P}}{d\ln a}{}_{aH=k}. \tag{50}$$

The spectral indices read[3]

$$n_{\mathcal{R}}-1\equiv\frac{d\ln\mathcal{P}_{\mathcal{R}}}{d\ln k}=-6\,\epsilon_{tot}+2\frac{\epsilon_\chi}{\epsilon_{tot}}\eta_{\chi\chi}+4\frac{(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}{\epsilon_{tot}}\eta_{\phi\chi}+2\frac{\epsilon_\phi}{\epsilon_{tot}}\eta_{\phi\phi}-8\frac{1}{1+\dfrac{|\tilde{P}^2|}{|f^2|}}\left[\frac{\dot{\beta}}{H}\right]_{l.o.}^2\frac{e^{-CN_k}}{C}(1-e^{-CN_k}), \tag{51}$$

$$n_S-1\equiv\frac{d\ln\mathcal{P}_S}{d\ln k}=-2\,\epsilon_{tot}+2\frac{\epsilon_\chi}{\epsilon_{tot}}\eta_{\phi\phi}-4\frac{(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}{\epsilon_{tot}}\eta_{\phi\chi}+2\frac{\epsilon_\phi}{\epsilon_{tot}}\eta_{\chi\chi}, \tag{52}$$

$$n_{\mathcal{C}}-1\equiv\frac{d\ln|\mathcal{P}_{\mathcal{C}}|}{d\ln k}=n_S-1-\frac{C}{e^{CN_k}-1}e^{CN_k}, \tag{53}$$

where all the slow-roll parameters are evaluated at $k=aH$.

As far as the tensor power spectrum is concerned, it is immediate to generalize the standard result for a single-field model (see, for example, [3]). To lowest order it is

[2]Note that the correlation can be positive or negative. In this case the spectral index can be defined as $n-1\equiv d\ln|\mathcal{P}|/d\ln k$. In the expressions below we have already taken it into account.

[3]Our definition of the isocurvature spectral index differs from n_{iso} as given, for instance, in Refs. [11]; one has $n_{\mathrm{iso}}=n_S-4$.

$$\mathcal{P}_T = \left(\frac{4}{\sqrt{\pi}} \frac{H}{m_{Pl}} \right)^2 \Bigg|_{k=aH}, \tag{54}$$

and thus the spectral index n_T reads

$$n_T = \frac{d \ln \mathcal{P}_T}{d \ln k} = -2\epsilon_{tot}. \tag{55}$$

As can be seen from Eq. (51), in the case of a single field (for example ϕ) we recover the standard result:

$$n_{\mathcal{R}} - 1 = -6\epsilon_\phi + 2\eta_{\phi\phi}. \tag{56}$$

It can be checked that, to lowest order in the slow-roll parameters, these spectral indices can be treated as independent of *k*, and so the spectra can be approximated, to the desired accuracy, as power laws.

B. Consistency relations

In order to get the consistency relations we have inverted the equations defining the observational quantities $\mathcal{P}_{\mathcal{R}}/\mathcal{P}_{\mathcal{S}}$, $\mathcal{P}_{\mathcal{R}}/\mathcal{P}_{\mathcal{C}}$, $\mathcal{P}_{\mathcal{R}}/\mathcal{P}_T$, $n_{\mathcal{R}}$, n_S, $n_{\mathcal{C}}$ and n_T in terms of the slow-roll parameters.

Let us multiply the power spectra by suitable coefficients which are conventional in literature (see Ref. [3]): $A_{\mathcal{R}}^2 = \frac{4}{25}\mathcal{P}_{\mathcal{R}}$, $A_{\mathcal{C}}^2 = \frac{2}{5}\mathcal{P}_{\mathcal{C}}$ and $A_T^2 = \frac{1}{100}\mathcal{P}_T$.

