Subtleties in *CPT* transformation for Majorana fermions

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We point out the relevance of the so-called Majorana creation phase in the *s*-channel matrix elements in connection with the *CPT* transformation of the latter.

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Majorana particles currently play an important role in particle models, either by invoking supersymmetry [1] or by speculating about the Majorana nature of massive neutrinos [2–4]. It is known that Majorana fermions are defined up to a phase λ_M, often called the "creation phase." This phase is conventional and therefore physical results should not depend upon it. It is, however, appreciated that in many situations the most convenient choice of this phase is often not ± 1 [3]. Once these phases are fixed, they will enter the individual expressions for the coupling constants of the Majorana particles, albeit, as stated above, the final physical results are independent of the choice of λ_M's. In this short note we point out that there exists yet another place where the creation phase makes its appearance. This is in connection with *s*-channel matrix elements where an overall dependence of the creation phase has to be taken into account. Neglecting this λ_M dependence of such matrix elements leads to contradictions with *CPT* invariance, as shown below. This is especially relevant when we parametrize in principle unknown matrix elements.

Let us start by introducing some definitions. The Majorana field Ψ_M is defined by demanding that Ψ_M be self-conjugate, up to a phase, i.e.,

$$\lambda_C C\overline{\Psi}_M^T(\mathbf{x},t)=\lambda_C\Psi_M^C(\mathbf{x},t)=\lambda_M'\Psi_M(\mathbf{x},t) \tag{1}$$

where C is the charge conjugation matrix satisfying $C^{-1}=C^\dagger=C^T=-C$ and λ_C as well as λ_M' are arbitrary phases. In Fock space we can write

$$\Psi(\mathbf{x},t)=\int[dk]\sum_\lambda\,[a(\mathbf{k},\lambda)u(\mathbf{k},\lambda)e^{-ikx}+\lambda_M a^\dagger(\mathbf{k},\lambda)v(\mathbf{k},\lambda)e^{ikx}] \tag{2}$$

where $[dk]$ is a three dimensional integration measure depending on the normalization of the spinors, the λ's are the particle helicities, and $\lambda_M\equiv-\lambda_C\lambda_M'^*$. The last is known as the Majorana creation phase. That it can be chosen at will can be best seen at the place in the Lagrangian where Majorana fields are defined. For instance, a Majorana mass term for a neutrino of the form $(-1/2)\overline{\nu_{iL}^c}M_{ij}\nu_{jL}$ can be diagonalized by a unitary matrix U with $M=(U^\dagger)^T mU^\dagger$ such that $m=\mathrm{diag}(m_i,m_j,\ldots)$. To obtain positive definite masses $|m_i|$ the possible phases in the parameters m_i can be absorbed in two ways. One way is to absorb the phases as creation phases in the definition of the Majorana fields. The other is simply to redefine the rotation matrix as $U'^\dagger=SU^\dagger$ with S being a phase diagonal matrix [2]. This, of course, also means that we are allowed to arbitrarily introduce phases in the definition of the Majorana fields which read in general as

$$n_i=n_{iL}-\lambda_M^{(i)}(n_{iL})^C \tag{3}$$

with $n_{iL}=U_{ij}'^\dagger\nu_{jL}$ and arbitrary phases $\lambda_M^{(i)}$. It can be shown that λ_M' in Eq. (1) has to be real, i.e., ± 1. As a result we have $\lambda_M^{(i)}=\pm\lambda_C^{(i)}$.

