Systematic study of the single instanton approximation in QCD

P. Faccioli* and E. V. Shuryak†
Department of Physics and Astronomy, State University of New York at Stony Brook, Stony Brook, New York 11794
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The single-instanton approximation (SIA) is often used to evaluate analytically instanton contributions to the Euclidean correlation function in QCD at small distances. We discuss how this approximation can be consistently derived from the theory of instanton ensemble and give precise definitions to a number of different "quark effective masses," generalizing the parameter m^*, which was introduced long ago to account for the collective contribution of the whole ensemble. We test numerically the range of applicability of the SIA for different quantities. Furthermore, we determine all the effective masses (for random and interacting instanton liquid models) as well as from phenomenology, and discuss to what extent those are universal.

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I. INTRODUCTION

The instanton liquid model of the QCD vacuum [1] is based on a semiclassical approximation, in which all gauge configurations are replaced by an ensemble of topologically nontrivial fields, instantons, and anti-instantons. It remains a model because we still do not understand why large-size instantons are not present in the ensemble. Fits to phenomenology and later lattice studies [23] showed that their total density is $n_0 \simeq 1\ \mathrm{fm}^{-4}$ with a typical size of about $\rho \sim 1/3$ fm, leading to a small diluteness parameter $n_0\rho^3 \sim 10^{-2}$ [1]. With these parameters, the model quantitatively explains such important phenomena as spontaneous $SU(N_f)$ chiral symmetry breaking for N_f quark flavors, explicit U(1) symmetry breaking, and many more other details of hadronic correlators and spectroscopy (for a recent example see the discussion of vector and axial correlators [2], for a review see [3]). The main feature of the instanton [4] ensemble is that each pseudoparticle is an effective vertex with $2N_f$ quark lines [5], which are exchanged between them and fill the vacuum. A theory is developed, called the interacting instanton liquid model (IILM), which includes these 't Hooft interactions *to all orders* [3].

If new sources (external currents) are added, they produce extra quarks which interact with those in vacuum and produce nontrivial correlation functions. In particular, many (Lorentz scalar) chirally odd local operators obtain nonzero vacuum expectation values. In general, all of those "condensates" and correlation functions are determined by the interaction of instantons and thus depend on the global (collective) properties of the ensemble.

On the other hand, as the instanton vacuum is fairly dilute, one may think that the correlation functions at distances short compared to instanton spacing $x \ll R = n^{-1/4} \sim 1$ fm may be dominated by a *single* instanton, the closest (or leading) one (LI). This framework [which we shall refer to as the single instanton approximation, (SIA)] has the advantage of allowing us to carry out calculations analytically. It is therefore possible to obtain closed expressions for an instanton contribution to Green's functions in momentum or in Borel space.

In SIA the collective contribution of all instantons *other than the leading one* is taken care of by a single effective parameter, usually called the effective mass, m^*. In the simplest approximation, it can be associated with an *average* value of the quark condesate [6]:

$$m^* = m - \frac{2}{3}\pi^2\rho^2\langle\bar{u}u\rangle, \tag{1}$$

which leads to the value $m^* \simeq 170$ MeV [1]. Note that it is already very different from what one infers from the same model for the long distance (or zero Euclidean momentum) limit of the quark propagator, which gives the *constituent quark mass* of the order of 400 MeV.

Furthermore, although the SIA has been used in several phenomenological studies (e.g., [1,7–9], and references therein), its derivation was never discussed in detail, its range of applicability was never quantitatively checked, and the values of relevant effective masses well specified. And indeed, if one uses the value $m^* \simeq 170$ MeV the correlation functions, evaluated in the SIA, do not agree with the results of the random and interacting instanton liquid [3].

In this paper we identify the origin of such a discrepancy and calculate the values of effective mass appropriate for different observables. This analysis reveals that the discrepancy between the SIA and full liquid calculations is due to an incorrect estimate of the effective mass, m^*. We also present a systematic study of the SIA in QCD by itself. We show that the approach is really accurate only for calculations that involve operators of dimension six or more, or correlators with more than one zero-mode propagator. We shall also prove that the mass terms, appearing in matrix elements involving different numbers of zero-mode propagators, are indeed independent parameters that have to be fixed separately. We provide with the definitions of all such mass factors in terms of averages of the instanton ensembles and prove that they are nearly universal, i.e., the same for all similar correlation functions.

*Electronic address: faccioli@tonic.physics.sunysb.edu
†Electronic address: shuryak@dau.physics.sunysb.edu

The paper is organized as follows. In Sec. II we derive the SIA from the theory of the instanton ensemble, in Sec. III we present the results of our numerical simulations that estimate the contribution from the leading instanton to several correlation functions. In Sec. IV we evaluate the effective mass terms both from the random and interacting instanton liquid and compare it with the values obtained phenomenologically from the pion sum rule. In Sec. V we compare our effective masses with the so-called ''determinantal masses,'' which are other effective parameters that can be defined in terms of averages of the fermionic determinant. The main results of our analysis are summarized in Sec. VI.