Defining

$$r_T \equiv \frac{A_T^2}{A_{\mathcal{R}}^2} \tag{57}$$

(not to be confused with the more traditional tensor-to-scalar quadrupole ratio) and

$$r_{\mathcal{C}} \equiv \frac{A_{\mathcal{C}}^2}{A_S A_{\mathcal{R}}}, \tag{58}$$

we have found the following consistency relations (the interested reader can find the details in Appendix B)

$$r_T = -\frac{1}{2} n_T (1 - r_{\mathcal{C}}^2), \tag{59}$$

$$(n_{\mathcal{C}} - n_S) r_T = -\frac{n_T}{4} (2n_{\mathcal{C}} - n_S - n_{\mathcal{R}}). \tag{60}$$

A few comments are in order at this point. From formula (59), one can easily recover the single-field model prediction, since in this case $A_{\mathcal{C}}^2$ vanishes and $n_T = -2A_T^2/A_{\mathcal{R}}^2$. We also learn from Eq. (59) that the tensor to adiabatic scalar amplitude ratio is smaller than $-n_T/2$ as soon as the adiabatic and entropy modes are cross-correlated. Equation (59) is a proof of the generic statement that $r_T \leqslant -n_T/2$ in the multi-component case (see, for example, Refs. [30,31]).

Equation (60) applies only when $r_{\mathcal{C}} \neq 0$; if the adiabatic and isocurvature modes are not correlated (as, for instance, in the case of assisted inflation [27]) there is only one consistency relation, which corresponds to the standard formula $r_T = -n_T/2$.

The consistency relation (60) can be further simplified if the slow-roll parameters are smaller than $1/N_k$. In such a case, to lowest order we get

$$r_T = -\frac{1}{2} n_T, \tag{61}$$

$$n_S = n_{\mathcal{R}}. \tag{62}$$

The consistency relations (59) and (60) [or (61) and (62)] are the main results of this paper.

V. CONCLUSIONS

In this paper we have considered the possibility that a cold dark matter (CDM) isocurvature perturbation mode can survive after an inflationary period in which two scalar fields are present. Linking the post inflationary epoch to the dynamics of inflation, under the slow-roll conditions, it is possible to get the expression for the spectra of the adiabatic, the isocurvature modes and their cross-correlation spectrum in terms of the slow-roll parameters defined for the two scalar fields. From these expressions two consistency relations follow, Eqs. (59) and (60), in analogy to what one finds in the single-field case. Thus these relations consitute a strong signature of inflation models with more than one scalar field. For an analysis of the CMB anisotropy measurements, these relations between observables must be taken into account, as a prediction of inflation. The main trend is actually to consider all the possible isocurvature modes (CDM, baryon, neutrino and neutrino velocity isocurvature modes) in a phenomenological way, considering all the amplitudes and the spectral indices as independent observables [22,23]. Even if the present treatment does not consider the possible origin of isocurvature modes different from the CDM one, it could be easily extended to the case of more than two scalar fields giving rise to many isocurvature modes. Our analysis clearly indicates that, in an inflationary scenario for the production of primordial perturbations, not all the observables have to be treated as independent. This has strong implications for our ability to accurately constrain cosmological parameters from CMB measurements in models where both adiabatic and isocurvature modes are present.

APPENDIX A: SLOW-ROLL EXPANSION

Here we report in more detail the calculations leading to Eqs. (29) and (33) to lowest order in the slow-roll parameters ϵ_I and η_{IJ}.

Using the definiton of the adiabatic and entropic fields, Eqs. (14) and (15), and Eq. (20), we obtain the initial conditions at the time t_k of horizon crossing:

$$Q_A \approx \frac{H_k}{\sqrt{2k^3}} e_A(\mathbf{k}), \quad \delta s \approx \frac{H_k}{\sqrt{2k^3}} e_s(\mathbf{k}). \tag{A1}$$

The solution of Eq. (29) will be

$$\delta s = B(k)\exp\left[\int_{t_k}^{t} -\frac{\mu_s^2}{3H^2} H dt\right], \tag{A2}$$

where $\mu_s^2 = V_{ss} + 3\dot{\beta}^2$. Let us recall the explicit expression of V_{ss}:

$$V_{ss} = (\sin^2\beta)V_{\phi\phi} - (\sin 2\beta)V_{\phi\chi} + (\cos^2\beta)V_{\chi\chi}. \tag{A3}$$