Let us now consider the electromagnetic matrix element of the current j_{em}^μ sandwiched between one particle Majorana states $\phi_{(\mathbf{k}_i,\lambda_i)}=a^\dagger(\mathbf{k}_i,\lambda_i)\Omega$, Ω being the vacuum state:

$$\begin{aligned}&(\phi_{(\mathbf{k}_1,\lambda_1)},j_{em}^\mu(x)\phi_{(\mathbf{k}_2,\lambda_2)})\\&\quad=e^{-i(k_1-k_2)x}\overline{u(\mathbf{k}_1,\lambda_1)}\Sigma_t^\mu(k_1,k_2)u(\mathbf{k}_2,\lambda_2)\end{aligned} \tag{4}$$

where the index t reminds us that we are in t-channel matrix elements (i.e., one outgoing and one incoming Majorana particle). Σ_t^μ can be as usual decomposed into form factors [4] in agreement with Lorentz covariance, Hermiticity of j_{em}^μ, and gauge invariance, viz.,

$$\begin{aligned}&\overline{u(\mathbf{k}_1,\lambda_1)}\Sigma_t^\mu(k_1,k_2)u(\mathbf{k}_2,\lambda_2)\\&\quad=\overline{u(\mathbf{k}_1,\lambda_1)}\Big[\gamma^\mu F_1(t)+i\sigma^{\mu\nu}q_\nu F_2(t)\\&\qquad+i\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}q_\nu F_3(t)\\&\qquad+\left(q^\mu-\frac{t}{2m}\gamma^\mu\right)\gamma_5F_4(t)\Big]u(\mathbf{k}_2,\lambda_2)\end{aligned} \tag{5}$$

with the standard meaning of the form factors at $q^2=t=0$.

It is known that for Majorana fields $\overline{\Psi}_M\gamma_\mu\Psi_M=\overline{\Psi}_M\sigma_{\mu\nu}\Psi_M=0$ and thus the only nonzero form factor for Majorana particles is the anapole form factor F_4. Interestingly, one can also exclude $F_{i=1,2,3}$ for Majorana states by imposing *CPT* invariance of the electromagnetic matrix element [5–9]. For this purpose we need the charge conjugation, parity, and time-reversal transformations acting on a mass-eigenstate Majorana field in Fock space

$$\mathcal{C}\Psi_M(\mathbf{x},t)\mathcal{C}^\dagger=\lambda'_M\Psi_M(\mathbf{x},t),$$
$$\mathcal{P}\Psi_M(\mathbf{x},t)\mathcal{P}^\dagger=\lambda_P\gamma^0\Psi_M(-\mathbf{x},t),$$
$$\mathcal{T}\Psi_M(\mathbf{x},t)\mathcal{T}^\dagger=\lambda_T\gamma_5 C\Psi_M(\mathbf{x},-t). \tag{6}$$

It is straightforward to deduce from this the combined transformation $\mathcal{S}\equiv\mathcal{CPT}$ for the annihilation and creation operators, respectively,

$$\mathcal{S}a(\mathbf{k},\lambda)\mathcal{S}^\dagger=-\lambda_S\lambda^*_M(-1)^{-(\lambda-1)/2}a(\mathbf{k},-\lambda),$$
$$\mathcal{S}a^\dagger(\mathbf{k},\lambda)\mathcal{S}^\dagger=+\lambda_S\lambda_M(-1)^{-(\lambda-1)/2}a^\dagger(\mathbf{k},-\lambda) \tag{7}$$

in which $\lambda_S=\lambda_C\lambda_P\lambda_T=\pm i$. Applying now the *CPT* transformation to the matrix element (4) gives

$$\begin{aligned}\overline{u}(\mathbf{k}_1,\lambda_1)\Sigma^\mu_t(k_1,k_2)u(\mathbf{k}_2,\lambda_2)\\ &=(\Omega a^\dagger(\mathbf{k}_1,\lambda_1),j^\mu_{em}(0)a^\dagger(\mathbf{k}_2,\lambda_2)\Omega)\\ &=(\Omega,\mathcal{S}^\dagger\mathcal{S}a(\mathbf{k}_1,\lambda_1)\mathcal{S}^\dagger\mathcal{S}j^\mu_{em}(0)\mathcal{S}^\dagger\mathcal{S}a^\dagger(\mathbf{k}_2,\lambda_2)\mathcal{S}^\dagger\mathcal{S}\Omega)\\ &=(-1)^{-(\lambda_1-\lambda_2)/2}(\Omega,a(\mathbf{k}_1,-\lambda_1)\\ &\quad\times j^\mu_{em}(0)a^\dagger(\mathbf{k}_2,-\lambda_2)\Omega)^*\\ &=(-1)^{-(\lambda_1-\lambda_2)/2}\\ &\quad\times[\overline{u}(\mathbf{k}_1,-\lambda_1)\Sigma^\mu_t(k_1,k_2)u(\mathbf{k}_2,-\lambda_2)]^\dagger.\end{aligned} \tag{8}$$