II. QUARK PROPAGATOR

In this section we review how the quark propagator in the instanton vacuum is obtained and present consistent derivation of the SIA. The quark propagator in the general background field is

$$S_I(x,y)=\langle x|(i\not{D}_I+im)^{-1}|y\rangle, \tag{2}$$

where $\not{D}_I$ denotes the Dirac operator. The inverse (2) can be formally represented as an expansion in eigenmodes of the Dirac operator:

$$S_I(x,y)=\sum_\lambda \frac{\psi_\lambda(x)\psi^\dagger_\lambda(y)}{\lambda+im}, \quad i\not{D}_I\psi_\lambda(x)=\lambda\psi_\lambda(x). \tag{3}$$

From Eq. (3) it follows that the propagator of light quarks is dominated by eigenmodes with small virtuality.

We begin by considering the academic case in which the vacuum contains only one isolated instanton. One eigenmode of $\not{D}_I$ with zero virtuality (zero modes) is given by 't Hooft [5,10]:

$$i\not{D}\psi_0(x)=0,$$

$$\psi_{0a\nu}(x;z)=\frac{\rho}{\pi}\frac{1}{[(x-z)^2+\rho^2]^{3/2}}\left[\frac{1-\gamma_5}{2}\frac{\not{x}-\not{z}}{\sqrt{(x-z)^2}}\right]_{\alpha\beta}\times U_{ab}\epsilon_{\beta b}, \tag{4}$$

where z denotes the instanton position, $\alpha,\beta=1,\dots,4$ are spinor indices and U_{ab} represents a general group element.

Isolating the contribution from zero modes we can write

$$S_I(x,y;z)=\frac{\psi_0(x-z)\psi_0^\dagger(y-z)}{im}+\sum_{\lambda\neq 0}\frac{\psi_\lambda(x-z)\psi^\dagger_\lambda(y-z)}{\lambda+im}$$
$$=S_I^{zm}(x,y;z)+S_I^{nzm}(x,y;z). \tag{5}$$

The zero-mode part of the propagator in the field of one instanton can be evaluated from Eqs. (5) and (4) to give [11]:

$$S_I^{zm}(x,y;z)=\frac{(\not{x}-\not{z})\gamma_\mu\gamma_\nu(\not{y}-\not{z})}{8m}\left[\tau^-_\mu\tau^+_\nu\frac{1-\gamma_5}{2}\right]\times\phi(x-z)\phi(y-z), \tag{6}$$

where

$$\phi(t):=\frac{\rho}{\pi}\frac{1}{|t|(t^2+\rho^2)^{3/2}}, \quad \tau^\pm_\mu:=(\tau,\mp i). \tag{7}$$

The corresponding expression in the field of one anti-instanton is obtained through the substitution:

$$\frac{1-\gamma_5}{2}\leftrightarrow\frac{1+\gamma_5}{2}, \quad \tau^-\leftrightarrow\tau^+. \tag{8}$$

In the chiral limit, $m\to 0$, the expression for $S_I^{nzm}(x,y;z)$ is also known exactly [12]. In the limit of small distances ($|x-y|\to 0$), or if the instanton is very far away ($|x-z|\to\infty$) one has

$$S_I^{nzm}(x,y;z)\simeq S_0(x,y), \tag{9}$$

where S_0 denotes the free propagator. Typically, corrections to Eq. (9) lead to small contributions and will be neglected in what follows. Once the propagator has been calculated, one can in principle evaluate any correlation function in the single-instanton background.

Now, let us turn to the realistic vacuum of QCD. Here, any configuration with a nonzero net topological charge would be highly disfavored by the small value of the θ angle. Therefore, one is lead to picture the vacuum as an ensemble with an equal density of instanton and anti-instantons. If the vacuum is dilute enough, the classical background field can be approximatively taken to be a superposition of separated instantons and anti-instantons [13]:

$$A_\mu(x,\{\Omega_i\}_i)=\sum_I A^I_\mu(x,\{\Omega^I_i\}_i)+\sum_A A^A_\mu(x,\{\Omega^A_i\}_i), \tag{10}$$

where $\{\Omega_i\}_i$ denotes the set of all collective coordinates.