Using Eq. (12), we get

$$\sin^2\beta = \frac{\dot{\chi}^2}{\dot{\phi}^2+\dot{\chi}^2} = \frac{\epsilon_\chi}{\epsilon_{tot}}, \tag{A4}$$

where $\epsilon_{tot} = \epsilon_\phi + \epsilon_\chi$ and we have used the following relations holding to lowest order:

$$H^2 = \frac{8\pi}{3m_{Pl}^2} V(\phi,\chi) \quad \text{and} \quad \dot{\phi}_I = -\frac{1}{3H}\frac{\partial V}{\partial \phi_I}. \tag{A5}$$

Since $V_{\phi\phi}/H^2 = 3\,\eta_{\phi\phi}$, we obtain

$$\sin^2\beta \frac{V_{\phi\phi}}{H^2} = 3\frac{\epsilon_\chi}{\epsilon_{tot}}\eta_{\phi\phi}. \tag{A6}$$

In the same way one calculates the other two terms on the right-hand side of Eq. (A3), leading to Eq. (30).

The quantity $\dot{\beta}^2/3H^2$ may be neglected since it is $O(\epsilon^2,\eta^2)$:

$$\frac{\dot{\beta}}{H} = \cos^2\beta \frac{1}{H}\frac{d(\tan\beta)}{dt} = \frac{\epsilon_\phi}{\epsilon_{tot}}\frac{1}{H}\frac{d(\tan\beta)}{dt}, \tag{A7}$$

with

$$\begin{aligned}\frac{1}{H}\frac{d(\tan\beta)}{dt} &= \frac{1}{H}\frac{\ddot{\chi}\dot{\phi}-\dot{\chi}\ddot{\phi}}{\dot{\phi}^2}\\ &= \frac{1}{\epsilon_\phi}[-\eta_{\chi\chi}(\pm\sqrt{\epsilon_\chi}) - \eta_{\phi\chi}(\pm\sqrt{\epsilon_\phi})\\ &\quad + \epsilon_{tot}(\pm\sqrt{\epsilon_\chi})](\pm\sqrt{\epsilon_\phi})\\ &\quad + \frac{1}{\epsilon_\phi}[\eta_{\phi\phi}(\pm\sqrt{\epsilon_\phi}) + \eta_{\phi\chi}\\ &\quad \times(\pm\sqrt{\epsilon_\chi}) - \epsilon_{tot}(\pm\sqrt{\epsilon_\phi})](\pm\sqrt{\epsilon_\chi}),\end{aligned} \tag{A8}$$

and thus

$$\begin{aligned}\left[\frac{\dot{\beta}}{H}\right]_{l.o.} &= \frac{1}{\epsilon_{tot}}[(\epsilon_\chi - \epsilon_\phi)\eta_{\phi\chi} + (\eta_{\phi\phi} - \eta_{\chi\chi})\\ &\quad \times(\pm\sqrt{\epsilon_\chi})(\pm\sqrt{\epsilon_\phi})].\end{aligned} \tag{A9}$$

The function $g(t)$ is

$$\begin{aligned}g(t) = \exp\Bigg[\Bigg(&-\frac{\epsilon_\chi}{\epsilon_{tot}}\eta_{\phi\phi} + 2\frac{(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}{\epsilon_{tot}}\eta_{\phi\chi} - \frac{\epsilon_\phi}{\epsilon_{tot}}\eta_{\chi\chi}\Bigg)\\ &\times[N_k - N(t)]\Bigg].\end{aligned} \tag{A10}$$