In Eq. (8) we have used $\mathcal{S}j^\mu_{em}(0)\mathcal{S}^\dagger=-j^\mu_{em}(0)$, $\lambda_S^2=-1$, and $\mathcal{S}\Omega=\Omega$. The details of the calculation in Eq. (8) follow essentially the steps given in [5–9]. It is now convenient to introduce the following notation for the linear independent γ matrices which we will in general denote by Γ:

$$\Gamma^\dagger=\eta_0[\Gamma]\gamma^0\Gamma\gamma^0,$$
$$\Gamma^T=\eta_C[\Gamma]C\Gamma C^{-1},$$
$$\gamma_5\Gamma\gamma_5=\eta_5[\Gamma]\Gamma, \tag{9}$$

where the η's are pure signs depending on the matrix Γ. We can now calculate the last expression in Eq. (8) for an arbitrary Γ matrix. The result reads

$$\begin{aligned}(-1)^{-(\lambda_1-\lambda_2)/2}\left[\overline{u}(\mathbf{k}_1,-\lambda_1)\left\{\begin{matrix}1\\ i\end{matrix}\right\}\Gamma u(\mathbf{k}_2,-\lambda_2)\right]^\dagger\\ =\eta_0[\Gamma]\eta_C[\Gamma]\eta_5[\Gamma]\overline{u}(\mathbf{k}_1,\lambda_1)\left\{\begin{matrix}-1\\ i\end{matrix}\right\}\Gamma u(\mathbf{k}_2,\lambda_2).\end{aligned} \tag{10}$$

Since $\eta_0[\Gamma]\eta_C[\Gamma]\eta_5[\Gamma]=+1,-1,-1,-1$ for $\Gamma=\gamma_\mu$, $\sigma_{\mu\nu},\gamma_5,\gamma_\mu\gamma_5$, respectively, we conclude that the only surviving electromagnetic form factor in Eq. (5) for Majorana fermions is the anapole form factor F_4 [5–9].

We next turn our attention to the same electromagnetic matrix element, but now calculated in the *s* channel (i.e., two outgoing Majorana particles). We write

$$\begin{aligned}(\Omega,j^\mu_{em}(0)a^\dagger(\mathbf{k}_1,\lambda_1)a^\dagger(\mathbf{k}_2,\lambda_2)\Omega)\\ =\overline{v}(\mathbf{k}_2,\lambda_2)\Sigma^\mu_s(k_1,k_2)u(\mathbf{k}_1,\lambda_1)\end{aligned} \tag{11}$$

and assume, in the first instance, that Σ^μ_s has exactly the same form-factor decomposition as in Eq. (5) with *t* replaced by *s*. We proceed then along the same lines as above. Since the steps in performing the *CPT* transformation are very similar to the *t*-channel case we only quote the final result. The *CPT* transformation gives now

$$\begin{aligned}\overline{v}(\mathbf{k}_2,\lambda_2)\Sigma^\mu_s(k_1,k_2)u(\mathbf{k}_1,\lambda_1)\\ &=(\lambda^*_M)^2(-1)^{(\lambda_1-\lambda_2)/2}(\Omega,j^\mu_{em}(0)a^\dagger(\mathbf{k}_1,-\lambda_1)\\ &\quad\times a^\dagger(\mathbf{k}_2,-\lambda_2)\Omega)^*\\ &=(\lambda^*_M)^2(-1)^{(\lambda_1-\lambda_2)/2}\\ &\quad\times[\overline{v}(\mathbf{k}_2,-\lambda_2)\Sigma^\mu_s(k_1,k_2)u(\mathbf{k}_1,-\lambda_1)]^\dagger.\end{aligned} \tag{12}$$