The propagator in such a background field can then be evaluated as follows [11]. Let us consider the expansion:

$$S=S_0+S_0\not{A}S_0+S_0\not{A}S_0\not{A}S_0+\cdots, \tag{11}$$

where integrations over the positions of each background field insertion is understood. The series (11) can be rearranged so that all terms depending on the collective coordinates of one instanton field only are summed up first, followed by all terms depending on two instantons and so on. One gets

$$S=S_0+\sum_I(S_I-S_0)+\sum_{I\neq J}(S_I-S_0)S_0^{-1}(S_J-S_0)+\cdots, \tag{12}$$

where S_I denotes the full propagator in the field of the instanton I so, in the approximation (9) one has

$$(S_I-S_0)_{ij}(x,y)\simeq\frac{\psi^I_{0i}(x)\psi^{\dagger I}_{0j}(y)}{im}, \tag{13}$$

where we have dropped all collective coordinates indices. Inserting Eq. (13) in Eq. (12) and dropping also all spinor indices we get

$$S(x,y) \simeq S^0(x,y) + \sum_I \frac{\psi_0(x)\psi_0^\dagger(y)}{im} + \sum_{I,J} \frac{\psi_{0I}(x)}{im} \times \left(\int d^4z\, \psi_{0I}^\dagger(z)(i\partial\!\!\!/_z + im)\psi_{J0}(z) - im\,\delta_{IJ} \right) \frac{\psi_{0J}^\dagger(y)}{im} + \cdots, \tag{14}$$

where $-im\delta_{I,J}$ has been added in order to relax the $J \neq I$ constraint in the summation. All the terms, starting from the second on, form a geometrical progression, which can be resummed to give

$$S(x,y) \simeq S^0(x,y) + \sum_{I,J} \psi_{0I}(x) \left(\frac{1}{T+o(m)} \right)_{IJ} \psi_{0J}^\dagger(y), \tag{15}$$

where T_{IJ} denotes the overlap matrix in zero-modes subspace

$$T_{IJ} = \int d^4z\, \psi^\dagger(z)_I (i\partial\!\!\!/) \psi(z)_J. \tag{16}$$

In Eq. (15), the zero-mode part of the quark propagator is approximatively written as a bilinear form in the space spanned by the quark zero-mode wave functions. From Eq. (4) it follows that the contribution coming from all the terms in the sum associated with instantons very far away from the points x and y will be negligible. In particular, the biggest term in Eq. (15) is associated with the closest instanton, I^*. Such an instanton is dominating if the average of the correlation function calculated retaining only the (I^*,I^*) term in Eq. (15) is much larger than the average of the same quantity calculated from all other terms in the sum (15). Notice that this is a much weaker assumption than demanding

$$\psi_{0I^*}(x) \left(\frac{1}{T+o(m)} \right)_{I^*I^*} \psi_{0I^*}^\dagger(y) \gg \sum_{I \neq I^*, J \neq I^*} \psi_{0I}(x) \left(\frac{1}{T+o(m)} \right)_{IJ} \psi_{0J}^\dagger(y), \tag{17}$$

for *each* configuration.

Let us summarize the framework developed so far. First of all, the inverse matrix $(1/T)_{IJ}$ contains all the information about the particular configuration of the instanton ensemble. In order to evaluate correlation functions, one needs to average over all possible configurations. Since contributions from distant instantons are suppressed by their zero-mode wave functions, one expects correlation functions with the highest number of zero modes to be most influenced by the leading instanton I^*. If it is possible to retain only the contribution from I^*, the global properties of the ensemble are present in the matrix element $(1/T)_{I^*I^*}$.

As it was suggested a long time ago by one of us [1], one can represent the collective contribution of all other instantons by introducing an effective mass associated with quark propagating in the zero modes. In other words, one assumes that for $|x-y| < 1$ fm, the quark propagator can be written as

$$S(x,y) = \frac{\psi_0(x)\psi_0^\dagger(y)}{im^*}. \tag{18}$$

With such a propagator all quark correlation functions in the instanton background could be evaluated simply by computing all relevant Feynman diagrams and then averaging over the instanton collective coordinates [14].

More specifically, in the random instanton liquid model (RILM) one introduces a model instanton density $n(\rho)$,

$$n_I(\rho) := \overline{n_I} d(\rho), \tag{19}$$

where $\overline{n_I} = \overline{n_A} \simeq \frac{1}{2}$ fm^{-4} and $d(\rho)$ represents the instanton size distribution. The latter is schematically taken to be

$$d(\rho) = \delta(\rho - \overline{\rho}) \tag{20}$$

with $\overline{\rho} \simeq 1/3$ fm. This approach has the advantage of being considerably simple and was also proven to be quite phenomenologically successful [7,8]. However, we show below that the effective mass defined in Eq. (18) is a quantity quite different from its naive estimate (1).