Recall now Eq. (33) for the adiabatic perturbation Q_A:

$$\dot{Q}_A = a(t)Q_A + b(t), \tag{A11}$$

where

$$a(t) = -\frac{1}{3H}\left[V_{AA} - \dot{\beta}^2 - \frac{8\pi G}{a^3}\left(\frac{a^3\dot{A}^2}{H}\right)^{\cdot}\right], \tag{A12}$$

and

$$b(t) = \frac{2}{3H}\left[(\dot{\beta}\delta s)^{\cdot} - \left(\frac{V_A}{\dot{A}} + \frac{\dot{H}}{H}\right)\dot{\beta}\delta s\right]. \tag{A13}$$

The homogenous solution $f(t)$ is given by $\exp[\int_{t_k}^{t} a(s)ds]$. We expand $a(s)$ to lowest order. The same procedure for V_{ss}/H^2 holds for V_{AA}/H^2, and $\dot{\beta}^2/H^2$ is again neglected. As far as the last term in $a(t)$ is concerned, one has

$$\frac{1}{H^2}\frac{8\pi G}{a^3}\left(\frac{a^3\dot{A}^2}{H}\right)^{\cdot} = 3\frac{8\pi}{m_{Pl}^2}\frac{\dot{A}^2}{H^2} + \frac{8\pi}{m_{Pl}^2}\frac{1}{H^2}\left(\frac{\dot{A}^2}{H}\right)^{\cdot}. \tag{A14}$$

Since $\dot{A} = (\cos\beta)\dot{\phi} + (\sin\beta)\dot{\chi}$, it follows that

$$3\frac{8\pi}{m_{Pl}^2}\frac{\dot{A}^2}{H^2} = 3\frac{8\pi}{m_{Pl}^2}\frac{\dot{\phi}^2+\dot{\chi}^2}{H^2} = 6\epsilon_{tot}, \tag{A15}$$

and

$$\frac{1}{H^2}\left(\frac{\dot{A}^2}{H}\right)^{\cdot} = -\frac{m_{Pl}^2}{4\pi}\left[\left(\frac{\dot{H}}{H^2}\right)^2 + \frac{1}{H}\frac{d}{dt}\left(\frac{\dot{H}}{H^2}\right)\right], \tag{A16}$$

where we have used the formula $\dot{A}^2/H^2 = (-4\pi)^{-1} m_{Pl}^2 \dot{H}/H^2$. Since $-\dot{H}/H^2 = \epsilon_{tot}$ to lowest order, the term in Eq. (A16) is negligible [it is easy to verify that the time derivative of ϵ_{tot} is $\mathcal{O}(\epsilon^2,\eta^2)$].

Thus $a(t)$ reads

$$a(t) = -\frac{\epsilon_\chi}{\epsilon_{tot}}\eta_{\chi\chi} - \frac{\epsilon_\phi}{\epsilon_{tot}}\eta_{\phi\phi} - 2\frac{(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}{\epsilon_{tot}}\eta_{\phi\chi} + 2\epsilon_{tot} \tag{A17}$$

and Eq. (31) follows.

Finally, we calculate the particular solution $\tilde{P}(t)$ of the full Eq. (33). This is given by

$$\exp\left[\int_{t_k}^{t}a(s)ds\right]\int_{t_k}^{t}\exp\left[-\int_{t_k}^{\tau}a(s)ds\right]b(\tau)d\tau.$$

Inserting $\dot{\delta s}$, Eq. (29) into $b(t)$, we find

$$\frac{2}{3}\left[\frac{\ddot\beta}{H}-\frac{\dot\beta}{H}\frac{1}{3H}(V_{ss}+3\dot\beta^2)-\left(\frac{V_A}{\dot A}+\frac{\dot H}{H}\right)\frac{\dot\beta}{H}\right]\delta s, \tag{A18}$$

where $V_A=(\cos\beta)V_\phi+(\sin\beta)V_\chi$. To lowest order the only term which survives is $(V_A/\dot A)(\dot\beta/H)$, which to lowest order is given by $-3H[\dot\beta/H]_{l.o.}$.