As before, we evaluate the last expression for an individual Γ matrix and get

$$\begin{aligned}(\lambda^*_M)^2(-1)^{(\lambda_1-\lambda_2)/2}\left[\overline{v}(\mathbf{k}_2,-\lambda_2)\left\{\begin{matrix}1\\ i\end{matrix}\right\}\Gamma u(\mathbf{k}_1,-\lambda_1)\right]^\dagger\\ =(\lambda^*_M)^2\eta_0[\Gamma]\eta_C[\Gamma]\eta_5[\Gamma]\overline{v}(\mathbf{k}_2,\lambda_2)\left\{\begin{matrix}-1\\ i\end{matrix}\right\}\Gamma u(\mathbf{k}_1,\lambda_1).\end{aligned} \tag{13}$$

It is obvious that with respect to *CPT* transformations our conclusion would depend now on the *choice* of λ_M. For instance, putting $\lambda_M=\pm i$ seemingly excludes F_4 in the *s* channel. Either *CPT* would be broken or the Majorana creation would not be conventional. Both conclusions are physically not acceptable. The remedy is at hand when we change Eq. (11) by multiplying the right hand side by λ^*_M: i.e.,

$$\begin{aligned}(\Omega,j^\mu_{em}(0)a^\dagger(\mathbf{k}_1,\lambda_1)a^\dagger(\mathbf{k}_2,\lambda_2)\Omega)\\ =\lambda^*_M\overline{v}(\mathbf{k}_2,\lambda_2)\Sigma^\mu_s(k_1,k_2)u(\mathbf{k}_1,\lambda_1)\end{aligned} \tag{14}$$

where Σ^μ_s has still the same form-factor decomposition as in Eq. (5) with *t* replaced by *s*. The consequence is that Eq. (13) now becomes

$$\begin{aligned}(\lambda^*_M)^2(-1)^{(\lambda_1-\lambda_2)/2}\left[\lambda^*_M\overline{v}(\mathbf{k}_2,-\lambda_2)\left\{\begin{matrix}1\\ i\end{matrix}\right\}\Gamma u(\mathbf{k}_1,-\lambda_1)\right]^\dagger\\ =\lambda^*_M\eta_0[\Gamma]\eta_C[\Gamma]\eta_5[\Gamma]\overline{v}(\mathbf{k}_2,\lambda_2)\left\{\begin{matrix}-1\\ i\end{matrix}\right\}\Gamma u(\mathbf{k}_1,\lambda_1),\end{aligned} \tag{15}$$

which leads to the same results now as in the case of the *t* channel, namely, excluding all form factors except F_4. This also means, however, that *s*-channel matrix elements with Majorana fermions pick up the complex conjugate of the creation phase as an overall phase, independent of the operator involved. Once the creation phase is fixed, neglecting this overall phase in *s*-channel matrix elements leads to disastrous consequences for *CPT* properties. For calculable matrix elements, such as those involving a normal product of free Majorana fields, this overall phase can be justified directly. We get

$$(\Omega,:\overline{\Psi}_M(x)gG\Psi_M(x):a^\dagger(\mathbf{k}_1,\lambda_1)a^\dagger(\mathbf{k}_2,\lambda_2)\Omega)$$
$$=\lambda_M^* e^{-i(k_1+k_2)x}(1+\eta_C[\Gamma])\overline{v}(\mathbf{k}_2,\lambda_2)\Gamma u(\mathbf{k}_1,\lambda_1). \tag{16}$$

Viewing Feynman rules as Fourier transforms of the functional derivatives of the action, it is clear that the appearance of the global phase in the last two equations is not part of these rules.

In summary, although the creation phase is conventional and unmeasurable, *s*-channel matrix elements have to be parametrized in the way indicated by Eqs. (14) and (16) to avoid conflict with *CPT* invariance.

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