In order to clarify the statement, let us first consider the quark condensate

$$\chi_{uu} = \langle 0|\mathrm{Tr}\, \overline{u}(x) u(x)|0\rangle = \langle \mathrm{Tr}\, S(x,x)\rangle, \tag{21}$$

where, in general, the average is done over all possible gauge field configurations. In the SIA the average is easily evaluated

$$\langle 0|\overline{u}(x)u(x)|0\rangle = \int d^4z \int d\rho\, \overline{n} d(\rho) \times \left[\frac{-2\rho^2}{[(z-x)^2+\rho^2]^3 \pi^2 m_{uu}} \right], \tag{22}$$

where, for reasons that will become clear shortly, we have denoted with m_{uu} the quark effective mass and $\overline{n} := \overline{n_I} + \overline{n_A}$. After performing the integrations one finds

$$\chi_{uu} = -\frac{\overline{n}}{m_{uu}}, \tag{23}$$

for any normalized $d(\rho)$.

Now, repeating the same calculation in the full liquid [15] gives

$$\chi_{uu} = \left\langle \mathrm{Tr} \left[\sum_{I,J} \psi_{0I}(x) \left(\frac{1}{T} \right)_{IJ} \psi_{0J}^\dagger(x) \right] \right\rangle, \tag{24}$$

where, again, the average is made over all possible configurations of the ensemble. A comparison between Eqs. (23) and (24) gives

$$m_{uu}:=-\frac{\overline{n}}{\left\langle \mathrm{Tr}\left[\sum_{I,J}\psi_{0I}(x)\left(\frac{1}{T}\right)_{IJ}\psi^{\dagger}_{0J}(x)\right]\right\rangle}. \tag{25}$$

Let us now consider another quark condensate

$$\chi_{uudd}:=\langle 0|\mathrm{Tr}[\overline{u}(x)u(x)]\mathrm{Tr}[\overline{d}(x)d(x)]|0\rangle=\langle[\mathrm{Tr}S(x,x)]^2\rangle. \tag{26}$$

Such a condensate receives a double contribution from zero modes. In the SIA one obtains

$$\chi_{uudd}=\int d\rho d(\rho)\frac{\overline{n}}{5\pi^2\rho^4 m^2_{uudd}}, \tag{27}$$

where we have now denoted with m_{uudd} the quark effective mass.

Comparing, as before, with the result of full liquid calculations leads to

$$m^2_{uudd}=\left(\int d\rho d(\rho)\frac{\overline{n}}{5\pi^2\rho^4}\right)\times\frac{1}{\left\langle\left[\mathrm{Tr}\sum_{I,J}\psi_{0I}(x)\left(\frac{1}{T}\right)_{IJ}\psi^{\dagger}_{0J}(x)\right]^2\right\rangle}. \tag{28}$$

Now, if the effective mass is universal, $(m_{uu})^2=m^2_{uudd}$, it would imply

$$\frac{\left\langle\mathrm{Tr}\left[\sum_{I,J}\psi_{0I}(x)\left(\frac{1}{T}\right)_{IJ}\psi^{\dagger}_{0J}(x)\right]\right\rangle^2}{\left\langle\left[\mathrm{Tr}\sum_{I,J}\psi_{0I}(x)\left(\frac{1}{T}\right)_{IJ}\psi^{\dagger}_{0J}(x)\right]^2\right\rangle}=\frac{5\pi^2\overline{n}}{\int d\rho d(\rho)\frac{1}{\rho^4}}\simeq 5\pi^2\overline{n\rho^4}\sim\frac{5}{8}, \tag{29}$$

where we have used the ansatz (20) [16]. Some comments on Eq. (29) are in order. First of all, in general the quark condensate is rather inhomogeneous, and for a parametrically dilute instanton ensemble this ratio is small. However, with empirical diluteness it happens to be not so small, about 0.6. In principle, by measuring the left-hand side and right-hand side of Eq. (29) on the lattice *separately*, one can estimate the accuracy of the universality of the effective mass.

However, since different configurations and even points have different leading instantons, the corresponding value T_{I*I*} fluctuates, and the average of its different powers in general leads to different effective masses. (This effect should not be confused with the inhomogeneity of the condensates discussed above.) Let us define a parameter R_m, such that $R_m=1$ means universal mass $(m_{uu})^2=m^2_{uudd}$:

TABLE I. Quark condensates evaluated in the full instanton ensemble and from the leading-instanton only.

Condensate	Complete calculation	LI
χ_{uu}	$(-232\pm5\ \mathrm{MeV})^3$	$(-198\pm1\ \mathrm{MeV})^3$
χ_{uudd}	$(310\pm7\ \mathrm{MeV})^6$	$(309\pm3\ \mathrm{MeV})^6$

$$R_m:=\frac{\left\langle\mathrm{Tr}\left[\sum_{I,J}\psi_{0I}(x)\left(\frac{1}{T}\right)_{IJ}\psi^{\dagger}_{0J}(x)\right]\right\rangle^2}{5\pi^2\overline{\rho^4}\,\overline{n}\left\langle\left[\mathrm{Tr}\sum_{I,J}\psi_{0I}(x)\left(\frac{1}{T}\right)_{IJ}\psi^{\dagger}_{0J}(x)\right]^2\right\rangle}. \tag{30}$$

III. NUMERICAL STUDY OF THE SINGLE INSTANTON APPROXIMATION

In general, reliability of the SIA depends on the vacuum diluteness. In this section we want to establish whether the QCD vacuum with realistic density is actually dilute enough for the leading instanton to be dominant, at least for some observables.