Thus $b(t)$ reads

$$b(t)=2H\left[\frac{\dot\beta}{H}\right]_{l.o.}g(t), \tag{A19}$$

and $\widetilde{P}(t)$ at the end of inflation becomes

$$\widetilde{P}(t)|_{t_f}=2\left[\frac{\dot\beta}{H}\right]_{l.o.}e^{\int_{t_k}^{t_f}a(s)ds}\int_{t_k}^{t_f}e^{-\int_{t_k}^{\tau}a(\tau)d\tau}Hg(\tau)d\tau$$

$$=2\left[\frac{\dot\beta}{H}\right]_{l.o.}g(t)|_{t_f}\int_{t_k}^{t_f}e^{CN(t)}H\,dt \tag{A20}$$

where C is given in Eq. (37) and we have extracted from the integral $[\dot\beta/H]_{l.o.}$ and $g(t)|_{t_f}$, since, to lowest order, they can be considered as being constant. The integral can be resolved by the change of variables $Hdt=-dN$ and it yields $C^{-1}(e^{CN_k}-1)$. Thus Eq. (35) follows.

APPENDIX B: CONSISTENCY RELATIONS

To calculate the formulas (59) and (60), we must take into account that there are *seven* observables expressed through *five* slow-roll parameters at horizon crossing. To invert the equations defining $\mathcal{P}_R/\mathcal{P}_S$, $\mathcal{P}_R/\mathcal{P}_C$, $\mathcal{P}_R/\mathcal{P}_T$, $n_{\mathcal{R}}$, n_S, $n_{\mathcal{C}}$ and n_T, we have made a change of variables using five combinations of the slow-roll parameters at horizon crossing which are always found in the expressions for the observables. They are

$$\left[\frac{\dot\beta}{H}\right]_{l.o.}\equiv x \tag{B1}$$

$$|f|^2|_{t_f}\equiv u>0 \tag{B2}$$

$$g^2|_{t_f}\equiv r>0 \tag{B3}$$

$$\epsilon_{tot}|_{k=aH} \tag{B4}$$

and

$$\left(2\frac{\epsilon_\chi}{\epsilon_{tot}}\eta_{\phi\phi}-4\frac{(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})}{\epsilon_{tot}}\eta_{\phi\chi}+2\frac{\epsilon_\phi}{\epsilon_{tot}}\eta_{\chi\chi}\right). \tag{B5}$$

Note that in our results for the spectra, Eqs. (46), (47) and (48), there appear also two expressions in the slow-roll parameters evaluated at the end of inflation, $t=t_f$, and not only slow-roll parameters at $k=aH$. They are $\epsilon_{tot}|_{t_f}$ and $(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})|_{t_f}$. However, we have taken ϵ_{tot} equal to one at the end of inflation. So one can verify that $|f|^2|_{t_f}\sim 1/\epsilon_{tot}|_{k=aH}$, and therefore there are *five* variables again: x, u, r, the one defined in Eq. (B6), plus $(\pm\sqrt{\epsilon_\phi})(\pm\sqrt{\epsilon_\chi})|_{t_f}\equiv s>0$.

In these new variables we have considered the following equations:

$$n_S-n_{\mathcal{C}}=\frac{1}{2N_k}\frac{\sqrt{p}}{\sqrt{p}-1}\ln p \tag{B6}$$

$$n_{\mathcal{R}}-n_S=-\frac{1}{N_k}\ln p-\left[16x^2\frac{N_k}{\ln p}\frac{\sqrt{p}-1}{p}\right]\frac{1}{F(x,p)} \tag{B7}$$

$$\frac{4}{9}\mathcal{P}_R/\mathcal{P}_S=\frac{4}{81}s^2p\,F(x,p) \tag{B8}$$

$$\frac{9}{4}\mathcal{P}_C/\mathcal{P}_R=\left[-\frac{1}{9}\frac{s}{x}\frac{1}{2N_k}\ln p\frac{p}{\sqrt{p}-1}F(x,p)\right]^{-1} \tag{B9}$$

$$\frac{9}{4}\mathcal{P}_T/\mathcal{P}_R=36\frac{1}{rp}\frac{1}{F(x,p)} \tag{B10}$$

$$n_T=-2\frac{1}{rp} \tag{B11}$$

where

$$F(x,p)=1+16\times N_k^2x^2\frac{1}{(\ln p)^2}\frac{(\sqrt{p}-1)^2}{p}, \tag{B12}$$

with $p\equiv u/r>0$, and we have used the fact that the quantity C given in Eq. (37) can be written as $C\approx(1/N_k)\ln\sqrt{p}$.