For this purpose we have performed a numerical analysis of several correlation functions, measured in the random instanton liquid model. In such an ensemble, the vacuum expectation values are obtained by averaging over configurations of randomly distributed instantons of size $\rho=1/3$ fm. The contribution from the leading instanton is evaluated by retaining only the largest term in Eq. (15) for each configuration.

We begin by considering two quark condensates χ_{uu} and χ_{uudd}, introduced in Eqs. (21) and (26). We will show later that they represent all generic observables which receive contributions from one and two zero-mode propagators, respectively.

In this calculation we average 5000 configurations of 20 instantons in a box of volume $3.4\times1.8^3\ \mathrm{fm}^4$. The results of this simulation are presented in Table I. From these results one can see how the accuracy of the SIA (keeping only the closest instanton) depends on the particular matrix element being evaluated. Naturally, the accuracy increases with the dimension of the operator involved because it diminishes the contribution of distant instantons. Specifically, the SIA for dimension-six local operators which receive contribution from two zero-mode propagators agree with full calculation within a few percent. On the other hand, prediction for operators and/or correlators with only one zero-mode propagators are not really accurate: the error in quark condesate is large ($\gtrsim 35\%$).

Next we consider two-point correlation functions. This allows us to determine the scale at which the closest instanton is no longer dominant. At this purpose we have measured the pion pseudoscalar two-point function

$$P(x):=\langle 0|J_5(x)J_5^{\dagger}(0)|0\rangle, \tag{31}$$

where

$$J_5(x):=\overline{u}(x)\gamma_5 d(x). \tag{32}$$

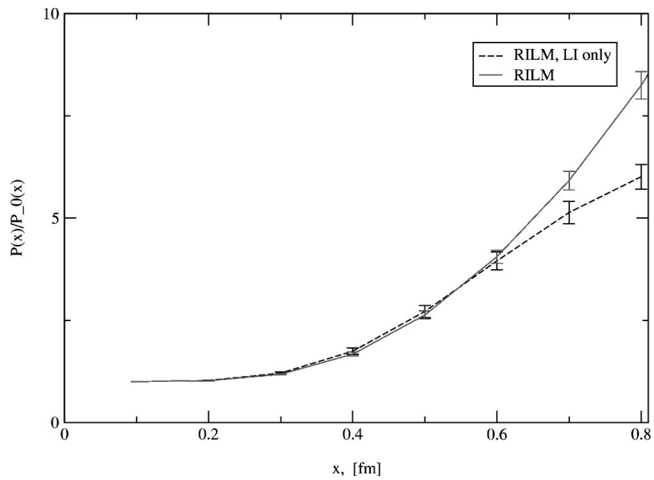

FIG. 1. Pion pseudoscalar correlation function in the RILM, normalized to the same correlation function in the free theory. The solid line corresponds to the full RILM simulation; the dashed line denotes the leading-instanton contribution.

This particular choice is motivated by the fact that such a correlation function is known to receive maximal contribution from quark zero modes [17]. One expects many instantons effects to become important for $|x|$ larger than the instanton size and smaller than the typical distance between two neighbor instantons

$$1/3\ \ \mathrm{fm} \lesssim |x| \lesssim 1\ \ \mathrm{fm}. \tag{33}$$

Results of simulations, including the contribution from all instantons and from the leading-instanton only, are reported in Fig. 1. One can see that the agreement is lost for rather large values of $|x|$ ($|x| \gtrsim 0.6$ fm).

IV. NUMERICAL STUDY OF THE QUARK EFFECTIVE MASS PARAMETERS

In Sec. II we argued that the universality of the effective mass, which collectively describes the effects of all nonleading instantons, can be set in relation to the fluctuations of the quark condensates through Eq. (29). Obviously, the accuracy of calculations in the SIA depends on the value of R_m [defined in Eq. (30)] in realistic ensembles.

We have have evaluated R_m and the corresponding effective masses, numerically [18] in the random instanton liquid and in the interacting liquid (for a review of these ensembles see [3]). Our results are summarized in Table II.

These results show that, in the instanton vacuum with realistic density, the *universality does not hold*

$$m_{uu}^2 \neq m_{uudd}^2 . \tag{34}$$

This implies that an effective mass extracted from the quark condesate cannot be used in calculations involving more than one zero-mode propagator.