Using Eqs. (B8), (B9), (B10) and (B11) one gets the first consistency relation Eq. (59) eliminating the variables x and r.

Equations (B7) and (B10), eliminating x, give the following equation:

$$\frac{9}{4}\mathcal{P}_T/\mathcal{P}_R=-\frac{n_T}{2}\left[36\sqrt{p}+36\,N_kn_S\frac{(1-\sqrt{p})}{\ln p}-36\,N_kn_{\mathcal{R}}\frac{(1-\sqrt{p})}{\ln p}\right]. \tag{B13}$$

Using Eqs. (B6) and (B13), one gets the second consistency relation (60). The procedure is as follows. We have defined $w \equiv \ln p$ and $1-\sqrt{p} \equiv z$. Thus the consistency relation is found from the equation $e^{w} \equiv (1-z)^2$, once the explicit expressions for w and z are obtained. This is straightforward leading to

$$1-z=\frac{-(n_{\mathcal{C}}-n_S)}{4\, n_T \dfrac{\mathcal{P}_R}{\mathcal{P}_T}} \tag{B14}$$

$$w=-2N_k\,(n_{\mathcal{C}}-n_S) -36\,N_k(n_{\mathcal{C}}-n_S)n_T\frac{4}{9}\,\frac{\mathcal{P}_R}{\mathcal{P}_T}-36\,N_k(n_{\mathcal{C}}-n_{\mathcal{R}})n_T\frac{4}{9}\,\frac{\mathcal{P}_R}{2\mathcal{P}_T}. \tag{B15}$$

From these equations we get Eq. (59) and

$$\ln\left[\frac{4(n_{\mathcal{C}}-n_S)r_T}{n_T(n_S+n_{\mathcal{R}}-2n_{\mathcal{C}})}\right]=N_k\left[(n_S-n_{\mathcal{C}})+\frac{n_T(n_S+n_{\mathcal{R}}-2n_{\mathcal{C}})}{4r_T}\right]. \tag{B16}$$

In order to make the solution time-independent, consistently with our first-order slow-roll expansion, both sides of Eq. (B16) have to vanish. Equation (60) then follows. The relation (62) in the limit $|C|N_k \ll 1$ can be easily derived by noting that $n_{\mathcal{C}}-n_S=-1/N_k$ and $2n_{\mathcal{C}}-n_S-n_{\mathcal{R}}=-2/N_k$.