On the other hand, the results of numerical simulations presented in Sec. III have shown that matrix elements involving only one zero-mode propagator (like the quark condensate) cannot be reliably evaluated in the SIA simply because the leading instanton is not dominant. As a consequence, one is forced to consider only correlation functions involving at least two such propagators and therefore m_{uu} is of no practical usefulness.

In more general terms, one may address the question whether the effective mass parameter depends on the particular correlation function being evaluated. If so, this feature would spoil much of the predictive power of the SIA. In such a pessimistic scenario the SIA would only allow us to work out the functional expressions of small-sized correlations, but not their overall normalization. However, we will show that the effective mass parameters depend essentially on the number of zero-mode propagators involved, and that m_{uudd}^2 is in a way universal for a number of applications. In this case, SIA is predictive including the normalization. To check that we have extracted m_2^2 from the analysis of several hadronic two-point functions evaluated in SIA and in the liquid. In particular, we considered the pion pseudoscalar the scalar diquark and the a nucleon scalar correlation functions:

$$P(x) = \langle 0 | J_5(x) J_5^\dagger(0) | 0 \rangle, \tag{35}$$

$$D(x) = \langle 0 | J_{C5}^a(x) J_{C5}^{a\dagger}(0) | 0 \rangle, \tag{36}$$

$$N(x) = \langle 0 | \mathrm{Tr}[\eta(x) \overline{\eta}(0) \gamma_4] | 0 \rangle, \tag{37}$$

where

$$J_5(x) := \overline{u}(x) \gamma_5 d(x), \tag{38}$$

$$J_{C5}^a(x) := \epsilon^{abc} u_b(x) C \gamma_5 d_c(x), \tag{39}$$

$$\eta_\alpha(x) := \epsilon^{abc} [u^a(x) C \gamma_5 u^b(x)] u_\alpha^c(x). \tag{40}$$

All these correlations function are known to receive a contribution from two propagators in the zero mode.

The comparison between results obtained in the SIA, in the random instanton liquid model (RILM), and in the interacting instanton liquid model (IILM) are reported in Figs. 2, 3, and 4. The corresponding values for $\sqrt{m_2^2}$ are presented in Table III. These values are indeed rather different from the traditionally adopted estimate $m^* = 170$ MeV, extracted from the quark condensate.

TABLE II. Universality parameter R_m and the effective masses evaluated in the RILM and in the IILM.

Quantity	RILM calculation	IILM calculation
R_m	0.4	0.2
m_{uu}	120 MeV	177 MeV
$\sqrt{m_{uudd}^2}$	65 MeV	91 MeV

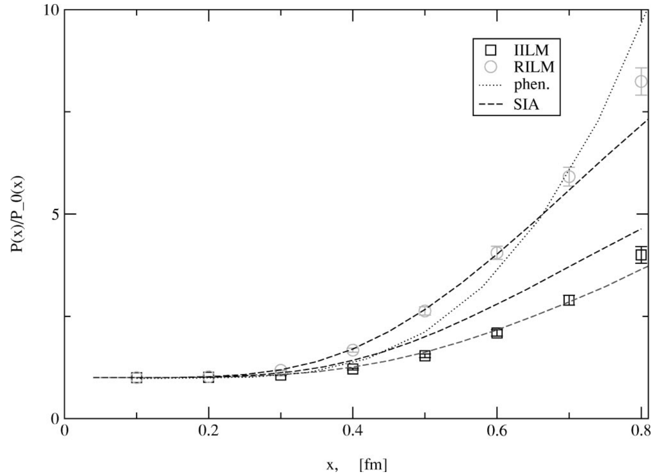

FIG. 2. Pion pseudoscalar two-point function normalized to the same correlation function in the free theory. The open circles (squares) represent RILM (IILM) points, the dashed lines represent SIA calculations with masses given in Table III and the dotted line is the phenomenological curve obtained from the spectral decomposition.

The general reason why these masses are rather small is the following. Instantons have fluctuating strength of interaction with others in the ensemble: some of them are "hermits" and have small matrix elements in the corresponding entries of the overlap matrix *T*. In all expressions, we average the *inverse* of this matrix, therefore the contribution of such hermits is enhanced. This phenomenon is effectively described by a lower value of the effective mass. Furthermore, because the random ensemble of RILM has more such hermits, as compared to IILM (where the fermionic determinant in the statistical weight suppresses them), these masses are smaller in RILM as compared to IILM. Such a discrepancy reflects the fact that the two ensembles give actually quite different correlation functions [3].