[1] A.H. Guth, Phys. Rev. D **23**, 347 (1981).
[2] For a recent review, see D.H. Lyth and A. Riotto, Phys. Rep. **314**, 1 (1999).
[3] J.E. Lidsey, A.R. Liddle, E.W. Kolb, E.J. Copeland, T. Barreiro, and M. Abney, Rev. Mod. Phys. **69**, 373 (1993), and references therein.
[4] A.R. Liddle, P. Parsons, and J.D. Barrow, Phys. Rev. D **50**, 7222 (1994).
[5] W.H. Kinney, A. Melchiorri, and A. Riotto, Phys. Rev. D **63**, 023505 (2001); J.R. Bond *et al.*, in Proceedings of the IAU Symposium 201 (PASP), CITA-2000-65, astro-ph/0011378; P. de Bernardis *et al.*, Proceedings of the CAPP2000 conference, Verbier, 2000, astro-ph/0011469.
[6] http://map.gsfc.nasa.gov
[7] http://astro.estec.esa.nl/SA-general/Projects/Planck
[8] J.R. Bond, G. Efstathiou, and M. Tegmark, astro-ph/9702100; E.J. Copeland, I.J. Grivell, and A.R. Liddle, astro-ph/9712028.
[9] S. Mollerach, Phys. Lett. B **242**, 158 (1990); S. Mollerach, Phys. Rev. D **42**, 313 (1990); J. Yokoyama and Y. Suto, Astrophys. J. **379**, 427 (1991); M. Kawasaki, N. Sugiyama, and T. Yanagida, Phys. Rev. D **54**, 2442 (1996).
[10] T.J. Allen, B. Grinstein, and M.B. Wise, Phys. Lett. B **197**, 66 (1987); J.R. Bond and D. Salopek, Phys. Rev. D **45**, 1139 (1992); K. Yamamoto *et al.*, *ibid.* **46**, 4206 (1992); A. Linde and V. Mukhanov, *ibid.* **56**, 535 (1997); M. Bucher and Y. Zhu, *ibid.* **55**, 7415 (1997).
[11] P.J.E. Peebles, Astrophys. J. Lett. **483**, L1 (1997); Astrophys. J. **510**, 523 (1999); **510**, 531 (1999).
[12] A.D. Linde, Phys. Lett. **158B**, 375 (1985); L.A. Kofman, Phys. Lett. B **173**, 400 (1986); L.A. Kofman and A. Linde, Nucl. Phys. **B282**, 555 (1987); S. Mollerach, S. Matarrese, and A. Ortolan, Phys. Rev. D **44**, 1670 (1991).
[13] For a recent review, see V.A. Rubakov, hep-ph/0104152.
[14] A.D. Linde, JETP Lett. **40**, 1333 (1984).
[15] A.A. Starobinsky, JETP Lett. **42**, 152 (1985).
[16] D. Polarsky and A.A. Starobinsky, Phys. Rev. D **50**, 6123 (1994).
[17] K. Enqvist and H. Kurki-Suonio, Phys. Rev. D **61**, 043002 (2000); E. Pierpaoli, J. Garcia-Bellido, and S. Borgani, J. High Energy Phys. **10**, 015 (1999).
[18] D. Langlois and A. Riazuelo, Phys. Rev. D **62**, 043504 (2000).
[19] D. Langlois, Phys. Rev. D **59**, 123512 (1999).
[20] C. Gordon, D. Wands, B.A. Bassett, and R. Maartens, Phys. Rev. D **63**, 023506 (2000).
[21] N. Bartolo, S. Matarrese, and A. Riotto, Phys. Rev. D **64**, 083514 (2001).
[22] R. Trotta, A. Riazuelo, and R. Durrer, astro-ph/0104017; L. Amendola, C. Gordon, D. Wands, and M. Sasaki, astro-ph/0107089.
[23] M. Bucher, K. Moodley, and N. Turok, Phys. Rev. D **62**, 083508 (2000); astro-ph/0007360; Phys. Rev. Lett. (to be published), astro-ph/0012141.
[24] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).
[25] V. Lukash, Zh. Éksp. Teor. Fiz. **79**, 1601 (1980) [Sov. Phys. JETP **52**, 807 (1980)]; D.H. Lyth, Phys. Rev. D **31**, 1792 (1985).
[26] M. Sasaki, Prog. Theor. Phys. **76**, 1036 (1986); V.F. Mukhanov, Zh. Éksp. Teor. Fiz. **94**, 1 (1988) [Sov. Phys. JETP **67**, 1297 (1988)].
[27] A.R. Liddle, A. Mazumdar, and F.E. Schunk, Phys. Rev. D **58**, 061301 (1998); K.A. Malik and D. Wands, *ibid.* **59**, 123501 (1999).
[28] J. Garcia-Bellido and D. Wands, Phys. Rev. D **52**, 6739 (1995); **53**, 5437 (1996).
[29] J.-C. Hwang and H. Noh, Phys. Lett. B **495**, 277 (2000).
[30] D. Polarsky and A.A. Starobinsky, Phys. Lett. B **356**, 196 (1995).
[31] M. Sasaki and E.D. Stewart, Prog. Theor. Phys. **95**, 71 (1996).