From these results we conclude that m_2^2 seems to be a universal parameter, describing the collective many-

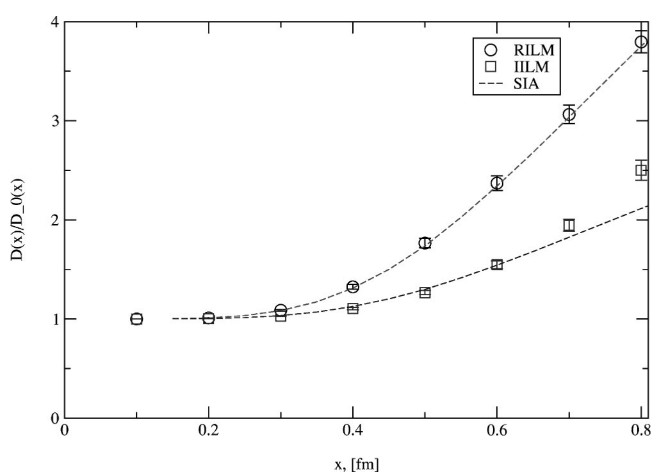

FIG. 3. Diquark scalar two-point function normalized to the same correlation function in the free theory. The open circles (squares) represent RILM (IILM) points and the dashed lines represent SIA calculations with the effective masses given in Table III.

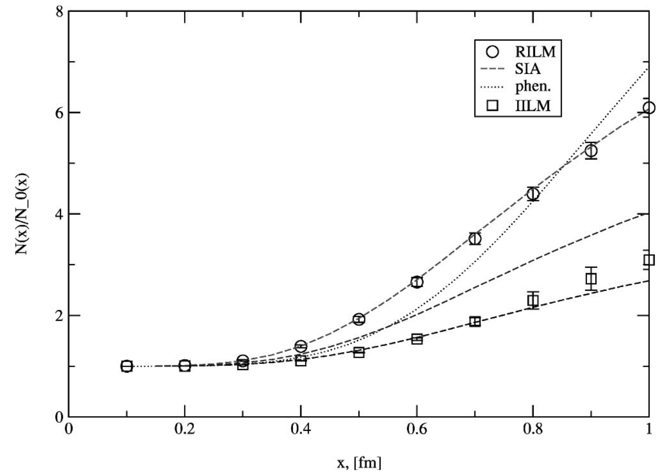

FIG. 4. Nucleon scalar two-point function normalized to the same correlation function in the free theory. The open circles (squares) represent RILM (IILM) points and the dashed lines represent SIA calculations with the effective masses given in Table III.

instanton effects. It is important to know what value of m_2^2 is suggested by the available phenomenology. As before, we chose to consider the pion pseudoscalar correlator, because it receives maximal contribution from instanton zero modes. The traditional "pole-plus-continuum" model for the spectral decomposition of $P(x)$, gives [19,17]

$$P(x)=\lambda_\pi^2 D(m_\pi;x)+\frac{3}{8\pi^2}\int_{s_0}^{\infty} ds s D(\sqrt{s};x), \qquad (41)$$

where $D(m;x)$ is the scalar propagator, s_0 is the threshold for the continuum ($\sqrt{s_0}\simeq 1.6$ GeV) and the pseudoscalar decay constant λ_π is given by

$$\lambda_\pi=\langle 0|\overline{u}\gamma_5 d|\pi\rangle=\frac{f_\pi m_\pi^2}{m_u+m_d}\simeq(480\ \ \mathrm{MeV})^2. \qquad (42)$$

We determined m_2^2 by fitting the SIA prediction to the phenomenological curve obtained from Eq. (41). We found (see Fig. 2)

$$m_{2phen.}^2=(86\ \ \mathrm{MeV})^2. \qquad (43)$$

To further check the approach, we have evaluated the scalar proton two-point function $N(x)$, using the value (43) and we have compared with the phenomenological curve (see Fig. 4 [20]). In summary: with this value we obtained very good agreement with phenomenology and therefore we suggest

TABLE III. Estimates of the quark effective mass m_2^2 from several correlation functions.

Correlation function	m_2^2 [MeV^2] (RILM)	m_2^2 [MeV^2] (IILM)
χ_{uudd} condensate	$(65)^2$	$(91)^2$
Pion pseudoscalar	$(65)^2$	$(105)^2$
Diquark scalar	$(69)^2$	$(105)^2$
Nucleon scalar	$(67)^2$	$(105)^2$

that Eq. (43) should be used for the applications of the SIA, when two zero-mode propagators are involved.

V. EVALUATION OF AN EFFECTIVE MASS IN THE FERMIONIC DETERMINANT

The propagator is not the only place where the Dirac operator appears: the QCD statistical sum contains its *determinant*, appearing in power given by the number of light quark flavors N_f. If one considers the academic vacuum with only one instanton, this determinat contains the product of "current" quark masses for all quarks [5]. If this would be the final answer for the instanton density, the instanton effect would be strongly suppressed by their small values.

However, in a physical vacuum there are sufficiently many instantons to break chiral symmetry and produce nonzero quark condesates and effective quark masses, which substitute for much smaller "current" masses and make instanton effects significantly stronger. The interplay between these effective masses and current quark masses is especially interesting for strange quark, since the former and the latter m_s are of comparable magnitude. This issue has been studied, e.g., in a recent paper [21], where it was concluded that the usual additive formula for the total effective quark mass of the strange quark $M_s^{tot}=m_{eff}(m_s=0)+m_s$ is wrong, and the true value of M_s^{tot} is not very different from that for u,d quarks because $m_{eff}(m_s)$ strongly decreases with m_s.

Apart from the role of the strange quark mass in general, there is also a general issue of correct connection of units and vacuum parameters (with the instanton density being one of them) for QCD with a different number of flavors (for example, between no-quark or quenched QCD and the physical world). In order to study all of this, it is important to know what is the absolute magnitude of the fermionic determinants in the instanton-based vacuum models considered. Some of those are reported in this section.

In the instanton-based model context, the fermionic determinant is usually represented by the determinant of the overlap matrix T (see description, e.g., in [3]) in the zero mode subspace. After averaging over the appropriate ensemble, one can define the so-called "determinantal masses":

$$m^i_{det}:=\frac{\langle(\det[\not{D}])^{i/N}\rho^i\rangle}{\langle\rho^i\rangle},\quad i=1,2,\ldots \tag{44}$$

where the index i refers to the number of flavors and N denotes the number of instantons. Their values tell us how much the presence of fermions reduces the instanton density, compared to the same ensemble without them.

Originally, in [6,22] an estimate for the determinant effective mass was extracted from the averaging of the 't Hooft Lagrangian assuming factorization of quark condesates, and using the same $m^*=170$ MeV. If so, each flavor reduces instanton density by the factor $m^*\bar{\rho}\approx 0.28$. As we will see shortly, the corresponding reduction factor is actually even smaller. In principle, there is no reason why the values of m_{det} and m^2_{det} should agree with m_1 and m_2^2, defined in the previous section: we now average the positive rather than negative powers of the overlap matrix.

TABLE IV. Determinantal masses evaluated in the RILM and in the IILM as compared to m_1 and m_2^2, defined in Sec. IV.

Mass	RILM calculation	IILM calculation
m_1	120 MeV	177 MeV
m_{det}	63 MeV	102 MeV
m_2^2	(65 MeV)^2	(103 MeV)^2
m^2_{det}	(64 MeV)^2	(103 MeV)^2

We have evaluated the determinantal masses in the RILM and in the IILM. The results are reported in Table IV. Some comments are in order. First of all note that, in both ensembles, the values of m^2_{det} turn out to be quite consistent with the values of m_2^2. Furthermore, the fluctuations of the determinantal mass m_{det} and $(m_{det})^2$ are very small

$$m^2_{det}-(m_{det})^2\ll m_2^2-(m_1)^2, \tag{45}$$

implying essentially that m_1 is inconsistent with m_{det}. This fact could have two possible explanations. On the one hand, one could argue that m_1 is a somewhat ill-defined parameter because the SIA cannot be used to evaluate quark condensate. On the other hand, one could observe that larger fluctuations for the effective masses defined in Sec. IV should not be surprising, since such parameters appear always in denominators of SIA calculations.

VI. CONCLUSIONS AND OUTLOOK

Summarizing our study of the SIA approximation in QCD, we first notice that this approach has been related to the theory of the full ensemble and all the effective parameters previously loosely called "effective masses" are defined. All of them describe different aspects of collective interaction between the "leading" instanton (the closest to the observation points) and all others, and related to the overlap matrix T. Different effective mass values simply follow from different ensemble averaging. In particular, the factor $1/m_1$, appearing in SIA calculations with one propagator in the zero mode, does not correspond to the square root of the factor $1/m_2^2$, appearing when two such propagators are involved.

We have made numerical simulations in the RILM and IILM and found that the contribution of the leading-instanton actually dominates all condensates of operators of dimension six or more, as well as short-distance correlation functions ($|x|\lesssim 0.6$ fm). This, however, is true only for correlation functions with at least two zero-mode propagators involved. Earlier estimates extracted from the quark condensate are not accurate.

Furthermore, the parameter $1/m_2^2$ is approximatively universal for several correlation functions with *two* zero-mode propagators involved. We have also extracted a phenomenological estimate of its value from the analysis of the pion pseudoscalar correlator. We found $m^2_{2phen.}\simeq(86\text{ MeV})^2$, much smaller than the value originally obtained from the quark condensate. Our new value should be used in many applications of the SIA.

Finally, we have compared our estimates for the effective mass parameters m_1 and m_2^2, with the measurements of the "determinantal" masses, introduced in [22]. We observed substantial agreement between m_2^2 and m_{det}^2 both in the RILM and the IILM, but different from m_1 extracted from the quark condesate alone. This implies that light quarks are about twice more effective (per flavor) in diluting the instanton vacuum density [23].

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